Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

The 2-dimensional stable homotopy hypothesis ☆

Nick Gurski [a,*], Niles Johnson [b], Angélica M. Osorno [c]

[a] *Department of Mathematics, Applied Mathematics, and Statistics, Case Western Reserve University, United States of America*
[b] *Department of Mathematics, The Ohio State University, Newark, United States of America*
[c] *Department of Mathematics, Reed College, United States of America*

A B S T R A C T

We prove that the homotopy theory of Picard 2-categories is equivalent to that of stable 2-types.

© 2019 Elsevier B.V. All rights reserved.

1. Introduction

Grothendieck's Homotopy Hypothesis posits an equivalence of homotopy theories between homotopy n-types and weak n-groupoids. We pursue a similar vision in the stable setting. Inspiration for a stable version of the Homotopy Hypothesis begins with [38,30] which show, for 1-categories, that symmetric monoidal structures give rise to infinite loop space structures on their classifying spaces. Thomason [42] proved this is an equivalence of homotopy theories, relative to stable homotopy equivalences. This suggests that the categorical counterpart to stabilization is the presence of a symmetric monoidal structure with all cells invertible, an intuition that is reinforced by a panoply of results from the group-completion theorem of May [31] to the Baez-Dolan stabilization hypothesis [1,6] and beyond. A stable homotopy n-type is a spectrum with nontrivial homotopy groups only in dimensions 0 through n. The corresponding symmetric

monoidal n-categories with invertible cells are known as Picard n-categories. We can thus formulate the Stable Homotopy Hypothesis.

Stable Homotopy Hypothesis. *There is an equivalence of homotopy theories between $\mathcal{Pic}^n$, Picard n-categories equipped with categorical equivalences, and $\mathcal{Sp}_0^n$, stable homotopy n-types equipped with stable equivalences.*

For $n = 0$, the Stable Homotopy Hypothesis is the equivalence of homotopy theories between abelian groups and Eilenberg–Mac Lane spectra. The case $n = 1$ is described by the second two authors in [25]. Beyond proving the equivalence of the homotopy theories, they constructed a dictionary in which the algebraic invariants of the stable homotopy 1-type (the two homotopy groups and the unique k-invariant) can be read directly from the Picard category. Moreover, they gave a construction of the stable 1-type of the sphere spectrum.

The main result of this paper, Theorem 6.1, is the Stable Homotopy Hypothesis for $n = 2$. In this case, the categorical equivalences are biequivalences. The advantage of being able to work with categorical equivalences is that the maps in the homotopy category between two stable 2-types modeled by strict Picard 2-categories are realized by symmetric monoidal pseudofunctors—not general zigzags. In fact, the set of homotopy classes of maps between two strict Picard 2-categories $\mathcal{D}$ and $\mathcal{D}'$ is the quotient of the set of symmetric monoidal pseudofunctors $\mathcal{D} \to \mathcal{D}'$ modulo monoidal pseudonatural equivalence.

In future work we will develop 2-categorical models for the 2-type of the sphere and for fiber/cofiber sequences of stable 2-types. We can apply these to give algebraic expressions for the secondary operations arising from a stable Postnikov tower and for the low-dimensional algebraic K-groups of a commutative ring. Moreover, via the theory of cofibers (cokernels) associated with a Postnikov tower, we may shed new light on the theory of symmetric monoidal tricategories.
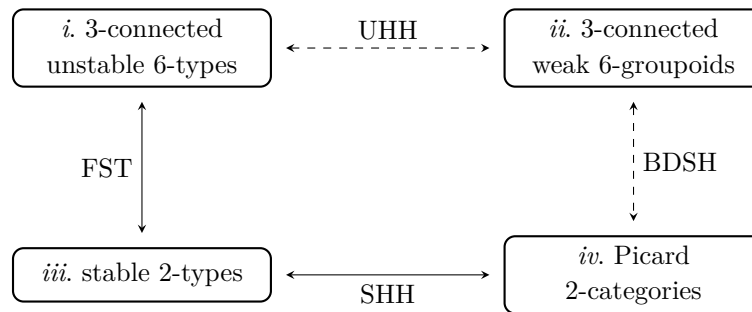
Our proof of the 2-dimensional stable homotopy hypothesis is a culmination of previous work in [20] and [21]. Although we have attempted to make the current account as self-contained as possible, we rely heavily on this and other previous work. We include selective reviews as needed. The proof of the main theorem functions as an executive summary of the paper, and the reader may find it helpful to begin reading there.

1.1. The homotopy hypothesis and categorical stability

The Stable Homotopy Hypothesis (SHH) in dimension 2 establishes an equivalence between items (iii) and (iv) of the following conjecturally equivalent homotopy theories:

  i. 3-connected topological 6-types and topological weak equivalences;
 ii. weak 6-groupoids with only a single cell in dimensions 0 to 3 and 6-categorical equivalences;
iii. stable topological 2-types and stable weak equivalences; and
iv. Picard 2-categories and 2-categorical equivalences.

The equivalence of homotopy theories between items (i) and (iii) is an immediate consequence of the Freudenthal Suspension Theorem (FST). Equivalences between (i) and (ii) or between (ii) and (iv) are the subjects of, respectively, the Unstable Homotopy Hypothesis (UHH) and the Baez-Dolan Stabilization Hypothesis (BDSH). Both of these have been studied in lower dimensions, but nothing approaching dimension 6 for fully algebraic notions of higher category has appeared as of this writing. The following schematic diagram sketches these statements.

```
┌─────────────────────┐      UHH       ┌─────────────────────┐
│ i. 3-connected      │ ←──────────→   │ ii. 3-connected     │
│ unstable 6-types    │                │ weak 6-groupoids    │
└─────────────────────┘                └─────────────────────┘
        ↑                                        ↑
   FST  │                                   BDSH │
        ↓                                        ↓
┌─────────────────────┐                ┌─────────────────────┐
│ iii. stable 2-types │ ←──────────→   │ iv. Picard          │
│                     │      SHH       │ 2-categories        │
└─────────────────────┘                └─────────────────────┘
```

Analogues of the Baez-Dolan Stabilization Hypothesis have been proved for Tamsamani's weak n-categories [39,40], for certain algebras over operads in (∞, n)-categories [12], and for Rezk's Θ_n-categories [6]. However, it is unknown how these homotopical notions of higher categories compare with fully algebraic weak n-categories for $n > 2$ (see [28] for a comparison in dimension 2). In particular, work of Cheng and the first author [9,10] shows clearly that, in codimension greater than 1, additional subtleties arise even in the *formulation* of the Stabilization Hypothesis for fully algebraic notions of weak n-category. In the absence of a comparison between algebraic and homotopical notions of higher category, using the homotopical definitions like Θ_n-categories does not address the core motivation for the (Stable) Homotopy Hypothesis: to compare fully topological notions with fully algebraic ones.

1.2. Outline

We begin with necessary topological background in Section 2, particularly recalling the theory of group-completion and an elementary consequence of the relative Hurewicz theorem. Next we recall the relevant algebra of symmetric monoidal structures on 2-categories in Section 3, including a discussion of both the fully weak case (symmetric monoidal bicategories) and what might be called the semi-strict case (permutative Gray-monoids).

The core construction in this paper is a "Picardification." That is, the construction of a Picard 2-category from a general permutative Gray-monoid, while retaining the same stable homotopy groups in dimensions 0, 1, 2. This entails a group-completion, and to apply previous work on group-completion we develop an independent theory of symmetric monoidal bicategories arising from E_∞ algebras in 2-categories in Section 4. This theory is an extension of techniques first developed by the first and third authors for the little n-cubes operad [22].

Our subsequent analysis in Section 5 uses the fundamental 2-groupoid of Moerdijk–Svensson [33] (Section 5.1), the K-theory for 2-categories developed in [20] (Section 5.2), and the topological group-completion theorem of May [31] (Section 5.3). We combine these to conclude with the proof of the main theorem in Section 6.

1.3. Acknowledgments

2. Topological background

In this section we review basic topological background needed for the work in this paper. We will use topological spaces built using the geometric realization of a simplicial nerve for 2-categories, and we begin by fixing notation and reviewing the monoidal properties of the relevant functors. We then turn to group completion and Postnikov truncations, both of which play a key role in this work.

2.1. Topological spaces and nerves of 2-categories

Notation 2.1. For spaces, we work in the category of compactly-generated weak Hausdorff spaces and denote this category *Top*.

Notation 2.2. We let *sSet* denote the category of simplicial sets.

Notation 2.3. We let $|-|$ and S denote, respectively, the geometric realization and singular functors between simplicial sets and topological spaces.

Notation 2.4. We let *Cat* denote the category of categories and functors, and let *2Cat* denote the category of 2-categories and 2-functors. Note that these are both 1-categories.

The category of 2-categories admits a number of morphism variants, and it will be useful for us to have separate notations for these.

Notation 2.5. We let $2Cat_{\mathrm{ps}}$ denote the category of 2-categories with pseudofunctors and let $2Cat_{\mathrm{nps}}$ denote the category of 2-categories with normal pseudofunctors, that is, pseudofunctors which preserve the identities strictly. We let $2Cat_{\mathrm{nop}}$ denote the category of 2-categories and normal oplax functors. Note that these are all 1-categories.

The well-known nerve construction extends to 2-categories (in fact to general bicategories) in a number of different but equivalent ways [16,7].

Notation 2.6. We let N denote the nerve functor from categories to simplicial sets. By abuse of notation, we also let N denote the 2-dimensional nerve on $2Cat_{\mathrm{ps}}$. This nerve has 2-simplices given by 2-cells whose target is a composite of two 1-cells, as in the display below.
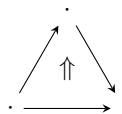
This is the nerve used by [33] in their study of the Whitehead 2-groupoid (see Section 5.1). A detailed study of this nerve, together with 9 other nerves for bicategories, appears in [7] with further work in [8].

Proposition 2.7. *The functors*

- $|-|\colon sSet \to Top$
- $S\colon Top \to sSet$
- $N\colon 2Cat_{ps} \to sSet$

are strong symmetric monoidal with respect to cartesian product. The adjunction between geometric realization and the singular functor is monoidal in the sense that the unit and the counit are monoidal natural transformations.

Proof. The fact that S and N preserve products follows from the fact that they are right adjoints. The statements about $|-|$ and the (co)unit are standard (see for example [11, Section 3.4]), but depend on good categorical properties of compactly-generated Hausdorff spaces. $\square$

Notation 2.8. We will write B for the classifying space of a 2-category, so $B\mathcal{C} = |N\mathcal{C}|$ for a 2-category $\mathcal{C}$.

2.2. Group-completions and Postnikov truncations

In this section we review and fix terminology for group-completion, Postnikov truncation, and the attendant notions of equivalence for spaces.

Definition 2.9. A map of homotopy associative and homotopy commutative H-spaces $f\colon X \to Y$ is a *(topological) group-completion* if

- $\pi_0(Y)$ is a group and $f_*\colon \pi_0(X) \to \pi_0(Y)$ is an algebraic group-completion, and
- for any field of coefficients k, the map

$$H_*(X;k)[\pi_0(X)^{-1}] \to H_*(Y;k)$$

induced by f_* is an isomorphism.

Remark 2.10. The group-completion of a given homotopy associative and homotopy commutative H-space X, if one exists, is unique up to weak homotopy equivalence by the Whitehead theorem. The definition was motivated by the work of Barratt and Barratt–Priddy [2,4] and of Quillen [35], who proved that for a homotopy commutative simplicial monoid M, the map $M \to \Omega BM$ satisfies the homology condition of Definition 2.9. The work of [31,38] constructs group-completions for E_∞-spaces, and both of these are foundational for results which we use in this paper (see Theorems 5.19 and 5.25).

Notation 2.11. Let X be a homotopy associative and homotopy commutative H-space. If a topological group-completion of X exists, we denote it by $X \to \Omega BX$.

Definition 2.12. Let P_n denote the nth Postnikov truncation on the category of spaces. This is a localizing functor, and the P_n-*equivalences* are those maps $f\colon X \to Y$ which induce isomorphisms on π_i for $0 \le i \le n$ and all choices of basepoint. We likewise define P_n-equivalences for maps of simplicial sets.

We will also require the slightly weaker, and more classical, notion of n-equivalence.

Definition 2.13. Let $n \ge 0$. A map of spaces $f\colon X \to Y$ is an *n-equivalence* if, for all choices of basepoint $x \in X$, the induced map

$$\pi_q(X, x) \to \pi_q(Y, f(x))$$

is a bijection for $0 \le q < n$ and a surjection for $q = n$. Note that this notion does not satisfy the 2-out-of-3 property in general.

Clearly every $(n+1)$-equivalence is a P_n-equivalence, and every P_n-equivalence can be replaced, via Postnikov truncation, by a zigzag of $(n+1)$-equivalences. Indeed, the collection of P_n-equivalences is the closure of the collection of $(n+1)$-equivalences with respect to the 2-out-of-3 property.

Remark 2.14. For a map of spaces $f\colon X \to Y$, the following are equivalent.

- The map f is an n-equivalence.
- For all choices of basepoint $x \in X$, the homotopy fiber of f over x is an $(n-1)$-connected space.

- The pair (Mf, X) is n-connected, where Mf denotes the mapping cylinder of f. That is, $\pi_0(X)$ surjects onto $\pi_0(Mf)$ and the relative homotopy groups $\pi_q(Mf, X)$ are trivial for $0 < q \leq n$.

We require the following result connecting n-equivalences with group-completion, particularly the subsequent corollary. The case $n = \infty$ follows from Whitehead's theorem, but we have not discovered a reference for finite n. We give a proof below.

Proposition 2.15. *Let $n \geq 0$ and let $f : X \to Y$ be an E_∞ map. If f is an n-equivalence then the group-completion $\Omega Bf : \Omega BX \to \Omega BY$ is an n-equivalence.*

Definition 2.16. A space Y is said to be *k-coconnected* if $\pi_i Y = 0$ for $i \geq k$.

Corollary 2.17. *Let $n \geq 0$ and let $f : X \to Y$ be an E_∞ map. If f is a P_n-equivalence and Y is $(n+1)$-coconnected, then $\Omega Bf : \Omega BX \to \Omega BY$ is a P_n-equivalence.*

Our proof of Proposition 2.15 makes use of the relative Hurewicz theorem, specifically a corollary below which we have also not discovered in the literature.

Theorem 2.18 *(Relative Hurewicz [23, Theorem 4.37]). Suppose (X, A) is an $(n - 1)$-connected pair of path-connected spaces with $n \geq 2$ and $x_0 \in A$, and suppose that $\pi_1(A, x_0)$ acts trivially on $\pi_1(X, A, x_0)$. Then the Hurewicz homomorphism*

$$\pi_i(X, A, x_0) \to H_i(X, A)$$

is an isomorphism for $i \leq n$.

Remark 2.19. This result can be extended to the case when the action of $\pi_1(A, x_0)$ is nontrivial, and is stated as such in [23]. We will not need that additional detail.

Definition 2.20. We say that a map $f : X \to Y$ is a *homology-n-equivalence* if $H_q(f)$ is an isomorphism for $q < n$ and a surjection for $q = n$.

Corollary 2.21 *(Hurewicz for maps). Let $f : X \to Y$ be a map of path-connected spaces. If f is an n-equivalence with $n \geq 1$ then f is a homology-n-equivalence. When X and Y are path-connected H-spaces and f is an H-map, then the converse also holds.*

Proof. Consider the comparison of long exact sequences below. The condition that f is an n-equivalence is equivalent to the condition that $\pi_i(Mf, X, x_0) = 0$ for $i \leq n$. The condition that f is a homology n-equivalence is equivalent to the condition that $H_i(Mf, X) = 0$ for $i \leq n$.

$$\cdots \to \pi_{i+1}(Mf, X, x_0) \longrightarrow \pi_i(X, x_0) \longrightarrow \pi_i(Mf, x_0) \longrightarrow \pi_i(Mf, X, x_0) \longrightarrow \cdots$$
$$\cdots \longrightarrow H_{i+1}(Mf, X) \longrightarrow H_i(X) \longrightarrow H_i(Mf) \longrightarrow H_i(Mf, X) \longrightarrow \cdots$$

Therefore the first statement is a direct consequence of the relative Hurewicz theorem. The second holds also by the relative Hurewicz theorem because the assumption that f is an H-map implies that the induced action of $\pi_1(X, x_0)$ on $\pi_1(Mf, X, x_0)$ is trivial [41, Section 9.2]. $\quad\square$

Proof of Proposition 2.15. If f is an n-equivalence, each path component of f is an n-equivalence, so by Corollary 2.21 each path component of f is a homology-n-equivalence. Therefore f itself is a homology-n-equivalence. This implies the group-completion, $\Omega B f$, is a homology-n-equivalence because on homology, the map

$$H_* \Omega B f \colon H_*(X)[\pi_0(X)^{-1}] \longrightarrow H_*(Y)[\pi_0(Y)^{-1}]$$

is the map induced by localizing $H_* f$ with respect to π_0, and localization is exact. Since f is an E_∞ map, it induces an H-map between the unit components of $\Omega B X$ and $\Omega B Y$. By the converse part of Corollary 2.21 for the unit component of $\Omega B f$, this unit component must be an n-equivalence. Lastly, in any group-complete H-space, translation by any point x induces a homotopy equivalence between the basepoint component and the component of x. Hence all of the components of are all homotopy equivalent, and therefore $\Omega B f$ is an n-equivalence. $\square$

Notation 2.22. We let $\mathcal{S}p_0$ denote the category of connective spectra, i.e., the full subcategory of spectra consisting of those objects X with $\pi_n X = 0$ for all $n < 0$.

Definition 2.23. We say that a map $f \colon X \to Y$ of connective spectra is a *stable P_n-equivalence* when the conditions of Definition 2.12 hold for stable homotopy groups; i.e., f induces an isomorphism on stable homotopy groups π_i for $0 \leq i \leq n$. We let *st P_n-eq* denote the class of stable P_n-equivalences.

3. Symmetric monoidal algebra in dimension 2

One has a number of distinct notions of symmetric monoidal algebra in dimension 2, and it will be necessary for us to work with several of these. The most general form is the notion of symmetric monoidal bicategory, and we outline essential details of this structure in Section 3.1. Several of our constructions make use of a stricter notion arising as monoids in *2Cat*, and these are reviewed in Section 3.2.

One also has various levels of strength for morphisms, both with respect to functoriality and with respect to the monoidal structure. In this paper, we can work solely with those morphisms of symmetric monoidal bicategories—either strict functors or pseudofunctors—which preserve the symmetric monoidal structure strictly (see Definition 3.10). In contrast with the weakest notion of morphism, that of symmetric monoidal pseudofunctor, these stricter variants all enjoy composition which is strictly associative and unital.

There are many good reasons to consider versions which are stricter than the most general possible notion. The most obvious is that the stricter structures are easier to work with, and in this case often allow the use of techniques from the highly-developed theory of 2-categories. The second reason we work with a variety of stricter notions is that many of these have equivalent *homotopy theories* to that of the fully weak version; we address this point in Section 3.3. Even if some construction does not preserve a particular strict variant of symmetric monoidal bicategory, but outputs a different variant with the same homotopy theory, we can still make use of the stricter setting. Finally, stricter notions usually admit more transparent constructions; the various K-theory functors for symmetric monoidal bicategories in [34,22,20] provide an excellent example, with stricter variants admitting simpler K-theory functors.

3.1. Background about symmetric monoidal bicategories

In this section we review the minimal necessary content from the theory of symmetric monoidal bicategories so that the reader can understand our construction of symmetric monoidal structure from operad actions in Section 4. More complete details can be found in [32,37,27,10].

Convention 3.1. We always use *transformation* to mean pseudonatural transformation (which we will only indicate via components) and *equivalence* to mean pseudonatural (adjoint) equivalence, that is, a pseudonatural transformation with inverse up to isomorphisms satisfying triangle identities.

**Definition 3.2** *(Sketch, see [32], [37, Definition 2.3], or [10]).* A *symmetric monoidal bicategory* consists of

- a bicategory $\mathcal{B}$,
- a tensor product pseudofunctor $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$, denoted by concatenation,
- a unit object $e \in \mathrm{ob}\mathcal{B}$,
- an associativity equivalence $\alpha\colon (xy)z \simeq x(yz)$,
- unit equivalences $l\colon ex \simeq x$ and $r\colon x \simeq xe$,
- invertible modifications π, μ, λ, ρ as follows,

- a braid equivalence $\beta\colon xy \simeq yx$,
- two invertible modifications (denoted $R_{-|--}$, $R_{--|-}$) which correspond to two instances of the third Reidemeister move,
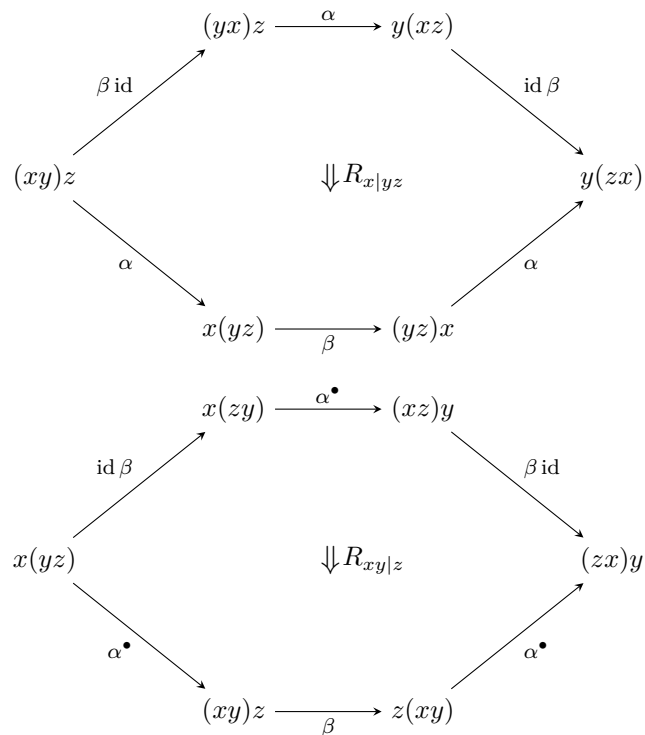
- and an invertible modification (the syllepsis, v)
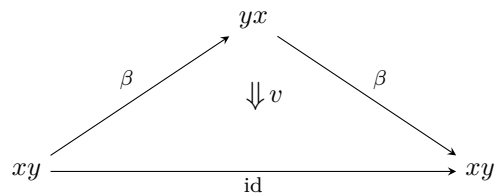
satisfying three axioms for the monoidal structure, four axioms for the braided structure, two axioms for the sylleptic structure, and one final axiom for the symmetric structure.

Definition 3.3 *(Sketch, see [37, Definition 2.5]).* A *symmetric monoidal pseudofunctor* $F \colon \mathcal{B} \to \mathcal{C}$ consists of

- a pseudofunctor $F \colon \mathcal{B} \to \mathcal{C}$,
- a unit equivalence $e_{\mathcal{C}} \simeq F(e_{\mathcal{B}})$,
- an equivalence for the tensor product $FxFy \simeq F(xy)$,
- three invertible modifications between composites of the unit and tensor product equivalences, and
- an invertible modification comparing the braidings in $\mathcal{B}$ and $\mathcal{C}$

satisfying two axioms for the monoidal structure, two axioms for the braided structure, and one axiom for the symmetric (and hence subsuming the sylleptic) structure.

Definition 3.4 *(Sketch, see [37, Definition 2.7]).* A *symmetric monoidal transformation* $\eta \colon F \longrightarrow G$ consists of

- a transformation $\eta \colon F \longrightarrow G$, and

- two invertible modifications concerning the interaction between η and the unit objects on the one hand and the tensor products on the other

satisfying two axioms for the monoidal structure and one axiom for the symmetric structure (and hence subsuming the braided and sylleptic structures).

The following is verified in [37]. Note that we have not defined symmetric monoidal modifications as we will not have any reason to use them in any of our constructions.

Lemma 3.5. *There is a tricategory* SMB *of symmetric monoidal bicategories, symmetric monoidal pseudofunctors, symmetric monoidal transformations, and symmetric monoidal modifications.*

We will need to know when symmetric monoidal pseudofunctors or transformations are invertible in the appropriate sense.

Definition 3.6. A *symmetric monoidal biequivalence* $F\colon \mathcal{B} \to \mathcal{C}$ is a symmetric monoidal pseudofunctor such that the underlying pseudofunctor F is a biequivalence of bicategories, i.e., has an inverse up to pseudonatural equivalence.

Definition 3.7. A *symmetric monoidal equivalence* $\eta\colon F \longrightarrow G$ between symmetric monoidal pseudofunctors is a symmetric monoidal transformation $\eta\colon F \longrightarrow G$ such that the underlying transformation η is an equivalence. This is logically equivalent to the condition that each component 1-cell $\eta_b\colon Fb \to Gb$ is an equivalence 1-cell in $\mathcal{C}$.

The results of [17] can be used to easily prove the following lemma, although the first part is also verified by elementary means in [37].

Lemma 3.8. *Let $F, G\colon \mathcal{B} \to \mathcal{C}$ be symmetric monoidal pseudofunctors, and $\eta\colon F \longrightarrow G$ a symmetric monoidal transformation between them.*

- $F\colon \mathcal{B} \to \mathcal{C}$ *is a symmetric monoidal biequivalence if and only if it is an internal biequivalence in the tricategory* SMB.
- $\eta\colon F \longrightarrow G$ *is a symmetric monoidal equivalence if and only if it is an internal equivalence in the bicategory* SMB$(\mathcal{B}, \mathcal{C})$.

We have defined a symmetric monoidal biequivalence to be a symmetric monoidal pseudofunctor which is also a biequivalence. The content of this lemma is that the weak inverse can also be chosen to be symmetric monoidal, as well as all the accompanying transformations and modifications.

Definition 3.9. Let Ho SMB denote the category of symmetric monoidal bicategories with morphisms given by equivalence classes of symmetric monoidal pseudofunctors under the relation given by symmetric monoidal pseudonatural equivalence. Note that in this category, every symmetric monoidal biequivalence is an isomorphism.

Definition 3.10. A *strictly symmetric monoidal pseudofunctor* $F\colon \mathcal{B} \to \mathcal{C}$ between symmetric monoidal bicategories is a pseudofunctor of the underlying bicategories that preserves the symmetric monoidal structure strictly, and for which all of the constraints are either the identity (when this makes sense) or the unique coherence isomorphism obtained from the coherence theorem for pseudofunctors [26,18]. A *strict functor* is a strictly symmetric monoidal pseudofunctor for which the underlying pseudofunctor is strict.

Remark 3.11. There is a monad on the category of 2-globular sets whose algebras are symmetric monoidal bicategories. Strict functors can then be identified with the morphisms in the Eilenberg–Moore category for this monad, and in particular symmetric monoidal bicategories with strict functors form a category. This point of view is crucial to the methods employed in [37].

Notation 3.12. The principal variants we will use are listed below.

- We let $\mathcal{SMBicat}_{ps}$ denote the category of symmetric monoidal bicategories and strictly symmetric monoidal pseudofunctors. Note that the composition of these is given by the composite of the underlying pseudofunctors and then the unique choice of coherence cells making them strictly symmetric.
- We let $\mathcal{SMBicat}_s$ denote the subcategory of $\mathcal{SMBicat}_{ps}$ whose morphisms are strict functors.
- We let $\mathcal{SM2Cat}_{ps}$, respectively $\mathcal{SM2Cat}_s$, denote the full subcategories of $\mathcal{SMBicat}_{ps}$, respectively $\mathcal{SMBicat}_s$, with objects whose underlying bicategory is a 2-category.

We note two subtleties regarding subcategories of strict functors. The first is that the inverse of a strictly symmetric monoidal strict biequivalence is not necessarily itself strict. However, we will see in Corollary 3.34 that the homotopy category obtained by inverting strict biequivalences in $\mathcal{SMBicat}_s$ is equivalent to $\mathsf{Ho}\,\mathsf{SMB}$.

Second, note that the multiplication map of a symmetric monoidal bicategory or 2-category $\mathcal{A}$ is a pseudofunctor

$$\mathcal{A} \times \mathcal{A} \to \mathcal{A}.$$

In both $\mathcal{SMBicat}_s$ and $\mathcal{SM2Cat}_s$, we consider strictly functorial morphisms which commute strictly with this multiplication pseudofunctor. The work in [20] shows that, relative to all stable equivalences, it is possible to restrict the structure further and still represent every stable homotopy type. Relative only to the categorical equivalences, however, we must retain some pseudofunctoriality in the multiplication.

3.2. Background on permutative Gray-monoids

In this section we give a definition that is a semi-strict version of symmetric monoidal bicategories. Here too we give the minimal necessary background for our current work. For details, see [14,13,18], or [20, Section 3].

Definition 3.13. Let $\mathcal{A}, \mathcal{B}$ be 2-categories. The *Gray tensor product* of $\mathcal{A}$ and $\mathcal{B}$, written $\mathcal{A} \otimes \mathcal{B}$ is the 2-category given by

- 0-cells consisting of pairs $a \otimes b$ with a an object of $\mathcal{A}$ and b an object of $\mathcal{B}$;
- 1-cells generated under composition by basic 1-cells of the form $f \otimes 1 : a \otimes b \to a' \otimes b$ for $f : a \to a'$ in $\mathcal{A}$ and $1 \otimes g : a \otimes b \to a \otimes b'$ for $g : b \to b'$ in $\mathcal{B}$; and
- 2-cells generated by basic 2-cells of the form $\alpha \otimes 1$ for 2-cells α in $\mathcal{A}$; $1 \otimes \delta$ for 2-cells δ in $\mathcal{B}$; and new 2-cells $\Sigma_{f,g} : (f \otimes 1)(1 \otimes g) \cong (1 \otimes g)(f \otimes 1)$.

These cells satisfy axioms related to composition, naturality and bilinearity; for a complete list, see [18, Section 3.1] or [20, Definition 3.16].

The assignment $(\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A} \otimes \mathcal{B}$ extends to a functor of categories

$$\mathcal{2Cat} \times \mathcal{2Cat} \longrightarrow \mathcal{2Cat}$$

which defines a symmetric monoidal structure on *2Cat*. The unit for this monoidal structure is the terminal 2-category. The Gray tensor product has a universal property that relates it to the notion of cubical functor.

Definition 3.14. Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{B}$ be 2-categories. A *cubical functor* $F\colon \mathcal{A}_1 \times \mathcal{A}_2 \longrightarrow \mathcal{B}$ is a normal pseudofunctor such that for all composable pairs (f_1, f_2), (g_1, g_2) of 1-cells in $\mathcal{A}_1 \times \mathcal{A}_2$, the comparison 2-cell

$$\phi\colon F(f_1, f_2) \circ F(g_1, g_2) \Rightarrow F(f_1 \circ g_1, f_2 \circ g_2)$$

is the identity whenever either f_1 or g_2 is the identity.

Theorem 3.15 *([18, Theorem 3.7], [20, Theorem 3.21]). Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be 2-categories. There is a cubical functor*

$$c\colon \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{A} \otimes \mathcal{B}$$

natural in $\mathcal{A}$ and $\mathcal{B}$, such that composition with c induces a bijection between cubical functors $\mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{C}$ and 2-functors $\mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{C}$.

Remark 3.16. There exists a 2-functor $i\colon \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{A} \times \mathcal{B}$ natural in $\mathcal{A}$ and $\mathcal{B}$ such that $i \circ c = \mathrm{id}$ and $c \circ i \cong \mathrm{id}$ (see [18, Corollary 3.22]). This map makes the identity functor Id on *2Cat* a lax symmetric monoidal functor

$$(2Cat, \times) \longrightarrow (2Cat, \otimes)$$

with the constraint

$$\mathrm{Id}(\mathcal{A}) \otimes \mathrm{Id}(\mathcal{B}) \to \mathrm{Id}(\mathcal{A} \times \mathcal{B})$$

given by i. Similarly, c gives the constraint that makes the inclusion

$$(2Cat, \otimes) \longrightarrow (2Cat_{\mathrm{ps}}, \times)$$

into a lax symmetric monoidal functor [19].

Definition 3.17. A *Gray-monoid* is a monoid object in $(2Cat, \otimes)$. This consists of a 2-category $\mathcal{C}$, a 2-functor

$$\oplus\colon \mathcal{C} \otimes \mathcal{C} \to \mathcal{C},$$

and an object e of $\mathcal{C}$ satisfying associativity and unit axioms.

Via the bijection in Theorem 3.15, we can view a Gray-monoid as a particular type of monoidal bicategory such that the monoidal product is a cubical functor and all the other coherence cells are identities [18, Theorem 8.12].

Definition 3.18. A *permutative Gray-monoid* $\mathcal{C}$ consists of a Gray-monoid $(\mathcal{C}, \oplus, e)$ together with a 2-natural isomorphism,

where $\tau \colon \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ is the symmetry isomorphism in *2Cat* for the Gray tensor product, such that the following axioms hold.

- The following pasting diagram is equal to the identity 2-natural transformation for the 2-functor $\oplus$.

- The following equality of pasting diagrams holds where we have abbreviated the tensor product to concatenation when labeling 1- or 2-cells.

Remark 3.19. In [20,21] the definition of permutative Gray-monoid includes a third axiom relating β to the unit e. This axiom is implied by the other two axioms and is therefore unnecessary.

Definition 3.20. A *strict functor* $F : \mathcal{C} \to \mathcal{D}$ of permutative Gray-monoids is a 2-functor $F : \mathcal{C} \to \mathcal{D}$ of the underlying 2-categories satisfying the following conditions.

- $F(e_{\mathcal{C}}) = e_{\mathcal{D}}$, so that F strictly preserves the unit object.
- The diagram

commutes, so that F strictly preserves the sum.
- The equation

$$\beta^{\mathcal{D}} * (F \otimes F) = F * \beta^{\mathcal{C}}$$

holds, so that F strictly preserves the symmetry. This equation is equivalent to requiring that

$$\beta^{\mathcal{D}}_{Fx,Fy} = F(\beta^{\mathcal{C}}_{x,y})$$

as 1-cells from $Fx \oplus Fy = F(x \oplus y)$ to $Fy \oplus Fx = F(y \oplus x)$.

Notation 3.21. The category of permutative Gray-monoids, $\mathcal{PGM}$, is the full subcategory of $\mathcal{SM2Cat}_s$ whose objects are permutative Gray-monoids.

The following two results follow from straightforward calculations and are used in Section 4.2.

Proposition 3.22. *The underlying 2-category functor $\mathcal{PGM} \to 2\mathcal{Cat}$ is monadic in the usual, 1-categorical sense.*

Lemma 3.23. *Let $F \colon X \times Y \to Z$ be a pseudofunctor between bicategories.*

- *For any object x of X, F induces a pseudofunctor $F(x, -) \colon Y \to Z$. The pseudofunctor $F(x, -)$ is strict if F is, hence a 2-functor if X, Y are 2-categories.*
- *For any 1-cell $f \colon x \to x'$, F induces a pseudonatural transformation $F(f, -)$ from $F(x, -)$ to $F(x', -)$; if f is an equivalence in X, then the pseudonatural transformation $F(f, -)$ is an equivalence. The transformation $F(f, -)$ is strict if F is, hence a 2-natural transformation if F is strict and X, Y are 2-categories; furthermore, if f is also an isomorphism then $F(f, -)$ is a 2-natural isomorphism.*
- *For any 2-cell $\alpha \colon f \Rightarrow f'$, F induces a modification $F(\alpha, -)$ from $F(f, -)$ to $F(f', -)$; if α is invertible in X, then the modification $F(\alpha, -)$ is an isomorphism.*

One uses the modifier "Picard" for symmetric monoidal algebra where all objects and morphisms are invertible. We have several notions in dimension 2, each consisting of those objects which have invertible 0-, 1-, and 2-cells.

Definition 3.24. Let $(\mathcal{D}, \oplus, e)$ be a Gray-monoid.

i. A 2-cell of $\mathcal{D}$ is invertible if it has an inverse in the usual sense.
ii. A 1-cell $f \colon x \to y$ is invertible if there exists a 1-cell $g \colon y \to x$ together with invertible 2-cells $g \circ f \cong \mathrm{id}_x$, $f \circ g \cong \mathrm{id}_y$. In other words, f is invertible if it is an internal equivalence (denoted with the $\simeq$ symbol) in $\mathcal{D}$.
iii. An object x of $\mathcal{D}$ is invertible if there exists another object y together with invertible 1-cells $x \oplus y \simeq e$, $y \oplus x \simeq e$.

Notation 3.25 *(Picard objects in dimension 2 [21, Definition 2.19]).*
- *$\mathcal{Pic}\,\mathcal{Bicat}_s$* denotes the full subcategory of $\mathcal{SMBicat}_s$ consisting of those symmetric monoidal bicategories with all cells invertible; we call these *Picard bicategories*.
- *$\mathcal{Pic}\,2\mathcal{Cat}_s$* denotes the full subcategory of $\mathcal{SM2Cat}_s$ consisting of symmetric monoidal 2-categories with all cells invertible; we call these *Picard 2-categories*.
- *$\mathcal{Pic}\,\mathcal{PGM}$* denotes the full subcategory of $\mathcal{PGM}$ consisting of those permutative Gray-monoids with all cells invertible; we call these *strict Picard 2-categories*.

3.3. The homotopy theory of symmetric monoidal bicategories

In this section we discuss the homotopy theories for symmetric monoidal algebra in dimension 2 and obtain a number of equivalence results. To begin, we recall quasistrictification results from [37] and [20], which show how to replace a symmetric monoidal bicategory with an appropriately equivalent permutative Gray-monoid.

Theorem 3.26 *([37, Theorem 2.97], [20, Theorem 3.14]). Let $\mathcal{B}$ be a symmetric monoidal bicategory.*

i. *There are two endofunctors, $\mathcal{B} \mapsto \mathcal{B}^c$ and $\mathcal{B} \mapsto \mathcal{B}^{qst}$, of $\mathcal{SMBicat}_s$. Any symmetric monoidal bicategory of the form $\mathcal{B}^{qst}$ is a permutative Gray-monoid.*

ii. *There are natural transformations $(-)^c \Rightarrow \mathrm{id}, (-)^c \Rightarrow (-)^{qst}$. When evaluated at a symmetric monoidal bicategory $\mathcal{B}$, these give natural strict biequivalences*

$$\mathcal{B} \leftarrow \mathcal{B}^c \rightarrow \mathcal{B}^{qst}.$$

iii. *For a symmetric monoidal pseudofunctor $F : \mathcal{B} \rightarrow \mathcal{C}$, there are strict functors $F^c : \mathcal{B}^c \rightarrow \mathcal{C}^c$, $F^{qst} : \mathcal{B}^{qst} \rightarrow \mathcal{C}^{qst}$ such that the right hand square below commutes and the left hand square commutes up to a symmetric monoidal equivalence.*

$$
\begin{array}{ccccc}
\mathcal{B} & \longleftarrow & \mathcal{B}^c & \longrightarrow & \mathcal{B}^{qst} \\
{\scriptstyle F}\downarrow & \simeq & \downarrow{\scriptstyle F^c} & & \downarrow{\scriptstyle F^{qst}} \\
\mathcal{C} & \longleftarrow & \mathcal{C}^c & \longrightarrow & \mathcal{C}^{qst}
\end{array}
$$

Theorem 3.27 *([20, Theorem 3.15]). Let $\mathcal{B}$ be a permutative Gray-monoid.*

i. *There is a strict functor $\nu \colon \mathcal{B}^{qst} \rightarrow \mathcal{B}$ such that*

$$
\begin{array}{ccc}
\mathcal{B}^c & \longrightarrow & \mathcal{B}^{qst} \\
& \searrow \quad \swarrow{\scriptstyle \nu} & \\
& \mathcal{B} &
\end{array}
$$

commutes, where the unlabeled morphisms are those from Theorem 3.26. In particular, ν is a strict symmetric monoidal biequivalence.

ii. *For a symmetric monoidal pseudofunctor $F : \mathcal{B} \rightarrow \mathcal{C}$ with $\mathcal{B}, \mathcal{C}$ with both permutative Gray-monoids, the square below commutes up to a symmetric monoidal equivalence.*

$$
\begin{array}{ccc}
\mathcal{B}^{qst} & \overset{\nu}{\longrightarrow} & \mathcal{B} \\
{\scriptstyle F^{qst}}\downarrow & \simeq & \downarrow{\scriptstyle F} \\
\mathcal{C}^{qst} & \underset{\nu}{\longrightarrow} & \mathcal{C}
\end{array}
$$

We also require an additional detail about quasistrictification which follows from the construction in [37, Theorem 2.97] and the proof of [37, Proposition 2.77].

Lemma 3.28. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be permutative Gray-monoids, and let $F \colon \mathcal{A} \rightarrow \mathcal{B}$ and $G \colon \mathcal{B} \rightarrow \mathcal{C}$ be symmetric monoidal pseudofunctors. Then $(GF)^{qst}$ is symmetric monoidal equivalent to $G^{qst}F^{qst}$. Moreover, if G is a strict functor, then $(GF)^{qst} = G^{qst}F^{qst}$.*

We now turn to equivalences of homotopy theories. In Proposition 3.31 and Corollary 3.34 we show that the homotopy category of $\mathcal{SMBicat}_s$ is equivalent to $\mathrm{Ho\,SMB}$. In Lemma 3.35 we show that the homotopy theory of $\mathcal{Pic\,PGM}$, respectively $\mathcal{PGM}$, is equivalent to that of $\mathcal{Pic\,SMBicat}_s$, respectively $\mathcal{SMBicat}_s$, with a wide choice of classes of weak equivalences.

Definition 3.29. Let PGM denote the tricategory whose objects are permutative Gray-monoids and whose 1-, 2-, and 3-cells are those of SMB. Likewise let HoPGM denote the full subcategory of $\mathrm{Ho\,SMB}$ whose objects are permutative Gray-monoids; its morphisms are given by equivalence classes of pseudofunctors.

Definition 3.30. For a category $\mathcal{C}$ and a class of morphisms $\mathcal{W}$ in $\mathcal{C}$, we write $\mathrm{Ho}(\mathcal{C}, \mathcal{W})$ for the localization of $\mathcal{C}$ with respect to $\mathcal{W}$; we note that this is not necessarily locally small. This category is the homotopy category of the homotopy theory $(\mathcal{C}, \mathcal{W})$. See [36] for the general theory supporting this notion, or [20, Section 2.1] for an overview.

Proposition 3.31. *There is an isomorphism of categories*

$$\mathrm{Ho}(\mathcal{PGM}, \mathit{cat\ eq}) \cong \mathrm{HoPGM}.$$

Proof. We define a functor

$$\Phi\colon \mathrm{HoPGM} \to \mathrm{Ho}(\mathcal{PGM}, \mathit{cat\ eq})$$

to be the identity on objects. For permutative Gray-monoids $\mathcal{A}$ and $\mathcal{B}$, let $[\mathcal{A}, \mathcal{B}]$ denote $\mathrm{HoPGM}(\mathcal{A}, \mathcal{B})$, and let $\{\mathcal{A}, \mathcal{B}\}$ denote $\mathrm{Ho}(\mathcal{PGM}, \mathit{cat\ eq})(\mathcal{A}, \mathcal{B})$. As a mnemonic, note that $[\mathcal{A}, \mathcal{B}]$ is a set of equivalence classes of morphisms, while $\{\mathcal{A}, \mathcal{B}\}$ is, *a priori*, defined by zigzags of strict functors. We define a function

$$\Phi_{\mathcal{A},\mathcal{B}}\colon [\mathcal{A}, \mathcal{B}] \to \{\mathcal{A}, \mathcal{B}\}$$

below, using the quasistrictification $(-)^{qst}$ of Theorem 3.26.

Recall that when $\mathcal{A}$ is a permutative Gray-monoid, there is a strict symmetric monoidal biequivalence $\nu\colon \mathcal{A}^{qst} \to \mathcal{A}$ which is natural in strict functors (Theorem 3.27). For a symmetric monoidal pseudofunctor $F\colon \mathcal{A} \to \mathcal{B}$, we let $\Phi_{\mathcal{A},\mathcal{B}}(F)$ be the class of the zigzag

$$\mathcal{A} \xleftarrow{\ \nu\ } \mathcal{A}^{qst} \xrightarrow{\ F^{qst}\ } \mathcal{B}^{qst} \xrightarrow{\ \nu\ } \mathcal{B}$$

in $\mathrm{Ho}(\mathcal{PGM}, \mathit{cat\ eq})$. We must check that this function is well-defined on equivalence classes of morphisms which will be used to prove that the functions $\Phi_{\mathcal{A},\mathcal{B}}$ assemble to define a functor. We will show that this functor is a bijection, hence Φ defines an isomorphism of categories.

We begin with an observation about the collection of functions $\Phi_{-,-}$. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a symmetric monoidal pseudofunctor and $S\colon \mathcal{B} \to \mathcal{C}$ is a strict functor. By Theorem 3.26, Lemma 3.28, and the naturality of ν with respect to strict functors, we have the following equalities in $\mathrm{Ho}(\mathcal{PGM}, \mathit{cat\ eq})$:

$$\Phi_{\mathcal{A},\mathcal{C}}(SF) = \nu(SF)^{qst}\nu^{-1} = \nu S^{qst} F^{qst}\nu^{-1} = S\nu F^{qst}\nu^{-1} = S\Phi_{\mathcal{A},\mathcal{B}}(F). \tag{3.32}$$

We now apply the observation in Display (3.32) to a path object construction. Given a symmetric monoidal bicategory $\mathcal{B}$, the path object $\mathcal{B}^I$ is another symmetric monoidal bicategory with

- objects (b, b', f) where b, b' are objects and $f\colon b \to b'$ is an equivalence,
- 1-cells $(p, p', \alpha)\colon (b, b', f) \to (c, c', g)$ where $p\colon b \to c$, $p'\colon b' \to c'$, and $\alpha\colon p'f \cong gp$, and
- 2-cells $(\Gamma, \Gamma')\colon (p, p', \alpha) \Rightarrow (q, q', \beta)$ where $\Gamma\colon p \Rightarrow q$, $\Gamma'\colon p' \Rightarrow q'$ commuting with α, β in the obvious way.

Tedious calculation shows that if $\mathcal{B}$ is a permutative Gray-monoid, then so is $\mathcal{B}^I$. There are strict projection functors to each coordinate which are both symmetric monoidal biequivalences $e_0, e_1\colon \mathcal{B}^I \to \mathcal{B}$, and the inclusion $i\colon \mathcal{B} \to \mathcal{B}^I$ which sends b to (b, b, id_b) which is also a strict symmetric monoidal biequivalence if $\mathcal{B}$ is a permutative Gray-monoid. Since $e_0 i = e_1 i = \mathrm{id}_{\mathcal{B}}$, we get that $e_0 = e_1$ in $\mathrm{Ho}(\mathcal{PGM}, \mathit{cat\ eq})$. Finally, given symmetric monoidal pseudofunctors $F, G\colon \mathcal{A} \to \mathcal{B}$ and a symmetric monoidal equivalence α between

them, we have a symmetric monoidal pseudofunctor $A_\alpha : \mathcal{A} \to \mathcal{B}^I$ such that $e_0 A_\alpha = F$, $e_1 A_\alpha = G$, and the values of A_α on the nontrivial morphism in I are given by the components of α.
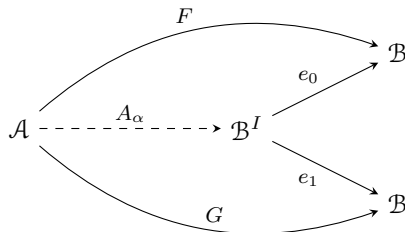
Given a symmetric monoidal equivalence α between F and G, we now apply the observation in Display (3.32) to the pseudofunctor A_α and deduce that the following holds in $\mathrm{Ho}(\mathcal{PGM}, \textit{cat eq})$:

$$
\begin{aligned}
\nu F^{qst}\nu^{-1} &= \nu\,(e_0 A_\alpha)^{qst}\,\nu^{-1} \\
&= e_0 \nu A_\alpha^{qst}\nu^{-1} \\
&= e_1 \nu A_\alpha^{qst}\nu^{-1} \\
&= \nu(e_1 A_\alpha)^{qst}\nu^{-1} \\
&= \nu G^{qst}\nu^{-1}.
\end{aligned}
$$

Thus if two symmetric monoidal pseudofunctors are equivalent, then their images in $\mathrm{Ho}(\mathcal{PGM}, \textit{cat eq})$ are equal and therefore $\Phi_{\mathcal{A},\mathcal{B}}$ is well-defined. This implies that Φ is functorial because $(GF)^{qst}$ is equivalent to $G^{qst}F^{qst}$ once again by Lemma 3.28. This finishes the construction of a functor $\mathrm{HoPGM} \to \mathrm{Ho}(\mathcal{PGM}, \textit{cat eq})$.

A functor in the other direction is even easier to construct. Once again we take the function on objects to be the identity. For a strict functor F, we take the image to be F considered as a symmetric monoidal pseudofunctor; for the formal inverse of a strict biequivalence F, we take the image to be the equivalence class of weak inverses for F in PGM. Note that all weak inverses are equivalent, so this is well-defined up to equivalence. Now $\nu F^{qst} = F\nu$ in $\mathcal{PGM}$, again by naturality of ν with respect to strict functors, so if G is a weak inverse for F in PGM then

$$
F\nu G^{qst}\nu^{-1} = \nu F^{qst}G^{qst}\nu^{-1} = \nu(FG)^{qst}\nu^{-1} = \nu\nu^{-1} = \mathrm{id}_{\mathcal{A}}
$$

in $\mathrm{Ho}(\mathcal{PGM}, \textit{cat eq})$ and $\nu G^{qst}\nu^{-1}$ is also the formal inverse for F. This calculation shows that the composite function

$$
\{\mathcal{A}, \mathcal{B}\} \to [\mathcal{A}, \mathcal{B}] \to \{\mathcal{A}, \mathcal{B}\}
$$

is the identity. The composite

$$
[\mathcal{A}, \mathcal{B}] \to \{\mathcal{A}, \mathcal{B}\} \to [\mathcal{A}, \mathcal{B}]
$$

is also the identity since F is equivalent to $\nu F^{qst}\nu^{-1}$ for any symmetric monoidal pseudofunctor F by Theorem 3.27 (ii). $\square$

Remark 3.33. The construction of the path object $\mathcal{B}^I$ is mentioned in [37, Remark 2.69] where it is claimed that this path object can be used to equip the category of symmetric monoidal bicategories and strict functors with a transferred Quillen model structure. As far as we are aware, this claim is incorrect, as the map $\mathcal{B} \to \mathcal{B}^I$ is not strict without further restrictions on $\mathcal{B}$. For example, this inclusion will not strictly preserve the monoidal structure unless $\mathrm{id}_a \oplus \mathrm{id}_b = \mathrm{id}_{a \oplus b}$ which does not hold in a generic symmetric monoidal bicategory.

Corollary 3.34. *There is an equivalence of categories between* $\mathrm{Ho}(\mathcal{SMBicat}_s, cat\ eq)$ *and* $\mathrm{Ho}\,\mathsf{SMB}$.

Proof. In both cases, every object is isomorphic to a permutative Gray-monoid, and thus we have equivalences for the vertical inclusions of subcategories in the following display.

$$
\begin{array}{ccc}
\mathrm{HoPGM} & \xrightarrow{\ \ \Phi\ \ } & \mathrm{Ho}(\mathcal{PGM}, cat\ eq) \\
\simeq \Big\downarrow & & \Big\downarrow \simeq \\
\mathrm{Ho}\,\mathsf{SMB} & \dashrightarrow & \mathrm{Ho}(\mathcal{SMBicat}_s, cat\ eq)
\end{array}
$$

The isomorphism Φ from Proposition 3.31 therefore induces an equivalence for the dashed arrow shown above. $\square$

Lemma 3.35. *The inclusions*

$$
\mathcal{PGM} \hookrightarrow \mathcal{SMBicat}_s, \quad \mathcal{P}ic\,\mathcal{PGM} \hookrightarrow \mathcal{P}ic\,\mathcal{B}icat_s
$$

induce equivalences of homotopy theories

$$
(\mathcal{PGM}, \mathcal{W} \cap \mathcal{PGM}) \hookrightarrow (\mathcal{SMBicat}_s, \mathcal{W}),
$$
$$
(\mathcal{P}ic\,\mathcal{PGM}, \mathcal{W} \cap \mathcal{P}ic\,\mathcal{PGM}) \hookrightarrow (\mathcal{P}ic\,\mathcal{B}icat_s, \mathcal{W})
$$

for any class of morphisms $\mathcal{W}$ *that includes all biequivalences.*

Proof. By Theorems 3.26 and 3.27, we have natural strict biequivalences

$$
\mathcal{A} \longleftarrow \mathcal{A}^c \longrightarrow \mathcal{A}^{qst}
$$

for any symmetric monoidal bicategory $\mathcal{A}$, and a natural strict biequivalence $\mathcal{B}^{qst} \to \mathcal{B}$ for any permutative Gray-monoid $\mathcal{B}$. This implies that the inclusion and quasistrictification induce weak equivalences between the Rezk nerves of $(\mathcal{PGM}, \mathcal{W} \cap \mathcal{PGM})$ and $(\mathcal{SMBicat}_s, \mathcal{W})$, and hence give an equivalence of homotopy theories (see, e.g., [20, Corollary 2.9]). The same argument implies the final equivalence of homotopy theories because the property of being Picard is invariant under biequivalences. $\square$

The most important case of such a $\mathcal{W}$ is the class of P_2-equivalences we define now.

Definition 3.36. A functor (of any type) $F \colon \mathcal{A} \to \mathcal{B}$ of bicategories is a P_2-*equivalence* if the induced map of topological spaces $NF \colon N\mathcal{A} \to N\mathcal{B}$ is a P_2-equivalence, i.e., induces an isomorphism on π_n for $n = 0, 1, 2$ and all choices of basepoint.

4. Symmetric monoidal structures from operad actions

In this section we describe how to extract symmetric monoidal structure from an operad action on a 2-category. We describe the motivating example of algebras over the Barratt–Eccles operad, but then abstract the essential features to a general theory. Our main applications appear in Section 5, where we use this theory and the topological group-completion theorem of May [31] to deduce information about the fundamental 2-groupoid of a group-completion.

4.1. Background about operads

Definition 4.1. Let $(\mathscr{V}, \star, e)$ be a symmetric monoidal category. An *operad* P in $\mathscr{V}$ is a sequence $\{P(n)\}_{n \geq 0}$ of objects in $\mathscr{V}$ such that $P(n)$ has a (right) Σ_n-action, together with morphisms

$$\gamma \colon P(n) \star P(k_1) \star \cdots \star P(k_n) \longrightarrow P(k_1 + \cdots + k_n)$$

and

$$\mathbf{1} \colon e \longrightarrow P(1)$$

that are appropriately equivariant and compatible. See [30] or [43] for a complete description.

A *map* of operads $f \colon P \longrightarrow Q$ is given by a Σ_n-map $f_n \colon P(n) \longrightarrow Q(n)$ for each $n \geq 0$ compatible with the operations and the identity.

A *P-algebra* is given by a pair (X, μ), where X is an object of $\mathscr{V}$, and μ is a collection of morphisms

$$\mu_n \colon P(n) \star X^{\star n} \longrightarrow X$$

in $\mathscr{V}$ that are appropriately equivariant and compatible with γ and $\mathbf{1}$. A morphism of P-algebras $(X, \mu) \longrightarrow (X', \mu')$ is given by a morphism $g \colon X \longrightarrow X'$ compatible with the maps μ_n and μ'_n.

In this paper we will be concerned with operads in $(s\mathcal{S}et, \times, *)$ and $(\mathcal{T}op, \times, *)$, as well as several variants for 2-categories, including $(2\mathcal{C}at, \times, *)$, $(2\mathcal{C}at, \otimes, *)$, and $(2\mathcal{C}at_{\mathrm{ps}}, \times, *)$.

Notation 4.2. We let $P\text{-}\mathbf{Alg}(\mathscr{V}, \star)$ denote the category of P-algebras in $(\mathscr{V}, \star)$ and their morphisms.[1] If there is no confusion over the ambient symmetric monoidal category we also write $P\text{-}\mathbf{Alg}$.

We now recall and fix notation for standard transfers of operadic structures; see e.g., [24,43].

Lemma 4.3. *A map of operads* $f \colon P \longrightarrow Q$ *induces a functor*

$$f^* \colon Q\text{-}\mathbf{Alg} \longrightarrow P\text{-}\mathbf{Alg}$$

that sends a Q-algebra (X, μ) to the P-algebra (X, ν), where ν_n is given by the composite

$$P(n) \star X^{\star n} \xrightarrow{f_n \star \mathrm{id}} Q(n) \star X^{\star n} \xrightarrow{\mu_n} X.$$

If f and g are composable maps of operads, then $(g \circ f)^* = f^* \circ g^*$.

Lemma 4.4. *Let P be an operad in $(\mathscr{V}, \star)$, and let $F, G \colon (\mathscr{V}, \star) \longrightarrow (\mathscr{W}, \Diamond)$ be lax symmetric monoidal functors, and $\alpha \colon F \Rightarrow G$ a monoidal natural transformation. Then*

  i. *FP is an operad in $(\mathscr{W}, \Diamond)$;*
  ii. *F induces a functor*

$$F \colon P\text{-}\mathbf{Alg}(\mathscr{V}, \star) \longrightarrow FP\text{-}\mathbf{Alg}(\mathscr{W}, \Diamond)$$

   that sends (X, μ) to (FX, ν), where ν_n is given by the composite

---

[1] We do not include the monoidal unit in the notation as in all of our cases it will be the terminal object.

$$FP\Diamond(FX)^{\Diamond n} \longrightarrow F(P \star X^{\star n}) \xrightarrow{F\mu_n} FX;$$

and

iii. α induces a map of operads $\alpha\colon FP \longrightarrow GP$, and a natural transformation $\alpha\colon F \Rightarrow \alpha^* \circ G$ of functors

$$P\text{-}\mathbf{Alg}(\mathcal{V}, \star) \longrightarrow FP\text{-}\mathbf{Alg}(\mathcal{W}, \Diamond),$$

whose component at (X, μ) is given by α_X.

Corollary 4.5. *Any operad P in $(2\mathcal{C}at, \times)$ gives rise to an operad (with the same underlying sequence of objects) in $(2\mathcal{C}at, \otimes)$ and $(2\mathcal{C}at_{\mathrm{ps}}, \times)$.*

There are inclusions

$$P\text{-}\mathbf{Alg}(2\mathcal{C}at, \times) \longrightarrow P\text{-}\mathbf{Alg}(2\mathcal{C}at, \otimes) \longrightarrow P\text{-}\mathbf{Alg}(2\mathcal{C}at_{\mathrm{ps}}, \times).$$

Proof. Following Remark 3.16, we can use the identity functor $(2\mathcal{C}at, \times) \longrightarrow (2\mathcal{C}at, \otimes)$ and the inclusion $(2\mathcal{C}at, \otimes) \longrightarrow (2\mathcal{C}at_{\mathrm{ps}}, \times)$ to transfer the algebra structures. □

Remark 4.6. Given the relationship between cubical functors and the Gray tensor product, we can specify the objects in the categories of algebras above as follows. In all cases, an algebra is given by a pair (X, μ), where X is a 2-category, and μ is a collection of morphisms $\mu_n\colon P(n) \times X^n \longrightarrow X$, which are 2-functors if working in $(2\mathcal{C}at, \times)$, cubical functors if working in $(2\mathcal{C}at, \otimes)$, or pseudofunctors if working in $(2\mathcal{C}at_{\mathrm{ps}}, \times)$. In the first two cases, a map of algebras is given by a 2-functor $X \longrightarrow X'$ commuting with μ, but for the latter case, a map of algebras is given by a pseudofunctor $X \longrightarrow X'$.

Notation 4.7. Let P be an operad in $(2\mathcal{C}at, \times)$. Let $P\text{-}2\mathcal{C}at$ denote the subcategory of $P\text{-}\mathbf{Alg}(2\mathcal{C}at_{\mathrm{ps}}, \times)$ given by all objects and those morphisms whose underlying pseudofunctor $X \longrightarrow X'$ is a 2-functor.

Definition 4.8 *(The Barratt–Eccles operad).* Let $\mathcal{O}$ denote the operad in $(\mathcal{C}at, \times)$ with $\mathcal{O}(n)$ being the translation category for Σ_n: the objects of $\mathcal{O}(n)$ are the elements of the symmetric group Σ_n, and there is a unique isomorphism between any two objects. By abuse of notation, we also denote by $\mathcal{O}$ the operad in $(2\mathcal{C}at, \times)$ obtained by adding identity 2-cells. Note that $\mathcal{O}$ is also an operad in $(2\mathcal{C}at, \otimes)$ and $(2\mathcal{C}at_{\mathrm{ps}}, \times)$.

Because the nerve and geometric realization functors are strong monoidal, we have an operad $|N\mathcal{O}| = B\mathcal{O}$ in $\mathcal{T}op$. Likewise, if $\mathcal{A}$ is an $\mathcal{O}$-algebra in $(2\mathcal{C}at_{\mathrm{ps}}, \times)$, then $B\mathcal{A}$ is a $B\mathcal{O}$-algebra in $\mathcal{T}op$.

Remark 4.9. The operad $B\mathcal{O}$ in $\mathcal{T}op$ was used implicitly in [3]. As an operad in $\mathcal{C}at$, $\mathcal{O}$ was independently introduced by May in [30,31]. In [31], May shows that a permutative category is an $\mathcal{O}$-algebra [31, Lemmas 4.3–4.5] and that $B\mathcal{O}$ is an E_∞ operad [31, Lemma 4.8].

4.2. Symmetric monoidal structures from arbitrary operads

The relationship between $\mathcal{O}$-algebras in $\mathcal{C}at$ and permutative categories extends to the 2-categorical level. We describe this now, and then abstract the key features for general operads. To begin, note that an operad P in $(2\mathcal{C}at, \times)$ induces a different monad $P_\otimes$ on $2\mathcal{C}at$ using a combination of the cartesian product and the Gray tensor product. Explicitly, $P_\otimes$ is defined via the formula

$$X \mapsto \coprod_{n \geq 0} P(n) \times_{\Sigma_n} X^{\otimes n}$$

using the natural 2-functor

$$(\mathcal{A} \times \mathcal{B}) \otimes (\mathcal{A}' \times \mathcal{B}') \to (\mathcal{A} \otimes \mathcal{A}') \times (\mathcal{B} \otimes \mathcal{B}') \to (\mathcal{A} \times \mathcal{A}') \times (\mathcal{B} \times \mathcal{B}').$$

Proposition 4.10. *The category $\mathcal{O}_\otimes Alg$ of algebras for the monad $\mathcal{O}_\otimes$ on $2Cat$ is isomorphic to PGM.*

Proof. Note that objects $e \in \mathcal{A}$ are in bijection with 2-functors $\mathcal{O}(0) \to \mathcal{A}$, and that 2-functors $\oplus\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ together with a 2-natural isomorphism

where $\tau\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the symmetry isomorphism in $2Cat$ for the Gray tensor product are in bijection with 2-functors

$$\tilde{\oplus}\colon \mathcal{O}(2) \times_{\Sigma_2} \mathcal{A}^{\otimes 2} \to \mathcal{A}$$

using the strict parts of Lemma 3.23. It is now straightforward to verify that the axioms for a permutative Gray-monoid are the same as those for an algebra over $\mathcal{O}_\otimes$. $\square$

Using Corollary 4.5 with Proposition 4.10, we can also regard a permutative Gray-monoid as an algebra with respect to the cartesian product. We will use this implicitly in our work below.

Corollary 4.11. *Every permutative Gray-monoid is an algebra for the operad $\mathcal{O}$ acting on $(2Cat_{\mathrm{ps}}, \times)$.*

Although we have no use for it here, there is an analogous result for $\mathcal{O}$-algebras in $(2Cat, \times)$ and the stricter notion of *permutative 2-category* described in [20], which we state in the following proposition. This is the Cat-enriched version of the statement that permutative *categories* are precisely the $\mathcal{O}$-algebras in $(Cat, \times)$ (see Remark 4.9).

Proposition 4.12. *The category of $\mathcal{O}$-algebras in $(2Cat, \times)$ is isomorphic to the category $Perm2Cat$ of permutative 2-categories and strict functors of such.*

We now turn to the general question of how symmetric monoidal structures arise from operad actions on 2-categories.

Definition 4.13. Let $\mathbb{P}$ be a property of 2-categories. We write $\mathbb{P}(\leq n)$ (including the case $n = \infty$) for the full subcategory of the category of operads consisting of those operads P for which $P(k)$ has $\mathbb{P}$ for all $k \leq n$.

Notation 4.14. Let $\mathbb{C}$ denote the property of being bicategorically contractible, i.e., X has $\mathbb{C}$ if the unique 2-functor $X \to *$ is a biequivalence.

Lemma 4.15. *A nonempty 2-category X is contractible if and only if the following four conditions hold.*
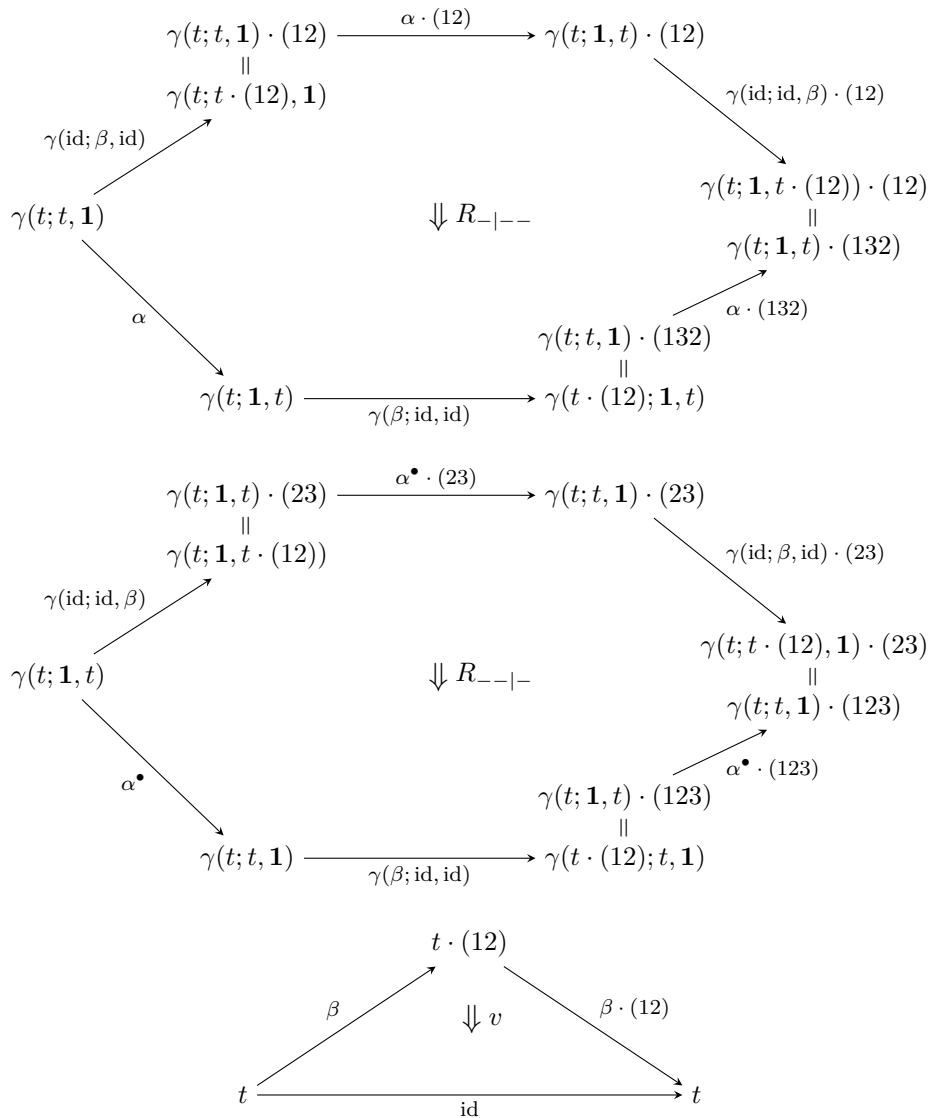
 i. Any two objects are connected by a 1-cell.
 ii. Every 1-cell is an equivalence.
 iii. Every 2-cell is invertible.
 iv. Any two parallel 1-cells are connected by a unique 2-isomorphism.

Example 4.16. The operad $\mathcal{O}$ of Definition 4.8 is in $\mathbb{C}(\leq \infty)$.

Definition 4.17. A *choice of multiplication* χ in P consists of the following:

- a choice of an object $i \in P(0)$;
- a choice of an object $t \in P(2)$;
- an adjoint equivalence $\alpha\colon \gamma(t; t, \mathbf{1}) \simeq \gamma(t; \mathbf{1}, t)$ in $P(3)$;
- adjoint equivalences $l\colon \gamma(t; i, \mathbf{1}) \simeq \mathbf{1}$ and $r\colon \mathbf{1} \simeq \gamma(t; \mathbf{1}, i)$ in $P(1)$;
- invertible 2-cells π in $P(4)$, μ in $P(3)$, and λ and ρ in $P(2)$, as depicted below;
- an adjoint equivalence $\beta\colon t \simeq t \cdot (12)$ in $P(2)$;
- invertible 2-cells $R_{-|--}$ and $R_{--|-}$ in $P(3)$, and v in $P(2)$, as depicted below.

$$\gamma(t;t,\mathbf{1})\cdot(12) \xrightarrow{\ \alpha\cdot(12)\ } \gamma(t;\mathbf{1},t)\cdot(12)$$

$$\|$$

$$\gamma(t;t\cdot(12),\mathbf{1})$$

$$\gamma(\mathrm{id};\beta,\mathrm{id})$$

$$\gamma(\mathrm{id};\mathrm{id},\beta)\cdot(12)$$

$$\gamma(t;t,\mathbf{1})$$

$$\Downarrow R_{-|--}$$

$$\gamma(t;\mathbf{1},t\cdot(12))\cdot(12)$$

$$\|$$

$$\gamma(t;\mathbf{1},t)\cdot(132)$$

$$\alpha$$

$$\alpha\cdot(132)$$

$$\gamma(t;t,\mathbf{1})\cdot(132)$$

$$\|$$

$$\gamma(t;\mathbf{1},t) \xrightarrow{\ \gamma(\beta;\mathrm{id},\mathrm{id})\ } \gamma(t\cdot(12);\mathbf{1},t)$$

$$\gamma(t;\mathbf{1},t)\cdot(23) \xrightarrow{\ \alpha^{\bullet}\cdot(23)\ } \gamma(t;t,\mathbf{1})\cdot(23)$$

$$\|$$

$$\gamma(t;\mathbf{1},t\cdot(12))$$

$$\gamma(\mathrm{id};\mathrm{id},\beta)$$

$$\gamma(\mathrm{id};\beta,\mathrm{id})\cdot(23)$$

$$\gamma(t;\mathbf{1},t)$$

$$\Downarrow R_{--|-}$$

$$\gamma(t;t\cdot(12),\mathbf{1})\cdot(23)$$

$$\|$$

$$\gamma(t;t,\mathbf{1})\cdot(123)$$

$$\alpha^{\bullet}$$

$$\alpha^{\bullet}\cdot(123)$$

$$\gamma(t;\mathbf{1},t)\cdot(123)$$

$$\|$$

$$\gamma(t;t,\mathbf{1}) \xrightarrow{\ \gamma(\beta;\mathrm{id},\mathrm{id})\ } \gamma(t\cdot(12);t,\mathbf{1})$$

$$t\cdot(12)$$

$$\beta \qquad \Downarrow v \qquad \beta\cdot(12)$$

$$t \xrightarrow[\mathrm{id}]{} t$$

Note that in all of the diagrams above, the equalities on objects follow from the axioms of an operad.

Remark 4.18. Our choice of multiplication is reminiscent of Batanin's notion of a system of compositions on a globular operad [5]. In both instances, this extra structure on an operad is intended to pick out preferred binary operations, ensuring they exist as needed. Proposition 4.20 below also reflects the modification made by Leinster [29] in which contractibility of the operad already ensures enough operations.

Example 4.19. Consider the Barratt–Eccles operad $\mathcal{O}$ of Definition 4.8. There is a canonical choice of multiplication κ on $\mathcal{O}$ given by $i = * \in \mathcal{O}(0)$ and t equal to the identity permutation in Σ_2. All the equivalences are given by the unique 1-morphisms between the corresponding objects, and all the 2-cells are the identity (noting that the boundaries are equal because there is a unique 1-morphism between any two objects).

The idea behind this example can be generalized to a larger class of operads, and we explain this now.

Proposition 4.20. *Let P be an operad in $\mathbb{C}(\leq 4)$. Then there exists a choice of multiplication on P.*

Proof. Since $P(0)$ and $P(2)$ are contractible, they are in particular non-empty, and hence we can pick objects $i \in P(0)$ and $t \in P(2)$. By Lemma 4.15 the equivalences α, l, r and β and the 2-isomorphisms π, μ, λ, ρ, $R_{-|--}$, $R_{--|-}$ and v can be picked using contractibility. $\square$

Proposition 4.21. *Let $f: P \to Q$ be a map of operads in $(2Cat_{\mathrm{ps}}, \times)$. If P has a choice of multiplication χ, then taking the images of all data involved gives a choice of multiplication $f(\chi)$ in Q.*

Theorem 4.22. *Let P be an operad in $\mathbb{C}(\leq 5)$. Then a choice of multiplication χ in P determines a functor*

$$\chi^*: P\text{-}\mathbf{Alg}(2Cat_{\mathrm{ps}}, \times) \to SM2Cat_{ps}$$

which is the identity on underlying 2-categories and pseudofunctors. We therefore have a functor

$$\chi^*: P\text{-}2Cat \to SM2Cat_s.$$

Proof. First, observe that the second statement is a refinement of the first because P-$2Cat$ denotes the subcategory of P-$\mathbf{Alg}(2Cat_{\mathrm{ps}}, \times)$ whose morphisms are 2-functors (see Notation 4.7). To prove the first statement, we use χ to construct a symmetric monoidal structure on an arbitrary P-algebra (X, μ).

We define the monoidal product as the pseudofunctor $\mu_2(t; -, -)$ obtained by applying Lemma 3.23 to the pseudofunctor $\mu_2: P(2) \times X^2 \to X$. Explicitly, the monoidal product on objects is defined as $xy = \mu_2(t; x, y)$. The unit object is defined as $e = \mu_0(i)$.

The equivalences α, l, r and β are defined as the appropriate images of their namesakes in χ, as given by Lemma 3.23. For example, $\alpha: (xy)z \to x(yz)$ is given by the pseudonatural equivalence $\mu_3(\alpha; -, -, -)$. Similarly, the invertible modifications π, μ, λ, ρ, $R_{-|--}$, $R_{--|-}$ and v are given by the appropriate images of their namesakes in χ. For example, v is defined as $\mu_2(v; -, -)$.

Contractibility of $P(n)$ for $n \leq 5$ implies that analogues of the axioms for the modifications in a symmetric monoidal bicategory are satisfied by the corresponding cells in P, which in turn implies that the same axioms are satisfied after applying μ.

Let $F: X \to Y$ be a P-algebra morphism, i.e., a pseudofunctor that commutes with the action of P. It is easy to check that this implies that F preserves strictly the symmetric monoidal structures on X and Y given by the choice of multiplication χ. Thus we have a functor $\chi^*: P\text{-}\mathbf{Alg}(2Cat_{\mathrm{ps}}, \times) \to SM2Cat_{ps}$. $\square$

We now record several results which follow from Theorem 4.22 and its proof.

Proposition 4.23. *Let $f: P \to Q$ in $\mathbb{C}(\leq 5)$, and assume P has choice of multiplication χ. As functors Q-$2Cat \to SM2Cat_s$, we have an equality $\chi^* \circ f^* = (f(\chi))^*$.*

Proposition 4.24. *Suppose that P_1 and P_2 are operads with choices of multiplication χ_1 and χ_2. Then the product $P = P_1 \times P_2$ has a choice of multiplication defined by the pointwise product of the data, $\chi = \chi_1 \times \chi_2$.*

Remark 4.25. As a special case of Proposition 4.23, the projections $\pi_i: P \to P_i$ identify $\pi_i(\chi)$ with χ_i.

Proposition 4.26. *If P is an operad in $\mathbb{C}(\leq 5)$, X is a P-algebra, and χ_1, χ_2 are two different choices of multiplication in P, the identity pseudofunctor on X can be equipped with the structure of a symmetric monoidal biequivalence $\chi_1^* X \to \chi_2^* X$ relating the two symmetric monoidal structures. Moreover, this assignment is natural in X.*

Proof. Note that contractibility of P gives 1-morphisms relating i_1 with i_2 and t_1 with t_2, which when applied to the algebra X give rise to the map that compares units and multiplications. The rest of the data and axioms of a symmetric monoidal biequivalence follow from contractibility as well. $\square$

Recall that κ denotes the canonical choice of multiplication for the Barratt–Eccles operad (Example 4.19), and we implicitly regard a permutative Gray-monoid as an $\mathcal{O}$-algebra by Corollary 4.11.

Proposition 4.27. *Let $\mathcal{A}$ be a permutative Gray-monoid. Then $\mathcal{A} = \kappa^*\mathcal{A}$. The composite*

$$\mathcal{PGM} \to \mathcal{O}\text{-}\mathbf{Alg}(\mathit{2Cat}_{\mathrm{ps}}, \times) \xrightarrow{\kappa^*} \mathcal{SM2Cat}_s$$

is the inclusion functor from $\mathcal{PGM}$ to $\mathcal{SM2Cat}_s$.

Proof. The underlying 2-categories of $\mathcal{A}$ and $\kappa^*\mathcal{A}$ are equal. It is clear by construction that the two symmetric monoidal structures are the same. $\square$

5. Symmetric monoidal structures and group-completion

In this section we show how to construct a Picard 2-category with the same stable 2-type as a given permutative Gray-monoid. Our construction begins with a strict version of the fundamental 2-groupoid in Section 5.1. In Section 5.2 we analyze its effect on stable equivalences, and in Section 5.3 we apply the theory of Section 4 together with topological group-completion to obtain the desired Picard 2-category.

Definition 5.1. A *strict 2-groupoid* is a 2-category in which every 1- and 2-cell is strictly invertible, i.e., for every 1-cell $f\colon a \to b$ there is a 1-cell $g\colon b \to a$ such that gf, fg are both identity 1-cells, and similarly for 2-cells. We define the category $\mathit{Str2Gpd}$ to have strict 2-groupoids as objects and 2-functors as morphisms.

Remark 5.2. We should note that any 2-category isomorphic to a strict 2-groupoid is itself a strict 2-groupoid, but that a 2-category which is biequivalent to a strict 2-groupoid will not in general have 1-cells which are strictly invertible in the sense above, but only satisfy the weaker condition that we have called invertible in Definition 3.24: given f there exists a g such that fg and gf are isomorphic to identity 1-cells.

5.1. Background on the Whitehead 2-groupoid and nerves

We now recall the construction of a strict fundamental 2-groupoid for simplicial sets due to Moerdijk–Svensson [33], known as the Whitehead 2-groupoid.

Definition 5.3 ([33, Section 1, Example (2)]). Let X be a topological space, $Y \subseteq X$ a subspace, and $S \subseteq Y$ a subset. We define the *Whitehead 2-groupoid $W(X, Y, S)$* to be the strict 2-groupoid with

- objects the set S,
- 1-cells $[f]\colon a \to b$ to be homotopy classes of paths f from a to b in Y, relative the endpoints, and
- 2-cells $[\alpha]\colon [f] \Rightarrow [g]$ to be homotopy classes of maps $\alpha\colon I \times I \to X$ such that
  - i. $\alpha(t, 0) = f(t)$,
  - ii. $\alpha(t, 1) = g(t)$,
  - iii. $\alpha(0, -)$ is constant at the source of f (and hence also g),
  - iv. $\alpha(1, -)$ is constant at the target of f (and hence also g), and
  - v. homotopies $H(s, t, -)$ between two such maps fix the vertical sides and map the horizontal sides into Y for each s.

Since the nerve functor $N\colon \mathit{Str2Gpd} \to \mathit{sSet}$ preserves limits and filtered colimits, it has a left adjoint. In [33], this left adjoint was explicitly computed using Whitehead 2-groupoids, and we recall their construction now.

Notation 5.4. For a simplicial set X, let $X^{(n)}$ denote the n-skeleton of X.

Theorem 5.5 *([33, Theorem 2.3]). The functor $W\colon sSet \to Str2Gpd$, defined by*

$$W(X) = W(|X|, |X^{(1)}|, |X^{(0)}|),$$

is left adjoint to the nerve functor, N.

We will need several key properties of W from [33]; we summarize these in the next proposition.

Proposition 5.6 *([33]).*

i. *If $X \to Y$ is a weak equivalence of simplicial sets, then $WX \to WY$ is a biequivalence of 2-groupoids [33, Proposition 2.2 (iii)].*

ii. *For a strict 2-groupoid $\mathcal{C}$, the strict 2-functor $\varepsilon\colon WN\mathcal{C} \to \mathcal{C}$ is a bijective-on-objects biequivalence, although its pseudoinverse is only a pseudofunctor [33, Displays (1.9) and (2.10)].*

iii. *For a simplicial set X, the unit $\eta_X\colon X \to NWX$ of the adjunction $W \dashv N$ is a P_2-equivalence. In particular, if $\mathcal{K}$ is a 2-category then $\eta_{N\mathcal{K}}\colon N\mathcal{K} \to NWN\mathcal{K}$ is a P_2-equivalence [33, Corollary 2.6].*

iv. *The functor W is strong monoidal with respect to the cartesian product [33, Proposition 2.2 (i)].*

Remark 5.7. For a strict 2-groupoid $\mathcal{C}$, the 1-cells of $WN\mathcal{C}$ are freely generated by the underlying graph of the 1-cells of $\mathcal{C}$ [33, Display (2.10)].

Now the nerve functor is defined on all of $2Cat_{\mathrm{nop}}$, not just the subcategory of 2-groupoids or strict 2-groupoids. Thus we can define a functor $2Cat_{\mathrm{nop}} \to Str2Gpd$ using the composite $W \circ N$. Perhaps surprisingly, this composite is also a left adjoint even though N is a right adjoint.

Proposition 5.8. *The functor WN extends to a functor $2Cat_{\mathrm{nop}} \to Str2Gpd$, and is left adjoint to the inclusion $i\colon Str2Gpd \hookrightarrow 2Cat_{\mathrm{nop}}$.*

Proof. First note that N extends to a full and faithful functor $2Cat_{\mathrm{nop}} \to sSet$ by [16]. Thus we have natural isomorphisms

$$Str2Gpd(WN\mathcal{A}, \mathcal{B}) \cong sSet(N\mathcal{A}, N\mathcal{B}) \cong 2Cat_{\mathrm{nop}}(\mathcal{A}, i\mathcal{B}). \quad \square$$

Remark 5.9. Note that, in the above proof, $\mathcal{B}$ is a strict 2-groupoid so in particular every normal oplax functor $\mathcal{A} \to i\mathcal{B}$ is in fact a normal pseudofunctor. Thus we have a natural isomorphism $Str2Gpd(WN\mathcal{A}, \mathcal{B}) \cong 2Cat_{\mathrm{nps}}(\mathcal{A}, i\mathcal{B})$ as well, so WN is also left adjoint to the inclusion $Str2Gpd \hookrightarrow 2Cat_{\mathrm{nps}}$.

Note since the nerve functor $N\colon 2Cat_{\mathrm{nop}} \to sSet$ is full and faithful, $\eta_{N\mathcal{K}}$ is in fact in the image of N.

Notation 5.10. Let $\varepsilon^{\bullet}_{\mathcal{K}}\colon \mathcal{K} \to WN\mathcal{K}$ be the unique normal pseudofunctor (Remark 5.9) with $N\varepsilon^{\bullet}_{\mathcal{K}} = \eta_{N\mathcal{K}}$. Note that $\varepsilon^{\bullet}$ is strictly natural in normal pseudofunctors.

Remark 5.11. The triangle identities for η and ε show that the composite $\varepsilon\varepsilon^{\bullet}$ is the identity 2-functor. The unit and counit of the adjunction $WN \dashv i$ are given, respectively, by $\varepsilon^{\bullet}$ and ε.

Lemma 5.12. *The transformations $\varepsilon^{\bullet}$ and ε are P_2-equivalences.*

Proof. Recall that η is a P_2-equivalence by Proposition 5.6 (*iii*). Since the nerve functor creates P_2-equivalences of 2-categories, $\varepsilon^{\bullet}$ is also a P_2-equivalence. This implies that ε is a P_2-equivalence by Remark 5.11 and 2-out-of-3. $\square$

This accomplishes the first goal of this section, to produce from a 2-category $\mathcal{C}$ a strict 2-groupoid $WN\mathcal{C}$ and a pseudofunctor $\mathcal{C} \to WN\mathcal{C}$ which is a natural P_2-equivalence. We now turn to incorporating the symmetric monoidal structure.

Proposition 5.13. *The adjunction $WN \dashv i$ of Proposition 5.8 is monoidal with respect to the cartesian product.*

Proof. Note that N preserves products since it is a right adjoint (Proposition 2.7), and therefore WN is strong monoidal by Proposition 5.6 (*iv*). Straightforward calculations show that $\varepsilon^{\bullet}$ and ε are monoidal transformations. $\square$

Notation 5.14. Let

$$h\colon WN(-) \Rightarrow WS|N(-)| = WSB(-)$$

denote the natural transformation induced by the unit $\mathrm{id} \Rightarrow S|-|$.

Applying symmetric monoidal functors to the operad $\mathcal{O}$, we have the following corollary (see Lemma 4.4).

Corollary 5.15. *There are operads $WN\mathcal{O}$ and $WSB\mathcal{O}$. The transformations ε, $\varepsilon^{\bullet}$, and h induce operad maps*

$$\varepsilon\colon WN\mathcal{O} \to \mathcal{O}, \qquad \varepsilon^{\bullet}\colon \mathcal{O} \to WN\mathcal{O}, \qquad and \qquad h\colon WN\mathcal{O} \to WSB\mathcal{O}.$$

Notation 5.16. Let $\widetilde{\kappa}$ denote the choice of multiplication in $WN\mathcal{O}$ given by applying $\varepsilon^{\bullet}$ to the canonical choice κ (Proposition 4.21).

Proposition 5.17. *Given a permutative Gray-monoid $\mathcal{A}$, there is a natural zigzag of strict functors of symmetric monoidal 2-categories as shown below. The left leg is a P_2-equivalence and the right leg is a biequivalence.*

$$
\begin{array}{ccc}
 & \widetilde{\kappa}^{*}WN(\mathcal{A}) & \\
\widetilde{\kappa}^{*}(\varepsilon_{\mathcal{A}}) \swarrow & & \searrow \widetilde{\kappa}^{*}(h_{\mathcal{A}}) \\
\mathcal{A} \;=\; \kappa^{*}\mathcal{A} \;=\; \widetilde{\kappa}^{*}\varepsilon^{*}\mathcal{A} & & \widetilde{\kappa}^{*}h^{*}WSB(\mathcal{A}).
\end{array}
$$

Proof. Recall that we implicitly regard $\mathcal{A}$ as an $\mathcal{O}$-algebra via Corollary 4.11. Therefore we have a zigzag of $WN\mathcal{O}$-algebra maps (note that these have underlying 2-functors) induced by the components of ε and h, respectively,

$$\varepsilon^{*}\mathcal{A} \longleftarrow WN(\mathcal{A}) \longrightarrow h^{*}WSB(\mathcal{A}).$$

We have $\mathcal{A} = \kappa^{*}\mathcal{A}$ by Proposition 4.27. Note $\widetilde{\kappa}^{*}\varepsilon^{*} = \kappa^{*}$ because $\varepsilon\varepsilon^{\bullet} = \mathrm{id}$ (Remark 5.11). This gives a zigzag of symmetric monoidal 2-categories and strict functors. Naturality follows from naturality of ε and h. Moreover, ε is a P_2-equivalence by Lemma 5.12 and h is a biequivalence because W sends weak equivalences to biequivalences by Proposition 5.6 (*i*). $\square$

It is clear that the property of being Picard is preserved by biequivalences and, moreover, every P_2-equivalence of Picard 2-categories is a biequivalence. Therefore we have the following corollary of Proposition 5.17.

Corollary 5.18. *If $\mathcal{A}$ is a strict Picard 2-category, then the span in Proposition 5.17 is a span of Picard 2-categories.*

5.2. E_∞-algebras and stable homotopy theory of symmetric monoidal bicategories

In this section, we show that the composite WS, combined with any choice of multiplication, sends stable equivalences of E_∞ spaces to stable P_2-equivalences of symmetric monoidal 2-groupoids.

Our notions of stable equivalence, stable n-equivalence, and P_n-equivalence for strict functors of symmetric monoidal bicategories are created by the K-theory functors of [20,22], which construct infinite loop spaces from bicategories and 2-categories. We begin with a review of these functors and then apply the theory of E_∞ algebras in $\mathcal{T}op$.

Theorem 5.19 *([22,20]).* *There is a functor $K\colon \mathcal{SMBicat}_s \to \mathcal{Sp}_0$. For a symmetric monoidal bicategory $\mathcal{A}$, $K\mathcal{A}$ is a positive Ω-spectrum, with the property that*

$$B\mathcal{A} \simeq K\mathcal{A}(0) \to \Omega K\mathcal{A}(1)$$

is a group-completion. In particular, we have that

$$\pi_n(K\mathcal{A}) \cong \pi_n(\Omega B(B\mathcal{A})),$$

where the latter are the unstable homotopy groups of the topological group-completion of the classifying space $B\mathcal{A}$.

Definition 5.20. A strict functor $F\colon \mathcal{A} \to \mathcal{B}$ of symmetric monoidal bicategories is a *stable equivalence* if the induced map of spectra $KF\colon K\mathcal{A} \to K\mathcal{B}$ is a stable equivalence. Similarly, F is a stable n-equivalence, respectively stable P_n-equivalence, if KF is so.

Lemma 5.21. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a strict functor such that $BF\colon B\mathcal{A} \to B\mathcal{B}$ is a weak equivalence. Then F is a stable equivalence, and hence, also a stable P_n-equivalence for all $n \geq 0$.*

Proof. The corresponding map of spectra KF is a level equivalence. □

Restricting to permutative Gray-monoids, we obtain the main result in [20].

Theorem 5.22 *([20]).* *There is a functor $K\colon \mathcal{PGM} \to \mathcal{Sp}_0$ which induces an equivalence of homotopy theories*

$$(\mathcal{PGM}, st\ eq) \simeq (\mathcal{Sp}_0, st\ eq)$$

between permutative Gray-monoids and connective spectra, working relative to the stable equivalences.

Proposition 5.23 *([20, Remark 6.4]).* *When restricted to the subcategory $\mathcal{PGM}$, the functor K of [22] is equivalent to that of [20].*

Definition 5.24. An operad $\mathscr{D}$ in $\mathcal{T}op$ is an E_∞ *operad* if for all $n \geq 0$, the Σ_n-action on $\mathscr{D}(n)$ is free, and $\mathscr{D}(n)$ is contractible.

The following theorem appeared first in [31]. A modern (equivariant) version is in [15].

Theorem 5.25 *([31, Theorem 2.3], [15, Theorem 1.14, Definition 2.7]). Let $\mathscr{D}$ be an E_∞ operad in $\mathcal{T}op$. There is a functor*

$$\mathbb{E} \colon \mathscr{D}\text{-}\mathbf{Alg} \to \mathcal{S}pectra$$

such for a $\mathscr{D}$-algebra X and all $n \geq 2$,

$$X = \mathbb{E}(X)(0) \to \Omega^n \mathbb{E}(X)(n)$$

is a group-completion.

Definition 5.26. Let $\mathscr{D}$ be an E_∞ operad in $\mathcal{T}op$. A map $f \colon X \to Y$ of $\mathscr{D}$-algebras is a *stable equivalence* if the associated map $\mathbb{E}(f)$ of spectra is so. Similarly, f is said to be a *stable P_n-equivalence* if $\mathbb{E}(f)$ is so.

We use Theorem 5.25 to recognize stable equivalences and stable P_n-equivalences of $\mathscr{D}$-algebras by their induced maps on group-completions, as in the following result.

Corollary 5.27. *A map $f \colon X \to Y$ of $\mathscr{D}$-algebras is a stable equivalence if and only if the associated map on group-completions*

$$\Omega B f \colon \Omega B X \to \Omega B Y$$

is an unstable equivalence.
   Similarly, f is a stable P_n-equivalence if and only if

$$\Omega B f \colon \Omega B X \to \Omega B Y$$

is an unstable P_n-equivalence.

Applying Theorem 5.19, we can recognize stable P_n-equivalences of symmetric monoidal bicategories in the same way.

Corollary 5.28.
   A strict functor $F \colon \mathcal{A} \to \mathcal{B}$ of symmetric monoidal bicategories is a stable P_n-equivalence if and only if the associated map on topological group-completions

$$\Omega B(B F) \colon \Omega B(B \mathcal{A}) \to \Omega B(B \mathcal{B})$$

is an unstable P_n-equivalence.

Proposition 5.29. *Let $\mathscr{D}$ be an E_∞ operad, and let χ denote any choice of multiplication for $WS(\mathscr{D})$. If $\alpha \colon X \to Y$ is a map of $\mathscr{D}$-algebras which is a stable equivalence, then $\chi^* WS\alpha \colon \chi^* WSX \to \chi^* WSY$ is a stable P_2-equivalence in $\mathcal{SM}2\mathcal{C}at_s$.*

Proof. By Corollary 5.28 and Corollary 5.27, it suffices to show that $BWS\alpha$ is a stable P_2-equivalence. Consider the following diagram of algebras over $|S\mathscr{D}|$, induced by naturality of the counit

$$|S(-)| \Rightarrow \mathrm{id}$$

and the transformation

$$|S(-)| \Rightarrow BWS(-)$$

induced by the unit of $W \dashv N$.

$$
\begin{array}{ccc}
X & \xrightarrow{\ \alpha\ } & Y \\
\uparrow & & \uparrow \\
|SX| & \xrightarrow{\ |S\alpha|\ } & |SY| \\
\downarrow & & \downarrow \\
BWSX & \xrightarrow{\ BWS\alpha\ } & BWSY
\end{array}
$$

The upper vertical arrows are unstable weak equivalences, therefore stable equivalences. The lower vertical arrows are unstable P_2-equivalences by Proposition 5.6 (*iii*). Since W takes values in 2-groupoids, $BWSX$ and $BWSY$ are 3-coconnected. Therefore by Corollary 2.17 and Corollary 5.27 the lower vertical morphisms are stable P_2-equivalences. The assumption that α is a stable equivalence means that $|S\alpha|$ must be too, and hence both are stable P_2-equivalences. The result then follows by 2-out-of-3 for stable P_2-equivalences. $\quad\square$

5.3. Group-completion for E_∞ algebras

In this section we recall the theory of group-completions of E_∞ algebras in *Top* and discuss its implications for the symmetric monoidal 2-groupoids studied above. Let $\mathscr{D}$ be an arbitrary E_∞ operad in *Top*.

Notation 5.30. Let $\mathscr{C}_n$ be the little n-cubes operad, and let $\mathscr{C}_\infty$ be the colimit (the maps are given by inclusions of $\mathscr{C}_n$ into $\mathscr{C}_{n+1}$). This is an E_∞ operad (see [30, Section 4]). Let $\mathscr{D}_\infty = \mathscr{D} \times \mathscr{C}_\infty$ and let p_1 and p_2 denote the two projections.

Theorem 5.31 ([31, Theorem 2.3]). *If X is a $\mathscr{D}$-algebra, then there is an algebra qX over $\mathscr{D}_\infty$ and a $\mathscr{C}_\infty$-algebra LX, together with $\mathscr{D}_\infty$-algebra maps*

$$p_1^* X \xleftarrow{\ \xi\ } qX \xrightarrow{\ \alpha\ } p_2^* LX$$

such that ξ is a homotopy equivalence and α is a group-completion. The assignments $X \mapsto qX$ and $X \mapsto LX$ are functorial, and ξ and α are natural.

Remark 5.32. The functors q and L are constructed explicitly in [31]. The homotopy inverse of ξ is also very explicit, but it is not a $\mathscr{D}_\infty$-algebra map.

Note that Corollary 5.27 implies that both ξ and α above are stable equivalences of $\mathscr{D}_\infty$-algebras. We now specialize to the E_∞ operad $B\mathcal{O}$, and we let $B\mathcal{O}_\infty = B\mathcal{O} \times \mathscr{C}_\infty$. Since the functors W and S are strong symmetric monoidal we obtain the following result.

Lemma 5.33. *There are operads $WS\mathscr{C}_\infty$ and $WSB\mathcal{O}_\infty$ in 2Cat, together with projections*

$$p_1 \colon WSB\mathcal{O}_\infty \to WSB\mathcal{O}$$

and

$$p_2 \colon WSB\mathcal{O}_\infty \to WS\mathcal{C}_\infty.$$

Corollary 5.34. *Given a permutative Gray-monoid $\mathcal{A}$, there exists a natural zigzag of maps of $WSB\mathcal{O}_\infty$-algebras in $(2\mathcal{Cat}, \times)$*

$$
\begin{array}{ccc}
& WS(qB\mathcal{A}) & \\
{}^{WS(\xi_{B\mathcal{A}})}\swarrow & & \searrow^{WS(\alpha_{B\mathcal{A}})} \\
p_1^* WS(B\mathcal{A}) & & p_2^* WS(LB\mathcal{A}).
\end{array}
$$

The left arrow is a biequivalence.

Proof. By Propositions 2.7 and 5.6 (*iv*), B, S and W are strong monoidal. For the biequivalence part, S sends homotopy equivalences to weak equivalences, and W sends weak equivalences to biequivalences (Proposition 5.6 (*i*)). $\quad\square$

Notation 5.35. Because $WS\mathcal{C}_\infty$ is in $\mathbb{C}(\le \infty)$, Proposition 4.20 guarantees that it has a choice of multiplication. For the rest of this paper, let ν denote a fixed such choice. For example, the content of [22, §2.2] provides one such choice.

Notation 5.36. Let $\mathfrak{c}$ denote the choice of multiplication in $WSB\mathcal{O}_\infty$ given by the product of $h(\widetilde{\kappa})$ (Notations 5.14 and 5.16) and ν. We call this the canonical choice of multiplication for $WSB\mathcal{O}_\infty$.

Proposition 5.37. *Given a permutative Gray-monoid $\mathcal{A}$, there is a natural zigzag of symmetric monoidal 2-categories and strict monoidal 2-functors*

$$
\begin{array}{ccc}
& \mathfrak{c}^* WS(qB\mathcal{A}) & \\
{}^{\mathfrak{c}^*(WS(\xi_{B\mathcal{A}}))}\swarrow & & \searrow^{\mathfrak{c}^*(WS(\alpha_{B\mathcal{A}}))} \\
\widetilde{\kappa}^* h^* WS(B\mathcal{A}) \;=\; \mathfrak{c}^* p_1^* WS(B\mathcal{A}) & & \mathfrak{c}^* p_2^* WS(LB\mathcal{A}) \;\;=\; \nu^* WS(LB\mathcal{A}).
\end{array}
$$

Moreover, $\nu^* WS(LB\mathcal{A})$ is a Picard 2-category, ξ is a biequivalence and α is a stable P_2-equivalence.

Proof. The existence of this natural zigzag follows by applying Corollary 5.34 with the canonical choice of multiplication $\mathfrak{c}$. By Proposition 4.23 we identify $\mathfrak{c}^* p_1^* = \widetilde{\kappa}^* h^*$ and $\mathfrak{c}^* p_2^* = \nu^*$.

We see that $\nu^* WS(LB\mathcal{A})$ is a 2-groupoid because WX is a 2-groupoid for every simplicial set X (see Theorem 5.5). We note that $\nu^* WS$ preserves π_0 and that $\pi_0(\nu^* WS(LB\mathcal{A})) \cong \pi_0(LB\mathcal{A})$ is a group because $LB\mathcal{A}$ is group-complete. The product is induced by the monoidal structure, and therefore it follows that objects have inverses up to equivalence. Thus $\nu^* WS(LB\mathcal{A})$ is a Picard 2-category.

The fact that $\mathfrak{c}^* WS(\xi)$ is a biequivalence is immediate from Corollary 5.34, and the claim about $\mathfrak{c}^* WS(\alpha)$ follows from Proposition 5.29 because the map α of Theorem 5.31 is a group-completion and hence a stable equivalence. $\quad\square$

Because the property of being Picard is preserved by biequivalences, we have the following corollary.

Corollary 5.38. *If $\mathcal{A}$ is a strict Picard 2-category, then the span in Proposition 5.37 is a span of Picard 2-categories.*

6. Proof of the 2-dimensional stable homotopy hypothesis

Our main theorem is the following.

Theorem 6.1. *There is an equivalence of homotopy theories*

$$(\mathcal{P}ic\,\mathcal{PGM},\ cat\ eq) \simeq (\mathcal{S}p_0^2,\ st\ eq).$$

Proof. The proof follows from putting together several results in this section. To be precise, we combine Propositions 6.2 and 6.4 below, which follow easily from previous work in [20,21], with Theorem 6.5, whose proof depends on the content of Sections 3 through 5. □

Proposition 6.2. *There is an equality of homotopy theories*

$$(\mathcal{P}ic\,\mathcal{PGM},\ cat\ eq) = (\mathcal{P}ic\,\mathcal{PGM},\ st\ P_2\text{-}eq).$$

Proof. Recall that a 2-functor is a biequivalence if and only if it is essentially surjective and a local equivalence. The formulas of [21, Lemma 3.2] show that the stable homotopy groups of a strict Picard 2-category are computed by the algebraic homotopy groups (i.e. equivalence classes of invertible morphisms) in each dimension. Therefore a strict functor between strict Picard 2-categories is a stable P_2-equivalence if and only if it is a biequivalence. □

Lemma 6.3. *The functor P of [20] preserves stable P_2-equivalences.*

Proof. Using the notation of [20], let $f\colon X \to Y$ be a stable P_2-equivalence of Γ-2-categories (certain diagrams of 2-categories indexed on finite pointed sets; see [20, Definition 2.11]). Since stable P_2-equivalences of permutative Gray-monoids are created by the K-theory functor of [20], it suffices to check that KPf is a stable P_2-equivalence. This is immediate from the naturality of the unit η with respect to strict Γ-maps ([20, Corollary 7.14]): we have

$$\eta \circ f = KPf \circ \eta.$$

Since η is a stable equivalence, then KPf is a stable P_2-equivalence by 2-out-of-3, and therefore Pf is too. □

Proposition 6.4. *There are equivalences of homotopy theories*

$$(\mathcal{PGM},\ st\ P_2\text{-}eq) \simeq (\mathcal{S}p_{\geq 0},\ st\ P_2\text{-}eq) \simeq (\mathcal{S}p_0^2,\ st\ eq).$$

Proof. The K-theory functor of [20, Proposition 6.13] creates stable P_2-equivalences by definition. Lemma 6.3 observes that the inverse P preserves stable P_2-equivalences as well. The first equivalence then follows from the equivalences of [20] relative to stable P_2-equivalences. The second equivalence is a reformulation of definitions. □

Theorem 6.5. *There is an equivalence of homotopy theories*

$$(\mathcal{P}ic\,\mathcal{PGM},\ st\ P_2\text{-}eq) \simeq (\mathcal{PGM},\ st\ P_2\text{-}eq).$$
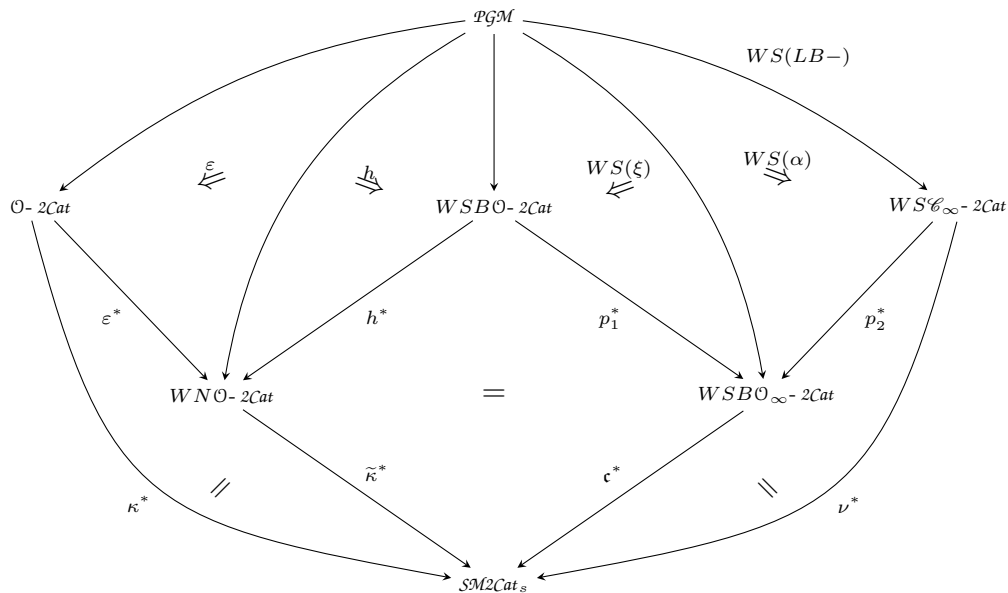
Fig. 6.11. This diagram of categories, functors, and natural transformations summarizes the zigzag constructed in the proof of Theorem 6.5.

To prove Theorem 6.5, we consider the serially-commuting diagram of homotopy theories and relative functors below. Lemma 3.35 shows that the inclusions j in this diagram are equivalences of homotopy theories, with inverse equivalences given by $r = (-)^{qst}$. We will show that the inclusions i are equivalences of homotopy theories.

$$
\begin{array}{ccc}
(\mathcal{P}ic\,\mathcal{P}\mathcal{G}\mathcal{M},\ st\ P_2\text{-}eq) & \xrightarrow{\quad i \quad} & (\mathcal{P}\mathcal{G}\mathcal{M},\ st\ P_2\text{-}eq) \\
j \Big\downarrow\Big\uparrow r & \quad\mathbb{G}\quad & j \Big\downarrow\Big\uparrow r \\
(\mathcal{P}ic\,\mathcal{B}icat_s,\ st\ P_2\text{-}eq) & \xrightarrow{\quad i \quad} & (\mathcal{S}\mathcal{M}\mathcal{B}icat_s,\ st\ P_2\text{-}eq)
\end{array}
\qquad (6.6)
$$

To do this, we first reduce to the problem of constructing a relative functor $\mathbb{G}$ which commutes with i and j up to natural zigzags of stable P_2-equivalences.

Lemma 6.7. *Suppose there is a relative functor $\mathbb{G}$ as shown in Display (6.6), and suppose that diagram involving $\mathbb{G}$, i, and j commutes up to a natural zigzag of stable P_2-equivalences. Then the inclusions labeled i are equivalences of homotopy theories.*

Proof. Because the square involving i and j commutes, it suffices to prove that the inclusion

$$
i\colon (\mathcal{P}ic\,\mathcal{B}icat_s,\ st\ P_2\text{-}eq) \to (\mathcal{S}\mathcal{M}\mathcal{B}icat_s,\ st\ P_2\text{-}eq)
$$

is an equivalence of homotopy theories. We do this by showing that the composite $\mathbb{G}r$ is an inverse for i up to natural zigzag of stable P_2-equivalences.

Let us write $\leftrightarrow\widetilde{\rightarrow}\leftrightarrow$ to denote a natural zigzag of stable P_2-equivalences. Then the proof of Lemma 3.35 shows we have

$$
jr \leftrightarrow\widetilde{\rightarrow}\leftrightarrow \mathrm{id} \qquad \text{and} \qquad rj \leftrightarrow\widetilde{\rightarrow}\leftrightarrow \mathrm{id}.
$$

By assumption, we have $i\mathbb{G} \leftrightarrow\tilde{\leftrightarrow}\rightarrow j$ and $\mathbb{G}i \leftrightarrow\tilde{\leftrightarrow}\rightarrow j$. Hence we have

$$i\mathbb{G}r \leftrightarrow\tilde{\leftrightarrow}\rightarrow jr \leftrightarrow\tilde{\leftrightarrow}\rightarrow \text{id} \qquad \text{and} \qquad \mathbb{G}ri = \mathbb{G}ir \leftrightarrow\tilde{\leftrightarrow}\rightarrow jr \leftrightarrow\tilde{\leftrightarrow}\rightarrow \text{id}. \quad \square$$

Now we describe $\mathbb{G}$ and show that it satisfies the hypotheses of Lemma 6.7.

Definition 6.8. Let $\mathbb{G} = \nu^* WS(LB-)$.

Recalling the relevant notation, this is the composite of the classifying space B, topological group completion L, singular simplicial set S, Whitehead 2-groupoid W, and choice of multiplication ν^* (applied to a permutative Gray-monoid considered as an $\mathcal{O}$-algebra via Corollary 4.11). By Proposition 5.37, this is a functor from permutative Gray-monoids to Picard 2-categories. We will confirm that $\mathbb{G}$ is a relative functor in the course of the proof of Theorem 6.5.

Remark 6.9. The attentive reader will note that ν^* takes values in the subcategory $\mathcal{P}ic\,2\mathcal{C}at_s \subset \mathcal{P}ic\,\mathcal{B}icat_s$. We implicitly compose with this inclusion because, although we suspect $\mathcal{P}\mathcal{G}\mathcal{M}$, $\mathcal{S}\mathcal{M}2\mathcal{C}at_s$, and $\mathcal{S}\mathcal{M}\mathcal{B}icat_s$ all have equivalent homotopy theories (and likewise for the Picard subcategories of each), the proof of Lemma 3.35 does not specialize to $\mathcal{S}\mathcal{M}2\mathcal{C}at_s$.

Proof of Theorem 6.5. The necessary zigzags to apply Lemma 6.7 have already been constructed; we review them now. Let $\mathcal{A}$ be a permutative Gray-monoid. To compare $j\mathcal{A}$ and $i\mathbb{G}(\mathcal{A})$, we require three operads: $B\mathcal{O}$ is the geometric realization of categorical Barratt–Eccles operad $\mathcal{O}$ (Definition 4.8); $\mathscr{C}_\infty$ is the little infinite cubes operad (Notation 5.30); and $B\mathcal{O}_\infty = B\mathcal{O} \times \mathscr{C}_\infty$ is their product.

We consider choices of multiplication induced by operad maps shown in Display (6.10) below (see Corollary 5.15 and Lemma 5.33).

$$
\begin{array}{ccccccc}
& WN\mathcal{O} & & & WSB\mathcal{O}_\infty & & \\
& \swarrow{\scriptstyle \varepsilon} \quad \searrow{\scriptstyle h} & & \swarrow{\scriptstyle p_1} & & \searrow{\scriptstyle p_2} & \\
\mathcal{O} & & WSB\mathcal{O} & & & & WS\mathscr{C}_\infty
\end{array}
\qquad (6.10)
$$

This is a diagram of operads in $2\mathcal{C}at$, that is, at level n the maps are given by 2-functors. With appropriate choices of multiplication, we construct the required zigzag in two stages. First, we use $\widetilde{\kappa}$, given by applying $\varepsilon^{\cdot}$ to the canonical choice κ (see Notation 5.16). By Proposition 5.17 we have the following zigzag of in $\mathcal{S}\mathcal{M}2\mathcal{C}at_s$, where the right leg is a biequivalence and the left leg is an unstable P_2-equivalence and therefore a stable P_2-equivalence by Corollaries 2.17 and 5.27.

$$
\begin{array}{ccc}
& \widetilde{\kappa}^* WN(\mathcal{A}) & \\
\swarrow{\scriptstyle \widetilde{\kappa}^*(\varepsilon_\mathcal{A})} & & \searrow{\scriptstyle \widetilde{\kappa}^*(h_\mathcal{A})} \\
\mathcal{A} = \kappa^*\mathcal{A} = \widetilde{\kappa}^*\varepsilon^*\mathcal{A} & & \widetilde{\kappa}^* h^* WSB(\mathcal{A})
\end{array}
$$

Second, we use $\mathfrak{c}$, described in Notation 5.36. By Proposition 5.37 we have the following zigzag in $\mathcal{S}\mathcal{M}2\mathcal{C}at_s$, where the left leg is a biequivalence, and the right leg is a stable P_2-equivalence.

$$\mathfrak{c}^* WS(qB\mathcal{A})$$

$$\mathfrak{c}^*(WS(\xi_{B\mathcal{A}})) \swarrow \qquad \searrow \mathfrak{c}^*(WS(\alpha_{B\mathcal{A}}))$$

$$\widetilde{\kappa}^* h^* WS(B\mathcal{A}) \;=\; \mathfrak{c}^* p_1^* WS(B\mathcal{A}) \qquad\qquad \mathfrak{c}^* p_2^* WS(LB\mathcal{A}) \;=\; \nu^* WS(LB\mathcal{A})$$

Thus we have a natural zigzag of stable P_2-equivalences between $i\mathbb{G}$ and j. This also shows that $\mathbb{G}$ is a relative functor since j preserves and i creates stable P_2-equivalences.

As noted in Corollaries 5.18 and 5.38, this is a zigzag of Picard 2-categories when $\mathcal{A}$ is a strict Picard 2-category. Thus we also have a natural zigzag of stable P_2-equivalences between $\mathbb{G}i$ and j. By Lemma 6.7, this completes the proof. $\square$

The key step, producing a zigzag of stable P_2-equivalences between $i\mathbb{G}$ and j, is summarized in Fig. 6.11.

Composing with the inclusion $\mathcal{SM2Cat}_s \subset \mathcal{SMBicat}_s$, the composite along the left hand side becomes the inclusion j. Likewise, the composite $\nu^* WSLB$ along the right hand side becomes $i\mathbb{G}$. The components of h and $WS(\xi)$ are biequivalences; the components of ε and $WS(\alpha)$ are stable P_2-equivalences.

References

[1] J.C. Baez, J. Dolan, Higher-dimensional algebra and topological quantum field theory, J. Math. Phys. 36 (11) (1995) 6073–6105, https://doi.org/10.1063/1.531236.
[2] M.G. Barratt, A note on the cohomology of semigroups, J. Lond. Math. Soc. 36 (1961) 496–498, https://doi.org/10.1112/jlms/s1-36.1.496.
[3] M.G. Barratt, A free group functor for stable homotopy, in: Algebraic Topology, Proc. Sympos. Pure Math., Vol. XXII, Univ. Wisconsin, Madison, Wis., 1970, Amer. Math. Soc., Providence, RI, 1971, pp. 31–35.
[4] M. Barratt, S. Priddy, On the homology of non-connected monoids and their associated groups, Comment. Math. Helv. 47 (1972) 1–14, https://doi.org/10.1007/BF02566785.
[5] M.A. Batanin, Monoidal globular categories as a natural environment for the theory of weak n-categories, Adv. Math. 136 (1) (1998) 39–103, https://doi.org/10.1006/aima.1998.1724.
[6] M.A. Batanin, An operadic proof of Baez–Dolan stabilization hypothesis, Proc. Am. Math. Soc. 145 (7) (2017) 2785–2798, https://doi.org/10.1090/proc/13423.
[7] P. Carrasco, A.M. Cegarra, A.R. Garzón, Nerves and classifying spaces for bicategories, Algebraic Geom. Topol. 10 (1) (2010) 219–274, https://doi.org/10.2140/agt.2010.10.219.
[8] A.M. Cegarra, B.A. Heredia, J. Remedios, Bicategorical homotopy pullbacks, Theory Appl. Categ. 30 (6) (2015) 147–205.
[9] E. Cheng, N. Gurski, The periodic table of n-categories II: degenerate tricategories, Cah. Topol. Géom. Différ. Catég. 52 (2) (2011) 82–125.
[10] E. Cheng, N. Gurski, Iterated icons, Theory Appl. Categ. 29 (32) (2014) 929–977.
[11] P. Gabriel, M. Zisman, Calculus of Fractions and Homotopy Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 35, Springer-Verlag, New York, Inc., New York, 1967.
[12] D. Gepner, R. Haugseng, Enriched ∞-categories via non-symmetric ∞-operads, Adv. Math. 279 (2015) 575–716, https://doi.org/10.1016/j.aim.2015.02.007.
[13] R. Gordon, A.J. Power, R. Street, Coherence for Tricategories, 558 edition, Mem. Am. Math. Soc., vol. 117, Amer. Math. Soc., 1995.
[14] J.W. Gray, Formal Category Theory: Adjointness for 2-Categories, Lecture Notes in Mathematics, vol. 391, Springer-Verlag, Berlin–New York, 1974.
[15] B. Guillou, J. May, Equivariant iterated loop space theory and permutative G-categories, Algebraic Geom. Topol. 17 (6) (2017) 3259–3339, https://doi.org/10.2140/agt.2017.17.325.
[16] N. Gurski, Nerves of bicategories as stratified simplicial sets, J. Pure Appl. Algebra 213 (6) (2009) 927–946, https://doi.org/10.1016/j.jpaa.2008.10.011.
[17] N. Gurski, Biequivalences in tricategories, Theory Appl. Categ. 26 (14) (2012) 349–384.
[18] N. Gurski, Coherence in Three-Dimensional Category Theory, Cambridge Tracts in Mathematics, vol. 201, Cambridge University Press, Cambridge, 2013.
[19] N. Gurski, The monoidal structure of strictification, Theory Appl. Categ. 28 (1) (2013) 1–23.
[20] N. Gurski, N. Johnson, A.M. Osorno, K-theory for 2-categories, Adv. Math. 322 (2017) 378–472, https://doi.org/10.1016/j.aim.2017.10.011.
[21] N. Gurski, N. Johnson, A.M. Osorno, M. Stephan, Stable Postnikov data of Picard 2-categories, Algebraic Geom. Topol. 17 (2017) 2763–2806, https://doi.org/10.2140/agt.2017.17.2763.
[22] N. Gurski, A.M. Osorno, Infinite loop spaces, and coherence for symmetric monoidal bicategories, Adv. Math. 246 (2013) 1–32, https://doi.org/10.1016/j.aim.2013.06.028.
[23] A. Hatcher, Algebraic Topology, Cambridge University Press, Cambridge, 2002.

[24] M.W. Johnson, D. Yau, A Foundation for PROPS, Algebras, and Modules, Mathematical Surveys and Monographs, vol. 203, American Mathematical Society, Providence, RI, 2015.

[25] N. Johnson, A.M. Osorno, Modeling stable one-types, Theory Appl. Categ. 26 (20) (2012) 520–537.

[26] A. Joyal, R. Street, Braided tensor categories, Adv. Math. 102 (1) (1993) 20–78, https://doi.org/10.1006/aima.1993.1055.

[27] S. Lack, Icons, Appl. Categ. Struct. 18 (3) (2010) 289–307, https://doi.org/10.1007/s10485-008-9136-5.

[28] S. Lack, S. Paoli, 2-nerves for bicategories, K-Theory 38 (2) (2008) 153–175, https://doi.org/10.1007/s10977-007-9013-2.

[29] T. Leinster, Higher Operads, Higher Categories, London Mathematical Society Lecture Note Series, vol. 298, Cambridge University Press, Cambridge, 2004.

[30] J.P. May, The Geometry of Iterated Loop Spaces, Lecture Notes in Mathematics, vol. 271, Springer-Verlag, Berlin, 1972.

[31] J.P. May, E_∞ spaces, group completions, and permutative categories, in: New Developments Topology, Proc. Sympos. Algebraic Topology, Oxford, 1972, in: Lond. Math. Soc. Lect. Note Ser., vol. 11, Cambridge Univ. Press, London, 1974, pp. 61–93.

[32] P. McCrudden, Balanced coalgebroids, Theory Appl. Categ. 7 (6) (2000) 71–147.

[33] I. Moerdijk, J. Svensson, Algebraic classification of equivariant homotopy 2-types. I, J. Pure Appl. Algebra 89 (1–2) (1993) 187–216, https://doi.org/10.1016/0022-4049(93)90094-A.

[34] A.M. Osorno, Spectra associated to symmetric monoidal bicategories, Algebraic Geom. Topol. 12 (1) (2012) 307–342, https://doi.org/10.2140/agt.2012.12.307.

[35] D. Quillen, On the group completion of a simplicial monoid, Mem. Am. Math. Soc. 110 (529) (1994) 89–105, https://doi.org/10.1090/memo/0529, MIT Preprint 1971.

[36] C. Rezk, A model for the homotopy theory of homotopy theory, Trans. Am. Math. Soc. 353 (3) (2001) 973–1007, https://doi.org/10.1090/S0002-9947-00-02653-2 (electronic).

[37] C. Schommer-Pries, The classification of two-dimensional extended topological field theories, arXiv:1112.1000v2, 2011.

[38] G. Segal, Categories and cohomology theories, Topology 13 (1974) 293–312, https://doi.org/10.1016/0040-9383(74)90022-6.

[39] C. Simpson, On the Breen–Baez–Dolan stabilization hypothesis for Tamsamani's weak n-categories, arXiv:math/9810058, 1998.

[40] C. Simpson, Homotopy Theory of Higher Categories, New Mathematical Monographs, vol. 19, Cambridge University Press, Cambridge, 2012.

[41] J. Strom, Modern Classical Homotopy Theory, Graduate Studies in Mathematics, vol. 127, American Mathematical Society, Providence, RI, 2011.

[42] R.W. Thomason, Symmetric monoidal categories model all connective spectra, Theory Appl. Categ. 1 (5) (1995) 78–118 (electronic).

[43] D. Yau, Colored Operads, Graduate Studies in Mathematics, vol. 170, American Mathematical Society, Providence, RI, 2016.