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Scattering profile for global solutions of the energy-critical wave equation

Received January 13, 2016

Abstract. Consider the focusing energy-critical wave equation in space dimension 3, 4 or 5. We prove that any global solution which is bounded in the energy space converges in the exterior of wave cones to a radiation term which is a solution of the linear wave equation.

Keywords. Wave equation, critical nonlinearity, global solution, radiation term

1. Introduction

In this note we consider the energy-critical nonlinear wave equation on $\mathbb{R}^N$, $N \in \{3, 4, 5\}$:

$$\begin{cases} \partial_t^2 u - \Delta u - |u|^{4/(N-2)} u = 0, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \dot{H}^1 \times L^2, \end{cases} \quad (1.1)$$

where $\dot{H}^1 = \dot{H}^1(\mathbb{R}^N)$, $L^2 = L^2(\mathbb{R}^N)$, and u is real-valued. If f is a function of space and time, we will denote $\vec{f} = (f, \partial_t f)$. It is known that the equation (1.1) is locally well posed in $\dot{H}^1 \times L^2$ and that the energy

$$E(\vec{u}(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx + \frac{1}{2} \int |\partial_t u(t, x)|^2 dx - \frac{N-2}{2N} \int |u(t, x)|^{\frac{2N}{N-2}} dx$$

is conserved.

All solutions of the defocusing equation ((1.1) with a + instead of a − sign in front of the nonlinearity) are global and scatter to a solution of the linear equation (see e.g. [26, 16, 14, 17, 24, 19, 25, 3, 23]). The dynamics of the focusing equation (1.1) are richer: small data solutions are global and scatter, but blow-up in finite time may occur [22]. Global,

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Mathematics Subject Classification (2010): Primary 35L71; Secondary 35B33, 35B40

nonscattering solutions also exist. Examples of such solutions are given by solutions of the elliptic equation

$$-\Delta Q = |Q|^{4/(N-2)} Q, \quad Q \in \dot{H}^1, \quad (1.2)$$

(see [6] for the existence of such solutions) and their Lorentz transforms

$$Q_\ell(t, x) = Q\left(\left(-\frac{t}{\sqrt{1-|\ell|^2}} + \frac{1}{|\ell|^2}\left(\frac{1}{\sqrt{1-|\ell|^2}} - 1\right)\ell \cdot x\right)\ell + x\right) = Q_\ell(0, x - t\ell) \quad (1.3)$$

where $\ell \in \mathbb{R}^N$, $|\ell| < 1$. Note that

$$Q_\ell(t, x) = Q_\ell(0, x - t\ell),$$

so that Q_ℓ is a solitary wave travelling at speed $|\ell|$. The energy of Q_ℓ is given by

$$E(\vec{Q}_\ell(0)) = \frac{1}{\sqrt{1-|\ell|^2}} E(\vec{Q}(0)) \xrightarrow{|\ell| \rightarrow 1} +\infty. \quad (1.4)$$

It is conjectured that any bounded, global solution of (1.1) is a sum of modulated, decoupled travelling waves and a scattering part. More precisely:

Conjecture 1 (Soliton resolution). *Let u be a solution of (1.1) on $[0, +\infty) \times \mathbb{R}^N$ such that*

$$\sup_{t \geq 0} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2} < \infty. \quad (1.5)$$

Then there exist a solution v_L of the linear wave equation

$$\begin{cases} (\partial_t^2 - \Delta)v_L = 0, \\ \vec{v}_L|_{t=0} = (v_0, v_1) \in \dot{H}^1 \times L^2, \end{cases} \quad (1.6)$$

an integer $J \geq 0$, and for $j \in \{1, \dots, J\}$, a (nonzero) travelling wave $Q_{\ell_j}^j$ (where $|\ell_j| < 1$ and Q^j is a solution of (1.2)), and parameters $x_j(t) \in \mathbb{R}^N$, $\lambda_j(t) \in \mathbb{R}$ such that

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \left[\vec{u}(t) - \vec{v}_L(t) \right. \\ & \left. - \sum_{j=1}^J \left(\frac{1}{\lambda_j(t)^{(N-2)/2}} Q_{\ell_j}^j \left(0, \frac{\cdot - x_j(t)}{\lambda_j(t)} \right), \frac{1}{\lambda_j(t)^{N/2}} \partial_t Q_{\ell_j}^j \left(0, \frac{\cdot - x_j(t)}{\lambda_j(t)} \right) \right) \right] = 0 \end{aligned} \quad (1.7)$$

in $\dot{H}^1 \times L^2$ and

$$\begin{aligned} & \forall j \in \{1, \dots, J\}, \quad \lim_{t \rightarrow +\infty} \frac{x_j(t)}{t} = \ell_j, \quad \lim_{t \rightarrow +\infty} \frac{\lambda_j(t)}{t} = 0, \\ & \forall j, k \in \{1, \dots, J\}, \quad j \neq k \implies \lim_{t \rightarrow +\infty} \left(\frac{|x_j(t) - x_k(t)|}{\lambda_j(t)} + \frac{\lambda_j(t)}{\lambda_k(t)} + \frac{\lambda_k(t)}{\lambda_j(t)} \right) = +\infty. \end{aligned}$$

This conjecture was proved by the authors in [10] for radial solutions in space dimension $N = 3$. In this case $x_j(t) \equiv 0$ and the only stationary solutions in the expansion (1.7) are W and $-W$, where

$$W(x) = \frac{1}{(1 + |x|^2/3)^{1/2}}.$$

In this article we extract the linear profile $v_L(t)$ which appears in the expansion (1.7). More precisely, we prove:

Theorem 2. *Let u be a solution of (1.1) on $[0, +\infty) \times \mathbb{R}^N$ that satisfies (1.5). Then there exists a solution v_L of the linear wave equation (1.6) such that, for all $A \in \mathbb{R}$,*

$$\lim_{t \rightarrow +\infty} \int_{|x| \geq t+A} \left(|\nabla(u - v_L)(t, x)|^2 + |\partial_t(u - v_L)(t, x)|^2 + \frac{1}{|x|^2} |u(t, x)|^2 + |u(t, x)|^{\frac{2N}{N-2}} \right) dx = 0. \quad (1.8)$$

We see Theorem 2 as a first step toward the proof of Conjecture 1. It is used in an article by Hao Jia and the authors to prove a weak form of the conjecture, that the expansion (1.7) holds for a sequence of times $t_n \rightarrow +\infty$ (see [7]). Note that Theorem 2 implies that the following limit exists:

$$\lim_{A \rightarrow -\infty} \lim_{t \rightarrow +\infty} \int_{|x| \geq t+A} \left(\frac{1}{2} |\nabla u(t, x)|^2 + \frac{1}{2} (\partial_t u(t, x))^2 - \frac{N-2}{2N} |u(t, x)|^{\frac{2N}{N-2}} \right) dx = E_L(v_0, v_1),$$

where

$$E_L(\vec{v}_L(t)) = \frac{1}{2} \int [|\nabla v_L(t, x)|^2 + (\partial_t v_L(t, x))^2] dx$$

is the conserved energy for the linear wave equation. One can also prove (using for example the profile decomposition of [2] recalled in §2.4), as a consequence of Theorem 2,

$$\vec{S}_L(-t)\vec{u}(t) \xrightarrow[t \rightarrow +\infty]{} (v_0, v_1), \quad (1.9)$$

where $\vec{S}_L$ denotes the linear evolution (see §2.1 below).

We next mention a few related works. Theorem 2 is proved in [8, Subsection 3.3] in the radial case in dimension 3 (where it is significantly simpler). A very close proof yields Theorem 2 for radial solutions of (1.1) in higher dimensions (see e.g. [5]) and of the defocusing analogue of (1.1) with an additional linear potential [18]. We refer to [5] for an analogue of Theorem 2 for equivariant wave maps. The spherically symmetric setting is much easier since any solution can only propagate in two directions, outward ($x/|x|$) or inward ($-x/|x|$), a fact that is crucially used in the previous proofs. Related results for energy-subcritical, mass-supercritical Schrödinger equations were proved by T. Tao [27, 28, 29].

Let us give a short outline of the paper. We first prove (Section 3), as a consequence of small data theory and finite speed of propagation, that (1.8) holds for large positive A . We

then argue by contradiction, assuming that (1.8) does not hold for all $A \in \mathbb{R}$, and defining $\bar{A}$ as the largest real number such that (1.8) does not hold. We divide (see Section 4) the elements of the sphere S^{N-1} between *regular directions* (in an angular neighbourhood of which (1.8) holds locally, for some $A < \bar{A}$), and *singular directions* (other elements of S^{N-1}), and prove, using geometrical considerations and again small data theory and finite speed of propagation, that the set of singular directions is finite. To conclude the proof, we show in Sections 5–7 that the set of singular directions is empty, which will contradict the definition of $\bar{A}$. More precisely, in Section 5, we prove that there exists a nonzero profile which concentrates close to the wave cone $\{t = |x| + \bar{A}\}$, along an arbitrary singular direction. The core of the proof is in Section 6, where we prove, using a local version of virial-type identities, that there are no nonlinear profiles remaining close to the wave cone $\{t = |x| + \bar{A}\}$. This is coherent with the intuition that nonlinear objects with finite energy travel at a speed strictly slower than 1, as the travelling waves (1.3) and their energies (1.4) suggest. Finally, Section 7 collects the results of the preceding two sections to prove by contradiction the emptiness of the set of singular directions. Let us mention that we never need to know, in all this proof, what happens inside the wave cone (that is, for (t, x) such that $|t| - |x| \gg 1$). This is of course made possible by finite speed of propagation.

In Section 2, we give some preliminary results on linear and nonlinear wave equations. We introduce in particular an isometry between the initial data and the asymptotic profile of a solution of the linear wave equation that is known (see e.g. the work of Friedlander [13]) but seems to have been somehow forgotten. We construct this isometry (which we use many times in the article) in Appendices A and B for the sake of completeness.

2. Preliminaries and notations

2.1. Linear wave equation

If $(v_0, v_1) \in \dot{H}^1 \times L^2$, we let

$$S_L(t)(v_0, v_1) := \cos(t\sqrt{-\Delta})v_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}v_1,$$

let v_L be the solution of (1.6), and

$$\vec{S}_L(t)(v_0, v_1) := \vec{v}_L(t) = \left(S_L(t)(v_0, v_1), \frac{\partial}{\partial t}(S_L(t)(v_0, v_1)) \right).$$

The linear energy

$$E_L(\vec{v}_L(t)) = \frac{1}{2} \int |\nabla v_L(t, x)|^2 dx + \frac{1}{2} \int (\partial_t v_L(t, x))^2 dx$$

is conserved. We will often use radial coordinates, denoting, for $x \in \mathbb{R}^N \setminus \{0\}$, $r = |x| > 0$ and $\omega = x/|x| \in S^{N-1}$. We also denote

$$\partial_r = \frac{\partial}{\partial r} = \frac{x}{|x|} \cdot \nabla, \quad \frac{1}{r} \nabla_\omega = \nabla - \frac{x}{|x|} \partial_r.$$

We will often use the following asymptotic property.

Theorem 2.1. Assume $N \geq 3$ and let v_L be a solution of the linear wave equation (1.6). Then

$$\lim_{t \rightarrow +\infty} \left(\left\| \frac{1}{r} \nabla_\omega v_L(t) \right\|_{L^2} + \left\| \frac{1}{r} v_L(t) \right\|_{L^2} \right) = 0 \quad (2.1)$$

and there exists a unique $G_+ \in L^2(\mathbb{R} \times S^{N-1})$ such that

$$\lim_{t \rightarrow +\infty} \int_0^{+\infty} \int_{S^{N-1}} \left| r^{\frac{N-1}{2}} \partial_t v_L(t, r\omega) - G_+(r-t, \omega) \right|^2 d\omega dr = 0, \quad (2.2)$$

$$\lim_{t \rightarrow +\infty} \int_0^{+\infty} \int_{S^{N-1}} \left| r^{\frac{N-1}{2}} \partial_r v_L(t, r\omega) + G_+(r-t, \omega) \right|^2 d\omega dr = 0. \quad (2.3)$$

Furthermore,

$$E_L(v_0, v_1) = \int_{\mathbb{R} \times S^{N-1}} |G_+(\eta, \omega)|^2 d\eta d\omega = \|G_+\|_{L^2}^2, \quad (2.4)$$

and the map

$$(\dot{H}^1 \times L^2)(\mathbb{R}^N) \rightarrow L^2(\mathbb{R} \times S^{N-1}), \quad (v_0, v_1) \mapsto \sqrt{2} G_+,$$

is a bijective isometry.

Remark 2.2. Applying Theorem 2.1 to the solution $(t, x) \mapsto v_L(-t, x)$ of (1.6), we find that

$$\lim_{t \rightarrow -\infty} \left(\left\| \frac{1}{r} \nabla_\omega v_L(t) \right\|_{L^2} + \left\| \frac{1}{r} v_L(t) \right\|_{L^2} \right) = 0,$$

there exists $G_- \in L^2(\mathbb{R} \times S^{N-1})$ such that

$$\lim_{t \rightarrow -\infty} \int_0^{+\infty} \int_{S^{N-1}} \left| r^{\frac{N-1}{2}} \partial_t v_L(t, r\omega) - G_-(t+r, \omega) \right|^2 d\omega dr = 0,$$

$$\lim_{t \rightarrow -\infty} \int_0^{+\infty} \int_{S^{N-1}} \left| r^{\frac{N-1}{2}} \partial_r v_L(t, r\omega) - G_-(t+r, \omega) \right|^2 d\omega dr = 0,$$

and the map $(v_0, v_1) \mapsto \sqrt{2} G_-$ is a bijective isometry from $(\dot{H}^1 \times L^2)(\mathbb{R}^N)$ to $L^2(\mathbb{R} \times S^{N-1})$.

We will call G_+ (respectively G_-) the *outgoing radiation field* (respectively the *incoming radiation field*) associated to v_L .

Theorem 2.1 is known (see in particular the works of Friedlander [12, 13]). We give a proof in Appendices A and B for the sake of completeness. Let us mention that the following identity for the conserved momentum is also available (but will not be used in this article):

$$\int \nabla v_0 v_1 = - \int_{\mathbb{R} \times S^{N-1}} \omega |G_+(\eta, \omega)|^2 d\eta d\omega.$$

2.2. Strichartz estimates

If Ω is a measurable subset of $\mathbb{R}_t \times \mathbb{R}_x^N$, of the form $\Omega = \bigcup_{t \in \mathbb{R}} \{t\} \times \Omega_t$, $\Omega_t \subset \mathbb{R}^N$ measurable, and u a measurable function defined on Ω , we denote

$$\|u\|_{S(\Omega)} := \left(\int_{-\infty}^{+\infty} \left(\int_{\Omega_t} |u(t, x)|^{\frac{2(N+2)}{N-2}} dx \right)^{\frac{1}{2}} dt \right)^{\frac{N-2}{N+2}}. \quad (2.5)$$

If $I \subset \mathbb{R}$ is measurable, we will abuse notation, writing $S(I)$ for $S(I \times \mathbb{R}^N)$.

The spaces $S(I)$ appear in the following Strichartz estimate (see [15]): Let $f \in L^1(\mathbb{R}, L^2(\mathbb{R}^N))$ and $(u_0, u_1) \in (\dot{H}^1 \times L^2)(\mathbb{R}^N)$, and let

$$u(t) = S_L(t)(u_0, u_1) + \int_0^t S_L(t-s)(0, f(s)) ds$$

be the solution of

$$(\partial_t^2 - \Delta)u = f, \quad \vec{u}|_{t=0} = (u_0, u_1) \in \dot{H}^1 \times L^2. \quad (2.6)$$

Then u is well defined, $\vec{u} \in C^0(\mathbb{R}, \dot{H}^1 \times L^2)$, $u \in S(\mathbb{R})$ and

$$\|u\|_{S(\mathbb{R})} + \sup_{t \in \mathbb{R}} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2} \leq C(\|f\|_{L^1(\mathbb{R}, L^2)} + \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}).$$

We will occasionally use other Strichartz estimates: in the preceding inequality, one can replace $S(\mathbb{R})$ by $L^{\frac{2(N+1)}{N-2}}(\mathbb{R}^{N+1})$, and also by $L^4(\mathbb{R}, L^{12})$ (if $N = 3$) or $L^2(\mathbb{R}, L^{\frac{2N}{N-3}})$ (if $N \geq 4$).

2.3. Miscellaneous properties of the critical nonlinear wave equation

We recall the Cauchy theory for equation (1.1):

Theorem 3 (see [20]). (a) (Small data theory) *There exists $\delta_0 > 0$ such that if I is an interval containing 0 and $(u_0, u_1) \in \dot{H}^1 \times L^2$ is such that $\|S_L(t)(u_0, u_1)\|_{S(I)} < \delta_0$, then there exists a unique solution $\vec{u} \in C^0(I, \dot{H}^1 \times L^2)$ of (1.1). Furthermore*

$$\sup_{t \in I} \|\vec{u}(t) - \vec{S}_L(t)(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|u - S_L(\cdot)(u_0, u_1)\|_{S(I)} \leq C \|S_L(\cdot)(u_0, u_1)\|_{S(I)}^{\frac{N+2}{N-2}}.$$

(b) *If $(u_0, u_1) \in \dot{H}^1 \times L^2$, there exists a unique maximal solution u of (1.1). Letting (T_-, T_+) be the maximal interval of existence of u , we have the following blow-up criterion:*

$$T_+ < \infty \implies \|u\|_{S((0, T_+))} = +\infty.$$

(c) *If u is a solution of (1.1) such that $u \in S((0, T_+))$, then $T_+ = +\infty$ and u scatters for positive times: there exists a solution u_L of (1.6) such that*

$$\lim_{t \rightarrow +\infty} \|\vec{u}(t) - \vec{u}_L(t)\|_{\dot{H}^1 \times L^2} = 0. \quad (2.7)$$

In the theorem, a *solution* of (1.1) on I is by definition a solution in the Duhamel sense which is in $S(J)$ for all $J \Subset I$. Point (c) can be seen as a consequence of the following result on the nonhomogeneous linear wave equation, which we will use repeatedly:

Claim 2.3. *Let $f \in L^1(\mathbb{R}, L^2(\mathbb{R}^N))$ and u the solution of (2.6) (in the Duhamel sense). Then there exists a solution v_L of (1.6) such that*

$$\lim_{t \rightarrow +\infty} \|\vec{u}(t) - \vec{v}_L(t)\|_{\dot{H}^1 \times L^2} = 0. \quad (2.8)$$

Proof. The existence of v_L follows from the formula

$$\vec{S}_L(-t)\vec{u}(t) = (u_0, u_1) + \int_0^t \vec{S}_L(-s)(0, f(s)) ds$$

and energy estimates. $\square$

We recall the *finite speed of propagation* property: if $R > 0$, $x_0 \in \mathbb{R}^N$ and u is a solution of (2.6) such that $(u_0, u_1)(x) = 0$ for $|x - x_0| \leq R$, and $f(t, x) = 0$ for $|x - x_0| \leq R - t$, $t \in [0, R]$, then $u(t, x) = 0$ for $|x - x_0| \leq R - t$, $t \in [0, R]$. As a consequence, if the initial data of two solutions of (1.1) coincide for $|x - x_0| \leq R$, then the two solutions coincide for $t \in [0, R]$, $|x - x_0| < R - t$, if t is in the domains of existence of both solutions. A consequence of finite speed of propagation and small data theory is the following claim:

Claim 2.4. *There exists $\delta_1 > 0$ with the following property. Let u be a solution of (1.1) such that $T_+(u) = +\infty$. Let $T > T_-(u)$ and $A > -T$.*

- (a) *If $\|u\|_{S(\{|x| \geq A+t, t \geq T\})} = \delta < \delta_1$ then $\|S_L(\cdot - T)\vec{u}(T)\|_{S(\{|x| \geq A+t, t \geq T\})} \leq 2\delta$.*
- (b) *If $\|S_L(\cdot - T)\vec{u}(T)\|_{S(\{|x| \geq A+t, t \geq T\})} = \delta' < \delta_1$ then $\|u\|_{S(\{|x| \geq A+t, t \geq T\})} \leq 2\delta'$.*

Proof. Let $\tilde{u}$ be the solution of

$$\begin{cases} \partial_t^2 \tilde{u} - \Delta \tilde{u} = |u|^{\frac{4}{N-2}} u \mathbb{1}_{\{|x| \geq A+t\}}, \\ (\tilde{u}, \partial_t \tilde{u})|_{t=T} = \vec{u}(T). \end{cases}$$

By Strichartz estimates, for all $T_1 > T$,

$$\begin{aligned} & \left| \|\tilde{u}\|_{S(\{|x| \geq A+t, T \leq t \leq T_1\})} - \|S_L(\cdot - T)\vec{u}(T)\|_{S(\{|x| \geq A+t, T \leq t \leq T_1\})} \right| \\ & \leq C \|u\|_{S(\{|x| \geq A+t, T \leq t \leq T_1\})}^{\frac{N+2}{N-2}}. \end{aligned}$$

By finite speed of propagation, $u(t, x) = \tilde{u}(t, x)$ if $t \geq T$, $|x| > A + t$, and thus

$$\begin{aligned} & \left| \|u\|_{S(\{|x| \geq A+t, T \leq t \leq T_1\})} - \|S_L(\cdot - T)\vec{u}(T)\|_{S(\{|x| \geq A+t, T \leq t \leq T_1\})} \right| \\ & \leq C \|u\|_{S(\{|x| \geq A+t, T \leq t \leq T_1\})}^{\frac{N+2}{N-2}}. \end{aligned}$$

For δ_1 small, point (a) follows immediately. An easy bootstrap argument yields (b). $\square$

2.4. Profile decomposition

We finally recall that any sequence $\{(u_{0,n}, u_{1,n})\}_n$ bounded in $\dot{H}^1 \times L^2$ has a subsequence (still denoted by $\{(u_{0,n}, u_{1,n})\}_n$) that admits a *profile decomposition* $(U_L^j, \{t_{j,n}, x_{j,n}, \lambda_{j,n}\}_{j \geq 1})$, where for all j , U_L^j is a solution of the linear wave equation (1.6) and for all j, n , $\lambda_{j,n} > 0$, $x_{j,n} \in \mathbb{R}^N$ and $t_{j,n} \in \mathbb{R}$, have the following properties:

$$j \neq k \implies \lim_{n \rightarrow \infty} \left(\frac{\lambda_{j,n}}{\lambda_{k,n}} + \frac{|x_{j,n} - x_{k,n}|}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{k,n}|}{\lambda_{j,n}} \right) = +\infty$$

(pseudo-orthogonality) and

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \|w_n^J\|_{S(\mathbb{R})} = 0,$$

where

$$w_n^J(t) = S_L(t)(u_{0,n}, u_{1,n}) - \sum_{j=1}^J U_{L,n}^j(t), \quad (2.9)$$

and

$$U_{L,n}^j(t, x) = \frac{1}{\lambda_{j,n}^{(N-2)/2}} U^j\left(\frac{t - t_{j,n}}{\lambda_{j,n}}, \frac{x - x_{j,n}}{\lambda_{j,n}}\right).$$

The existence of the profile decomposition was established in [2] for $N = 3$ (see [4] for higher dimensions). We refer to [2] for the properties of this profile decomposition (see also [11, Section 3] for a review).

3. Scattering to a linear solution outside a large wave cone

We let $\tau_n \rightarrow +\infty$ and (after extraction),

$$(v_0, v_1) = \text{wlim}_{n \rightarrow \infty} \vec{S}_L(-\tau_n) \vec{u}(\tau_n) \quad \text{in } \dot{H}^1 \times L^2, \quad (3.1)$$

$$v_L(t) = S_L(t)(v_0, v_1), \quad (3.2)$$

where wlim stands for weak limit.

Proposition 3.1. *Let u be as in Theorem 2, and assume that the conclusion of Theorem 2 does not hold. Then there exists $\bar{A} \in \mathbb{R}$ with the following properties, for all $A > \bar{A}$:*

$$\|u\|_{S(\{t>0, |x|>t+A\})} < \infty, \quad (3.3)$$

$$\lim_{t \rightarrow \infty} \left(\| |x|^{-1} u(t) \|_{L^2(\{|x|>t+A\})} + \|u(t)\|_{L^{\frac{2N}{N-2}}(\{|x|>t+A\})} \right) = 0, \quad (3.4)$$

$$\lim_{t \rightarrow +\infty} \|\nabla_{t,x}(u - v_L)(t)\|_{L^2(\{|x|>t+A\})} = 0, \quad (3.5)$$

$$\|u\|_{S(\{t>0, |x|>t+\bar{A}\})} = \infty. \quad (3.6)$$

Proof. Step 1. We note that $\|u\|_{S(\{|x|>|t|+A\})}$ is finite for large $A > 0$. Indeed, since

$$\|S_L(\cdot)(u_0, u_1)\|_{S(\mathbb{R})} < \infty,$$

we have, for large A ,

$$\|S_L(\cdot)(u_0, u_1)\|_{S(\{|x|\geq|t|+A\})} < \delta_1,$$

where δ_1 is given by Claim 2.4. The conclusion follows from Claim 2.4.

Step 2. We let $\bar{A} \in \mathbb{R} \cup \{-\infty\}$ be defined by

$$\bar{A} := \inf \{A \in \mathbb{R} : \|u\|_{S(\{t>0, |x|>t+A\})} < \infty\}. \quad (3.7)$$

In particular (3.3) holds. We prove that (3.4) and (3.5) holds. Let $A > \bar{A}$. Let v be the solution of

$$\begin{cases} \partial_t^2 v - \Delta v = |u|^{\frac{4}{N-2}} u \mathbb{1}_{|x|>t+A}, \\ \vec{v}|_{t=0} = (u_0, u_1). \end{cases} \quad (3.8)$$

By the definition of $\bar{A}$, we see that the right-hand side of the first equation in (3.8) is in $L^1([0, +\infty), L^2(\mathbb{R}^N))$. By Claim 2.3, there exists a solution v_L^A of (1.6) such that

$$\lim_{t \rightarrow +\infty} \int |\nabla_{t,x}(v - v_L^A)(t, x)|^2 dx = 0. \quad (3.9)$$

By finite speed of propagation, $u(t, x) = v(t, x)$ for (t, x) such that $t > 0, |x| > A + t$, and thus

$$\lim_{t \rightarrow +\infty} \int_{|x|>t+A} |\nabla_{t,x}(u - v_L^A)(t, x)|^2 dx = 0 \quad (3.10)$$

and

$$\lim_{t \rightarrow +\infty} \int_{|x|>t+A} \left(\frac{1}{|x|^2} |u(t, x)|^2 + |u(t, x)|^{\frac{2N}{N-2}} \right) dx = 0.$$

It remains to prove

$$\lim_{t \rightarrow +\infty} \|\nabla_{t,x}(v_L^A(t) - v_L(t))\|_{L^2(|x|>t+A)} = 0.$$

We let $G_+, G_+^A \in L^2(\mathbb{R} \times S^{N-1})$ be the outgoing radiation fields associated to v_L and v_L^A respectively (see Theorem 2.1). We will prove

$$\forall \omega \in S^{N-1}, \forall \eta > A, \quad G_+(\eta, \omega) = G_+^A(\eta, \omega), \quad (3.11)$$

which, in view of (3.10) and Theorem 2.1, will yield the conclusion of Step 2.

Fix $\varepsilon > 0$ and $\Phi \in L^2(\mathbb{R} \times S^{N-1})$ such that $\Phi(\eta, \omega) = 0$ if $\eta \leq A + \varepsilon$. Let $w_L(t) = S_L(t)(w_0, w_1)$ be the solution of the linear wave equation (1.6) whose associated outgoing radiation field is Φ (see Theorem 2.1). In other words,

$$\lim_{t \rightarrow +\infty} \int_0^{+\infty} \int_{S^{N-1}} \left| r^{\frac{N-1}{2}} \partial_t w_L(t, r\omega) - \Phi(r-t, \omega) \right|^2 d\omega dr = 0, \quad (3.12)$$

$$\lim_{t \rightarrow +\infty} \int_0^{+\infty} \int_{S^{N-1}} \left| r^{\frac{N-1}{2}} \partial_r w_L(t, r\omega) + \Phi(r-t, \omega) \right|^2 d\omega dr = 0. \quad (3.13)$$

On the one hand, we have

$$\begin{aligned} (\vec{u}(\tau_n), S_L(\tau_n)(w_0, w_1))_{\dot{H}^1 \times L^2} &= (S_L(-\tau_n)\vec{u}(\tau_n), (w_0, w_1))_{\dot{H}^1 \times L^2} \\ &\xrightarrow{n \rightarrow \infty} ((v_0, v_1), (w_0, w_1))_{\dot{H}^1 \times L^2}, \end{aligned} \quad (3.14)$$

by the definition (3.1) of (v_0, v_1) , and thus, using the isometry property of radiation fields, we get

$$\lim_{n \rightarrow \infty} (\vec{u}(\tau_n), S_L(\tau_n)(w_0, w_1))_{\dot{H}^1 \times L^2} = 2 \int_{-\infty}^{+\infty} \int_{S^{N-1}} G_+(\eta, \omega) \Phi(\eta, \omega) d\omega d\eta. \quad (3.15)$$

On the other hand,

$$\begin{aligned} (\vec{u}(\tau_n), S_L(\tau_n)(w_0, w_1))_{\dot{H}^1 \times L^2} &= \int \nabla_{t,x} u(\tau_n, x) \cdot \nabla_{t,x} w_L(\tau_n, x) dx \\ &= \int_0^{+\infty} \int_{S^{N-1}} r^{\frac{N-1}{2}} (\partial_r u(\tau_n, r\omega) - \partial_r u(\tau_n, r\omega)) \Phi(r - \tau_n, \omega) d\omega dr + o_n(1) \\ &= 2 \int_0^{+\infty} \int_{S^{N-1}} G_+^A(r - \tau_n, \omega) \Phi(r - \tau_n, \omega) d\omega dr + o_n(1), \end{aligned}$$

where at the last line we have used $\Phi(r - \tau_n, \omega) = 0$ if $r - \tau_n \leq A$, (3.10), and the definition of G_+^A . Hence

$$\lim_{n \rightarrow \infty} (\vec{u}(\tau_n), \vec{S}_L(\tau_n)(w_0, w_1))_{\dot{H}^1 \times L^2} = 2 \int_{-\infty}^{+\infty} \int_{S^{N-1}} G_+^A(\eta, \omega) \Phi(\eta, \omega) d\omega d\eta.$$

Combining this with (3.15), we see that for all $\Phi \in L^2(\mathbb{R} \times S^{N-1})$ such that $\Phi(\eta) = 0$ if $\eta \leq A + \varepsilon$,

$$\int_{-\infty}^{+\infty} (G_+^A - G_+) \Phi = 0.$$

Using this equality with

$$\Phi(\eta, \omega) = (G_+^A(\eta, \omega) - G_+(\eta, \omega)) \mathbb{1}_{\eta \geq A + \varepsilon}$$

yields (3.11) (since $\varepsilon > 0$ can be taken arbitrarily small), which concludes Step 2.

Step 3. By Step 2, and since we are assuming that the conclusion of Theorem 2 does not hold, $\bar{A} \in \mathbb{R}$. In this step we prove

$$\|u\|_{S(\{t>0, |x|>\bar{A}+t\})} = \infty, \quad (3.16)$$

which will conclude the proof of Proposition 3.1. We argue by contradiction, assuming

$$\|u\|_{S(\{t>0, |x|\geq\bar{A}+t\})} < \infty. \quad (3.17)$$

Let δ_1 be given by Claim 2.4 and $T \gg 1$ such that

$$\|u\|_{S(\{t>T, |x|\geq\bar{A}+t\})} < \delta_1/4.$$

Then by Claim 2.4,

$$\|S_L(t - T)\vec{u}(T)\|_{S(\{t>T, |x|>\bar{A}+t\})} < \delta_1/2.$$

Let $\varepsilon > 0$ be such that

$$\|S_L(t - T)\vec{u}(T)\|_{S(\{t>T, |x|>\bar{A}-\varepsilon+t\})} < \delta_1. \quad (3.18)$$

Then, again by Claim 2.4,

$$\|u\|_{S(\{t>T, |x|>\bar{A}-\varepsilon+t\})} < 2\delta_1,$$

which contradicts the definition of $\bar{A}$, concluding the proof. $\square$

4. Singular and regular directions

In this section, we still assume that u is a solution of (1.1) that satisfies the assumptions of Theorem 2 and not its conclusion. We let $\bar{A} \in \mathbb{R}$ be defined by Proposition 3.1.

Definition 4.1. The set $\mathcal{R}$ of *regular directions* is the set of $\omega \in S^{N-1}$ such that there exists $\varepsilon > 0$ with

$$\|u\|_{S(\{t>0, |x|>\bar{A}-\varepsilon+t \text{ and } |\widehat{(\omega, x)}|<\varepsilon\})} < \infty, \quad (4.1)$$

where $\widehat{(\omega, x)} \in [-\pi, \pi]$ is the angle between ω and x . The set $\mathcal{S}$ of *singular directions* is defined as $\mathcal{S} := S^{N-1} \setminus \mathcal{R}$.

In this section we prove:

Proposition 4.2. *Under the above assumptions, the set $\mathcal{S}$ is finite and nonempty.*

Proposition 4.3. *There exists $\delta_2 > 0$ such that if $\omega \in S^{N-1}$ satisfies, for some $\varepsilon > 0$,*

$$\liminf_{T \rightarrow +\infty} \|S_L(\cdot - T)\vec{u}(T)\|_{S(\{t>T, \bar{A}-\varepsilon+t \leq |x| \leq \bar{A}+\varepsilon+t, |\widehat{(\omega, x)}| \leq 4/\sqrt{T}\})} < \delta_2, \quad (4.2)$$

then $\omega \in \mathcal{R}$.

We first prove Proposition 4.2 assuming Proposition 4.3.

4.1. Finiteness of the set of singular directions

We first prove that $\mathcal{S}$ is nonempty. Assume otherwise. Thus $\mathcal{R} = S^{N-1}$ and for all ω belonging to S^{N-1} , there exists $\varepsilon(\omega)$ such that

$$\|u\|_{S(\{t>0, |x|>\bar{A}-\varepsilon(\omega)+t, |\widehat{(\omega, x)}|<\varepsilon(\omega)\})} < \infty. \quad (4.3)$$

By the compactness of S^{N-1} , we can find $\omega_1, \dots, \omega_J$ in S^{N-1} such that

$$S^{N-1} = \bigcup_{j=1}^J \{\omega \in S^{N-1} : |\widehat{(\omega, \omega_j)}| < \varepsilon(\omega_j)\}. \quad (4.4)$$

Letting $\varepsilon = \min_{j=1, \dots, J} \varepsilon(\omega_j)$, we see that

$$\|u\|_{S(\{t>0, |x|>\bar{A}-\varepsilon+t\})} < \infty,$$

contradicting the definition of $\bar{A}$.

We next prove that $\mathcal{S}$ is finite. Let $\omega_1, \dots, \omega_J$ be *pairwise distinct* elements of $\mathcal{S}$. Then for any fixed $\varepsilon > 0$ and large T , the sets

$$\Omega_j = \{(t, x) : t > T, \bar{A} - \varepsilon + t \leq |x| \leq \bar{A} + \varepsilon + t \text{ and } |\widehat{(\omega_j, x)}| \leq 4/\sqrt{T}\}$$

are pairwise disjoint. As a consequence, for large T ,

$$\sum_{j=1}^J \|S_L(\cdot - T)\vec{u}(T)\|_{L^{\frac{2(N+1)}{N-2}}(\Omega_j)}^{\frac{2(N+1)}{N-2}} \leq \|S_L(\cdot - T)\vec{u}(T)\|_{L^{\frac{2(N+1)}{N-2}}([\bar{A}-\varepsilon+T, +\infty) \times \mathbb{R}^N)}^{\frac{2(N+1)}{N-2}}. \quad (4.5)$$

Since $\vec{u}(T)$ is bounded in $\dot{H}^1 \times L^2$, the right-hand side of (4.5) is bounded independently of $T > 0$.

We will prove that $\mathcal{S}$ is finite, distinguishing between $N = 3$ and $N = 4, 5$. If $N = 3$, we have

$$\|S_L(\cdot - T)\vec{u}(T)\|_{S(\Omega_j)} \leq \|S_L(\cdot - T)\vec{u}(T)\|_{L_t^4 L_x^{12}(\Omega_j)}^{3/5} \|S_L(\cdot - T)\vec{u}(T)\|_{L^8(\Omega_j)}^{2/5}. \quad (4.6)$$

Since $\omega_j \in \mathcal{S}$, the left-hand side of (4.6) is, according to Proposition 4.3, bounded from below by $\delta_2/2$ for large T . Combining this with the boundedness of $\vec{u}(t)$ in $\dot{H}^1 \times L^2$ and Strichartz estimates, we deduce that for large T ,

$$\delta_2 \leq C \|S_L(\cdot - T)\vec{u}(T)\|_{L^8(\Omega_j)}^{2/5}.$$

Hence by (4.5),

$$J\delta_2^{20} \leq C \|S_L(\cdot - T)\vec{u}(T)\|_{L^8([\bar{A}-\varepsilon+T, +\infty) \times \mathbb{R}^N)}^8 \leq C,$$

where the right inequality is a consequence of the Strichartz inequality and the boundedness of $\vec{u}$ in $\dot{H}^1 \times L^2$. This proves that $\mathcal{S}$ is finite.

If $N = 4$ or $N = 5$, the proof is very close, using

$$\|S_L(\cdot - T)\|_{S(\Omega_j)} \leq C \|S_L(\cdot - T)\|_{L_t^2 L^{\frac{2N}{N-3}}(\Omega_j)}^{1-\theta} \|S_L(\cdot - T)\vec{u}(T)\|_{L^{\frac{2(N+1)}{N-2}}(\Omega_j)}^{\theta} \quad (4.7)$$

with $\theta = \frac{(N+1)(6-N)}{3(N+2)}$ instead of (4.6). We omit the details. The proof of Proposition 4.2 is complete. $\square$

4.2. A geometrical lemma

We now turn to some elementary geometrical properties that will be useful in the proof of Proposition 4.3. Without loss of generality, we will assume $\omega = e_1 := (1, 0, \dots, 0)$.

For $\theta \in (0, \pi/2)$, we let

$$\Gamma_\theta := \{x \in \mathbb{R}^N \setminus \{0\} : |\widehat{(e_1, x)}| \geq \pi/2 + \theta\} \cup \{0\},$$

where as before $\widehat{(e_1, x)}$ is the angle between e_1 and x . If $\tau > 0$, we define

$$D_{\tau, \theta} = \{x \in \mathbb{R}^N : d(x, \Gamma_\theta) > \tau\},$$

where $d(x, \Gamma_\theta) = \inf \{|y - x| : y \in \Gamma_\theta\}$ is the distance between x and Γ_θ (see Figure 1).

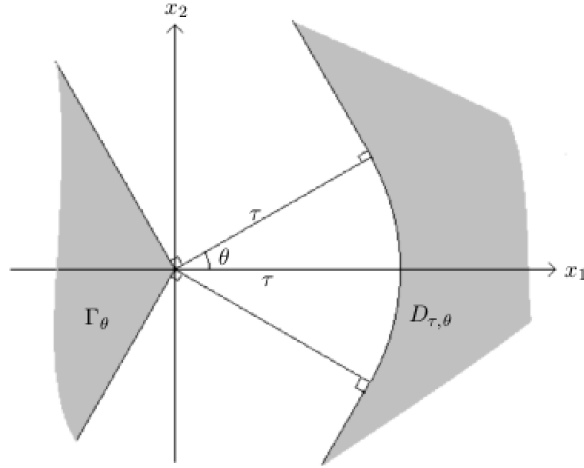


Fig. 1

Lemma 4.4. (a) $D_{\tau, \theta} \subset \{|x| > \tau\}$.

(b) $(|x| > \tau \text{ and } |\widehat{(x, e_1)}| < \theta) \Rightarrow x \in D_{\tau, \theta}$.

(c) If $\ell > 0$, $x \in D_{\tau, \theta}$ and $|x| \leq \tau + \ell$, then

$$|\widehat{(x, e_1)}| \leq \theta + \sqrt{\frac{\ell}{\tau + \ell}}.$$

Proof. (a) Since $0 \in \Gamma_\theta$, we have $|x| \geq d(x, \Gamma_\theta)$ and (a) follows from the definition of $D_{\tau, \theta}$.

(b) Let x be such that $|x| > \tau$ and $|\widehat{(x, e_1)}| < \theta$. Rotating around the axis Oe_1 , we can assume $x = (x_1, x_2, 0, \dots, 0)$. But then (b) is clear from Figure 1.

(c) We will use the elementary inequality

$$\forall s \in [0, \pi/2], \quad 1 - \cos s \geq s^2/4. \quad (4.8)$$

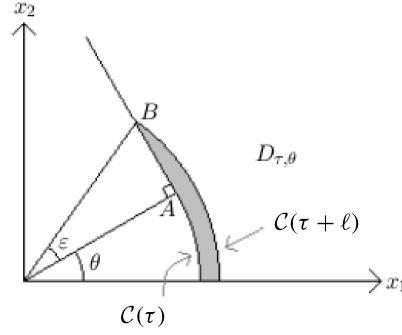


Fig. 2

Let $x \in D_{\tau, \theta}$ be such that $|x| < \tau + \ell$. As before, we can assume $x = (x_1, x_2, 0, \dots, 0)$. Denote by $\mathcal{C}(R)$ the circle of radius $R > 0$ centred at the origin.

Let $A = (\tau \cos \theta, \tau \sin \theta, 0, \dots, 0)$. The tangent at A to the circle $\mathcal{C}(\tau)$ intersects the circle $\mathcal{C}(\tau + \ell)$ at a point B . Let ε be the angle $(\widehat{OA, OB})$.

From the conditions $|x| < \tau + \ell$ and $x \in D_{\tau, \theta}$, we get $|\widehat{(x, e_1)}| \leq \varepsilon + \theta$. Indeed, in Figure 2, x must be in the dark region.

We have $\cos \varepsilon = \tau/(\tau + \ell)$ and hence, by (4.8),

$$\frac{\tau}{\tau + \ell} \leq 1 - \frac{\varepsilon^2}{4},$$

i.e. $\varepsilon^2/4 \leq \ell/(\tau + \ell)$, which yields the conclusion of (c).

4.3. Sufficient condition to be a regular direction

In this subsection we prove Proposition 4.3. We assume that (4.2) holds with $\omega = e_1$. Thus there exist $\varepsilon > 0$ and a sequence $\{t_n\}_n \rightarrow +\infty$ such that

$$\lim_{n \rightarrow \infty} \|S_L(t - t_n)\vec{u}(t_n)\|_{S(\{t > t_n, \bar{A} - \varepsilon + t \leq |x| \leq \bar{A} + \varepsilon + t, |\widehat{(e_1, x)}| \leq 4/\sqrt{t_n}\})} < \delta_2. \quad (4.9)$$

We denote

$$\mathcal{D}_n = \bigcup_{t > t_n} \{t\} \times D_{\bar{A} - \varepsilon + t, 1/\sqrt{t_n}}, \quad (4.10)$$

where $\varepsilon > 0$ is as in (4.9). We claim that $\mathcal{D}_n$ has the following causality property:

Claim 4.5. *Let $(t, x) \in \mathcal{D}_n$ and $(t', x') \in \mathbb{R} \times \mathbb{R}^N$ with $t_n < t' < t$ and $|x' - x| < |t - t'|$. Then $(t', x') \in \mathcal{D}_n$.*

Proof. Indeed, we must check that $x' \in D_{\bar{A} - \varepsilon + t', 1/\sqrt{t_n}}$, i.e. $d(x', \Gamma_{1/\sqrt{t_n}}) > \bar{A} - \varepsilon + t'$. By the triangle inequality,

$$d(x', \Gamma_{1/\sqrt{t_n}}) > d(x, \Gamma_{1/\sqrt{t_n}}) - |x - x'| > d(x, \Gamma_{1/\sqrt{t_n}}) - |t - t'|.$$

Since $x \in \mathcal{D}_{\bar{A}-\varepsilon+t, 1/\sqrt{t_n}}$, we obtain

$$d(x', \Gamma_{1/\sqrt{t_n}}) > \bar{A} - \varepsilon + t - (t - t') = \bar{A} - \varepsilon + t'. \quad \square$$

We divide the proof of Proposition 4.3 into two steps.

Step 1. We prove

$$\limsup_{n \rightarrow \infty} \|S_L(\cdot - t_n)\vec{u}(t_n)\|_{S(\mathcal{D}_n)} < 2\delta_2. \quad (4.11)$$

Indeed, we first note that by the definition of $\bar{A}$,

$$\|u\|_{S(\{t \geq 0, |x| \geq \bar{A} + \varepsilon + t\})} < \infty.$$

Thus

$$\lim_{n \rightarrow \infty} \|u\|_{S(\{t > t_n, |x| \geq \bar{A} + \varepsilon + t\})} = 0. \quad (4.12)$$

By Claim 2.4,

$$\lim_{n \rightarrow \infty} \|S_L(\cdot - t_n)\vec{u}(t_n)\|_{S(\{t > t_n, |x| \geq \bar{A} + \varepsilon + t\})} = 0. \quad (4.13)$$

We are thus reduced to proving

$$\limsup_{n \rightarrow \infty} \|S_L(\cdot - t_n)\vec{u}(t_n)\|_{S(\{(t, x) \in \mathcal{D}_n : |x| < \bar{A} + \varepsilon + t\})} < 2\delta_2. \quad (4.14)$$

By Lemma 4.4(c), if $t > t_n$ and $x \in D_{\bar{A}-\varepsilon+t, 1/\sqrt{t_n}}$ satisfies $|x| < \bar{A} + \varepsilon + t$, then

$$|\widehat{(x, e_1)}| \leq \frac{1}{\sqrt{t_n}} + 2\sqrt{\frac{2\varepsilon}{\bar{A} + \varepsilon + t}},$$

so that, if $\varepsilon > 0$ is small enough and n large,

$$|\widehat{(x, e_1)}| \leq 2/\sqrt{t_n}.$$

As a consequence,

$$\begin{aligned} & ((t, x) \in \mathcal{D}_n \text{ and } |x| < \bar{A} + \varepsilon + t) \\ & \implies (t > t_n, \bar{A} - \varepsilon + t < |x| < \bar{A} + \varepsilon + t \text{ and } |\widehat{(x, e_1)}| \leq 2/\sqrt{t_n}), \end{aligned}$$

and (4.14) follows from (4.9).

Step 2. We prove that for large n , $\|u\|_{S(\mathcal{D}_n)}$ is finite, more precisely,

$$\limsup_{n \rightarrow \infty} \|u\|_{S(\mathcal{D}_n)} \leq 3\delta_2. \quad (4.15)$$

Since, by Lemma 4.4(b),

$$(t > t_n, |x| > \bar{A} - \varepsilon + t \text{ and } |\widehat{(x, e_1)}| \leq 1/\sqrt{t_n}) \implies (t, x) \in \mathcal{D}_n,$$

we see that (4.15) implies $e_1 \in \mathcal{R}$ as desired (see Definition 4.1).

For any $T > t_n$, we let $\mathcal{D}_{n,T} = \{(t, x) \in \mathcal{D}_n : t \in [t_n, T]\}$. We will prove by a bootstrap argument that $\|u\|_{S(\mathcal{D}_{n,T})} \leq 3\delta_2$ for large n . By finite speed of propagation, Claim 4.5 and Strichartz estimates,

$$\forall T > t_n, \quad \|u\|_{S(\mathcal{D}_{n,T})} \leq \|S_L(\cdot - t_n)\vec{u}(t_n)\|_{S(\mathcal{D}_{n,T})} + C\|u\|_{S(\mathcal{D}_{n,T})}^{\frac{N+2}{N-2}}.$$

Thus for large n , in view of Step 1,

$$\|u\|_{S(\mathcal{D}_{n,T})} \leq \frac{5}{2}\delta_2 + C\|u\|_{S(\mathcal{D}_{n,T})}^{\frac{N+2}{N-2}},$$

and the result follows if δ_2 is small enough. $\square$

5. Conditions on the profiles

In this section we consider again a solution u as in Proposition 3.1, i.e. one that satisfies the assumptions of Theorem 2 but not its conclusion. We let $\bar{A}$ be given by Proposition 3.1, and consider profile decompositions for sequences $\{\vec{u}(t_n)\}$, where $t_n \rightarrow +\infty$. Lemma 5.1 excludes concentrating profiles in the region $\{|x| > \bar{A} + t\}$. Lemma 5.2 means that for any singular direction $\omega \in S^{N-1}$, one can find a nonzero profile concentrating close to $\{x = (\bar{A} + t)\omega\}$. This profile will be used, together with the main result of Section 7, to prove by contradiction that the set $\mathcal{S}$ is empty.

Lemma 5.1. *Let $\{t_n\}_n \rightarrow +\infty$ be such that $\vec{u}(t_n)$ has a profile decomposition. Then there is no profile such that $U_L^j \not\equiv 0$ and the following three conditions hold:*

$$\lim_{n \rightarrow \infty} (|x_{j,n}| - t_n) > \bar{A}, \quad (5.1)$$

$$\lim_{n \rightarrow \infty} (|x_{j,n}| - t_n - |t_{j,n}|) \geq \bar{A}, \quad (5.2)$$

$$\lim_{n \rightarrow \infty} \lambda_{j,n} = 0. \quad (5.3)$$

Lemma 5.2. *There exists $\delta_3 > 0$ with the following property. Let $\{t_n\}_n \rightarrow +\infty$ and $\omega \in \mathcal{S}$. Then there exists a subsequence of $\{t_n\}_n$ (still denoted by $\{t_n\}_n$) such that $\{u(t_n)\}_n$ has a profile decomposition $(U_L^j, \{t_{j,n}, x_{j,n}, \lambda_{j,n}\}_{j \geq 1})$ such that*

$$\lim_{n \rightarrow \infty} (x_{1,n} - (\bar{A} + t_n)\omega) = 0, \quad (5.4)$$

$$\lim_{n \rightarrow \infty} \lambda_{1,n} = 0, \quad (5.5)$$

$$\lim_{n \rightarrow \infty} t_{1,n} = 0, \quad (5.6)$$

$$\|U_L^1\|_{S(\mathbb{R})} \geq \delta_3. \quad (5.7)$$

Proof of Lemma 5.1. We argue by contradiction. Let $j \geq 1$ be such that $U_L^j \not\equiv 0$, and (5.1)–(5.3) hold. Using the implications

$$j \neq k \implies \lim_{n \rightarrow \infty} \int \nabla_{t,x} U_{L,n}^j(0, x) \cdot \nabla_{t,x} U_{L,n}^k(0, x) dx = 0,$$

$$j \leq J \implies \lim_{n \rightarrow \infty} \int \nabla_{t,x} U_{L,n}^j(0, x) \cdot \nabla_{t,x} w_n^J(0, x) dx = 0,$$

we obtain

$$\lim_{n \rightarrow \infty} \int \nabla_{t,x} u(t_n, x) \cdot \nabla_{t,x} U_{L,n}^j(0, x) dx = \int |\nabla_{t,x} U_L^j(0, x)|^2 dx > 0. \quad (5.8)$$

We will deduce a contradiction from (5.8), using the localization properties of $U_{L,n}^j$.

Case 1. Assume that $\{-t_{j,n}/\lambda_{j,n}\}_n$ is bounded. Extracting subsequences and time translating U_L^j if necessary, we can assume $t_{j,n} = 0$ for all n . By (5.1), there exists $\varepsilon > 0$ such that

$$|x_{j,n}| - t_n \geq \bar{A} + 2\varepsilon \quad (5.9)$$

for large n . Since

$$\nabla_{t,x} U_{L,n}^j(0, x) = \frac{1}{\lambda_{j,n}^{N/2}} \nabla_{t,x} U_L^j\left(0, \frac{x - x_{j,n}}{\lambda_{j,n}}\right), \quad (5.10)$$

we obtain, using also assumption (5.3),

$$\begin{aligned} \int \nabla_{t,x} u(t_n, x) \cdot \nabla_{t,x} U_{L,n}^j(0, x) dx &= \int_{|x| > \bar{A} + \varepsilon + t_n} \nabla_{t,x} u(t_n, x) \cdot \nabla_{t,x} U_{L,n}^j(0, x) dx + o(1) \\ &= \int_{|x| > \bar{A} + \varepsilon + t_n} \nabla_{t,x} v_L(t_n, x) \cdot \nabla_{t,x} U_{L,n}^j(0, x) dx + o(1) \\ &= \int \nabla_{t,x} v_L(t_n, x) \cdot \nabla_{t,x} U_{L,n}^j(0, x) dx + o(1) \end{aligned}$$

as $n \rightarrow \infty$. We have used (3.5) and, at the last line, (5.3), (5.9) and (5.10). As a consequence, using the asymptotic behavior of $v_L(t)$ as $t \rightarrow +\infty$ (see Theorem 2.1), we obtain

$$\lim_{n \rightarrow \infty} \int \nabla_{t,x} u(t_n, x) \cdot \nabla_{t,x} U_{L,n}^j(0, x) dx = 0,$$

contradicting (5.8).

Case 2. We assume (after extracting a subsequence),

$$\lim_{n \rightarrow \infty} \frac{-t_{j,n}}{\lambda_{j,n}} = -\infty \quad (5.11)$$

(the proof in the case where this limit is $+\infty$ is the same). Let G_-^j be the incoming radiation field associated to $U_L^j(t)$ (see Theorem 2.1 and Remark 2.2). Then

$$\lim_{t \rightarrow -\infty} \int_0^{+\infty} \int_{S^{N-1}} |r^{N-1} |\nabla_{t,x} U_L^j(t, r\omega)|^2 - 2|G_-^j(r+t, \omega)|^2| d\omega dr = 0. \quad (5.12)$$

Subcase 2a. We assume, in addition to (5.11),

$$\lim_{n \rightarrow \infty} (|x_{j,n}| - t_n - |t_{j,n}|) > \bar{A}, \quad (5.13)$$

i.e. (5.2) holds with strict inequality. Fix a small $\varepsilon > 0$ and an $R \gg 1$ such that

$$\int_{|\eta| \geq R} \int_{S^{N-1}} |G_-^j(\eta, \omega)|^2 d\omega d\eta < \varepsilon. \quad (5.14)$$

Using (5.12) and (5.14), we see that for large n ,

$$\int_{||x_{j,n}-x|-|t_{j,n}|| \geq R\lambda_{j,n}} |\nabla_{t,x} U_{L,n}^j(0, x)|^2 dx < C\varepsilon. \quad (5.15)$$

Hence for large n ,

$$\begin{aligned} & \int \nabla_{t,x} u(t_n) \cdot \nabla_{t,x} U_{L,n}^j(0, x) dx \\ & \leq \int_{||x_{j,n}-x|-|t_{j,n}|| \leq R\lambda_{j,n}} \nabla_{t,x} u(t_n) \cdot \nabla_{t,x} U_{L,n}^j(0, x) dx + C\sqrt{\varepsilon} \end{aligned} \quad (5.16)$$

(where the constant $C > 0$ might depend on U^j and on $\sup_{t \geq 0} \|\nabla_{t,x} u\|_{\dot{H}^1 \times L^2}$ but is of course independent of n). By the triangle inequality, (5.3) and (5.13), there exists $\varepsilon' > 0$ such that for large n ,

$$|x_{j,n}-x|-|t_{j,n}| \leq R\lambda_{j,n} \implies |x| \geq |x_{j,n}|-|t_{j,n}|-R\lambda_{j,n} \implies |x| \geq t_n + \bar{A} + \varepsilon'. \quad (5.17)$$

Using (3.5) and (5.16), we find that for large n ,

$$\begin{aligned} & \int \nabla_{t,x} u(t_n, x) \cdot \nabla_{t,x} U_{L,n}^j(0, x) dx \\ & \leq \int_{||x_{j,n}-x|-|t_{j,n}|| \leq R\lambda_{j,n}} \nabla_{t,x} v_L(t_n, x) \cdot \nabla_{t,x} U_{L,n}^j(0, x) dx + C\sqrt{\varepsilon} \\ & \leq \int \nabla_{t,x} v_L(t_n, x) \cdot \nabla_{t,x} U_{L,n}^j(0, x) dx + C\sqrt{\varepsilon}. \end{aligned}$$

Since

$$\lim_{n \rightarrow \infty} \int \nabla_{t,x} v_L(t_n, x) \cdot \nabla_{t,x} U_{L,n}^j(0, x) dx = 0,$$

we deduce (as $\varepsilon > 0$ is arbitrarily small)

$$\limsup_{n \rightarrow \infty} \int \nabla_{t,x} u(t_n, x) \cdot \nabla_{t,x} U_{L,n}^j(0, x) dx \leq 0, \quad (5.18)$$

which contradicts (5.8).

Subcase 2b. We assume, in addition to (5.11),

$$\lim_{n \rightarrow \infty} (|x_{j,n}| - t_n - |t_{j,n}|) = \bar{A}, \quad (5.19)$$

i.e. (5.2) holds with equality. In view of (5.1), we must have

$$\lim_{n \rightarrow \infty} |t_{j,n}| > 0. \quad (5.20)$$

Extracting subsequences, we can assume

$$\lim_{n \rightarrow \infty} \frac{x_{j,n}}{|x_{j,n}|} = \omega_0 \in S^{N-1}. \quad (5.21)$$

Let $\varepsilon > 0$. Choose $R \gg 1$ such that (5.14) holds, and $\alpha > 0$ such that

$$\int_{\mathbb{R}} \int_{\substack{\omega \in S^{N-1} \\ |\omega + \omega_0| \leq \alpha}} |G_-^j(\eta, \omega)|^2 d\omega d\eta < \varepsilon. \quad (5.22)$$

In view of (5.14) and (5.22) we have, for large n ,

$$\int_{||x - x_{j,n}| - |t_{j,n}|| \geq R\lambda_{j,n}} |\nabla_{t,x} U_{L,n}^j(0, x)|^2 dx + \int_{\substack{|x - x_{j,n}| \\ |x - x_{j,n}| + \omega_0| \leq \alpha}} |\nabla_{t,x} U_{L,n}^j(0, x)|^2 dx \leq C\varepsilon, \quad (5.23)$$

and thus, for large n again,

$$\int \nabla_{t,x} u(t_n, x) \cdot \nabla_{t,x} U_{L,n}^j(0, x) dx \leq \int_{Z_n} \nabla_{t,x} u(t_n, x) \cdot \nabla_{t,x} U_{L,n}^j(0, x) dx + C\sqrt{\varepsilon},$$

where

$$Z_n := \left\{ x \in \mathbb{R}^N : ||x - x_{j,n}| - |t_{j,n}|| \leq R\lambda_{j,n} \text{ and } \left| \frac{x - x_{j,n}}{|x - x_{j,n}|} + \omega_0 \right| \geq \alpha \right\}.$$

We next prove that there exists $\varepsilon' > 0$ such that for large n ,

$$x \in Z_n \implies |x| \geq \bar{A} + \varepsilon + t_n. \quad (5.24)$$

Assuming (5.24), we can prove (5.18) exactly as in Subcase 2a, obtaining again a contradiction.

We have $x = x_{j,n} + x - x_{j,n}$ and

$$\begin{aligned} x \cdot \frac{x_{j,n}}{|x_{j,n}|} &= |x_{j,n}| + (x - x_{j,n}) \cdot \frac{x_{j,n}}{|x_{j,n}|} \\ &= |x_{j,n}| + |x - x_{j,n}| \frac{(x - x_{j,n}) \cdot x_{j,n}}{|x - x_{j,n}| |x_{j,n}|} \\ &= |x_{j,n}| + |x - x_{j,n}| \left(\frac{x - x_{j,n}}{|x - x_{j,n}|} \omega_0 + o_n(1) \right). \end{aligned}$$

Taking the square of the inequality $\left| \frac{x-x_{j,n}}{|x-x_{j,n}|} + \omega_0 \right| \geq \alpha$ and expanding, we deduce that $\omega_0 \cdot \frac{x-x_{j,n}}{|x-x_{j,n}|} \geq \frac{\alpha^2}{2} - 1$ for $x \in Z_n$. Hence for large n , if $x \in Z_n$,

$$\begin{aligned} |x| &\geq x \cdot \frac{x_{j,n}}{|x_{j,n}|} \geq |x_{j,n}| + \left(\frac{\alpha^2}{4} - 1 \right) |x - x_{j,n}| \\ &\geq \bar{A} + t_n + |t_{j,n}| + \left(\frac{\alpha^2}{4} - 1 \right) (|t_{j,n}| + R\lambda_{j,n} + o_n(1)) \geq t_n + \bar{A} + \frac{\alpha^2}{8} |t_{j,n}|, \end{aligned}$$

by (5.3) and (5.20). In view of (5.20), the desired conclusion (5.24) follows. This concludes the proof of Lemma 5.1. $\square$

Proof of Lemma 5.2. Assume without loss of generality that $\omega = e_1$.

Let (after extraction) $(U_L^j, \{t_{j,n}, x_{j,n}, \lambda_{j,n}\}_{j \geq 1})$ be a profile decomposition of $\{\vec{u}(t_n)\}_n$. Let $\delta_3 > 0$, to be specified later, and let J_1 be such that $\|U_L^j\|_{S(\mathbb{R})} \geq \delta_3$ for $1 \leq j \leq J_1$, and $\|U_L^j\|_{S(\mathbb{R})} < \delta_3$ for $j > J_1$. Since

$$\sum_{j \geq 1} \|\vec{U}_L^j(0)\|_{\dot{H}^1 \times L^2}^2 < \infty,$$

it is easy to see, using Strichartz estimates, that such a (finite) J_1 exists. Note that if δ_3 is small enough, then $J_1 \geq 1$, because otherwise u scatters, which contradicts our assumption that A is finite.

Note that $(U_L^j, \{t_{j,n}, x_{j,n}, \lambda_{j,n}\}_{j \geq 1+J_1})$ is a profile decomposition for the sequence of remainders $\{w_n^{J_1}\}_n$. By Claim C.1 in Appendix C, there exists $\theta > 0$ such that

$$\limsup_{n \rightarrow \infty} \|w_n^{J_1}\|_{S(\mathbb{R})} \leq C\delta_3^\theta, \quad (5.25)$$

where C depends only on the bound

$$M := \limsup_{t \rightarrow +\infty} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2}.$$

We choose δ_3 such that $C\delta_3^\theta < \delta_2/2$, where δ_2 is given by Proposition 4.3. Extracting subsequences, we can assume that the following limits exist for all $j \in \{1, \dots, J_1\}$:

$$\lim_{n \rightarrow \infty} \lambda_{j,n} \in [0, +\infty], \quad (5.26)$$

$$\lim_{n \rightarrow \infty} t_{j,n} \in \mathbb{R} \cup \{\pm\infty\}, \quad (5.27)$$

$$\lim_{n \rightarrow \infty} |x_{j,n} - (\bar{A} + t_n)e_1| \in [0, +\infty]. \quad (5.28)$$

We argue by contradiction, assuming that for all $j \in \{1, \dots, J_1\}$, one of the above limits is not 0.

By Proposition 4.3 and Claim C.1, it is sufficient to prove that for all $j \in \{1, \dots, J_1\}$ there exists $\varepsilon > 0$ such that

$$\limsup_{n \rightarrow \infty} |I_{n,\varepsilon}^j| < \frac{\delta_2}{4} J_1, \quad (5.29)$$

where

$$I_{n,\varepsilon}^j := \left(\int_{t_n}^{+\infty} \left(\int_{\substack{\bar{A}-\varepsilon+|t|\leq|x|\leq\bar{A}+\varepsilon+|t| \\ |\widehat{(e_1,x)}|\leq 4/\sqrt{t_n}}} |U_{L,n}^j(t-t_n, x)|^{\frac{2(N+2)}{N-2}} dx \right)^{1/2} dt \right)^{\frac{N-2}{N+2}}.$$

This would imply, by the triangle inequality, (5.25) and the bound $C\delta_3^\theta < \delta_2/2$,

$$\limsup_{n \rightarrow \infty} \|S_L(\cdot - t_n) \vec{u}(t_n)\|_{S(\{\bar{A}-\varepsilon+|t|\leq|x|\leq\bar{A}+\varepsilon+|t|, |\widehat{(e_1,x)}|\leq 4/\sqrt{t_n}\})} < \delta_2,$$

and thus, by Proposition 4.3, that e_1 is a regular point, a contradiction.

By the change of variables

$$s = \frac{t - t_n - t_{j,n}}{\lambda_{j,n}}, \quad y = \frac{x - x_{j,n}}{\lambda_{j,n}},$$

we obtain

$$I_{n,\varepsilon} = \int_{-t_{j,n}/\lambda_{j,n}}^{+\infty} \left(\int_{Q_{n,\varepsilon}(s)} |U_L^j(s, y)|^{\frac{2(N+2)}{N-2}} dy \right)^{1/2} ds, \quad (5.30)$$

where $Q_{n,\varepsilon}(s)$ is the set of $y \in \mathbb{R}^N$ such that the absolute value of the angle between $\lambda_{j,n}y + x_{j,n}$ and e_1 is $\leq 4/\sqrt{t_n}$ and

$$\bar{A} - \varepsilon + |\lambda_{j,n}s + t_n + t_{j,n}| < |x_{j,n} + \lambda_{j,n}y| < \bar{A} + \varepsilon + |\lambda_{j,n}s + t_n + t_{j,n}|.$$

If $\lim_{n \rightarrow \infty} -t_{j,n}/\lambda_{j,n} = +\infty$, then we see that $\lim_{n \rightarrow \infty} I_{n,\varepsilon} = 0$ and (5.29) follows. We are thus reduced to the case

$$\lim_{n \rightarrow \infty} \frac{-t_{j,n}}{\lambda_{j,n}} = -\infty \quad \text{or} \quad \forall n, t_{j,n} = 0. \quad (5.31)$$

We note that if for almost every $s \in \mathbb{R}$,

$$\mathbb{1}_{Q_{n,\varepsilon}(s)}(y) \xrightarrow{n \rightarrow \infty} 0 \quad \text{for a.a. } y, \quad (5.32)$$

then $\lim_n I_{n,\varepsilon} = 0$ by dominated convergence and we are done.

Case 1. If $\lim_{n \rightarrow \infty} \lambda_{j,n} = +\infty$, then $Q_{n,\varepsilon}(s)$ is for all s included in an annulus of width $\leq C/\lambda_{j,n}$, which proves that (5.32) holds.

Case 2. If $\lim_{n \rightarrow \infty} \lambda_{j,n} = 0$, we distinguish three subcases according to the limit

$$\ell = \lim_{n \rightarrow \infty} (|x_{j,n}| - (\bar{A} + t_n)). \quad (5.33)$$

- If $\ell > 0$, then by Lemma 5.1 we must have

$$\lim_{n \rightarrow \infty} (|x_{j,n}| - t_n - t_{j,n}) < \bar{A} \quad (5.34)$$

and we see, fixing ε small enough, that for any s , $Q_{n,\varepsilon}(s)$ is empty for large n , yielding (5.32).

- If $\ell < 0$, then since $t_{j,n} \geq 0$ for large n , we again obtain (5.34) and thus (5.32).

• In the case $\ell = 0$ and $\lim_n |t_{j,n}| > 0$, we again obtain (5.34) and (5.32). Finally, we assume $\ell = 0$ and

$$\lim_{n \rightarrow \infty} |t_{j,n}| = 0.$$

Letting, after extraction, $\omega_\infty = \lim_n x_{j,n}/|x_{j,n}|$, we see, fixing y , and using the fact that $|x_{j,n}|$ goes to infinity, that

$$\lim_{n \rightarrow \infty} |(\lambda_{j,n} y + \widehat{x_{j,n}}, e_1)| = |(\widehat{\omega_\infty}, e_1)|. \quad (5.35)$$

Since $\lim_n t_{j,n} = \lim_n \lambda_{j,n} = 0$, we must have

$$\lim_{n \rightarrow \infty} |x_{j,n} - (\bar{A} + t_n)e_1| > 0$$

and thus (using $\ell = 0$), $\omega_\infty \neq e_1$. By (5.35), we see that (5.32) holds again.

Case 3. Suppose

$$\lim_{n \rightarrow \infty} \lambda_{j,n} = \lambda_\infty \in (0, +\infty). \quad (5.36)$$

In this case we cannot prove (5.32) for a fixed ε . We prove (5.29) by contradiction, assuming that for all $\varepsilon > 0$,

$$\limsup_{n \rightarrow \infty} |I_{n,\varepsilon}| \geq \frac{\delta_2}{4J_1}. \quad (5.37)$$

Then we can find a sequence $\{\varepsilon_k\}_k \rightarrow 0$ of positive numbers and a sequence $\{n_k\}_k \rightarrow +\infty$ of integers such that

$$\forall k, \quad I_{n_k, \varepsilon_k} \geq \frac{\delta_2}{8J_1}. \quad (5.38)$$

As a consequence, we see that $Q_{n_k, \varepsilon_k}(s)$ is included in an annulus of width $\lesssim \varepsilon_k$ (using (5.36)) and thus $\mathbb{1}_{Q_{n_k, \varepsilon_k}(s)}(y)$ goes to 0 for almost every y , which contradicts (5.37) and concludes the proof. $\square$

6. Concentration in a direction and a virial-type identity

In this part we prove the following:

Proposition 6.1. *Let u be as in Theorem 2, and assume that the conclusion of Theorem 2 does not hold. Let $\bar{A}$ be given by Proposition 3.1. Then for any $\varepsilon > 0$, there exists a sequence $\{t'_n\}_n \rightarrow +\infty$ such that, for some $\alpha > 0$,*

$$\limsup_{n \rightarrow \infty} \int_{|x - (t'_n + \bar{A}e_1)| < \alpha} \left(|\partial_{x_1} u(t'_n) + \partial_t u(t'_n)|^2 + \sum_{j=2}^N |\partial_{x_j} u(t'_n)|^2 \right) dx < \varepsilon. \quad (6.1)$$

Note that $\partial_{x_1} u - \partial_t u = \partial_{x_2} u = \dots = \partial_{x_N} u = 0$ implies that $u(t, x)$ is of the form $\varphi(t - x_1)$, i.e. travels at speed 1 in the direction e_1 . Proposition 6.1 says heuristically that for a sequence of times t'_n , there is no component of $\vec{u}(t'_n)$ that concentrates close to the direction $(t'_n, t'_n + \bar{A}e_1)$ and travels at speed 1 in the direction e_1 .

Remark 6.2. We will use Proposition 6.1 to prove that $e_1 \notin \mathcal{S}$. Of course one could write an analogue of Proposition 6.1 adapted to another direction than e_1 . However, since we will use the rotation invariance of equation (1.1) to reduce to the direction e_1 , Proposition 6.1 will be sufficient for our purpose.

We start with a few lemmas.

Lemma 6.3. *Let u be as in Proposition 6.1. Let $\{\tau_n\}_n \rightarrow +\infty$ be any sequence. Denote*

$$\rho(t, x) = |u(t, (t + \bar{A})e_1 + x)|^{\frac{2N}{N-2}} + |\nabla_{t,x} u(t, (t + \bar{A})e_1 + x)|^2 + \frac{1}{|x|^2} |u(t, (t + \bar{A})e_1 + x)|^2. \quad (6.2)$$

Then, after extraction of a subsequence, there exists a nonnegative Radon measure $\tilde{\mu}$ on $\mathbb{R}^N$ such that

$$\rho(\tau_n, \cdot) \xrightarrow{n \rightarrow \infty} \tilde{\mu}, \quad (6.3)$$

$$\text{supp } \tilde{\mu} \subset \{x_1 \leq 0\}. \quad (6.4)$$

Proof. Since the sequence $\{\rho(\tau_n, \cdot)\}_n$ is bounded in $L^1(\mathbb{R}^N)$, one can always extract a subsequence such that (6.3) holds for some positive finite Radon measure $\tilde{\mu}$ on $\mathbb{R}^N$. We just need to prove (6.4). Let $\varepsilon > 0$. By Proposition 3.1,

$$\lim_{n \rightarrow \infty} \int_{x_1 \geq \varepsilon} |\nabla_{t,x} u(t_n, x + (t_n + \bar{A})e_1) - \nabla_{t,x} v_L(t_n, x + (t_n + \bar{A})e_1)|^2 dx = 0, \quad (6.5)$$

and

$$\lim_{n \rightarrow \infty} \int_{x_1 \geq \varepsilon} \left(\frac{1}{|x|^2} |u(t_n, x + (t_n + \bar{A})e_1)|^2 + |u(t_n, x + (t_n + \bar{A})e_1)|^{\frac{2N}{N-2}} \right) dx = 0. \quad (6.6)$$

We are thus reduced to proving that for all $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\lim_{n \rightarrow \infty} \int |\nabla_{t,x} v_L(t_n, x + (t_n + \bar{A})e_1)|^2 \varphi(x) dx = 0. \quad (6.7)$$

Consider the outgoing radiation field $G_+ \in L^2(\mathbb{R} \times S^{N-1})$ associated to v_L (see Theorem 2.1). We have

$$\begin{aligned} & \int |\nabla_{t,x} v_L(t_n, x + (t_n + \bar{A})e_1)|^2 \varphi(x) dx \\ &= \int |\nabla_{t,x} v_L(t_n, x)|^2 \varphi(x - (t_n + \bar{A})e_1) dx \\ &= 2 \int_0^{+\infty} \int_{S^{N-1}} |G_+(r - t_n, \omega)|^2 \varphi(r\omega - (t_n + \bar{A})e_1) d\omega dr + o(1) \\ &= 2 \int_{-t_n}^{+\infty} \int_{S^{N-1}} |G_+(\eta, \omega)|^2 \varphi((t_n + \eta)\omega - (t_n + \bar{A})e_1) d\omega d\eta + o(1). \end{aligned}$$

Next, notice that $\varphi((t_n + \eta)\omega - (t_n + \bar{A})e_1)$ goes to 0 for all $(\eta, \omega) \in \mathbb{R} \times (S^{N-1} \setminus \{e_1\})$. This proves (6.7), and thus (6.4), in view of (6.5) and (6.6) and since ε can be taken arbitrarily small. $\square$

In the following, we will decompose $\tilde{\mu}$ as

$$\tilde{\mu} = c_0 \delta_{\{x=0\}} + \mu, \quad (6.8)$$

where $\delta_{\{x=0\}}$ is the Dirac measure at $x = 0$, $c_0 = \tilde{\mu}(\{0\}) \geq 0$, and μ is a nonnegative Radon measure such that $\mu(\{0\}) = 0$. If e_1 is a singular direction, we can prove, using Lemma 5.2, that $c_0 > 0$, but this will not be used.

Lemma 6.4. *Let u be as in Proposition 6.1. Let $\{t_n\}_n$ be a sequence of times going to $+\infty$ as $n \rightarrow \infty$, and $\varepsilon > 0$. Then (after extraction of a subsequence from $\{t_n\}_n$), there exists $\alpha > 0$ and nonnegative Radon measures μ_0 and μ_1 on $\mathbb{R}^N$ and nonnegative real numbers c_0 and c_1 such that*

$$\mu_0(\{0\}) = \mu_1(\{0\}) = 0, \quad (6.9)$$

$$\rho(t_n, \cdot) \xrightarrow{n \rightarrow \infty} \mu_0 + c_0 \delta_{\{x=0\}}, \quad (6.10)$$

$$\rho(t_n - \alpha/10, \cdot) \xrightarrow{n \rightarrow \infty} \mu_1 + c_1 \delta_{\{x=0\}}, \quad (6.11)$$

$$\text{supp } \mu_j \subset \{x_1 \leq 0\}, \quad j = 0, 1, \quad (6.12)$$

$$\mu_0(\{|x| \leq \alpha\}) < \varepsilon, \quad (6.13)$$

$$\mu_1(\{|x| \leq 7\alpha/10\}) < \varepsilon. \quad (6.14)$$

Proof. By Lemma 6.3, there exist a subsequence of $\{t_n\}_n$ and a nonnegative measure μ_0 that satisfies (6.9), (6.10) and (6.12). Since μ_0 is outer regular, we have

$$0 = \mu_0(\{0\}) = \inf_{R>0} \mu_0(B(0, R)),$$

and we can find $\alpha > 0$ such that

$$\mu_0(\{|x| < \alpha\}) < \varepsilon/C_0 \quad (6.15)$$

for some large constant $C_0 > 0$ to be specified. By Lemma 6.3 again, with $\tau_n = t_n - \alpha/10$, there exists (after extracting subsequences) a measure μ_1 that satisfies (6.9), (6.11) and (6.12). It remains to check that (6.14) holds.

Let $\delta > 0$. Since $\mu_0(\{|x| < \alpha\}) < \varepsilon/C_0$, we have

$$\limsup_{n \rightarrow \infty} \int_{\delta \leq |x| \leq 19\alpha/20} \rho(t_n, x) dx < \varepsilon/C_0. \quad (6.16)$$

As a consequence, if φ is a C^∞ function equal to 1 for $|x| \geq 2$ and to 0 for $|x| \leq 1$, and ψ is a C^∞ function equal to 1 for $|x| \leq 9$ and to 0 for $|x| \geq 9.5$, we have

$$\limsup_{n \rightarrow \infty} \int \left| \nabla_{t,x} \left(\varphi \left(\frac{x}{\delta} \right) \psi \left(\frac{x}{\alpha} \right) u(t_n, (\bar{A} + t_n)e_1 + x) \right) \right|^2 dx < C\varepsilon/C_0.$$

Using finite speed of propagation and small data theory, we deduce (choosing C_0 large enough)

$$\limsup_{n \rightarrow \infty} \sup_{t_n - \alpha/10 \leq t \leq t_n} \int_{\alpha/10 + 3\delta \leq |x| \leq 4\alpha/5} \rho(t, t_n, x) dx < \varepsilon, \quad (6.17)$$

where

$$\begin{aligned} \rho(t, t_n, x) &= |u(t, (t_n + \bar{A})e_1 + x)|^{\frac{2N}{N-2}} + |\nabla_{t,x} u(t, (t_n + \bar{A})e_1 + x)|^2 \\ &\quad + \frac{1}{|x|^2} |u(t, (t_n + \bar{A})e_1 + x)|^2. \end{aligned} \quad (6.18)$$

This proves, since $\rho(t_n - \frac{\alpha}{10}, t_n, x - \frac{\alpha}{10}e_1) = \rho(t_n - \frac{\alpha}{10}, x)$, that

$$\mu_1\left(\left\{\frac{\alpha}{10} + 3\delta \leq \left|x - \frac{\alpha}{10}e_1\right| \leq \frac{4\alpha}{5}\right\}\right) < \varepsilon. \quad (6.19)$$

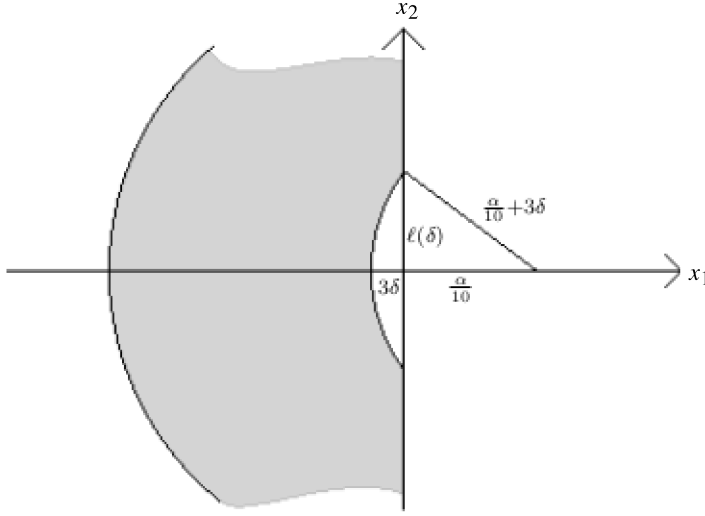


Fig. 3

The distance $\ell(\delta)$ in Figure 3 is equal to

$$\ell(\delta) = \sqrt{\left(\frac{\alpha}{10} + 3\delta\right)^2 - \left(\frac{\alpha}{10}\right)^2} \xrightarrow{\delta \rightarrow 0} 0.$$

Hence, from the figure,

$$\left\{x : 0 < |x| < \frac{7\alpha}{10} \text{ and } x_1 \leq 0\right\} \subset \bigcup_{\delta > 0} \left\{3\delta + \frac{\alpha}{10} < \left|x - \frac{\alpha}{10}e_1\right| < \frac{4\alpha}{5}\right\}.$$

Thus by (6.19), (6.9) and (6.12),

$$\mu_1(\{|x| \leq 7\alpha/10\}) < \varepsilon. \quad (6.20)$$

□

Remark 6.5. Let

$$R_n(\varepsilon) := \sup_{t_n - \alpha/10 \leq t \leq t_n} \int_{\alpha/5 \leq |x| \leq 4\alpha/5} \rho(t, t_n, x) dx, \quad (6.21)$$

where $\rho(t, t_n, x)$ is defined in (6.18). It follows from the proof of Lemma 6.4 that for large n ,

$$R_n(\varepsilon) < 2\varepsilon. \quad (6.22)$$

(See inequality (6.17) with $\delta = \alpha/30$.)

Proof of Proposition 6.1. Let $\{t_n\}_n \rightarrow +\infty$. Let α, μ_0, μ_1 be given by Lemma 6.4, corresponding to ε . Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that

$$\varphi(x) = 0 \text{ if } |x| \geq 1/2, \quad \varphi(x) = 1 \text{ if } |x| \leq 1/4. \quad (6.23)$$

Let

$$\begin{aligned} u_n(t, x) &= u(t, x + (t_n + \bar{A})e_1), \quad \varphi_\alpha(x) = \varphi(x/\alpha), \\ e(u_n)(t, x) &= \frac{1}{2}|\nabla u_n|^2 + \frac{1}{2}(\partial_t u_n)^2 - \frac{N-2}{2N}|u_n|^{\frac{2N}{N-2}}, \\ a_n(t) &= \int (\partial_t u_n)^2 \varphi_\alpha, \quad b_n(t) = \int |\nabla u_n|^2 \varphi_\alpha, \\ c_n(t) &= \int |u_n|^{\frac{2N}{N-2}} \varphi_\alpha, \quad d_n(t) = \int \partial_{x_1} u_n \partial_t u_n \varphi_\alpha. \end{aligned}$$

Observe that

$$|\nabla_{t,x} u_n(t_n, x)|^2 + \frac{1}{|x|^2} |u_n(t_n, x)|^2 + |u_n(t_n, x)|^{\frac{2N}{N-2}} \xrightarrow{n \rightarrow \infty} c_0 \delta_{\{x=0\}} + \mu_0, \quad (6.24)$$

$$\begin{aligned} &\left| \nabla_{t,x} u_n \left(t_n - \frac{\alpha}{10}, x - \frac{\alpha}{10} e_1 \right) \right|^2 + \frac{1}{|x|^2} \left| u_n \left(t_n - \frac{\alpha}{10}, x - \frac{\alpha}{10} e_1 \right) \right|^2 \\ &\quad + \left| u_n \left(t_n - \frac{\alpha}{10}, x - \frac{\alpha}{10} e_1 \right) \right|^{\frac{2N}{N-2}} \xrightarrow{n \rightarrow \infty} c_1 \delta_{\{x=0\}} + \mu_1. \end{aligned} \quad (6.25)$$

We denote the average values of a_n, b_n, c_n and d_n between $t_n - \alpha/10$ and t_n by $\bar{a}_n, \bar{b}_n, \bar{c}_n$ and $\bar{d}_n$ respectively:

$$\bar{a}_n := \frac{10}{\alpha} \int_{t_n - \alpha/10}^{t_n} a_n(t) dt,$$

and similarly for $\bar{b}_n, \bar{c}_n$ and $\bar{d}_n$.

Step 1. By explicit computations using (6.21) we obtain, for $t \in [t_n - \alpha/10, t_n]$,

$$\frac{d}{dt} \int u_n \partial_t u_n \varphi_\alpha = a_n(t) - b_n(t) + c_n(t) + \mathcal{O}(\varepsilon), \quad (6.26)$$

$$\frac{d}{dt} \int x \cdot \nabla u_n \partial_t u_n = -\frac{N}{2} a_n(t) + \left(\frac{N}{2} - 1 \right) b_n(t) - \left(\frac{N}{2} - 1 \right) c_n(t) + \mathcal{O}(\varepsilon), \quad (6.27)$$

$$\frac{d}{dt} \int x_1 e(u_n) \varphi_\alpha = -d_n + \mathcal{O}(\varepsilon), \quad (6.28)$$

$$\frac{d}{dt} \int \partial_{x_1} u_n \partial_t u_n \varphi_\alpha = d'_n(t) = \mathcal{O}(\varepsilon), \quad (6.29)$$

$$\frac{d}{dt} \int e(u_n) \varphi_\alpha = \frac{1}{2} a'_n(t) + \frac{1}{2} b'_n(t) - \frac{N-2}{2N} c'_n(t) = \mathcal{O}(\varepsilon), \quad (6.30)$$

where $\mathcal{O}(\varepsilon)$ is uniform with respect to $t \in [t_n - \alpha/10, t_n]$.

These computations are classical. The only thing to check is the bound on the remainder. For example, we have (using Einstein's summation convention)

$$\begin{aligned} \frac{d}{dt} \int x_k \partial_{x_k} u_n \partial_t u_n \varphi_\alpha &= -\frac{N}{2} a_n + \left(\frac{N}{2} - 1\right) b_n - \left(\frac{N}{2} - 1\right) c_n \\ &\quad + \frac{1}{2} \int x_k \partial_{x_k} \varphi_\alpha (\partial_{x_j} u_n)^2 - \frac{1}{2} \int x_k \partial_{x_k} \varphi_\alpha (\partial_t u_n)^2 \\ &\quad - \int \partial_{x_k} \varphi_\alpha x_j \partial_{x_j} u_n \partial_{x_k} u_n - \frac{N-2}{2N} \int x_j \partial_{x_j} \varphi_\alpha |u|^{\frac{2N}{N-2}}, \end{aligned}$$

and (6.27) follows from (6.22) and the bound $|x_j \partial_{x_k} \varphi_\alpha| = \left| \frac{x_j}{\alpha} \partial_{x_k} \varphi\left(\frac{x}{\alpha}\right) \right| \lesssim 1$.

Step 2. Approximate conservation laws. We prove that for large n ,

$$\sup_{t_n - \alpha/10 \leq t \leq t_n} |d_n(t) - \bar{d}_n| \lesssim \varepsilon \alpha, \quad (6.31)$$

$$\sup_{t_n - \alpha/10 \leq t \leq t_n} \left| \frac{1}{2} a_n(t) + \frac{1}{2} b_n(t) - \frac{N-2}{2N} c_n(t) - \frac{1}{2} \bar{a}_n - \frac{1}{2} \bar{b}_n + \frac{N-2}{2N} \bar{c}_n \right| \lesssim \varepsilon \alpha. \quad (6.32)$$

Indeed, by (6.29), there exists a constant $C > 0$ such that for all τ_1, τ_2 with $t_n - \alpha/10 \leq \tau_1, \tau_2 \leq t_n$,

$$|d_n(\tau_1) - d_n(\tau_2)| \leq C \varepsilon \alpha, \quad (6.33)$$

and (6.31) follows. The proof of (6.32) is similar, using (6.30) instead of (6.29).

Step 3. We prove that for large n ,

$$-\bar{d}_n = \frac{1}{2} \bar{a}_n + \frac{1}{2} \bar{b}_n - \frac{N-2}{2N} \bar{c}_n + \mathcal{O}(\varepsilon), \quad (6.34)$$

$$\bar{a}_n - \bar{b}_n + \bar{c}_n = \mathcal{O}(\varepsilon), \quad (6.35)$$

$$-\frac{N}{2} \bar{a}_n + \left(\frac{N}{2} - 1\right) (\bar{b}_n - \bar{c}_n) = \bar{d}_n + \mathcal{O}(\varepsilon). \quad (6.36)$$

Proof of (6.34). Integrate (6.28) between $t_n - \alpha/10$ and t_n to obtain

$$-\bar{d}_n = \frac{10}{\alpha} \int \varphi_\alpha(x) x_1 e(u_n)(t_n, x) dx - \frac{10}{\alpha} \int \varphi_\alpha(x) x_1 e(u_n)\left(t_n - \frac{\alpha}{10}, x\right) dx + \mathcal{O}(\varepsilon). \quad (6.37)$$

We have, for large n ,

$$\int \varphi(x) x_1 e(u_n)(t_n, x) dx = \mathcal{O}(\alpha \varepsilon). \quad (6.38)$$

Indeed, by the definition of μ_0 (see (6.24)),

$$\begin{aligned} \lim_{n \rightarrow \infty} \int \varphi_\alpha(x) |x_1| \left(|\nabla_{t,x} u_n|(t_n, x) + \frac{1}{|x|^2} |u(t_n, x)|^2 + |u(t_n, x)|^{\frac{2N}{N-2}} \right) dx \\ = \int |x_1| \varphi_\alpha(x) d\mu_0(x), \end{aligned}$$

and $|\int |x_1| \varphi_\alpha(x) d\mu_0(x)| \lesssim \varepsilon \alpha$ by Lemma 6.4 and the bound $|\varphi_\alpha(x)| \lesssim \alpha$.

Furthermore, by the change of variable $x = y - \frac{\alpha}{10} e_1$,

$$\begin{aligned} \int \varphi_\alpha(x) x_1 e(u_n) \left(t_n - \frac{\alpha}{10}, x \right) dx \\ = \int \varphi_\alpha \left(y - \frac{\alpha}{10} e_1 \right) \left(y_1 - \frac{\alpha}{10} \right) e(u_n) \left(t_n - \frac{\alpha}{10}, y - \frac{\alpha}{10} e_1 \right) dy \\ = \int y_1 \varphi_\alpha \left(y - \frac{\alpha}{10} e_1 \right) e(u_n) \left(t_n - \frac{\alpha}{10}, y - \frac{\alpha}{10} e_1 \right) dy \\ - \frac{\alpha}{10} \int \varphi_\alpha \left(y - \frac{\alpha}{10} e_1 \right) e(u_n) \left(t_n - \frac{\alpha}{10}, y - \frac{\alpha}{10} e_1 \right) dy. \end{aligned}$$

By (6.25) and Lemma 6.4, the limit of the first term on the right-hand side is bounded (up to a multiplicative constant) by

$$\left| \int |y_1| \varphi_\alpha(y - (\alpha/10)e_1) d\mu_1(y) \right| \lesssim \varepsilon \alpha.$$

The second term can be rewritten as

$$\begin{aligned} - \frac{\alpha}{10} \int \varphi_\alpha \left(y - \frac{\alpha}{10} e_1 \right) e(u_n) \left(t_n - \frac{\alpha}{10}, y - \frac{\alpha}{10} e_1 \right) dy \\ = - \frac{\alpha}{10} \int \varphi_\alpha(x) e(u_n) \left(t_n - \frac{\alpha}{10}, x \right) dx \\ = - \frac{\alpha}{10} \left(\frac{1}{2} a_n \left(t_n - \frac{\alpha}{10} \right) + \frac{1}{2} b_n \left(t_n - \frac{\alpha}{10} \right) - \frac{N-2}{2N} c_n \left(t_n - \frac{\alpha}{10} \right) \right). \end{aligned}$$

Going back to (6.37), we obtain

$$-\bar{d}_n = \frac{1}{2} a_n \left(t_n - \frac{\alpha}{10} \right) + \frac{1}{2} b_n \left(t_n - \frac{\alpha}{10} \right) - \frac{N-2}{2N} c_n \left(t_n - \frac{\alpha}{10} \right) + \mathcal{O}(\varepsilon),$$

which yields (6.34) in view of the approximate conservation of energy (6.32) proved in Step 2.

Proof of (6.35). Integrating (6.26) between $t_n - \alpha/10$ and t_n , we are reduced to proving

$$\left| \int \varphi_\alpha(x) u_n(t_n, x) \partial_t u_n(t_n, x) dx - \int \varphi_\alpha(x) u_n \left(t_n - \frac{\alpha}{10}, x \right) \partial_t u_n \left(t_n - \frac{\alpha}{10}, x \right) dx \right| \lesssim \varepsilon \alpha \quad (6.39)$$

for large n . This follows easily from Lemma 6.4. For example

$$\begin{aligned} & \left| \int \varphi_\alpha(x) u_n \left(t_n - \frac{\alpha}{10}, x \right) \partial_t u_n \left(t_n - \frac{\alpha}{10}, x \right) dx \right| \\ & \leq \left(\int \varphi_\alpha(x) \frac{1}{|x + \frac{\alpha}{10} e_1|} |u_n|^2 \left(t_n - \frac{\alpha}{10}, x \right) dx \right)^{1/2} \\ & \quad \times \left(\int \varphi_\alpha(x) \left| x + \frac{\alpha}{10} e_1 \right| |\partial_t u_n|^2 \left(t_n - \frac{\alpha}{10}, x \right) dx \right)^{1/2} \end{aligned}$$

and

$$\begin{aligned} & \int \varphi_\alpha(x) \frac{1}{|x + \frac{\alpha}{10} e_1|} |u_n|^2 \left(t_n - \frac{\alpha}{10}, x \right) dx \\ & = \int \varphi_\alpha \left(y - \frac{\alpha}{10} e_1 \right) |y| \frac{1}{|y|^2} |u_n|^2 \left(t_n - \frac{\alpha}{10}, y - \frac{\alpha}{10} e_1 \right) dy \\ & \lesssim \left| \int |y| \varphi_\alpha \left(y - \frac{\alpha}{10} e_1 \right) d\mu_1(y) \right| \lesssim \alpha \varepsilon \end{aligned}$$

by (6.25) and Lemma 6.4. The estimates of the other terms are similar.

Proof of (6.36). We integrate (6.27) between $t_n - \alpha/10$ and t_n , obtaining

$$\begin{aligned} & -\frac{N}{2} \bar{a}_n + \left(\frac{N}{2} - 1 \right) (\bar{b}_n - \bar{c}_n) \\ & = \mathcal{O}(\varepsilon) + \frac{10}{\alpha} \left(- \int \varphi_\alpha x \cdot \nabla u_n \left(t_n - \frac{\alpha}{10} \right) \partial_t u_n \left(t_n - \frac{\alpha}{10} \right) + \int \varphi_\alpha x \cdot \nabla u_n(t_n) \partial_t u_n(t_n) \right). \end{aligned} \tag{6.40}$$

By computations similar to the ones above, the right-hand side of (6.40) is given by

$$\mathcal{O}(\varepsilon) + \int \varphi_\alpha(x) \partial_{x_1} u_n \left(t_n - \frac{\alpha}{10}, x \right) \partial_t u_n \left(t_n - \frac{\alpha}{10}, x \right) = \mathcal{O}(\varepsilon) + d_n \left(t_n - \frac{\alpha}{10} \right) = \mathcal{O}(\varepsilon) + \bar{d}_n.$$

At the last line we have used Step 2 to replace $d_n(t_n - \alpha/10)$ by $\bar{d}_n + \mathcal{O}(\varepsilon)$.

Step 4. End of the proof. Subtracting (6.34) and (6.36), we obtain

$$\left(\frac{N}{2} - \frac{1}{2} \right) (\bar{b}_n - \bar{a}_n) - \left(\frac{N}{2} - \frac{1}{2} - \frac{1}{N} \right) \bar{c}_n = \mathcal{O}(\varepsilon)$$

for large n . Adding $(N/2 - 1/2)(6.35)$ we deduce

$$\bar{c}_n = \mathcal{O}(\varepsilon). \tag{6.41}$$

Combining this with (6.34), we obtain

$$\frac{1}{2} \bar{a}_n + \frac{1}{2} \bar{b}_n + \bar{d}_n = \mathcal{O}(\varepsilon). \tag{6.42}$$

Hence, for large n ,

$$\frac{1}{\alpha} \int_{t_n - \alpha/10}^{t_n} \int ((\partial_t u_n(t, x) + \partial_{x_1} u_n(t, x))^2 + |\nabla_{x'} u_n(t, x)|^2) \varphi_\alpha(x) dx dt = \mathcal{O}(\varepsilon), \quad (6.43)$$

where $\nabla_{x'} = (\partial_{x_2}, \dots, \partial_{x_N})$.

As a consequence, we obtain a sequence $\{t'_n\} \rightarrow \infty$ such that for all n , $t_n - \frac{\alpha}{10} \leq t'_n \leq t_n$ and

$$\int \left(((\partial_t u + \partial_{x_1} u)^2(t'_n, x) + |\nabla_{x'} u|^2(t'_n, x)) \varphi\left(\frac{x - (\bar{A} + t_n)e_1}{\alpha}\right) \right) dx \leq C\varepsilon. \quad (6.44)$$

Since

$$|x - (t'_n + A)e_1| \leq \frac{\alpha}{10} \implies |x - (A + t_n)e_1| \leq \frac{\alpha}{5} \implies \varphi\left(\frac{x - (A + t_n)e_1}{\alpha}\right) = 1,$$

we obtain (6.1) (renormalizing α and ε). The proof of Proposition 6.1 is complete. $\square$

7. Elimination of singular points and end of the proof

We are now ready to conclude the proof of Theorem 2. We will prove that $\mathcal{S}$ is empty, contradicting Proposition 4.2.

We argue by contradiction assuming (after a rotation in the space variable) $e_1 \in \mathcal{S}$. We let, for $(f, g) \in (\dot{H}^1 \times L^2)(\mathbb{R}^N)$,

$$\|(f, g)\|_{e_1}^2 = \|g + \partial_{x_1} f\|_{L^2}^2 + \sum_{j=2}^N \|\partial_{x_j} f\|_{L^2}^2. \quad (7.1)$$

Remark 7.1. If u_L is a solution to the linear wave equation with initial data in $\dot{H}^1 \times L^2$, then $\|(u_L(t), \partial_t u_L(t))\|_{e_1}$ is independent of t . Indeed,

$$\begin{aligned} \|(u_L(t), \partial_t u_L(t))\|_{e_1}^2 &= \|\partial_{x_1} u_L(t)\|_{L^2}^2 + \|\partial_t u_L(t)\|_{L^2}^2 \\ &\quad + 2 \int_{\mathbb{R}^N} \partial_{x_1} u_L(t) \partial_t u_L(t) + \sum_{k=2}^N \|\partial_{x_k} u_L(t)\|_{L^2}^2, \end{aligned} \quad (7.2)$$

and the conservation of $\|(u_L(t), \partial_t u_L(t))\|_{e_1}$ follows from energy and momentum conservation.

Remark 7.2. Denote by $(\cdot, \cdot)_{e_1}$ the scalar product associated to the norm $\|\cdot\|_{e_1}$. Let $\{(v_0^n, v_1^n)\}_n$ be a sequence bounded in $\dot{H}^1 \times L^2$ that has a profile decomposition $(U_L^j, \{t_{j,n}, x_{j,n}, \lambda_{j,n}\}_{j \geq 1})$. Then

$$\begin{aligned} j \neq k &\implies \lim_{n \rightarrow \infty} (\vec{U}_{L,n}^k(0), \vec{U}_{L,n}^j(0))_{e_1} = 0, \\ 1 \leq j \leq J &\implies \lim_{n \rightarrow \infty} (\vec{U}_{L,n}^j(0), \vec{w}_n^J(0))_{e_1} = 0, \end{aligned}$$

where w_n^J is the remainder of the profile decomposition (see (2.9)). This follows from Remark 7.1 and the same argument as the one used to prove the orthogonality of the energy of the profiles (see [7, Lemma 2.3] for a proof).

As a consequence, the following Pythagorean expansion holds: for all $J \geq 1$,

$$\|(v_0^n, v_1^n)\|_{e_1}^2 = \sum_{j=1}^J \|U_L^j(0)\|_{e_1}^2 + \|w_n^J(0)\|_{e_1}^2 + o(1) \quad \text{as } n \rightarrow \infty. \quad (7.3)$$

We will use the following claim, proved in Appendix D:

Claim 7.3. *Let $\beta, M > 0$. Then there exists $\varepsilon = \varepsilon(M, \beta) > 0$ such that if $(v_0, v_1) \in \dot{H}^1 \times L^2$ satisfies*

$$\|(v_0, v_1)\|_{e_1} \leq \varepsilon \quad \text{and} \quad \|(v_0, v_1)\|_{\dot{H}^1 \times L^2} \leq M, \quad (7.4)$$

then

$$\|S_L(t)(v_0, v_1)\|_{S(\mathbb{R})} \leq \beta. \quad (7.5)$$

Let $\varepsilon > 0$ be given by Claim 7.3 with $\beta = \delta_3/2$, δ_3 be given by Lemma 5.2 and

$$M = \sup_{t \geq 0} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2}.$$

By Proposition 6.1, there exists a sequence $\{t'_n\}_n \rightarrow +\infty$ and $\alpha > 0$ such that

$$\limsup_{n \rightarrow \infty} \int_{|x - (t'_n + \bar{A})e_1| < \alpha} \left((\partial_{x_1} u(t'_n) + \partial_t u(t'_n))^2 + \sum_{j=2}^N |\partial_{x_j} u(t'_n)|^2 \right) dx \leq \varepsilon. \quad (7.6)$$

By Lemma 5.2, there exists a subsequence of $\{t'_n\}_n$, still denoted by $\{t'_n\}_n$, such that $\{u(t'_n)\}_n$ has a profile decomposition $(U_L^j, \{t_{j,n}, x_{j,n}, \lambda_{j,n}\}_{j \geq 1})$ with the following properties:

$$\lim_{n \rightarrow \infty} (x_{1,n} - (\bar{A} + t'_n)e_1) = 0, \quad (7.7)$$

$$\lim_{n \rightarrow \infty} \lambda_{1,n} = \lim_{n \rightarrow \infty} t_{1,n} = 0, \quad (7.8)$$

$$\|U_L^1\|_{S(\mathbb{R})} \geq \delta_3. \quad (7.9)$$

By Remark 7.2,

$$\lim_{n \rightarrow \infty} (\vec{u}(t'_n), \vec{U}_{L,n}^1(0))_{e_1} = \|\vec{U}_L^1(0)\|_{e_1}^2. \quad (7.10)$$

Next, notice that it follows from (7.7) and (7.8) that

$$\lim_{n \rightarrow \infty} \int_{|x - (t'_n + \bar{A})e_1| \geq \alpha} |\nabla_{t,x} U_{L,n}^1(0)|^2 dx = 0. \quad (7.11)$$

Indeed, this integral can be rewritten as

$$\int_{|\lambda_{1,n}y + o(1)| \geq \alpha} \left| \nabla_{t,x} U_{L,n}^1 \left(\frac{-t_{1,n}}{\lambda_{1,n}}, y \right) \right|^2 dy, \quad (7.12)$$

where $o(1) = x_{1,n} - (\bar{A} + t'_n)e_1$ goes to 0 as $n \rightarrow \infty$ by (7.7). The desired limit (7.11) follows immediately if $-t_{1,n}/\lambda_{1,n}$ is bounded. If not, say if $\lim_{n \rightarrow \infty} -t_{1,n}/\lambda_{1,n} = -\infty$ after extraction of a subsequence, we can rewrite (7.12) as

$$2 \int_{|\lambda_{1,n}r + o(1)| \geq \alpha} \int_{S^{N-1}} \left| G_- \left(r + \frac{t_{1,n}}{\lambda_{1,n}}, \omega \right) \right|^2 d\omega dr$$

where $G_- \in L^2(\mathbb{R} \times S^{N-1})$ is the incoming radiation field associated to U_L^1 (see Theorem 2.1 and Remark 2.2). This integral goes to 0 as $n \rightarrow \infty$ since

$$\left| r + \frac{t_{1,n}}{\lambda_{1,n}} \right| \geq \frac{\alpha + o(1)}{\lambda_{1,n}} \xrightarrow{n \rightarrow \infty} +\infty$$

on the domain of integration (using $\lim_n t_{1,n} = 0$). This proves (7.11).

By (7.11), as $n \rightarrow \infty$,

$$\begin{aligned} & (\vec{u}(t'_n), \vec{U}_L^1(0))_{e_1} \\ & \leq \|\vec{U}_L^1(0)\|_{e_1} \sqrt{\int_{|x - (t'_n + \bar{A})e_1| \leq \alpha} (|\partial_{x_1} u(t'_n) + \partial_t u(t'_n)|^2 + \sum_{k=2}^N |\partial_{x_k} u(t'_n)|^2) + o(1)}. \end{aligned}$$

Combining with (7.6), we obtain

$$(\vec{u}(t'_n), \vec{U}_{L,n}^1(0))_{e_1} \leq \varepsilon \|U_L^1(0)\|_{e_1} + o(1) \quad \text{as } n \rightarrow \infty.$$

Hence by (7.10),

$$\|\vec{U}_L^1(0)\|_{e_1} \leq \varepsilon. \quad (7.13)$$

But then by the definition of ε (from Claim 7.3 with $\beta = \delta_3/2$ and $M = \sup_{t \geq 0} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2}$),

$$\|U_L^1\|_{S(\mathbb{R})} \leq \delta_3/2,$$

contradicting (7.9). The proof is complete. $\square$

Appendix A. Radiation field for linear wave equations

In this appendix we prove Theorem 2.1.

A.1. Introduction of a function space

We start by reformulating this theorem in terms of a space of functions on $\mathbb{R} \times S^{N-1}$ that we will define now. Let

$$\dot{H}_\eta^1(\mathbb{R} \times S^{N-1}) = \left\{ g \in C^0(\mathbb{R}, L^2(S^{N-1})) : \int_{\mathbb{R} \times S^{N-1}} |\partial_\eta g(\eta, \omega)|^2 d\eta d\omega < \infty \right\}.$$

Let $g \in \dot{H}_\eta^1$. We note that $\|\partial_\eta g\|_{L^2(\mathbb{R} \times S^{N-1})} = 0$ if and only if there exists $a \in L^2(S^{N-1})$ such that $g(\eta, \omega) = a(\omega)$ for all $\eta \in \mathbb{R}$ and almost all $\omega \in S^{N-1}$. We define $\dot{\mathcal{H}}_\eta^1$ as the quotient space of $\dot{H}_\eta^1$ by the equivalence relation

$$g \sim \tilde{g} \iff \exists a \in L^2(S^{N-1}), \forall \eta \in \mathbb{R}, g(\eta, \omega) - \tilde{g}(\eta, \omega) = a(\omega) \text{ for a.a. } \omega \in S^{N-1}.$$

We denote by $\bar{g} \in \dot{\mathcal{H}}_\eta^1$ the equivalence class of $g \in \dot{H}_\eta^1$, and we define the following norm on $\dot{\mathcal{H}}_\eta^1$:

$$\|\bar{g}\|_{\dot{\mathcal{H}}_\eta^1} = \|\partial_\eta g\|_{L^2(\mathbb{R} \times S^{N-1})}.$$

Then:

Proposition A.1. *The normed space $\dot{\mathcal{H}}_\eta^1$ is a Hilbert space, and $C_0^\infty(\mathbb{R} \times S^{N-1})$ is dense in $\dot{\mathcal{H}}_\eta^1$. The map $\bar{g} \mapsto \partial_\eta g$ is a bijective isometry from $\dot{\mathcal{H}}_\eta^1$ to $L^2(\mathbb{R} \times S^{N-1})$.*

We note that the proposition implies that $\dot{\mathcal{H}}_\eta^1$ is the closure of $C_0^\infty(\mathbb{R} \times S^{N-1})$ for the norm $\|\partial_\eta \cdot\|_{L^2(\mathbb{R} \times S^{N-1})}$. In view of Proposition A.1, the following is equivalent to Theorem 2.1:

Theorem A.2. *Assume $N \geq 3$ and let v be a solution of the linear wave equation (1.6) with initial data $(v_0, v_1) \in \dot{H}^1 \times L^2$. Then*

$$\lim_{t \rightarrow +\infty} \left(\left\| \frac{1}{r} \nabla_\omega v(t) \right\|_{L^2} + \left\| \frac{1}{r} v(t) \right\|_{L^2} \right) = 0 \quad (\text{A.1})$$

and there exists a unique $\bar{g} \in \dot{\mathcal{H}}_\eta^1$ such that

$$\lim_{t \rightarrow +\infty} \int_0^{+\infty} \int_{S^{N-1}} \left| \partial_{r,t} \left(r^{\frac{N-1}{2}} v(t, r\omega) - g(r-t, \omega) \right) \right|^2 d\omega dr = 0. \quad (\text{A.2})$$

Furthermore,

$$E_L(v_0, v_1) = \int_{\mathbb{R} \times S^{N-1}} |\partial_\eta g(\eta, \omega)|^2 d\eta d\omega = \|\bar{g}\|_{\dot{\mathcal{H}}_\eta^1}^2 \quad (\text{A.3})$$

and the map

$$\dot{H}^1 \times L^2 \rightarrow \dot{\mathcal{H}}_\eta^1, \quad (v_0, v_1) \mapsto \sqrt{2} \bar{g},$$

is a bijective isometry.

In this Appendix A we prove Theorem A.2 assuming Proposition A.1. We postpone the proof of Proposition A.1 to Appendix B.

A.2. The case of smooth, compactly supported functions

Lemma A.3. *Assume $(v_0, v_1) \in (C_0^\infty(\mathbb{R}^N))^2$ and let v be the corresponding solution of (1.6). Then there exists $F \in C^\infty(\mathbb{R} \times S^{N-1} \times [0, +\infty))$ such that*

$$\forall r > 0, \forall \omega \in S^{N-1}, \quad v(t, r\omega) = \frac{1}{r^{(N-1)/2}} F\left(r-t, \omega, \frac{1}{r}\right).$$

Proof. This is classical (see [12], [21], and also [1] for this exact statement), and can be proved using the explicit form of the solution of (1.6), distinguishing between even and odd dimensions. We give a proof relying on the conformal transformation which is independent of the dimension.

For $\rho \in \mathbb{R}$, $\omega \in S^{N-1}$, $\sigma \in (0, +\infty)$ we let

$$F(\rho, \omega, \sigma) = \frac{1}{\sigma^{(N-1)/2}} v\left(\frac{1}{\sigma} - \rho, \frac{\omega}{\sigma}\right).$$

Since v is smooth, F is smooth on $\mathbb{R} \times S^{N-1} \times (0, +\infty)$. We must prove that F can be extended to a smooth function on $\mathbb{R} \times S^{N-1} \times [0, +\infty)$.

Let $t_0 > 0$. We claim that there exists a C^∞ solution w of the linear wave equation (1.6), depending on t_0 , such that, for all x, t with $|x| > t - t_0$ and $t > t_0$,

$$v(t, x) = \frac{1}{(|x|^2 - (t - t_0)^2)^{(N-1)/2}} w\left(\frac{t - t_0}{|x|^2 - (t - t_0)^2}, \frac{x}{|x|^2 - (t - t_0)^2}\right). \quad (\text{A.4})$$

Indeed, let w be the solution of (1.6) with C^∞ initial data (w_0, w_1) at $t = 0$ given by

$$\left. \begin{aligned} w_0(y) &= \frac{1}{|y|^{N-1}} v\left(t_0, \frac{y}{|y|^2}\right) \\ w_1(y) &= \frac{1}{|y|^{N+1}} \partial_t v\left(t_0, \frac{y}{|y|^2}\right) \end{aligned} \right\} \text{ if } y \neq 0$$

and $w_0(0) = w_1(0) = 0$. Notice that since, by finite speed of propagation, $(v(t_0), \partial_t v(t_0))$ is compactly supported, the above definition yields C^∞ functions on $\mathbb{R}^N$ which are constant, equal to 0, in a neighbourhood of the origin.

We note that

$$\tilde{w}(\tau, y) = \frac{1}{(|y|^2 - \tau^2)^{(N-1)/2}} v\left(t_0 + \frac{\tau}{|y|^2 - \tau^2}, \frac{y}{|y|^2 - \tau^2}\right)$$

defines, for $|y| > \tau$, a C^∞ solution of the linear wave equation whose initial data at $\tau = 0$ equals (w_0, w_1) (at least for $|y| \neq 0$). Hence, by finite speed of propagation,

$$|y| > \tau \implies w(\tau, y) = \tilde{w}(\tau, y).$$

It remains to check that for $|x| > t - t_0$, $t > t_0$,

$$v(t, x) = \frac{1}{(|x|^2 - |t - t_0|^2)^{(N-1)/2}} \tilde{w}\left(\frac{t - t_0}{|x|^2 - (t - t_0)^2}, \frac{x}{|x|^2 - (t - t_0)^2}\right).$$

This follows easily from the definition of $\tilde{w}$ and the change of variables

$$t = t_0 + \frac{\tau}{|y|^2 - \tau^2}, \quad x = \frac{y}{|y|^2 - \tau^2}.$$

As a consequence of (A.4), going back to the definition of F , we obtain

$$\begin{aligned} (\sigma > 0, 0 < 1 - \sigma(\rho + t_0) \text{ and } \rho > -t_0) \implies \\ F(\rho, \omega, \sigma) = \frac{1}{(2(\rho + t_0) - (\rho + t_0)^2\sigma)^{(N-1)/2}} w \\ \times \left(\frac{1 - \sigma(\rho + t_0)}{2(\rho + t_0) - (\rho + t_0)^2\sigma}, \frac{\omega}{2(\rho + t_0) - (\rho + t_0)^2\sigma} \right) \end{aligned} \quad (\text{A.5})$$

However, the right-hand side of the above formula for F can be extended to a C^∞ function in the open set

$$\{(\rho, \omega, \sigma) \in \mathbb{R} \times S^{N-1} \times \mathbb{R} : \sigma(\rho + t_0) < 2 \text{ and } \rho > -t_0\}$$

which includes the set $(-t_0, +\infty) \times S^{N-1} \times \{0\}$. As a consequence, F can be extended to a C^∞ function in a neighbourhood of the set $(1 - t_0, +\infty) \times S^{N-1} \times [0, +\infty)$ and, since t_0 is arbitrarily large, to a neighbourhood of $\mathbb{R} \times S^{N-1} \times [0, +\infty)$. $\square$

Lemma A.4. *Let $N \geq 3$. Let $(v_0, v_1) \in (C_0^\infty(\mathbb{R}^N))^2$. Then*

$$\lim_{t \rightarrow +\infty} \left(\left\| \frac{1}{r} \nabla_\omega v(t) \right\|_{L^2} + \left\| \frac{1}{r} v(t) \right\|_{L^2} \right) = 0 \quad (\text{A.6})$$

and there exists $g \in \dot{H}_\eta^1$ such that

$$\lim_{t \rightarrow +\infty} \left\| \partial_{r,t} \left(r^{\frac{N-1}{2}} v(t, r\omega) - g(r - t, \omega) \right) \right\|_{L^2((0, +\infty) \times S^{N-1})} = 0. \quad (\text{A.7})$$

Let us mention that Lemma A.4 follows quite easily from Lemma A.3 if $N \geq 3$ is odd (in this case, $g \in C_0^\infty(\mathbb{R} \times S^{N-1})$). In the general case, we will need the following claim:

Claim A.5. *Let $(v_0, v_1) \in (C_0^\infty(\mathbb{R}^N))^2$, $N \geq 3$, and v the solution to the wave equation (1.6). Then $\|v(t)\|_{L^2}$ is bounded,*

$$\lim_{t \rightarrow +\infty} \int \frac{1}{|x|^2} |v(t, x)|^2 dx = 0 \quad (\text{A.8})$$

and

$$\lim_{R \rightarrow \infty} \limsup_{t \rightarrow +\infty} \int_{|x| \leq t-R} [|\nabla v(t, x)|^2 + (\partial_t v(t, x))^2] dx = 0. \quad (\text{A.9})$$

This is classical. We postpone the proof after the proof of Lemma A.4 (see also [7, Lemma 2.1]).

Proof of Lemma A.4. We let F be as in Lemma A.3 and $g(\eta, \omega) = F(\eta, \omega, 0)$. Note that $g \in C^\infty(\mathbb{R} \times S^{N-1})$.

Step 1. We prove that $g \in \dot{H}_\eta^1$, i.e.

$$\int_{\mathbb{R} \times S^{N-1}} |\partial_\eta g|^2 < \infty. \quad (\text{A.10})$$

We have $F(r - t, \omega, 1/r) = r^{\frac{N-1}{2}} v(t, r\omega)$. As a consequence, $F(\eta, \omega, \sigma) = 0$ if $\eta \geq M$ by finite speed of propagation, where $M > 0$ is such that

$$\text{supp}(v_0, v_1) \subset \{x \in \mathbb{R}^N : |x| \leq M\}. \quad (\text{A.11})$$

Let $A > 0$. We have

$$\begin{aligned} \int_{S^{N-1}} \int_{t-A}^{t+M} \left| \partial_\eta F\left(r - t, \omega, \frac{1}{r}\right) \right|^2 dr d\omega &= \int_{S^{N-1}} \int_{t-A}^{t+M} r^{N-1} (\partial_t v(t, r\omega))^2 dr d\omega \\ &\leq 2E_L(v_0, v_1). \end{aligned}$$

Hence

$$\int_{S^{N-1}} \int_{-A}^M \left| \partial_\eta F\left(\eta, \omega, \frac{1}{\eta + t}\right) \right|^2 d\eta d\omega \leq 2E_L(v_0, v_1).$$

Letting $t \rightarrow +\infty$, we obtain

$$\int_{S^{N-1}} \int_{-A}^{+\infty} |\partial_\eta g(\eta, \omega)|^2 d\eta d\omega = \int_{S^{N-1}} \int_{-A}^M |\partial_\eta F(\eta, \omega, 0)|^2 d\eta d\omega \leq 2E_L(v_0, v_1).$$

Since A can be taken arbitrarily large, we obtain (A.10).

Step 2. Proof of (A.6). By Claim A.5 and Step 1, it is sufficient to prove

$$\forall R > 0, \quad \lim_{t \rightarrow +\infty} \int_{|x| \geq t-R} \frac{1}{|x|^2} |\nabla_\omega v|^2 dx = 0, \quad (\text{A.12})$$

i.e.

$$\forall R > 0, \quad \lim_{t \rightarrow +\infty} \int_{S^{N-1}} \int_{t-R}^{t+M} \frac{1}{r^2} \left| \nabla_\omega F\left(r - t, \omega, \frac{1}{r}\right) \right|^2 dr d\omega = 0,$$

which follows easily from the fact that $F \in C^\infty(\mathbb{R} \times S^{N-1} \times [0, +\infty))$ and the change of variable $\eta = r - t$.

Step 3. Proof of (A.7). By Claim A.5 and Step 1, it is sufficient to prove

$$\begin{aligned} \forall R > 0, \quad \lim_{t \rightarrow +\infty} \int_{S^{N-1}} \int_{t-R}^{t+M} \left| \frac{\partial}{\partial r} \left(F\left(r - t, \omega, \frac{1}{r}\right) - F(r - t, \omega, 0) \right) \right|^2 dr d\omega \\ + \int_{S^{N-1}} \int_{t-R}^{t+M} \left| \frac{\partial}{\partial t} \left(F\left(r - t, \omega, \frac{1}{r}\right) - F(r - t, \omega, 0) \right) \right|^2 dr d\omega = 0. \end{aligned} \quad (\text{A.13})$$

As in Step 2, this follows from the change of variable $\eta = r - t$. $\square$

Lemma A.6. *Let (v_0, v_1) and g be as in Lemma A.4. Then*

$$E_L(v_0, v_1) = \|\bar{g}\|_{\dot{H}_\eta}^2.$$

Proof. We have

$$\begin{aligned} E_L(v, \partial_t v) &= \frac{1}{2} \int_0^\infty \int_{S^{N-1}} (\partial_r v)^2 r^{N-1} d\omega dr \\ &\quad + \frac{1}{2} \int_0^\infty \int_{S^{N-1}} \frac{1}{r^2} |\nabla_\omega v|^2 r^{N-1} d\omega dr + \frac{1}{2} \int_0^\infty \int_{S^{N-1}} (\partial_t v)^2 r^{N-1} d\omega dr. \end{aligned}$$

Notice that

$$\int_0^\infty \int_{S^{N-1}} (\partial_r v)^2 r^{N-1} d\omega dr = \int_0^{+\infty} \int_{S^{N-1}} (\partial_r (r^{\frac{N-1}{2}} v))^2 dr + o(1) \text{ as } t \rightarrow +\infty. \quad (\text{A.14})$$

Indeed, $\partial_r (r^{\frac{N-1}{2}} v) = r^{\frac{N-1}{2}} \partial_r v + \frac{N-1}{2} r^{\frac{N-3}{2}} v$, and (A.14) follows in view of (A.6). Using (A.6) again, we see that

$$\begin{aligned} E_L(v, \partial_t v) &= \frac{1}{2} \int_0^\infty \int_{S^{N-1}} (\partial_r (r^{\frac{N-1}{2}} v))^2 d\omega dr \\ &\quad + \frac{1}{2} \int_0^\infty \int_{S^{N-1}} (\partial_t v)^2 r^{N-1} d\omega dr + o(1) \text{ as } t \rightarrow +\infty. \end{aligned}$$

By (A.7),

$$\begin{aligned} E_L(v, \partial_t v) &= \int_0^\infty \int_{S^{N-1}} |\partial_\eta g(r-t, \omega)|^2 dr d\omega + o(1) \\ &= \int_{-t}^\infty \int_{S^{N-1}} |\partial_\eta g(\eta, \omega)|^2 d\eta d\omega + o(1) \\ &\xrightarrow{t \rightarrow +\infty} \int_{-\infty}^\infty \int_{S^{N-1}} |\partial_\eta g(\eta, \omega)|^2 d\eta d\omega, \end{aligned}$$

and we conclude using conservation of energy. $\square$

It remains to prove Claim A.5. We use the identity

$$\frac{d}{dt} \left(\int x \cdot \nabla v \partial_t v + \frac{N-1}{2} \int v \partial_t v \right) = -E_L(v_0, v_1).$$

(In all the proof of the claim, $\int$ denotes the integral on $\mathbb{R}^N$). Integrating between 0 and $t > 0$, we obtain

$$\int x \cdot \nabla v \partial_t v + \frac{N-1}{2} \int v \partial_t v = -t E_L(v_0, v_1) + \int x \cdot \nabla v_0 v_1 + \frac{N-1}{2} \int v_0 v_1. \quad (\text{A.15})$$

Since $N \geq 3$, $\dot{H}^{-1}(\mathbb{R}^N)$ is a Hilbert space, included in the space of tempered distributions, and $C_0^\infty(\mathbb{R}^N) \subset \dot{H}^{-1}(\mathbb{R}^N)$. By conservation of the $L^2 \times \dot{H}^{-1}$ norm of $(v, \partial_t v)$, we deduce that the L^2 norm of v remains bounded. Using (A.15), we deduce

$$\left| \int x \cdot \nabla v \partial_t v \right| \geq t E_L(v_0, v_1) - C \quad (\text{A.16})$$

for some constant $C > 0$ depending on (v_0, v_1) .

Let $M > 0$ be such that $|x| \leq M$ on the support of (v_0, v_1) . By finite speed of propagation, $|x| \leq M + |t|$ on the support of $(v(t), \partial_t v(t))$. Let $R > 0$. By the Cauchy–Schwarz inequality,

$$\begin{aligned} \left| \int x \cdot \nabla v \partial_t v \right| &\leq (t + M) \int_{|x| \geq t-R} |\nabla v \partial_t v| + (t - R) \int_{|x| \leq t-R} |\nabla v \partial_t v| \\ &\leq t E_L(v_0, v_1) + M E_L(v_0, v_1) - \frac{R}{2} \int_{|x| \leq t-R} |\nabla_{t,x} v|^2. \end{aligned}$$

Combining this with (A.16), we obtain

$$C + M E_L(v_0, v_1) \geq \frac{R}{2} \int_{|x| \leq t-R} |\nabla_{t,x} v|^2, \quad (\text{A.17})$$

which yields as announced

$$\lim_{R \rightarrow \infty} \limsup_{t \rightarrow +\infty} \int_{|x| \leq t-R} |\nabla_{t,x} v|^2 = 0. \quad (\text{A.18})$$

It remains to prove

$$\lim_{t \rightarrow +\infty} \int \frac{1}{|x|^2} |v|^2 = 0. \quad (\text{A.19})$$

Since

$$\sup_{t \geq 0} \int |v(t)|^2 < \infty,$$

it is sufficient to prove

$$\forall A > 0, \quad \lim_{t \rightarrow +\infty} \int_{|x| \leq A} \frac{1}{|x|^2} |v|^2 = 0.$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that $\varphi(x) = 1$ if $|x| \leq 1$ and $\varphi(x) = 0$ if $|x| \geq 2$. Then

$$\int \frac{1}{|x|^2} \varphi\left(\frac{x}{\sqrt{t}}\right) |v|^2 \leq \int \left| \nabla \left(\varphi\left(\frac{x}{\sqrt{t}}\right) u \right) \right|^2 \leq \frac{C}{t} \int |v|^2 + 2 \int \left| \varphi\left(\frac{x}{\sqrt{t}}\right) \nabla v \right|^2.$$

The first term on the right-hand side goes to 0 as $t \rightarrow \infty$ since the L^2 norm of v is bounded. The second one goes to 0 by (A.18). The proof is complete. $\square$

A.3. General case

Here we prove Theorem A.2.

A.3.1. Existence and uniqueness. Let $(v_0, v_1) \in \dot{H}^1 \times L^2$. We argue by density, considering a sequence $\{(v_{0,n}, v_{1,n})\}_n$ in $(C_0^\infty(\mathbb{R}^N))^2$ such that

$$\lim_{n \rightarrow \infty} \|(v_{0,n} - v_0, v_{1,n} - v_1)\|_{\dot{H}^1 \times L^2} = 0. \quad (\text{A.20})$$

Let $\bar{g}_n \in \dot{\mathcal{H}}_\eta^1$ be given by Lemma A.4, corresponding to $(v_{0,n}, v_{1,n})$. By the energy identity in Lemma A.6, the sequence $\{\bar{g}_n\}_n$ is a Cauchy sequence in $\dot{\mathcal{H}}_\eta^1$. Since, by Proposition A.1, $\dot{\mathcal{H}}_\eta^1$ is complete, the sequence has a limit $\bar{g}$ in $\dot{\mathcal{H}}_\eta^1$.

Let $\varepsilon > 0$. Choose n such that

$$\|v_{0,n} - v_0\|_{\dot{H}^1}^2 + \|v_{1,n} - v_1\|_{L^2}^2 < \varepsilon. \quad (\text{A.21})$$

By conservation of energy and (A.6) in Lemma A.4, we deduce

$$\lim_{t \rightarrow +\infty} \left(\left\| \frac{1}{r} v(t) \right\|_{L^2}^2 + \left\| \frac{1}{r} \nabla_\omega v(t) \right\|_{L^2}^2 \right) < C\varepsilon, \quad (\text{A.22})$$

which yields (A.1) by letting $\varepsilon \rightarrow 0$.

Fixing again ε and n such that (A.21) holds and

$$\|\bar{g}_n - \bar{g}\|_{\dot{\mathcal{H}}_\eta^1}^2 < \varepsilon, \quad (\text{A.23})$$

we obtain, by (A.21), conservation of energy and (A.1),

$$\limsup_{t \rightarrow +\infty} \int_{[0, +\infty) \times S^{N-1}} \left(\left| \frac{\partial}{\partial r} (r^{\frac{N-1}{2}} v_n - r^{\frac{N-1}{2}} v) \right|^2 + \left| \frac{\partial}{\partial t} (r^{\frac{N-1}{2}} v_n - r^{\frac{N-1}{2}} v) \right|^2 \right) dr d\omega < \varepsilon. \quad (\text{A.24})$$

Hence, by the definition of g_n ,

$$\limsup_{t \rightarrow +\infty} \int_{[0, +\infty) \times S^{N-1}} |\partial_{r,t} (g_n(r-t, \omega) - r^{\frac{N-1}{2}} v(t, r\omega))|^2 dr d\omega < \varepsilon. \quad (\text{A.25})$$

By (A.23) we can replace g_n by g in (A.25) (changing ε to 4ε on the right-hand side). Letting $\varepsilon \rightarrow 0$, we obtain (A.2).

Using

$$\lim_{n \rightarrow \infty} E_L(v_{0,n}, v_{1,n}) = E_L(v_0, v_1) \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\bar{g}_n\|_{\dot{\mathcal{H}}_\eta^1} = \|\bar{g}\|_{\dot{\mathcal{H}}_\eta^1}$$

we see that Lemma A.6 implies the energy identity (A.3). Of course (A.3) implies the uniqueness of $\bar{g}$ and that the map $(v_0, v_1) \mapsto \sqrt{2} \bar{g}$ is an isometry from $\dot{H}^1 \times L^2$ to $\dot{\mathcal{H}}_\eta^1$.

A.3.2. Proof of surjectivity. We next let $\bar{g} \in \dot{\mathcal{H}}_\eta^1$ and construct v satisfying (1.6) and such that (A.2) holds. Since $E_L(v_0, v_1) = \|\bar{g}\|_{\dot{\mathcal{H}}_\eta^1}^2$, the image of the map $(v_0, v_1) \mapsto \bar{g}$ is closed in $\dot{\mathcal{H}}_\eta^1$. Using the density of $C_0^\infty(\mathbb{R} \times S^{N-1})$ in $\dot{\mathcal{H}}_\eta^1$, we see that it is sufficient to prove the existence of v for $g \in C_0^\infty(\mathbb{R} \times S^{N-1})$.

We look for v of the form

$$v(t, x) = \frac{1}{r^{(N-1)/2}} g(t-r, \omega) + \epsilon(t, x) \quad (\text{A.26})$$

for large t , with $\omega = x/|x|$, and

$$\lim_{t \rightarrow +\infty} \|(\epsilon, \partial_t \epsilon)(t)\|_{\dot{H}^1 \times L^2} = 0. \quad (\text{A.27})$$

Since

$$\begin{aligned} (\partial_t^2 - \Delta) \left(\frac{1}{r^{(N-1)/2}} g(t-r, \omega) \right) &= \frac{N^2 - 4N + 3}{4r^{(N+3)/2}} g(t-r, \omega) - \frac{1}{r^{(N+3)/2}} \Delta_\omega g(t-r, \omega) \\ &=: H(t, x), \end{aligned}$$

equation (1.6) is equivalent to

$$(\partial_t^2 - \Delta) \epsilon = -H.$$

Let $L \gg 1$ be such that $\text{supp } g \subset [-L, L] \times S^{N-1}$. Then $H \in L^1([L+1, +\infty), L^2(\mathbb{R}^N))$. Indeed, if $t \geq L+1$, then

$$\begin{aligned} &\left\| \frac{1}{r^{(N+3)/2}} g(t-r, \omega) \right\|_{L^2(\mathbb{R}^N)}^2 + \left\| \frac{1}{r^{(N+3)/2}} \Delta_\omega g(t-r, \omega) \right\|_{L^2(\mathbb{R}^N)}^2 \\ &\lesssim \int_{t-L}^{t+L} \frac{1}{r^{N+3}} r^{N-1} dr \lesssim \frac{1}{(t-L)^3} - \frac{1}{(t+L)^3} \approx \frac{1}{t^4} \quad \text{as } t \rightarrow +\infty. \end{aligned}$$

Letting

$$\epsilon(t, x) = \int_t^{+\infty} \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} H(s, x) ds,$$

we obtain ϵ satisfying (A.27) and such that v defined by (A.26) is a solution of (1.6).

By (A.26) and (A.27),

$$\left\| \partial_{r,t} \left(r^{\frac{N-1}{2}} v - g(t-r, \omega) \right) \right\|_{L^2} = \left\| \partial_{t,r} \left(r^{\frac{N-1}{2}} \epsilon \right) \right\|_{L^2} \xrightarrow{t \rightarrow +\infty} 0,$$

which concludes the proof. We have used the fact that

$$\int_0^{+\infty} (\partial_r (r^{\frac{N-1}{2}} \epsilon))^2 dr = \int_0^{+\infty} (\partial_r \epsilon)^2 r^{N-1} dr$$

if $\epsilon \in \dot{H}^1(\mathbb{R}^N)$.

A.4. Radiation fields and channels of energy

We conclude this appendix with a remark on the relation between exterior energy estimates (see [9, Proposition 2.8]) and radiation fields. Assume that N is odd. It follows from the explicit formula for the solution of the wave equation in terms of spherical means that the outgoing radiation field G_+ of a solution v of the linear wave equation with initial data $(v_0, v_1) \in \dot{H}^1 \times L^2$ has the following symmetry properties:

- if $v_1 = 0$ then $G_+(\eta, \omega) = (-1)^{(N+1)/2} G_+(-\eta, -\omega)$,
- if $v_0 = 0$ then $G_+(\eta, \omega) = (-1)^{(N-1)/2} G_+(-\eta, -\omega)$

(see for example the asymptotic formulas in [9, proof of Lemma 2.9]). In both cases, we obtain

$$\begin{aligned} \lim_{t \rightarrow +\infty} \int_{|x| \geq |t|} |\nabla_{t,x} v(t, x)|^2 dx &= \int_{\mathbb{R}} \int_{S^{N-1}} |G_+(\eta, \omega)|^2 d\omega d\eta \\ &= \frac{1}{2} \int [|\nabla v_0(x)|^2 + |v_1(x)|^2] dx, \end{aligned}$$

which yields the exterior energy estimate for the linear wave equation in odd space dimension (see [9, Proposition 2.8]).

Appendix B. Study of a function space

Here we prove Proposition A.1.

B.1. Completeness

Let $\{\bar{g}_n\}_n$ be a Cauchy sequence in $\dot{\mathcal{H}}_\eta^1$. Replacing $g_n(\eta, \omega)$ by $g_n(\eta, \omega) - g_n(0, \omega)$ we can always assume

$$\forall \omega \in S^{N-1}, \quad g_n(0, \omega) = 0.$$

As a consequence, if $\eta \in \mathbb{R}$ and $n, p \in \mathbb{N}$, then

$$\begin{aligned} \int_{S^{N-1}} |g_n(\eta, \omega) - g_p(\eta, \omega)|^2 d\omega &= \int_{S^{N-1}} \left| \int_0^\eta (\partial_\eta g_n(\eta', \omega) - \partial_\eta g_p(\eta', \omega)) d\eta' \right|^2 d\omega \\ &\leq |\eta| \int_{S^{N-1}} \int_0^\eta (\partial_\eta g_n(\eta', \omega) - \partial_\eta g_p(\eta', \omega))^2 d\eta' d\omega. \end{aligned}$$

Thus, for all $M > 0$, $\{g_n\}_n$ is a Cauchy sequence in $C^0([-M, M], L^2(S^{N-1}))$. As a consequence, there exists $g \in C^0(\mathbb{R}, L^2(S^{N-1}))$ such that $\{g_n\}_n$ converges to g locally in $L^\infty(\mathbb{R}, L^2(S^{N-1}))$.

Since $\{\partial_\eta g_n\}_n$ is a Cauchy sequence in $L^2(\mathbb{R} \times S^{N-1})$, it converges to some $h \in L^2(\mathbb{R} \times S^{N-1})$ in this space. Letting $n \rightarrow \infty$ in the equality

$$g_n(\eta, \omega) = \int_0^\eta \partial_\eta g_n(\eta', \omega) d\eta' \quad \text{for a.a. } \omega \in S^{N-1}$$

at fixed η , and taking the limit in $L^2(S^{N-1})$, we see that

$$g(\eta, \omega) = \int_0^\eta h(\eta', \omega) d\eta' \quad \text{for a.a. } \omega \in S^{N-1}.$$

Thus $\partial_\eta g = h \in L^2(\mathbb{R} \times S^{N-1})$ and, by the definition of h ,

$$\lim_{n \rightarrow \infty} \|\partial_\eta g - \partial_\eta g_n\|_{L^2(\mathbb{R} \times S^{N-1})} = 0.$$

B.2. Density of compactly supported, smooth functions

Step 1. We let $\varphi \in C_0^\infty(\mathbb{R})$ be such that $\varphi(\eta) = 1$ if $|\eta| \leq 1$ and $\varphi(\eta) = 0$ if $|\eta| \geq 2$. Let $g \in \dot{H}_\eta^1$. In this step we prove

$$\lim_{R \rightarrow \infty} \left\| \frac{\partial}{\partial \eta} \left(g - \varphi\left(\frac{\cdot}{R}\right)g \right) \right\|_{L^2(\mathbb{R} \times S^{N-1})} = 0.$$

Indeed,

$$\begin{aligned} & \left\| \frac{\partial}{\partial \eta} \left(g - \varphi\left(\frac{\cdot}{R}\right)g \right) \right\|_{L^2(\mathbb{R} \times S^{N-1})} \\ & \leq \left\| \left(1 - \varphi\left(\frac{\cdot}{R}\right) \right) \partial_\eta g \right\|_{L^2(\mathbb{R} \times S^{N-1})} + \frac{1}{R} \left\| \varphi'\left(\frac{\cdot}{R}\right)g \right\|_{L^2(\mathbb{R} \times S^{N-1})}. \end{aligned} \quad (\text{B.1})$$

The first term of the right-hand side goes to 0 as $R \rightarrow \infty$ by dominated convergence. We next treat the second term. We have

$$\frac{1}{R^2} \left\| \varphi'\left(\frac{\cdot}{R}\right)g \right\|_{L^2}^2 \leq \frac{C}{R^2} \int_R^{2R} \int_{S^{N-1}} |g(\eta, \omega)|^2 d\omega d\eta. \quad (\text{B.2})$$

Let $A \gg 1$. Then, for $\eta > A$,

$$g(\eta, \omega) = g(A, \omega) + \int_A^\eta \partial_\eta g(\eta', \omega) d\eta' \quad \text{for a.a. } \omega \in S^{N-1}.$$

Hence,

$$|g(\eta, \omega)|^2 \leq 2\eta \int_A^{+\infty} |\partial_\eta g(\eta', \omega)|^2 d\eta' + 2|g(A, \omega)|^2 \quad \text{for a.a. } \omega \in S^{N-1}.$$

Integrating on S^{N-1} , we obtain

$$\int_{S^{N-1}} |g(\eta, \omega)|^2 d\omega \leq 2\eta \|\partial_\eta g\|_{L^2([A, +\infty) \times S^{N-1})}^2 + 2\|g(A, \cdot)\|_{L^2(S^{N-1})}^2.$$

For $R > A$, we have

$$\frac{1}{R^2} \int_R^{2R} \int_{S^{N-1}} |g(\eta, \omega)|^2 d\omega d\eta \leq 4\|\partial_\eta g\|_{L^2([A, +\infty) \times S^{N-1})}^2 + \frac{2}{R} \|g(A, \cdot)\|_{L^2(S^{N-1})}^2.$$

Letting $R \rightarrow \infty$, we obtain

$$\limsup_{R \rightarrow \infty} \frac{1}{R^2} \int_R^{2R} \int_{S^{N-1}} |g(\eta, \omega)|^2 d\omega d\eta \leq 4\|\partial_\eta g\|_{L^2([A, +\infty) \times S^{N-1})}^2,$$

and thus, since A is arbitrarily large,

$$\lim_{R \rightarrow \infty} \frac{1}{R^2} \int_R^{2R} \int_{S^{N-1}} |g(\eta, \omega)|^2 d\omega d\eta = 0.$$

In view of (B.1) and (B.2), this concludes Step 1.

Step 2. It remains to prove that a compactly supported function $g \in C^0(\mathbb{R}, L^2(S^{N-1}))$ such that $\partial_\eta g \in L^2(\mathbb{R} \times S^{N-1})$ can be approximated in the $\dot{\mathcal{H}}_\eta^1$ norm by smooth functions. This can be done using convolution with approximations of the identity (in the variable η) and projecting on the n first eigenspaces of the Laplace–Beltrami operator Δ_ω on S^{N-1} . We leave the details to the reader.

B.3. Isometry with L^2

Let $\Phi : \bar{g} \mapsto \partial_\eta g$. It follows obviously from the definition of $\dot{H}_\eta^1$ that Φ is a well defined, injective isometry. If $G \in L^2(\mathbb{R} \times S^{N-1})$, then

$$g(\eta, \omega) = \int_0^\eta G(\omega, \eta') d\omega$$

is an element of $\dot{H}_\eta^1$ which satisfies $\Phi(\bar{g}) = G$, which proves that the map is also surjective. The proof of Proposition A.1 is complete.

Appendix C. Profiles and estimates on Strichartz norms

Recall that $S(\mathbb{R}) = L^{\frac{N+2}{N-2}}(\mathbb{R}, L^{\frac{2(N+2)}{N-2}}(\mathbb{R}^N))$.

Claim C.1. *Let $\{(u_{0,n}, u_{1,n})\}_n$ be a sequence in $\dot{H}^1 \times L^2$ such that*

$$\limsup_{n \rightarrow \infty} \|(u_{0,n}, u_{1,n})\|_{\dot{H}^1 \times L^2} = M. \quad (\text{C.1})$$

Assume furthermore

$$\forall j, \quad \|U_L^j\|_{S(\mathbb{R})} \leq \delta_0. \quad (\text{C.2})$$

Then

$$\limsup_{n \rightarrow \infty} \|S_L(\cdot)(u_{0,n}, u_{1,n})\|_{S(\mathbb{R})} \leq \begin{cases} CM^{3/4}\delta_0^{1/4} & \text{if } N = 3, \\ CM^{\frac{2N-4}{N+2}}\delta_0^{\frac{6-N}{N+2}} & \text{if } N = 4, 5. \end{cases}$$

Proof. We use the following Pythagorean expansion:

$$\lim_{n \rightarrow \infty} \|S_L(\cdot)(u_{0,n}, u_{1,n})\|_{L^{\frac{2(N+1)}{N-2}}(\mathbb{R} \times \mathbb{R}^N)}^{\frac{2(N+1)}{N-2}} = \sum_{j \geq 1} \|U_L^j\|_{L^{\frac{2(N+1)}{N-2}}(\mathbb{R} \times \mathbb{R}^N)}^{\frac{2(N+1)}{N-2}}. \quad (\text{C.3})$$

Next, we notice that by the Hölder and Sobolev inequalities,

$$\|U_L^j\|_{L^{\frac{2(N+1)}{N-2}}(\mathbb{R} \times \mathbb{R}^N)}^{\frac{2(N+1)}{N-2}} \leq \|U_L^j\|_{L^\infty(\mathbb{R}, L^{\frac{2N}{N-2}})}^{\frac{N}{N-2}} \|U_L^j\|_{S(\mathbb{R})}^{\frac{N+2}{N-2}}. \quad (\text{C.4})$$

If $N = 3$, we deduce

$$\|U_L^j\|_{L_{t,x}^8}^8 \leq C \|\tilde{U}_L^j(0)\|_{\dot{H}^1 \times L^2}^2 M \delta_0^5.$$

Hence, by the Pythagorean expansion (C.3),

$$\limsup_{n \rightarrow \infty} \|S_L(\cdot)(u_{0,n}, u_{1,n})\|_{L_{t,x}^8}^8 \leq CM \delta_0^5 \sum_{j \geq 1} \|\tilde{U}_L^j(0)\|_{\dot{H}^1 \times L^2}^2 \leq CM^3 \delta_0^5.$$

By Hölder's inequality and the Strichartz inequality,

$$\begin{aligned} \|S_L(\cdot)(u_{0,n}, u_{1,n})\|_{L_t^5 L_x^{10}} &\leq \|S_L(\cdot)(u_{0,n}, u_{1,n})\|_{L_t^4 L_x^{12}}^{3/5} \|S_L(\cdot)(u_{0,n}, u_{1,n})\|_{L_{t,x}^8}^{2/5}, \\ \limsup_{n \rightarrow \infty} \|S_L(\cdot)(u_{0,n}, u_{1,n})\|_{L_t^5 L_x^{10}} &\leq C M^{3/5} M^{3/8 \times 2/5} \delta_0^{5/8 \times 2/5} = C M^{3/4} \delta_0^{1/4}. \end{aligned}$$

If $N \in \{4, 5\}$, we combine (C.4) with the Strichartz and Sobolev inequalities to obtain

$$\|U_L^j\|_{L_{t,x}^{\frac{2(N+1)}{N-2}}}^{\frac{2(N+1)}{N-2}} \leq C \|\tilde{U}_L^j(0)\|_{\dot{H}^1 \times L^2}^2 \|U_L^j\|_{S(\mathbb{R})}^{\frac{6}{N-2}} \leq C \|\tilde{U}_L^j(0)\|_{\dot{H}^1 \times L^2}^2 \delta_0^{\frac{6}{N-2}}. \quad (\text{C.5})$$

Summing up, we deduce

$$\sum_{j \geq 1} \|U_L^j\|_{L_{t,x}^{\frac{2(N+1)}{N-2}}}^{\frac{2(N+1)}{N-2}} \leq C M^2 \delta_0^{\frac{6}{N-2}}. \quad (\text{C.6})$$

Hence

$$\limsup_{n \rightarrow \infty} \|S_L(\cdot)(u_{0,n}, u_{1,n})\|_{L_{t,x}^{\frac{2(N+1)}{N-2}}}^{\frac{2(N+1)}{N-2}} \leq C M^2 \delta_0^{\frac{6}{N-2}}. \quad (\text{C.7})$$

By Hölder's inequality,

$$\|S_L(\cdot)(u_{0,n}, u_{1,n})\|_{S(\mathbb{R})} \leq \|S_L(\cdot)(u_{0,n}, u_{1,n})\|_{L_{t,x}^{\frac{2(N+1)}{N-2}}}^{\theta} \|S_L(\cdot)(u_{0,n}, u_{1,n})\|_{L^2(\mathbb{R}, L^{\frac{2N}{N-3}})}^{1-\theta}, \quad (\text{C.8})$$

where $\theta = \frac{(N+1)(6-N)}{3(N+2)}$, $1 - \theta = \frac{N(N-2)}{3(N+2)}$.

Hence

$$\|S_L(\cdot)(u_{0,n}, u_{1,n})\|_{S(\mathbb{R})} \leq C \delta_0^{\frac{6-N}{N+2}} M^{\frac{2N-4}{N+2}}. \quad \square$$

Appendix D. Dispersion for solutions with small e_1 -norm

In this appendix we prove Claim 7.3. We argue by contradiction. Assume that for all $n > 0$, there exists $(v_{0,n}, v_{1,n}) \in \dot{H}^1 \times L^2$ such that

$$\|(v_{0,n}^n, v_{1,n}^n)\|_{e_1} \leq 1/n, \quad \|(v_{0,n}, v_{1,n})\|_{\dot{H}^1 \times L^2} \leq M \quad (\text{D.1})$$

and

$$\|v_L^n\|_{S(\mathbb{R})} > \beta, \quad (\text{D.2})$$

where $v_L^n(t) = S_L(t)(v_{0,n}^n, v_{1,n}^n)$. Extracting subsequences, we can assume that $\{(v_{0,n}^n, v_{1,n}^n)\}$ has a profile decomposition $(U_L^j, \{t_{j,n}, x_{j,n}, \lambda_{j,n}\}_{j \geq 1})$. By the Pythagorean expansion of the norm $\|\cdot\|_{e_1}$ (see Remark 7.2), for all $J \geq 1$,

$$\frac{1}{n} \geq \|(v_{0,n}^n, v_{1,n}^n)\|_{e_1}^2 = \sum_{j=1}^J \|\tilde{U}_L^j(0)\|_{e_1}^2 + \|w_n^J(0)\|_{e_1}^2 + o(1)$$

as $n \rightarrow \infty$. As a consequence, $U_L^j = 0$ for all $j \geq 1$ and $\lim_n \|v_L^n\|_{S(\mathbb{R})} = 0$, which contradicts (D.2).

Acknowledgments. Research of T. Duyckaerts was partially supported by ERC Grant Blowdisol 291214 and French ANR Grant SchEq ANR-12-JS01-0005-01.

Research of C. Kenig was partially supported by NSF Grants DMS-1265429 and DMS-1463746.

Research of F. Merle was partially supported by ERC Grant Blowdisol 291214.

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