INVERTING THE LOCAL GEODESIC RAY TRANSFORM OF HIGHER RANK TENSORS

MAARTEN V. DE HOOP *, GUNTHER UHLMANN †, AND JIAN ZHAI ‡

Abstract. Consider a Riemannian manifold in dimension $n \geq 3$ with strictly convex boundary. We prove the local invertibility, up to potential fields, of the geodesic ray transform on tensor fields of rank four near a boundary point. This problem is closely related with elastic qP-wave tomography. Under the condition that the manifold can be foliated with a continuous family of strictly convex hypersurfaces, the local invertibility implies a global result. One can straightforwardedly adapt the proof to show similar results for tensor fields of arbitrary rank.

1. Introduction. We let $M \subset \mathbb{R}^3$ be a bounded domain with smooth boundary ∂M and $x = (x^1, x^2, x^3)$ be the Cartesian coordinates. The system of equations describing elastic waves reads

$$\rho \partial_t^2 u = \operatorname{div}(\mathbf{C}\varepsilon(u)). \tag{1.1}$$

Here, u denotes the displacement vector and

$$\varepsilon(u) = (\nabla u + (\nabla u)^T)/2 = (\varepsilon_{ij}(u)) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right)$$

the linear strain tensor which is the symmetric part of ∇u. Furthermore, $\mathbf{C} = (C_{ijkl}) = (C_{ijkl}(x))$ is the stiffness tensor and $\rho = \rho(x)$ is the density of mass.

The stiffness tensor is assumed to have the symmetries

$$C_{ijkl} = C_{jikl} = C_{klij}.$$

The operator $\operatorname{div}(\mathbf{C}\varepsilon(\cdot))$ is elliptic if we additionally assume that there exists a $\delta > 0$ such that for any 3×3 real-valued symmetric matrix (ε_{ij}),

$$\sum_{i,j,k,l=1}^{3} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq \delta \sum_{i,j=1}^{3} \varepsilon_{ij}^2.$$

If the stiffness tensor $\mathbf{C}$ is isotropic, we have

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{1.2}$$

where λ, μ are called the Lamé parameters. For isotropic elasticity there are two different wave-speeds, namely, P-wave (longitudinal wave) speed $c_P = \sqrt{\frac{\lambda+2\mu}{\rho}}$ and S-wave (transverse wave) speed $c_S = \sqrt{\frac{\mu}{\rho}}$. Then we can consider M as a manifold with metric $c_P^{-2}\mathrm{d}s^2$ or $c_S^{-2}\mathrm{d}s^2$. Correspondingly, we can view P waves traveling along geodesics in Riemannian manifold $(M, c_P^{-2}\mathrm{d}s^2)$, and S waves traveling along geodesics in $(M, c_S^{-2}\mathrm{d}s^2)$.

If there is an anisotropic perturbation a_{ijkl} around isotropy, that is,

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + a_{ijkl},$$

the perturbation in travel time of P-waves along a geodesic γ gives the following quantity [2]:

$$\int_\gamma \frac{a_{ijkl}}{\rho c_P^6}\dot\gamma^i\dot\gamma^j\dot\gamma^k\dot\gamma^l \mathrm{d}t. \tag{1.3}$$

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Here γ is a geodesic in $(M, c_P^{-2}\mathrm{d}s^2)$. The same quantity has been derived by a different perturbation analysis [14]. Equation (1.3) represents a geodesic ray transform of a 4-tensor $b_{ijkl} = \frac{a_{ijkl}}{\rho c_P^6}$ in $(M, c_P^{-2}\mathrm{d}s^2)$.

Let (M, g) be a compact Riemannian manifold with boundary ∂M. The geodesic ray transform of a symmetric tensor field f of order m is given by

$$I_m f(\gamma) = \int_\gamma \langle f(\gamma(t)), \dot{\gamma}^m(t)\rangle \mathrm{d}t, \tag{1.4}$$

where, in local coordinates, $\langle f, v^m(t)\rangle = f_{i_1,\cdots,i_m} v^{i_1}\cdots v^{i_m}$, and γ runs over all geodesics with endpoints on ∂M. We note, here, that the tensor b in (1.3) is not fully symmetric. Thus, we introduce f that is the symmetrization of b, and study the geodesic X-ray transform $I_4 f$. A general tensor with symmetry (1.2) has 21 unknowns, while a symmetric 4-tensor has 15 unknowns. Therefore we have already lost 6 components of $\mathbf{C}$ in the formulation of the problem.

It is known that potential vector fields, i.e., $f = \mathrm{d}^s v$ with v a symmetric field of order $m-1$ vanishing on ∂M $(m \geq 1)$, are in the kernel of I_m. Here, d^s is the symmetric part of the covariant derivative ∇, which will be defined in (2.6). We say that I_m is s-injective if $I_m f = 0$ implies $f = \mathrm{d}^s v$ with $v|_{\partial M} = 0$. The s-injectivity of I_m has been extensively investigated, and we refer to [5, 19] for detailed reviews.

Assuming that M is simple, when ∂M is strictly convex and any two points in M are connected by a unique minimizing geodesic smoothly depending on the endpoints, it has been proved that I_0 is injective [8, 9], and I_1 is s-injective [1]. In dimension two, the s-injectivity of I_m for arbitrary m is proved in [10]. In dimension three or higher, the s-injectivity of $I_m, m \geq 2$ is still open. When (M, g) has negative sectional curvature [13], or under certain other curvature conditions [3, 12, 14], the s-injectivity has been established. Without any curvature condition, it has been proved that the problem is Fredholm [16] (modulo potential fields) with a finite-dimensional smooth kernel. For analytic simple metrics, the uniqueness is proved using microlocal analytic continuation. With the Fredholm property, the uniqueness can be extended to an open and dense set of simple metrics in $C^k, k \gg 1$, containing analytic simple metrics.

In [20], Uhlmann and Vasy proved that, if ∂M is strictly convex at $p \in \partial M$ in dimension three or higher, $I_0 f(\gamma)$, for all geodesics localized in some suitable Ω near p, determine f near p. Furthermore, under some global convex "foliation condition", it gives a global result via layer stripping techniques. Then, Stefanov, Uhlmann and Vasy gave corresponding results for I_1 and I_2 [18]. The key point is to show the ellipticity (under a suitable gauge condition) of a different version of the normal operator $I_m^* I_m$ as a scattering pseudodifferential operator. The calculation for I_1, I_2, which is already massive, is not observed to have an easy extension to I_m, $m \geq 3$. In this paper, we will prove parallel results for I_4 for two main reasons: (1) it arises naturally from elastic qP-wave tomography; (2) the scheme of calculation needs to be general enough so that one can easily adapt the procedure to prove similar results for I_m with arbitrary m.

For an open set $O \subset M$, $O \cap \partial M \neq \emptyset$, we call γ an O-local geodesic if γ is a geodesic contained in O with endpoints in ∂M. We denote the set of O-local geodesics by $\mathcal{M}_O$. Note that $\mathcal{M}_O$ is an open subset of the set of all geodesics $\mathcal{M}$. The introduction of $\mathcal{M}$ and $\mathcal{M}_O$ can be found in [20]. We define the local geodesic ray transform of f as the collection $(I_m f)(\gamma)$ along all geodesics $\gamma \in \mathcal{M}_O$, that is, as the restriction of the geodesic ray transform to $\mathcal{M}_O$. We will restrict ourselves to the problem (1.4) with $m = 4$ from now on.

First, we consider M as a strictly convex domain in a Riemannian manifold $(\tilde{M}, g)$ (without boundary), with boundary defining function ρ, such that $\rho \geq 0$ on M. As in [20, 18], we first study the invertibility of I_4 in a neighborhood of a point $p \in \partial M$ of the form $\{\tilde{x} > -c\}$, $c > 0$. Here $\tilde{x}$ is a function with $\tilde{x}(p) = 0$, $\mathrm{d}\tilde{x}(p) = -\mathrm{d}\rho(p)$. We denote $\Omega = \Omega_c = \{x \geq 0, \rho \geq 0\}$, $x = x_c = \tilde{x} + c$. Using the local geodesic ray transform with Ω-local geodesics, we have the local injectivity result

THEOREM 1.1. *With $\Omega = \Omega_c$ as above, there is $c_0 > 0$ such that for $c \in (0, c_0)$, if $f \in L^2(\Omega)$ is a symmetric 4-tensor. then $f = u + \mathrm{d}^s v$, where $v \in \dot{H}^1_{loc}(\Omega \setminus \{x = 0\})$, while $u \in L^2_{loc}(\Omega \setminus \{x = 0\})$*

can be stably determined from $I_4 f$ restricted to Ω-local geodesics in the following sense. There is a continuous map $I_4 f \mapsto u$, where for $s \geq 0$, $f \in H^s(\Omega)$, the H^{s-1} norm of u restricted to any compact subset of $\Omega \setminus \{x=0\}$ is controlled by the H^s norm of $I_4 f$ restricted to the set of Ω-local geodesics.

Replacing $\Omega_c = \{\tilde{x} > -c\} \cap M$ by $\Omega_{\tau,c} = \{\tau > \tilde{x} > -c+\tau\} \cap M$, c can be taken uniform in τ for τ in a compact set on which the strict concavity assumption on level sets of $\tilde{x}$ holds.

The Sobolev spaces $\dot{H}^1_{loc}$ will be defined in Section 3. As in [18, 20], the above theorem can be applied to obtain the following global result. Now, assume $\tilde{x}$ is a globally defined function with level sets $\Sigma_t = \{\tilde{x} = t\}$ strictly concave (viewed from $\tilde{x}^{-1}(0,t)$) for $t \in (-T, 0]$, with $\tilde{x} \leq 0$ on the manifold M with boundary. Assume further that $\Sigma_0 = \partial M$ and $M \setminus \cup_{t\in(-T,0]}\Sigma_t$ has measure 0 or has an empty interior. Will say such an M satisfies the foliation condition.

Theorem 1.2. *Suppose M is compact. The geodesic ray transform is injective and stable modulo potentials on the restriction of symmetric 4-tensors f to $\tilde{x}^{-1}((-T,0])$ in the following sense. For all $\tau > -T$ there is $v \in \dot{H}^1_{loc}(\tilde{x}^{-1}((\tau,0]))$ such that $f - \mathrm{d}^s v \in L^2_{loc}(\tilde{x}^{-1}((\tau,0]))$ can be stably recovered from $I_4 f$. Here for stability we assume that $s \geq 0$, f is in an H^s-space, the norm on $I_4 f$ is an H^s-norm, while the norm for v is an H^{s-1}-norm.*

The foliation condition can be satisfied even in the presence of caustics. A Riemannian manifold $(M, c^{-2}(|x|)\mathrm{d}s^2)$ satisfying the Herglotz [4] and Wiechert and Zoeppritz [21] condition $\frac{\mathrm{d}}{\mathrm{d}r}\frac{r}{c(r)} > 0$ satisfies the foliation condition. The Euclidean spheres $|x| = r$ form a strictly convex foliation. With the PREM (Preliminary Reference Earth Model) model for Earth, this condition is a realistic one. We note here that it does not exclude the existence of conjugate points. More discussion on the foliation condition can be found in [11] and the references therein.

2. Pseudodifferential property. In Ω, we can use local coordinates (x, y), with x introduced above. We are interested in geodesics "almost tangent" to level sets of $\tilde{x}$.

Let $\gamma_{x,y,\lambda,\omega}$ be a geodesic in $\widetilde{M}$ such that

$$\gamma_{x,y,\lambda,\omega}(0) = (x,y), \quad \dot{\gamma}_{x,y,\lambda,\omega}(0) = (\lambda, \omega),$$

with $(x,y,\lambda,\omega) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{S}^{n-2}$. We need that for $x \geq 0$ and λ sufficiently small the geodesic $\gamma_{x,y,\lambda,\omega}(t)$ stays in $x \geq 0$ as long as it is in M. Thus for $x = 0$, λ can only be 0. This is guaranteed if $|\lambda| < C_1\sqrt{x}$, for sufficiently small C_1. For convenience, we use a smaller range $|\lambda| \leq C_2 x$. We take χ to be a smooth, even, non-negative function with compact support (to be specified).

We denote

$$(I_4 f)(x,y,\lambda,\omega) = \int_{\mathbb{R}} \langle f(\gamma_{x,y,\lambda,\omega}(t)), \dot{\gamma}^4_{x,y,\lambda,\omega}(t)\rangle \mathrm{d}t. \tag{2.1}$$

We note here that we are only interested in f supported in $\overline{M}$, whence the above integration is actually along the segment of $\gamma_{x,y,\lambda,\omega}$ in M. On $u(x,y,\lambda,\omega)$, we define

$$\begin{aligned}(L_4 u)(x,y) = x^4 \int \chi(\lambda/x) u(x,y,\lambda,\omega) g_{sc}(\lambda\partial_x + \omega\partial_y) \otimes g_{sc}(\lambda\partial_x + \omega\partial_y) \\ \otimes g_{sc}(\lambda\partial_x + \omega\partial_y) \otimes g_{sc}(\lambda\partial_x + \omega\partial_y)\mathrm{d}\lambda \mathrm{d}\omega.\end{aligned} \tag{2.2}$$

We will carry out the calculation on $X = \{x \geq 0\}$. Here, u is a (locally defined in the support of χ) function on the space of geodesics parametrized by (x,y,λ,ω), and g_{sc} maps vectors to covectors; g_{sc} is the scattering metric of the form

$$g_{sc} = x^{-4}\mathrm{d}x^2 + x^{-2}h, \tag{2.3}$$

where $h(x,y)$ is a standard 2-cotensor on X.

As in [18], we will show that $L_4 I_4$, conjugated by an exponential weight, is in Melrose's scattering pseudodifferential algebra (cf. [6] for an introduction). The ellipticity of the scattering pseudodifferential operator will be the main subject of this section. In local coordinates $(x, y^1, \cdots, y^{n-1})$, the scattering tangent bundle ${}^{sc}TX$, has a local basis $x\partial_x, x\partial_{y^1}, \cdots, x\partial_{y^{n-1}}$, and the dual bundle ${}^{sc}T^*X$ correspondingly has a local basis $\frac{\mathrm{d}x}{x^2}, \frac{\mathrm{d}y^1}{x}, \cdots, \frac{\mathrm{d}y^{n-1}}{x}$. We adopt the notation $\Psi_{sc}^{m,l}(X)$ for the scattering pseudodifferential algebra introduced in [18]. We also use the notation ${}^{sc}TX$, ${}^{sc}T^*X$ and $\mathrm{Sym}^{k\,sc}T^*X$ defined there in the following analogue of [18, Proposition 3.1]

PROPOSITION 2.1. *On symmetric* 4*-tensors, the operator* $N_{\mathsf{F}} = e^{-\mathsf{F}/x} L I_4 e^{\mathsf{F}/x}$*, lies in*

$$\Psi_{sc}^{-1,0}(X; \mathrm{Sym}^{4sc}T^*X, \mathrm{Sym}^{4sc}T^*X),$$

for $\mathsf{F} > 0$.

Proof. This proposition is analogous to . Use the map introduced in [20],

$$\Gamma_+ : S\tilde{M} \times [0, \infty] \to [\tilde{M} \times \tilde{M}; \mathrm{diag}],\ \Gamma_+(x, y, \lambda, \omega, t) = (x, y, |y' - y|, \frac{x' - x}{|y' - y|}, \frac{y' - y}{|y' - y|}),$$

where $(x', y') = \gamma_{x,y,\lambda,\omega}(t)$. Here $[\tilde{M} \times \tilde{M}; \mathrm{diag}]$ is the *blow-up* of $\tilde{M}$ at the diagonal $(x, y) = (x', y')$. Similarly, we can also define Γ_- in which $(-\infty, 0]$ takes the place of $[0, \infty)$.

We write

$$(\gamma_{x,y,\lambda,\omega}(t), \dot{\gamma}_{x,y,\lambda,\omega}(t)) = (\mathsf{X}_{x,y,\lambda,\omega}(t), \mathsf{Y}_{x,y,\lambda,\omega}(t), \Lambda_{x,y,\lambda,\omega}(t), \Omega_{x,y,\lambda,\omega}(t)),$$

in coordinates (x, y, λ, ω) for lifted geodesic $\gamma_{x,y,\lambda,\omega}(t)$. We use the coordinates,

$$x, y, X = \frac{x' - x}{x^2}, Y = \frac{y' - y}{x},$$

as in [20], and obtain the Schwartz kernel of N_{F} on symmetric 4-tensors (with $\hat{Y} = \frac{Y}{|Y|}$):

$$\begin{aligned} K^\flat(x, y, X, Y) = \sum_{\pm} e^{-\mathsf{F}X/(1+xX)} \chi \left(\frac{X - \alpha(x, y, x|Y|, \frac{xX}{|Y|}, \hat{Y})|Y|^2}{|Y|} + x\tilde{\Lambda}_{\pm} \left(x, y, x|Y|, \frac{x|X|}{|Y|}, \hat{Y} \right) \right) \\ \left[x^{-1}(\Lambda \circ \Gamma_{\pm}^{-1}) \frac{\mathrm{d}x}{x^2} + (\Omega \circ \Gamma_{\pm}^{-1}) \frac{h(\partial_y)}{x} \right]^4 \left[x^{-1}(\Lambda' \circ \Gamma_{\pm}^{-1}) x^2 \partial_{x'} + (\Omega' \circ \Gamma_{\pm}^{-1}) x \partial_{y'} \right]^4 \\ |Y|^{-n+1} J_{\pm} \left(x, y, \frac{X}{|Y|}, |Y|, \hat{Y} \right). \end{aligned} \tag{2.4}$$

□

We denote

$$\nabla : T^m M \to T^{m+1} M$$

being the connection defined componentwise as

$$\begin{aligned} \nabla_k u_{j_1, \cdots, j_m} &= u_{j_1, \cdots, j_m; k} \\ &= \frac{\partial}{\partial x^k} u_{j_1, \cdots, j_m} - \sum_{p=1}^{m} \Gamma^q_{k, j_p} u_{j_1, \cdots, j_{p-1}, q, j_{p+1}, \cdots, j_m}, \end{aligned} \tag{2.5}$$

where Γ is the Christoffel symbol with respect to the metric g. For $u \in T^m M$, we define its symmetrization as

$$\begin{aligned} \mathscr{S} : T^m M &\to S^m M \\ u &\mapsto f, \end{aligned}$$

with

$$f(v_1, \cdots, v_m) = \frac{1}{m!} \sum_{\sigma} u(v_{\sigma(1)}, \cdots, v_{\sigma(m)}),$$

where σ runs over all permutation group of $(1, \cdots, m)$, and $v_j \in C^\infty(TM)$, $j = 1, \cdots, m$.

We define the symmetric differential $\mathrm{d}^s \in S^m M \to S^{m+1} M$ to be

$$\mathrm{d}^s = \mathscr{S}\nabla. \tag{2.6}$$

and note that d^s is different from the exterior differential d defined on the bundle of k-forms $\Lambda^k M$. We also define $\mathrm{d}^s_{\mathsf{F}} = e^{-\mathsf{F}/x}\mathrm{d}^s e^{\mathsf{F}/x}$ and denote its adjoint with respect to the scattering metric g_{sc} (not g) as δ^s_{F}.

For convenience of calculation, we will use the basis

$$\begin{aligned}
&\frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}x}{x^2},\\
&\frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}y}{x},\ \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}x}{x^2},\ \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}x}{x^2},\\
&\frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}y}{x},\ \ \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}y}{x},\ \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}x}{x^2},\\
&\frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}y}{x}
\end{aligned}$$

for 3-tensors, and the basis

$$\begin{aligned}
&\frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}x}{x^2},\\
&\frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}y}{x},\ \ \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}x}{x^2},\ \ \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}x}{x^2},\ \ \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}x}{x^2},\\
&\frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}y}{x},\ \ \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}y}{x},\ \ \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}x}{x^2},\\
&\qquad\qquad \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}y}{x},\ \ \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}x}{x^2},\ \ \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}x}{x^2},\\
&\frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}y}{x},\ \ \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}x}{x^2},\ \ \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}y}{x},\ \ \frac{\mathrm{d}x}{x^2} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}y}{x},\\
&\frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}y}{x} \otimes \frac{\mathrm{d}y}{x},
\end{aligned}$$

for 4-tensors. For symmetric 3-tensors, we use the basis

$$\frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}x}{x^2},\ \ 2 \times \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}y}{x},\ \ 2 \times \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}y}{x} \otimes_s \frac{\mathrm{d}y}{x},\ \ \frac{\mathrm{d}y}{x} \otimes_s \frac{\mathrm{d}y}{x} \otimes_s \frac{\mathrm{d}y}{x};$$

for symmetric 4-tensors, we use the basis

$$\begin{aligned}
\frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}x}{x^2},\ \ 4 \times \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}y}{x},\ \ 6 \times \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}y}{x} \otimes_s \frac{\mathrm{d}y}{x},\\
4 \times \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}y}{x} \otimes_s \frac{\mathrm{d}y}{x} \otimes_s \frac{\mathrm{d}y}{x},\ \ \frac{\mathrm{d}y}{x} \otimes_s \frac{\mathrm{d}y}{x} \otimes_s \frac{\mathrm{d}y}{x} \otimes_s \frac{\mathrm{d}y}{x}.
\end{aligned}$$

In the above, $\otimes_s$ denotes the symmetric product, for example, $a \otimes_s b = \mathscr{S}(a \otimes b)$.

Lemma 2.2. *On symmetric 4-tensors,* $\mathrm{d}^s_{\mathsf{F}}\delta^s_{\mathsf{F}} \in \mathrm{Diff}^{2,0}_{sc}(X; \mathrm{Sym}^{4sc}T^*X, \mathrm{Sym}^{4sc}T^*X)$ *has princi-*

pal symbol

$$\mathfrak{D}(x,y,\xi,\eta)=\begin{pmatrix}\xi+\mathsf{i}\mathsf{F} & 0 & 0 & 0\\ \frac{1}{4}\eta\otimes & \frac{3}{4}(\xi+\mathsf{i}\mathsf{F}) & 0 & 0\\ a^\flat & \frac{1}{2}\eta\otimes_s & \frac{1}{2}(\xi+\mathsf{i}\mathsf{F}) & 0\\ 0 & b^\flat & \frac{3}{4}\eta\otimes_s & \frac{1}{4}(\xi+\mathsf{i}\mathsf{F})\\ 0 & 0 & c^\flat & \eta\otimes_s\end{pmatrix}\begin{pmatrix}\xi-\mathsf{i}\mathsf{F} & \iota_\eta & 6\langle a^\flat,\cdot\rangle & 0 & 0\\ 0 & (\xi-\mathsf{i}\mathsf{F}) & \iota_\eta^s & \frac{4}{3}\langle b^\flat,\cdot\rangle & 0\\ 0 & 0 & (\xi-\mathsf{i}\mathsf{F}) & \iota_\eta^s & \frac{1}{3}\langle c^\flat,\cdot\rangle\\ 0 & 0 & 0 & (\xi-\mathsf{i}\mathsf{F}) & \iota_\eta^s\end{pmatrix}$$

$$=\begin{pmatrix}|\xi|^2+\mathsf{F}^2 & (\xi+\mathsf{i}\mathsf{F})\iota_\eta & 6(\xi+\mathsf{i}\mathsf{F})\langle a^\flat,\cdot\rangle & 0 & 0\\ \frac{1}{4}(\xi-\mathsf{i}\mathsf{F})\eta\otimes & \frac{1}{4}(\eta\otimes)\iota_\eta+\frac{1}{4}(|\xi|^2+\mathsf{F}^2) & \mathfrak{D}_{23} & \mathfrak{D}_{24} & 0\\ (\xi-\mathsf{i}\mathsf{F})a^\flat & a^\flat\iota_\eta+\frac{1}{2}(\xi-\mathsf{i}\mathsf{F})\eta\otimes_s & \mathfrak{D}_{33} & \mathfrak{D}_{34} & \mathfrak{D}_{35}\\ 0 & (\xi-\mathsf{i}\mathsf{F})b^\flat & \mathfrak{D}_{43} & \mathfrak{D}_{44} & \mathfrak{D}_{45}\\ 0 & 0 & \mathfrak{D}_{53} & \mathfrak{D}_{54} & \mathfrak{D}_{55}\end{pmatrix}$$

with

$$\begin{aligned}
\mathfrak{D}_{23}&=\frac{3}{2}\eta\otimes\langle a^\flat,\cdot\rangle+\frac{3}{4}(\xi+\mathsf{i}\mathsf{F})\iota_\eta^s,\\
\mathfrak{D}_{24}&=(\xi+\mathsf{i}\mathsf{F})\otimes\langle b^\flat,\cdot\rangle,\\
\mathfrak{D}_{33}&=6a^\flat\langle a^\flat,\cdot\rangle+\frac{1}{2}(\eta\otimes)\iota_\eta+\frac{1}{2}(|\xi|^2+\mathsf{F}^2),\\
\mathfrak{D}_{34}&=\frac{2}{3}\eta\otimes\langle b^\flat,\cdot\rangle+\frac{1}{2}(\xi+\mathsf{i}\mathsf{F})\otimes\iota_\eta^s,\\
\mathfrak{D}_{35}&=\frac{1}{6}(\xi+\mathsf{i}\mathsf{F})\otimes\langle c^\flat,\cdot\rangle,\\
\mathfrak{D}_{43}&=b^\flat\iota_\eta^s+\frac{3}{4}(\xi-\mathsf{i}\mathsf{F})\eta\otimes_s,\\
\mathfrak{D}_{44}&=\frac{4}{3}b^\flat\langle b^\flat,\cdot\rangle+\frac{3}{4}(\eta\otimes)\iota_\eta^s+\frac{1}{4}(|\xi|^2+\mathsf{F}^2),\\
\mathfrak{D}_{45}&=\frac{1}{4}\eta\otimes\langle c^\flat,\cdot\rangle+\frac{1}{4}(\xi+\mathsf{i}\mathsf{F})\otimes\iota_\eta^s,\\
\mathfrak{D}_{53}&=(\xi-\mathsf{i}\mathsf{F})c^\flat,\\
\mathfrak{D}_{54}&=c^\flat\iota_\eta^s+(\xi-\mathsf{i}\mathsf{F})\eta\otimes_s,\\
\mathfrak{D}_{55}&=\frac{1}{3}c^\flat\langle c^\flat,\cdot\rangle+\eta\otimes_s\iota_\eta.
\end{aligned}$$

The quantities $a^\flat, b^\flat, c^\flat$ are defined in the proof

Proof. we denote

$$\begin{aligned}
f=&f_{xxx}\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}+3\times f_{xxy^i}\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y^i}{x}\\
&+3\times f_{xy^iy^j}\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y^i}{x}\otimes_s\frac{\mathrm{d}y^j}{x}+f_{y^iy^jy^k}\frac{\mathrm{d}y^i}{x}\otimes_s\frac{\mathrm{d}y^j}{x}\otimes_s\frac{\mathrm{d}y^k}{x}.
\end{aligned}$$

By calculation

$$\begin{aligned}
(\nabla f)_{xxxx}&=x^{-6}\partial_x f_{xxx}+O(x^{-7}),\\
(\nabla f)_{xxxy^i}&=x^{-6}\partial_{y^i} f_{xxx}+O(x^{-6}),\\
(\nabla f)_{xxy^ix}&=x^{-5}\partial_x f_{xxy^i}+O(x^{-6}),\\
(\nabla f)_{xxy^iy^j}&=x^{-5}\partial_{y^j} f_{xxy^i}+x^{-6}a_1(f_{xxx})+O(x^{-5}),\\
(\nabla f)_{xy^iy^jx}&=x^{-4}\partial_x f_{xy^iy^j}+O(x^{-5}),\\
(\nabla f)_{xy^iy^jy^k}&=x^{-4}\partial_{y^k} f_{xy^iy^j}+x^{-5}b_1(f_{xxy})+O(x^{-4}),\\
(\nabla f)_{y^iy^jy^kx}&=x^{-3}\partial_x f_{y^iy^jy^k}+O(x^{-4}),\\
(\nabla f)_{y^iy^jy^ky^l}&=x^{-3}\partial_{y^l} f_{y^iy^jy^k}+x^{-4}c_1(f_{xyy})+O(x^{-3}).
\end{aligned}$$

Here a_1, b_1, c_1 come from the contributions of Christoffel symbol Γ in equation (2.5). Then, we derive

$$
\begin{aligned}
\mathrm{d}^s f =& x^2 \partial_x f_{xxx} \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}x}{x^2} \\
&+ 4 \times \left(\frac{1}{4} x \partial_{y^i} f_{xxx} + \frac{3}{4} x^2 \partial_x f_{xxy^i} \right) \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}y^i}{x} \\
&+ 6 \times \left(\frac{1}{2} \mathrm{Sym}_y (x \partial_{y^j} f_{xxy^i}) + \frac{1}{2} x^2 \partial_x f_{xy^i y^j} + a^\flat (f_{xxx}) \right) \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}y^i}{x} \otimes_s \frac{\mathrm{d}y^j}{x} \\
&+ 4 \times \left(\frac{3}{4} \mathrm{Sym}_y (x \partial_{y^k} f_{xy^i y^j}) + \frac{1}{4} x^2 \partial_x f_{y^i y^j y^k} + b^\flat (f_{xxy}) \right) \frac{\mathrm{d}x}{x^2} \otimes_s \frac{\mathrm{d}y^i}{x} \otimes_s \frac{\mathrm{d}y^j}{x} \otimes_s \frac{\mathrm{d}y^k}{x} \\
&+ \left(\mathrm{Sym}_y (x \partial_{y^l} f_{y^i y^j y^k}) + c^\flat (f_{xyy}) \right) \frac{\mathrm{d}y^i}{x} \otimes_s \frac{\mathrm{d}y^j}{x} \otimes_s \frac{\mathrm{d}y^k}{x} \otimes_s \frac{\mathrm{d}y^l}{x} + \text{l.o.t.}.
\end{aligned}
\tag{2.7}
$$

In the above, Sym_y is defined as

$$
\mathrm{Sym}_y (v_{y^{k_1}, \cdots, y^{k_m}}) = \frac{1}{m!} \sum_\sigma v_{y^{k_{\sigma(1)}}, \cdots, y^{k_{\sigma(m)}}}.
$$

It follows that d^s has principal symbol

$$
\begin{pmatrix}
\xi & 0 & 0 & 0 \\
\frac{1}{4}\eta\otimes & \frac{3}{4}\xi & 0 & 0 \\
a^\flat & \frac{1}{2}\eta\otimes_s & \frac{1}{2}\xi & 0 \\
0 & b^\flat & \frac{3}{4}\eta\otimes_s & \frac{1}{4}\xi \\
0 & 0 & c^\flat & \eta\otimes_s
\end{pmatrix}.
$$

The term $\eta\otimes_s$ in the (32)-block has (iji')-entry (corresponding to the (ij) entry of the symmetric 2-tensor on Y and the i' entry of the 1-tensor)

$$
\frac{1}{2}(\eta_i \delta_{ji'} + \eta_j \delta_{ii'}).
$$

The term $\eta\otimes_s$ in the (43)-block has $(ijki'j')$-entry (corresponding to the (ijk) entry of the symmetric 3-tensor and the $i'j'$ entry of the 2-tensor)

$$
\frac{1}{6}(\eta_i \delta_{ji'} \delta_{kj'} + \eta_i \delta_{ki'} \delta_{jj'} + \eta_j \delta_{ii'} \delta_{kj'} + \eta_j \delta_{ki'} \delta_{jj'} + \eta_k \delta_{ii'} \delta_{jj'} + \eta_k \delta_{ji'} \delta_{ij'}).
$$

The term $\eta\otimes_s$ in the (54)-block has $(ijkli'j'k')$-entry (corresponding to the $(ijkl)$ entry of the symmetric 4-tensor and the $i'j'k'$ entry of the 3-tensor)

$$
\begin{aligned}
\frac{1}{24}\Big(&\sum_\sigma \eta_i \delta_{j\tau(\sigma(1))} \delta_{k\tau(\sigma(2))} \delta_{l\tau(\sigma(3))} + \sum_\sigma \eta_j \delta_{i\tau(\sigma(1))} \delta_{k\tau(\sigma(2))} \delta_{l\tau(\sigma(3))} \\
&+ \sum_\sigma \eta_k \delta_{i\tau(\sigma(1))} \delta_{j\tau(\sigma(2))} \delta_{l\tau(\sigma(3))} + \sum_\sigma \eta_l \delta_{i\tau(\sigma(1))} \delta_{j\tau(\sigma(2))} \delta_{k\tau(\sigma(3))}\Big).
\end{aligned}
$$

Here, σ runs over all permutations of (123), and $\tau(1) = i', \tau(2) = j', \tau(3) = k'$.

We note that $a^\flat$ maps a 0-tensor (smooth function) to a symmetric 2-tensor, $b^\flat$ maps a symmetric 1-tensor to a symmetric 3-tensor, $c^\flat$ maps a symmetric 2-tensor to a symmetric 4-tensor. They are symmetrizations of a, b, c respectively. Then the symbol of $\mathrm{d}^s_{\mathsf{F}} = e^{-\mathsf{F}/x} \mathrm{d}^s e^{\mathsf{F}/x}$ is given by

$$
\begin{pmatrix}
\xi + \mathrm{i}\mathsf{F} & 0 & 0 & 0 \\
\frac{1}{4}\eta\otimes & \frac{3}{4}(\xi + \mathrm{i}\mathsf{F}) & 0 & 0 \\
a^\flat & \frac{1}{2}\eta\otimes_s & \frac{1}{2}(\xi + \mathrm{i}\mathsf{F}) & 0 \\
0 & b^\flat & \frac{3}{4}\eta\otimes_s & \frac{1}{4}(\xi + \mathrm{i}\mathsf{F}) \\
0 & 0 & c^\flat & \eta\otimes_s
\end{pmatrix}.
$$

We use the inner product

$$M(4) = \begin{pmatrix} 1 & & & & \\ & 4\times \mathrm{Id} & & & \\ & & 6\times \mathrm{Id} & & \\ & & & 4\times \mathrm{Id} & \\ & & & & \mathrm{Id} \end{pmatrix} \tag{2.8}$$

on symmetric 4-tensors, and

$$M(3) = \begin{pmatrix} 1 & & & \\ & 3\times \mathrm{Id} & & \\ & & 3\times \mathrm{Id} & \\ & & & \mathrm{Id} \end{pmatrix} \tag{2.9}$$

on symmetric 3-tensors. If A maps a symmetric m_1-tensor to a symmetric m_2-tensor, we call B the (m_2, m_1)-adjoint of A if

$$\langle By, x\rangle_{M(m_1)} = \langle y, Ax\rangle_{M(m_2)}.$$

It is easy to check that

$$B = M(m_1)^{-1} A^* M(m_2).$$

If $m_1 = m_2 = m$, we call A is (m,m)-self-adjoint if $B = A$.

It follows that δ^s_{F} has a symbol given by the (3,4)-adjoint of that of $\mathrm{d}^s_{\mathsf{F}}$,

$$\begin{pmatrix} \xi - \mathrm{i}\mathsf{F} & \iota_\eta & 6\langle a^\flat, \cdot\rangle & 0 & 0 \\ 0 & (\xi - \mathrm{i}\mathsf{F}) & \iota^s_\eta & \frac{4}{3}\langle b^\flat, \cdot\rangle & 0 \\ 0 & 0 & (\xi - \mathrm{i}\mathsf{F}) & \iota^s_\eta & \frac{1}{3}\langle c^\flat, \cdot\rangle \\ 0 & 0 & 0 & (\xi - \mathrm{i}\mathsf{F}) & \iota^s_\eta \end{pmatrix}.$$

Remaining tedious calculations complete the proof. □

LEMMA 2.3. *On symmetric 4-tensors, N_{F} is elliptic at fiber infinity in ${}^{sc}T^*X$ when restricted to the kernel of the principal symbol of δ^s_{F}.*

Proof. With the notation,

$$S = \frac{X - \alpha(\hat{Y})|Y|^2}{|Y|}, \quad \hat{Y} = \frac{Y}{|Y|},$$

by (2.4), the Schwartz kernel of N_{F} at the scattering front face $x = 0$ is given by

$$e^{-\mathsf{F}X}|Y|^{-n+1}\chi(S)\left[S\frac{\mathrm{d}x}{x^2} + \hat{Y}\cdot\frac{\mathrm{d}y}{x}\right]^4 \left[(S + 2\alpha|Y|)(x^2\partial_x) + \hat{Y}\cdot(x\partial_y)\right]^4. \tag{2.10}$$

On a symmetric 4-tensor of the form

$$\begin{aligned} f = & f_{xxxx}\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2} + 4f_{xxxy}\cdot\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x} \\ & + 6f_{xxyy}\cdot\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x} + 4f_{xyyy}\cdot\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x} \\ & + f_{yyyy}\cdot\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}, \end{aligned} \tag{2.11}$$

we have

$$\begin{aligned}&\Big[(S+2\alpha|Y|)(x^2\partial_x)+\hat{Y}\cdot(x\partial_y)\Big]^4 f\\ =&(S+2\alpha|Y|)^4 f_{xxxx}+4(S+2\alpha|Y|)^3\langle\hat{Y},f_{xxxy}\rangle+6(S+2\alpha|Y|)^2\langle\hat{Y}\otimes\hat{Y},f_{xxyy}\rangle\\ &+4(S+2\alpha|Y|)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},f_{xyyy}\rangle+\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},f_{yyyy}\rangle.\end{aligned}$$

On a scalar a,

$$\begin{aligned}\left[S\frac{\mathrm{d}x}{x^2}+\hat{Y}\cdot\frac{\mathrm{d}y}{x}\right]^4 a=&\,aS^4\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}+4aS^3\hat{Y}\cdot\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x}\\ &+6aS^2\hat{Y}\otimes\hat{Y}\cdot\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}+4aS\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\cdot\frac{\mathrm{d}x}{x^2}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\\ &+a\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\cdot\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}\otimes_s\frac{\mathrm{d}y}{x}.\end{aligned}$$

Thus, under the basis of symmetric 4-tensors, we have

$$\begin{aligned}&\left[S\frac{\mathrm{d}x}{x^2}+\hat{Y}\cdot\frac{\mathrm{d}y}{x}\right]^4\Big[(S+2\alpha|Y|)(x^2\partial_x)+\hat{Y}\cdot(x\partial_y)\Big]^4\\ =&\begin{pmatrix}S^4\\ S^3\hat{Y}\\ S^2\hat{Y}\otimes\hat{Y}\\ S\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix}(S+2\alpha|Y|)^4\\ 4(S+2\alpha|Y|)^3\langle\hat{Y},\cdot\rangle\\ 6(S+2\alpha|Y|)^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle\\ 4(S+2\alpha|Y|)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T.\end{aligned}$$

The above matrix is $(4,4)$-self-adjoint. In coordinates on the support of χ,

$$x,y,|Y|,\frac{X}{|Y|},\hat{Y},$$

we can rewrite the kernel as

$$e^{-\mathsf{F}X}|Y|^{-n+1}\chi(S)\begin{pmatrix}S^4\\ S^3\hat{Y}\\ S^2\hat{Y}\otimes\hat{Y}\\ S\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix}(S+2\alpha|Y|)^4\\ 4(S+2\alpha|Y|)^3\langle\hat{Y},\cdot\rangle\\ 6(S+2\alpha|Y|)^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle\\ 4(S+2\alpha|Y|)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T.$$

The principal symbol associated with $K^\flat$ defined in (2.4) is the (X,Y)-Fourier transform of

$$\chi(\tilde{S})|Y|^{-n+1}\begin{pmatrix}\tilde{S}^4\\ \tilde{S}^3\hat{Y}\\ \tilde{S}^2\hat{Y}\otimes\hat{Y}\\ \tilde{S}\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix}\tilde{S}^4\\ 4\tilde{S}^3\langle\hat{Y},\cdot\rangle\\ 6\tilde{S}^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle\\ 4\tilde{S}\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T,$$

with $\tilde{S}=\frac{X}{|Y|}$. The equatorial sphere is

$$\tilde{S}\xi+\hat{Y}\cdot\eta=0. \tag{2.12}$$

Following the discussion around (3.8) in [20], we need to integrate

$$\chi(\tilde{S})\begin{pmatrix} \tilde{S}^4 \\ \tilde{S}^3\hat{Y} \\ \tilde{S}^2\hat{Y}\otimes\hat{Y} \\ \tilde{S}\hat{Y}\otimes\hat{Y}\otimes\hat{Y} \\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y} \end{pmatrix} \otimes \begin{pmatrix} \tilde{S}^4 \\ 4\tilde{S}^3\langle\hat{Y},\cdot\rangle \\ 6\tilde{S}^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle \\ 4\tilde{S}\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle \\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle \end{pmatrix}^T \tag{2.13}$$

on this sphere.

For a symmetric 4-tensor of the form (2.11) in the kernel of the principal symbol of δ_{F}^s, we have by Lemma 2.2 that

$$\begin{aligned} \xi f_{xxxx} + \langle\eta, f_{xxxy}\rangle &= 0,\\ \xi f_{xxxy} + \langle\eta, f_{xxyy}\rangle &= 0,\\ \xi f_{xxyy} + \langle\eta, f_{xyyy}\rangle &= 0,\\ \xi f_{xyyy} + \langle\eta, f_{yyyy}\rangle &= 0. \end{aligned} \tag{2.14}$$

Moreover, f is in the kernel of (2.13) if and only if

$$\begin{aligned} \tilde{S}^4 f_{xxxx} + 4\tilde{S}^3\langle\hat{Y}, f_{xxxy}\rangle + 6\tilde{S}^2\langle\hat{Y}\otimes\hat{Y}, f_{xxyy}\rangle \\ +4\tilde{S}\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{xyyy}\rangle + \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle = 0. \end{aligned} \tag{2.15}$$

Suppose a symmetric 4-tensor f satisfies (2.14) and (2.15) for $(\tilde{S},\hat{Y})$ such that (2.12) holds. We will consider two cases, $\xi = 0$ and $\xi \neq 0$.

Case 1: $\xi \neq 0$. If $\eta = 0$, we have directly form (2.14) that

$$f_{xxxx}\ , f_{xxxy},\ f_{xxyy},\ f_{xyyy}$$

all vanish. Then from (2.15), we have

$$\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle = 0.$$

Therefore, $f_{yyyy} = 0$, since $\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}$ spans the space of all symmetric 4-tensors with $\eta = 0$. If $\eta \neq 0$, we calculate successively,

$$\begin{aligned} &f_{xyyy} = -\langle\frac{\eta}{\xi}, f_{yyyy}\rangle,\\ &\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{xyyy}\rangle = -\langle\frac{\eta}{\xi}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle,\\ &f_{xxyy} = -\langle\frac{\eta}{\xi}, f_{xyyy}\rangle = \langle\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}, f_{yyyy}\rangle,\\ &\langle\hat{Y}\otimes\hat{Y}, f_{xxyy}\rangle = \langle\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle,\\ &f_{xxxy} = -\langle\frac{\eta}{\xi}, f_{xxyy}\rangle = -\langle\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}, f_{yyyy}\rangle,\\ &\langle\hat{Y}, f_{xxxy}\rangle = -\langle\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\hat{Y}, f_{yyyy}\rangle,\\ &f_{xxxx} = -\langle\frac{\eta}{\xi}, f_{xxxy}\rangle = \langle\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}, f_{yyyy}\rangle. \end{aligned}$$

With $\tilde{S} = -\frac{\hat{Y}\cdot\eta}{\xi}$, (2.15) gives

$$\begin{aligned} \Big\langle \left(\frac{\hat{Y}\cdot\eta}{\xi}\right)^4 \frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\frac{\eta}{\xi} + 4\left(\frac{\hat{Y}\cdot\eta}{\xi}\right)^3 \frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\hat{Y} + 6\left(\frac{\hat{Y}\cdot\eta}{\xi}\right)^2 \frac{\eta}{\xi}\otimes\frac{\eta}{\xi}\otimes\hat{Y}\otimes\hat{Y} \\ +4\left(\frac{\hat{Y}\cdot\eta}{\xi}\right)\frac{\eta}{\xi}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y} + \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\Big\rangle = 0. \end{aligned} \tag{2.16}$$

Now we take $\hat{Y} = \epsilon\hat{\eta} + (1-\epsilon^2)^{1/2}\hat{Y}^\perp$, where $\hat{Y}^\perp$ is a unit vector orthogonal to $\hat{\eta}$, and substituting it into (2.16), we find that

$$\begin{aligned}\Big\langle \epsilon^4 &\left(\frac{|\eta|^8}{\xi^8} + \frac{4|\eta|^6}{\xi^6} + \frac{6|\eta|^4}{\xi^4} + \frac{4|\eta|^2}{\xi^2} + 1\right)\hat{\eta}\otimes\hat{\eta}\otimes\hat{\eta}\otimes\hat{\eta}\\ &+ 4\epsilon^3(1-\epsilon^2)^{1/2}\left(\frac{|\eta|^6}{\xi^6} + \frac{3|\eta|^4}{\xi^4} + \frac{3|\eta|^2}{\xi^2} + 1\right)\hat{\eta}\otimes\hat{\eta}\otimes\hat{\eta}\otimes\hat{Y}^\perp\\ &+ 6\epsilon^2(1-\epsilon^2)\left(\frac{|\eta|^4}{\xi^4} + \frac{2|\eta|^2}{\xi^2} + 1\right)\hat{\eta}\otimes\hat{\eta}\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp\\ &+ 4\epsilon(1-\epsilon^2)^{3/2}\left(\frac{|\eta|^2}{\xi^2} + 1\right)\hat{\eta}\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp\\ &+ (1-\epsilon^2)^2\hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp, f_{yyyy}\Big\rangle = 0.\end{aligned} \tag{2.17}$$

Taking $\epsilon = 0$ in (2.17), we have

$$\langle \hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp, f_{yyyy}\rangle = 0.$$

Since $\hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp\otimes\hat{Y}^\perp$ spans $\eta^\perp\otimes\eta^\perp\otimes\eta^\perp\otimes\eta^\perp$, we conclude that f_{yyyy} is orthogonal to every element of $\eta^\perp\otimes\eta^\perp\otimes\eta^\perp\otimes\eta^\perp$. Taking 1st, 2nd, 3rd and 4th order derivatives of (2.17) at $\epsilon = 0$, it follows that f_{yyyy} is orthogonal to

$$\hat{\eta}\otimes\hat{\eta}^\perp\otimes\eta^\perp\otimes\hat{\eta}^\perp, \qquad \hat{\eta}\otimes\hat{\eta}\otimes\hat{\eta}^\perp\otimes\hat{\eta}^\perp,$$
$$\hat{\eta}\otimes\hat{\eta}\otimes\hat{\eta}\otimes\hat{\eta}^\perp, \qquad \hat{\eta}\otimes\eta\otimes\hat{\eta}\otimes\hat{\eta},$$

respectively. We then finally conclude that f_{yyyy} vanishes, and then the whole tensor f vanishes by (2.14).

Case 2: $\xi = 0$ (and so $\eta \neq 0$). Now (2.12) is equivalent to $\eta\cdot\hat{Y} = 0$, and (2.14) reduces to

$$\begin{aligned}\langle\hat{\eta}, f_{xxxy}\rangle &= 0,\\ \langle\hat{\eta}, f_{xxyy}\rangle &= 0,\\ \langle\hat{\eta}, f_{xyyy}\rangle &= 0,\\ \langle\hat{\eta}, f_{yyyy}\rangle &= 0.\end{aligned} \tag{2.18}$$

We differentiate (2.15) with respect to $\tilde{S}$ up to four times, evaluated at $\tilde{S} = 0$, and find that

$$\begin{aligned}f_{xxxx} &= 0,\\ \langle\hat{Y}, f_{xxxy}\rangle &= 0,\\ \langle\hat{Y}\otimes\hat{Y}, f_{xxyy}\rangle &= 0,\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{xyyy}\rangle &= 0,\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle &= 0.\end{aligned} \tag{2.19}$$

Combining the identities in (2.18) and (2.19), we conclude that $f = 0$. □

Lemma 2.4. *There exists $\mathsf{F}_0 > 0$ such that on symmetric 4-tensors N_F is elliptic at a finite set of points in ${}^{sc}T^*X$ when restricted to the kernel of the principal symbol of δ^s_F for any $\mathsf{F} > \mathsf{F}_0$.*

Proof. Taking $\chi(s) = e^{-s^2/(2\nu(\hat{Y}))}$, so $\hat{\chi}(\cdot) = c\sqrt{\nu}e^{-\nu|\cdot|^2/2}$. We get the X-Fourier transform of the Schwartz kernel at the front face $x=0$:

$$
\begin{aligned}
&\mathcal{F}_X K^\flat(0,y,|Y|,\frac{\xi}{|Y|},\hat{Y})\\
=&|Y|^{2-n}e^{-\mathrm{i}\alpha(-\xi-\mathrm{iF})|Y|^2}\begin{pmatrix} D_\sigma^4\\ -D_\sigma^3\hat{Y}\\ D_\sigma^2\hat{Y}\otimes\hat{Y}\\ -D_\sigma\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix}(-D_\sigma+2\alpha|Y|)^4\\ 4(-D_\sigma+2\alpha|Y|)^3\langle\hat{Y},\cdot\rangle\\ 6(-D_\sigma+2\alpha|Y|)^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle\\ 4(-D_\sigma+2\alpha|Y|)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T\hat{\chi}((-\xi-\mathrm{iF})|Y|)\\
=&c\sqrt{\nu}|Y|^{2-n}e^{\mathrm{i}\alpha(\xi+\mathrm{iF})|Y|^2}\begin{pmatrix} D_\sigma^4\\ -D_\sigma^3\hat{Y}\\ D_\sigma^2\hat{Y}\otimes\hat{Y}\\ -D_\sigma\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix}(-D_\sigma+2\alpha|Y|)^4\\ 4(-D_\sigma+2\alpha|Y|)^3\langle\hat{Y},\cdot\rangle\\ 6(-D_\sigma+2\alpha|Y|)^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle\\ 4(-D_\sigma+2\alpha|Y|)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T e^{-\nu(\xi+\mathrm{iF})^2|Y|^2/2}.
\end{aligned}
$$

Here D_σ denotes the differentiation of the argument of $\hat{\chi}$. Then we compute the Y-Fourier transform, which in polar coordinates takes the form,

$$
\begin{aligned}
&\int_{\mathbb{S}^{n-2}}\int_0^\infty e^{-\mathrm{i}|Y|\hat{Y}\cdot\eta}|Y|^{2-n}e^{\mathrm{i}\alpha(\xi+\mathrm{iF})|Y|^2}\\
&\qquad\begin{pmatrix} D_\sigma^4\\ -D_\sigma^3\hat{Y}\\ D_\sigma^2\hat{Y}\otimes\hat{Y}\\ -D_\sigma\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix}(-D_\sigma+2\alpha|Y|)^4\\ 4(-D_\sigma+2\alpha|Y|)^3\langle\hat{Y},\cdot\rangle\\ 6(-D_\sigma+2\alpha|Y|)^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle\\ 4(-D_\sigma+2\alpha|Y|)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T e^{-\nu(\xi+\mathrm{iF})^2|Y|^2/2}|Y|^{n-2}\mathrm{d}|Y|\mathrm{d}\hat{Y}.
\end{aligned}
$$

We denote

$$\phi(\xi,\hat{Y})=\nu(\hat{Y})(\xi+\mathrm{iF})^2-2\mathrm{i}\alpha(\xi+\mathrm{iF}).$$

By explicitly evaluating the derivates, the above integral yields

$$
\begin{aligned}
\int_{\mathbb{S}^{n-2}}\int_0^\infty e^{-\mathrm{i}|Y|\hat{Y}\cdot\eta}\begin{pmatrix}\mathrm{i}^4\nu^4(\xi+\mathrm{iF})^4|Y|^4\\ \mathrm{i}^3\nu^3(\xi+\mathrm{iF})^3|Y|^3\hat{Y}\\ \mathrm{i}^2\nu^2(\xi+\mathrm{iF})^2|Y|^2\hat{Y}\otimes\hat{Y}\\ \mathrm{i}\nu(\xi+\mathrm{iF})|Y|\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix}(\mathrm{i}\nu(\xi+\mathrm{iF})+2\alpha)^4|Y|^4\\ 4(\mathrm{i}\nu(\xi+\mathrm{iF})+2\alpha)^3|Y|^3\langle\hat{Y},\cdot\rangle\\ 6(\mathrm{i}\nu(\xi+\mathrm{iF})+2\alpha)^2|Y|^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle\\ 4(\mathrm{i}\nu(\xi+\mathrm{iF})+2\alpha)|Y|\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T&\\
\times e^{-\phi|Y|^2/2}\mathrm{d}|Y|\mathrm{d}\hat{Y}&.
\end{aligned}
$$

We extend the integral in $|Y|$ to $\mathbb{R}$, replacing it by a variable t, and using that the integrand is invariant under the joint change of variables $t\to -t$ and $\hat{Y}\to-\hat{Y}$. This gives

$$
\begin{aligned}
\int_{\mathbb{S}^{n-2}}\int_{-\infty}^\infty e^{-\mathrm{i}t\hat{Y}\cdot\eta}\begin{pmatrix}\mathrm{i}^4\nu^4(\xi+\mathrm{iF})^4t^4\\ \mathrm{i}^3\nu^3(\xi+\mathrm{iF})^3t^3\hat{Y}\\ \mathrm{i}^2\nu^2(\xi+\mathrm{iF})^2t^2\hat{Y}\otimes\hat{Y}\\ \mathrm{i}\nu(\xi+\mathrm{iF})t\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix}(\mathrm{i}\nu(\xi+\mathrm{iF})+2\alpha)^4t^4\\ 4(\mathrm{i}\nu(\xi+\mathrm{iF})+2\alpha)^3t^3\langle\hat{Y},\cdot\rangle\\ 6(\mathrm{i}\nu(\xi+\mathrm{iF})+2\alpha)^2t^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle\\ 4(\mathrm{i}\nu(\xi+\mathrm{iF})+2\alpha)t\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T&\\
\times e^{-\phi t^2/2}\mathrm{d}t\mathrm{d}\hat{Y}&.
\end{aligned}
$$

Now the t integral is a Fourier transform evaluated at $-\hat{Y}\cdot\eta$, under which multiplication by t becomes $D_{\hat{Y}\cdot\eta}$. We also note that the Fourier transform of $e^{-\phi(\xi,\hat{Y})t^2/2}$ is a constant multiple of

$$\phi(\xi,\hat{Y})^{-1/2}e^{-(\hat{Y}\cdot\eta)^2/(2\phi(\xi,\hat{Y}))}. \tag{2.20}$$

Thus we are left with

$$\int_{\mathbb{S}^{n-2}}\phi(\xi,\hat{Y})^{-1/2}\begin{pmatrix} \mathrm{i}^4\nu^4(\xi+\mathrm{i}\mathsf{F})^4 D^4_{\hat{Y}\cdot\eta} \\ \mathrm{i}^3\nu^3(\xi+\mathrm{i}\mathsf{F})^3 D^3_{\hat{Y}\cdot\eta}\hat{Y} \\ \mathrm{i}^2\nu^2(\xi+\mathrm{i}\mathsf{F})^2 D^2_{\hat{Y}\cdot\eta}\hat{Y}\otimes\hat{Y} \\ \mathrm{i}\nu(\xi+\mathrm{i}\mathsf{F}) D_{\hat{Y}\cdot\eta}\hat{Y}\otimes\hat{Y}\otimes\hat{Y} \\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix} (\mathrm{i}\nu(\xi+\mathrm{i}\mathsf{F})+2\alpha)^4 D^4_{\hat{Y}\cdot\eta} \\ 4(\mathrm{i}\nu(\xi+\mathrm{i}\mathsf{F})+2\alpha)^3\langle\hat{Y},\cdot\rangle D^3_{\hat{Y}\cdot\eta} \\ 6(\mathrm{i}\nu(\xi+\mathrm{i}\mathsf{F})+2\alpha)^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle D^2_{\hat{Y}\cdot\eta} \\ 4(\mathrm{i}\nu(\xi+\mathrm{i}\mathsf{F})+2\alpha)\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle D_{\hat{Y}\cdot\eta} \\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T \times e^{-(\hat{Y}\cdot\eta)^2/(2\phi(\xi,\hat{Y}))}\mathrm{d}\hat{Y}.$$

$$=\int_{\mathbb{S}^{n-2}}\phi(\xi,\hat{Y})^{-1/2}\begin{pmatrix} \nu^4(\xi+\mathrm{i}\mathsf{F})^4(\frac{\hat{Y}\cdot\eta}{\phi})^4 \\ -\nu^3(\xi+\mathrm{i}\mathsf{F})^3(\frac{\hat{Y}\cdot\eta}{\phi})^3\hat{Y} \\ \nu^2(\xi+\mathrm{i}\mathsf{F})^2(\frac{\hat{Y}\cdot\eta}{\phi})^2\hat{Y}\otimes\hat{Y} \\ -\nu(\xi+\mathrm{i}\mathsf{F})(\frac{\hat{Y}\cdot\eta}{\phi})\hat{Y}\otimes\hat{Y}\otimes\hat{Y} \\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix} (\nu(\xi+\mathrm{i}\mathsf{F})-2\mathrm{i}\alpha)^4(\frac{\hat{Y}\cdot\eta}{\phi})^4 \\ -4(\nu(\xi+\mathrm{i}\mathsf{F})-2\mathrm{i}\alpha)^3(\frac{\hat{Y}\cdot\eta}{\phi})^3\langle\hat{Y},\cdot\rangle \\ 6(\nu(\xi+\mathrm{i}\mathsf{F})-2\mathrm{i}\alpha)^2(\frac{\hat{Y}\cdot\eta}{\phi})^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle \\ -4(\nu(\xi+\mathrm{i}\mathsf{F})-2\mathrm{i}\alpha)(\frac{\hat{Y}\cdot\eta}{\phi})\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle \\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T \times e^{-(\hat{Y}\cdot\eta)^2/(2\phi(\xi,\hat{Y}))}\mathrm{d}\hat{Y}.$$

We note that

$$\begin{pmatrix} \nu^4(\xi+\mathrm{i}\mathsf{F})^4(\frac{\hat{Y}\cdot\eta}{\phi})^4 \\ -\nu^3(\xi+\mathrm{i}\mathsf{F})^3(\frac{\hat{Y}\cdot\eta}{\phi})^3\hat{Y} \\ \nu^2(\xi+\mathrm{i}\mathsf{F})^2(\frac{\hat{Y}\cdot\eta}{\phi})^2\hat{Y}\otimes\hat{Y} \\ -\nu(\xi+\mathrm{i}\mathsf{F})(\frac{\hat{Y}\cdot\eta}{\phi})\hat{Y}\otimes\hat{Y}\otimes\hat{Y} \\ \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\end{pmatrix}\otimes\begin{pmatrix} (\nu(\xi+\mathrm{i}\mathsf{F})-2\mathrm{i}\alpha)^4(\frac{\hat{Y}\cdot\eta}{\phi})^4 \\ -4(\nu(\xi+\mathrm{i}\mathsf{F})-2\mathrm{i}\alpha)^3(\frac{\hat{Y}\cdot\eta}{\phi})^3\langle\hat{Y},\cdot\rangle \\ 6(\nu(\xi+\mathrm{i}\mathsf{F})-2\mathrm{i}\alpha)^2(\frac{\hat{Y}\cdot\eta}{\phi})^2\langle\hat{Y}\otimes\hat{Y},\cdot\rangle \\ -4(\nu(\xi+\mathrm{i}\mathsf{F})-2\mathrm{i}\alpha)(\frac{\hat{Y}\cdot\eta}{\phi})\langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle \\ \langle\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y},\cdot\rangle\end{pmatrix}^T \tag{2.21}$$

is a multiple of a projection and is $(4,4)$-self-adjoint. We let $\nu=\mathsf{F}^{-1}\alpha$, with

$$\nu(\xi+\mathrm{i}\mathsf{F})-2\mathrm{i}\alpha=\nu(\xi-\mathrm{i}\mathsf{F}),$$

and

$$\phi=(\xi+\mathrm{i}\mathsf{F})(\nu(\xi+\mathrm{i}\mathsf{F})-2\mathrm{i}\alpha)=\nu(\xi^2+\mathsf{F}^2).$$

We then denote

$$\begin{aligned}
\mathfrak{C}_4&=\nu^4(\xi+\mathrm{i}\mathsf{F})-2\mathrm{i}\alpha)^4(\frac{\hat{Y}\cdot\eta}{\phi})^4=\nu^4(\xi-\mathrm{i}\mathsf{F})^4(\frac{\hat{Y}\cdot\eta}{\phi})^4,\\
\mathfrak{C}_3&=-\nu^3(\xi+\mathrm{i}\mathsf{F})-2\mathrm{i}\alpha)^3(\frac{\hat{Y}\cdot\eta}{\phi})^3=-\nu^3(\xi-\mathrm{i}\mathsf{F})^3(\frac{\hat{Y}\cdot\eta}{\phi})^3,\\
\mathfrak{C}_2&=\nu^2(\xi+\mathrm{i}\mathsf{F})-2\mathrm{i}\alpha)^2(\frac{\hat{Y}\cdot\eta}{\phi})^2=\nu^2(\xi-\mathrm{i}\mathsf{F})^2(\frac{\hat{Y}\cdot\eta}{\phi})^2,\\
\mathfrak{C}_1&=-\nu(\xi+\mathrm{i}\mathsf{F})-2\mathrm{i}\alpha)(\frac{\hat{Y}\cdot\eta}{\phi})=-\nu(\xi-\mathrm{i}\mathsf{F})(\frac{\hat{Y}\cdot\eta}{\phi}).
\end{aligned}$$

For a symmetric 4-tensor of the form (2.11) in the kernel of the principal symbol of δ^s_{F}, we have by Lemma 2.2 that

$$\begin{aligned}
&(\xi-\mathrm{i}\mathsf{F})f_{xxxx}+\langle\eta,f_{xxxy}\rangle+6\langle a^\flat,f_{xxyy}\rangle=0,\\
&(\xi-\mathrm{i}\mathsf{F})f_{xxxy}+\langle\eta,f_{xxyy}\rangle+4\langle b^\flat,f_{xyyy}\rangle=0,\\
&(\xi-\mathrm{i}\mathsf{F})f_{xxyy}+\langle\eta,f_{xyyy}\rangle+\langle c^\flat,f_{yyyy}\rangle=0,\\
&(\xi-\mathrm{i}\mathsf{F})f_{xyyy}+\langle\eta,f_{yyyy}\rangle=0.
\end{aligned}\tag{2.22}$$

Moreover, f is in the kernel of (2.21) if and only if

$$\begin{aligned}\mathfrak{C}_4 f_{xxxx} + 4\mathfrak{C}_3\langle \hat{Y}, f_{xxxy}\rangle + 6\mathfrak{C}_2\langle \hat{Y}\otimes\hat{Y}, f_{xxyy}\rangle \\ + 4\mathfrak{C}_1\langle \hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{xyyy}\rangle + \langle \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle = 0.\end{aligned} \tag{2.23}$$

We now take a semiclassical point of viewm setting $h = \mathsf{F}^{-1}$ and rescaling

$$\xi_{\mathsf{F}} = \mathsf{F}^{-1}\xi, \quad \eta_{\mathsf{F}} = \mathsf{F}^{-1}\eta.$$

Using these semiclassical variables, we calculate, successively,

$$\begin{aligned}
&f_{xyyy} = -(\xi_{\mathsf{F}} - \mathrm{i})^{-1}\langle \eta_{\mathsf{F}}, f_{yyyy}\rangle,\\
&\langle \hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{xyyy}\rangle = -(\xi_{\mathsf{F}} - \mathrm{i})^{-1}\langle \eta_{\mathsf{F}}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle,\\
&f_{xxyy} = -(\xi_{\mathsf{F}} - \mathrm{i})^{-1}\langle \eta_{\mathsf{F}}, f_{xyyy}\rangle + O(h) = (\xi_{\mathsf{F}} - \mathrm{i})^{-2}\langle \eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}, f_{yyyy}\rangle + O(h),\\
&\langle \hat{Y}\otimes\hat{Y}, f_{xxyy}\rangle = (\xi_{\mathsf{F}} - \mathrm{i})^{-2}\langle \eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle + O(h),\\
&f_{xxxy} = -(\xi_{\mathsf{F}} - \mathrm{i})^{-1}\langle \eta_{\mathsf{F}}, f_{xxyy}\rangle + O(h) = -(\xi_{\mathsf{F}} - \mathrm{i})^{-3}\langle \eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}, f_{yyyy}\rangle + O(h),\\
&\langle \hat{Y}, f_{xxxy}\rangle = -(\xi_{\mathsf{F}} - \mathrm{i})^{-3}\langle \eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}\otimes\hat{Y}, f_{yyyy}\rangle + O(h),\\
&f_{xxxx} = -(\xi_{\mathsf{F}} - \mathrm{i})^{-1}\langle \eta_{\mathsf{F}}, f_{xxxy}\rangle + O(h) = (\xi_{\mathsf{F}} - \mathrm{i})^{-4}\langle \eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}\otimes\eta_{\mathsf{F}}, f_{yyyy}\rangle.
\end{aligned}$$

we observe that

$$\mathfrak{C}_j = (-1)^j(\xi_{\mathsf{F}}^2 + 1)^{-j}(\xi_{\mathsf{F}} - \mathrm{i})^j\rho^j, \quad \text{with } \rho = \hat{Y}\cdot\eta_{\mathsf{F}}.$$

By calculation, and letting $h \to 0$, we have by (2.22) that

$$\langle \otimes_{i=1}^4((\xi_{\mathsf{F}}^2 + 1)^{-1}\rho\eta_{\mathsf{F}} + \hat{Y}), f_{yyyy}\rangle = 0.$$

If $\eta = 0$, then

$$\langle \hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}, f_{yyyy}\rangle = 0.$$

Since $\hat{Y}\otimes\hat{Y}\otimes\hat{Y}\otimes\hat{Y}$ span the space of all symmetric 4-tensors, we conclude that $f_{yyyy} = 0$ and thus $f = 0$.

If $\eta_{\mathsf{F}} \neq 0$, we take $\hat{Y} = \epsilon\hat{\eta}_{\mathsf{F}} + (1-\epsilon^2)^{1/2}\hat{Y}^{\perp}$, where $\hat{Y}^{\perp}$ is orthogonal to $\hat{\eta}_{\mathsf{F}}$. Then by (2.22), we have

$$\begin{aligned}
&\Big\langle \epsilon^4\left(1 + \frac{|\eta_{\mathsf{F}}|^2}{\xi_{\mathsf{F}}^2 + 1}\right)^4 \hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\\
&\quad + 4\epsilon^3(1-\epsilon^2)^{1/2}\left(1 + \frac{|\eta_{\mathsf{F}}|^2}{\xi_{\mathsf{F}}^2 + 1}\right)^3 \hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{Y}^{\perp}\\
&\quad + 6\epsilon^2(1-\epsilon^2)\left(1 + \frac{|\eta_{\mathsf{F}}|^2}{\xi_{\mathsf{F}}^2 + 1}\right)^2 \hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{Y}^{\perp}\otimes\hat{Y}^{\perp}\\
&\quad + 4\epsilon(1-\epsilon^2)^{3/2}\left(1 + \frac{|\eta_{\mathsf{F}}|^2}{\xi_{\mathsf{F}}^2 + 1}\right) \hat{\eta}_{\mathsf{F}}\otimes\hat{Y}^{\perp}\otimes\hat{Y}^{\perp}\otimes\hat{Y}^{\perp}\\
&\quad + (1-\epsilon^2)^2\hat{Y}^{\perp}\otimes\hat{Y}^{\perp}\otimes\hat{Y}^{\perp}\otimes\hat{Y}^{\perp}, f_{yyyy}\Big\rangle = 0.
\end{aligned} \tag{2.24}$$

Similar to the proof of Lemma 2.3, we take derivatives of (2.24) up to order four at $\epsilon = 0$; it follows that f_{yyyy} is orthogonal to

$$\begin{aligned}\hat{Y}^{\perp}\otimes\hat{Y}^{\perp}\otimes\hat{Y}^{\perp}\otimes\hat{Y}^{\perp}, \quad \hat{\eta}_{\mathsf{F}}\otimes\hat{Y}^{\perp}\otimes\hat{Y}^{\perp}\otimes\hat{Y}^{\perp}, \quad \hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{Y}^{\perp}\otimes\hat{Y}^{\perp},\\ \hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{Y}^{\perp}, \quad \hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}\otimes\hat{\eta}_{\mathsf{F}}.\end{aligned}$$

Then, $f=0$.

We conclude that for sufficiently large $\mathsf{F}>0$, one has ellipticity at all finite points. ∎

With Lemma 2.3 and Lemma 2.4, we obtain the following proposition by similar arguments as in the proof of [18, Proposition 3.3],

Proposition 2.5. *There exists $\mathsf{F}_0>0$ such that for $\mathsf{F}>\mathsf{F}_0$ the following holds. Given $\tilde\Omega$, a neighborhood of $X\cap M=\{x\geq 0,\rho>0\}$ in X; for a suitable choice of the cutoff $\chi\in C_c^\infty(\mathbb{R})$ and of $M\in\Psi_{sc}^{-3,0}(X;\mathrm{Sym}^3{}^{sc}T^*X,\mathrm{Sym}^3{}^{sc}T^*X)$, the operator*

$$A_{\mathsf{F}}=N_{\mathsf{F}}+\mathrm{d}^s_{\mathsf{F}}M\delta^s_{\mathsf{F}},\quad N_{\mathsf{F}}=e^{-\mathsf{F}/x}LI_4e^{\mathsf{F}/x},\quad \mathrm{d}^s_{\mathsf{F}}=e^{-\mathsf{F}/x}\mathrm{d}^se^{\mathsf{F}/x},$$

is elliptic in $\Psi_{sc}^{-1,0}(X;\mathrm{Sym}^4{}^{sc}T^*X,\mathrm{Sym}^4{}^{sc}T^*X)$ in $\tilde\Omega$.

3. Proofs of the main results. We prove the injectivity of I_4 with the gauge condition $\delta^s_{\mathsf{F}}f_{\mathsf{F}}=0$ in $\Omega=\Omega_c$, where $f_{\mathsf{F}}=e^{-\mathsf{F}/x}f$. Based on the discussion in [18, Section 4], we first need to check the invertibility of $\Delta_{\mathsf{F},s}$. Here $\Delta_{\mathsf{F},s}=\delta^s_{\mathsf{F}}\mathrm{d}^s_{\mathsf{F}}$ is the 'solenoidal Witten Laplacian' which we will show to be invertible with the desired boundary condition. The similar results for I_1 and I_2 are provided in Section 4 of [18].

Lemma 3.1. *There exists $\mathsf{F}_0>0$ such that for $\mathsf{F}\geq\mathsf{F}_0$ the operator $\Delta^s_F=\delta^s_{\mathsf{F}}\mathrm{d}^s_{\mathsf{F}}$ is (joint) elliptic in $\mathrm{Diff}^{2,0}_{sc}(X;\mathrm{Sym}^3{}^{sc}T^*X,\mathrm{Sym}^3{}^{sc}T^*X)$ on symmetric 3-tensors. In fact, on symmetric 3-tensors*

$$\delta^s_{\mathsf{F}}\mathrm{d}^s_{\mathsf{F}}=\frac{1}{4}\nabla^*_{\mathsf{F}}\nabla_{\mathsf{F}}+\frac{3}{4}\mathrm{d}^s_{\mathsf{F}}\delta^s_{\mathsf{F}}+A+R,\tag{3.1}$$

where $R\in x\mathrm{Diff}^1_{sc}(X;\mathrm{Sym}^3{}^{sc}T^*X,\mathrm{Sym}^3{}^{sc}T^*X)$, $A\in\mathrm{Diff}^1_{sc}(X;\mathrm{Sym}^3{}^{sc}T^*X,\mathrm{Sym}^3{}^{sc}T^*X)$ and $\nabla_{\mathsf{F}}=e^{-\mathsf{F}/x}\nabla e^{\mathsf{F}/x}$, with ∇ gradient relative to g_{sc} (not g), $\mathrm{d}_{\mathsf{F}}=e^{-\mathsf{F}/x}\mathrm{d}e^{\mathsf{F}/x}$ the exterior derivative on symmetric 3-tensors, while δ_{F} is its adjoint on symmetric 3-tensors.

Proof. By calculation and Lemma 2.2, Δ^s_{F} has symbol

$$\begin{pmatrix}\xi^2+\mathsf{F}^2+\frac14|\eta|^2 & \frac34(\xi+\mathrm{i}\mathsf{F})\iota_\eta & 0 & 0\\ \frac14(\xi-\mathrm{i}\mathsf{F})\eta\otimes & \frac34(\xi^2+\mathsf{F}^2)+\frac12\iota^s_\eta\eta\otimes_s & \frac12(\xi+\mathrm{i}\mathsf{F})\iota^s_\eta & 0\\ 0 & \frac12(\xi-\mathrm{i}\mathsf{F})\eta\otimes_s & \frac12(\xi^2+\mathsf{F}^2)+\frac34\iota^s_\eta\eta\otimes_s & \frac14(\xi+\mathrm{i}\mathsf{F})\iota^s_\eta\\ 0 & 0 & \frac34(\xi-\mathrm{i}\mathsf{F})\eta\otimes_s & \frac14(\xi^2+\mathsf{F}^2)+\iota^s_\eta\eta\otimes_s\end{pmatrix}$$
$$+\begin{pmatrix}6\langle a^\flat,\cdot\rangle a^\flat & 3\langle a,\cdot\rangle\eta\otimes_s & 3(\xi+\mathrm{i}\mathsf{F})\langle a^\flat,\cdot\rangle & 0\\ \iota^s_\eta a & \frac43\langle b^\flat,\cdot\rangle b^\flat & \langle b^\flat,\cdot\rangle\eta\otimes_s & \frac13(\xi+\mathrm{i}\mathsf{F})\langle b^\flat,\cdot\rangle\\ (\xi-\mathrm{i}\mathsf{F})a^\flat & \iota^s_\eta b^\flat & \langle c^\flat,\cdot\rangle c^\flat & \frac13\langle c^\flat,\cdot\rangle\eta\otimes_s\\ 0 & (\xi-\mathrm{i}\mathsf{F})b^\flat & \iota^s_\eta c^\flat & 0\end{pmatrix}.$$

Here, $\iota^s_\eta\eta\otimes_s$ at $(2,2)$-block has the (i_1',i_2')-entry

$$\frac12(|\eta|^2\delta_{i_1',i_2'}+\eta_{i_1'}\eta_{i_2'}).$$

$\iota^s_\eta\eta\otimes_s$ at $(3,3)$-block has (i_1',j_1',i_2',j_2')-entry

$$\frac16(|\eta|^2\delta_{i_1',i_2'}\delta_{j_1',j_2'}+|\eta|^2\delta_{i_1',j_2'}\delta_{j_1',i_2'}+\eta_{i_1'}\eta_{i_2'}\delta_{j_1'j_2'}+\eta_{i_1'}\eta_{j_2'}\delta_{i_1'j_2'}+\eta_{i_2'}\eta_{j_1'}\delta_{i_1'j_2'}+\eta_{j_1'}\eta_{j_2'}\delta_{i_1'i_2'})$$

and $\iota_\eta^s \eta \otimes_s$ at $(4,4)$-block has $(i_1', j_1', k_1', i_2', j_2', k_2')$-entry

$$\begin{aligned}
\frac{1}{24}\Big(|\eta|^2 (&\delta_{i_1' i_2'} \delta_{j_1' j_2'} \delta_{k_1' k_2'} + \delta_{i_1' j_2'} \delta_{j_1' i_2'} \delta_{k_1' k_2'} + \delta_{i_1' k_2'} \delta_{j_1' j_2'} \delta_{k_1' i_2'} \\
&+ \delta_{i_1' i_2'} \delta_{j_1' k_2'} \delta_{k_1' j_2'} + \delta_{i_1' j_2'} \delta_{j_1' k_2'} \delta_{k_1' i_2'} + \delta_{i_1' k_2'} \delta_{j_1' i_2'} \delta_{k_1' j_2'}) \\
&+ \eta_{i_1'} \eta_{i_2'} (\delta_{j_1' j_2'} \delta_{k_1' k_2'} + \delta_{j_1' k_2'} \delta_{k_1' j_2'}) + \eta_{i_1'} \eta_{j_2'} (\delta_{j_1' i_2'} \delta_{k_1' k_2'} + \delta_{j_1' k_2'} \delta_{i_1' k_2'}) \\
&+ \eta_{i_1'} \eta_{k_2'} (\delta_{j_1' j_2'} \delta_{k_1' i_2'} + \delta_{j_1' i_2'} \delta_{k_1' j_2'}) + \eta_{j_1'} \eta_{k_2'} (\delta_{i_1' j_2'} \delta_{k_1' i_2'} + \delta_{i_1' i_2'} \delta_{k_1' j_2'}) \\
&+ \eta_{j_1'} \eta_{j_2'} (\delta_{i_1' i_2'} \delta_{k_1' k_2'} + \delta_{i_1' k_2'} \delta_{i_1' k_2'}) + \eta_{k_1'} \eta_{k_2'} (\delta_{i_1' i_2'} \delta_{j_1' j_2'} + \delta_{i_1' j_2'} \delta_{j_1' i_2'}) \\
&+ \eta_{j_1'} \eta_{i_2'} (\delta_{i_1' j_2'} \delta_{k_1' k_2'} + \delta_{i_1' k_2'} \delta_{j_1' k_2'}) + \eta_{k_1'} \eta_{i_2'} (\delta_{j_1' j_2'} \delta_{i_1' k_2'} + \delta_{j_1' k_2'} \delta_{i_1' j_2'}) \\
&+ \eta_{j_1'} \eta_{j_2'} (\delta_{i_1' k_2'} \delta_{j_1' i_2'} + \delta_{i_1' i_2'} \delta_{j_1' k_2'})\Big).
\end{aligned}$$

We note that the gradient ∇ maps a symmetric 3-tensor to a (not symmetric) 4-tensor.

We introduce some further notation. We let A be a matrix of blocks, with

$$A_{\downarrow \times k}$$

representing

$$\left.\begin{pmatrix} A \\ \vdots \\ A \end{pmatrix}\right\} k - \text{tuple}.$$

Also, we write

$$A_{\rightarrow \times k}$$

representing

$$\begin{pmatrix} A & \cdots & A \end{pmatrix}.$$

Then we use the basis for 4-tensors (not the symmetric ones) and symmetric 3-tensors, under which the principal symbol of ∇_{F} relative to g_{sc} (not g) is

$$\begin{pmatrix}
\begin{pmatrix} \xi + \mathrm{i}\mathsf{F} \\ \eta\otimes \end{pmatrix} & & & \\
& \begin{pmatrix} \xi + \mathrm{i}\mathsf{F} \\ \eta\otimes \end{pmatrix}_{\downarrow\times 3} & & \\
& & \begin{pmatrix} \xi + \mathrm{i}\mathsf{F} \\ \eta\otimes \end{pmatrix}_{\downarrow\times 3} & \\
& & & \begin{pmatrix} \xi + \mathrm{i}\mathsf{F} \\ \eta\otimes \end{pmatrix}
\end{pmatrix}.$$

The number of rows is 16. Thus ∇_{F}^* has principal symbol,

$$\begin{pmatrix}
\begin{pmatrix} \xi - \mathrm{i}\mathsf{F} & \iota_\eta \end{pmatrix} & & & \\
& \frac{1}{3}\begin{pmatrix} \xi - \mathrm{i}\mathsf{F} & \iota_\eta \end{pmatrix}_{\rightarrow\times 3} & & \\
& & \frac{1}{3}\begin{pmatrix} \xi - \mathrm{i}\mathsf{F} & \iota_\eta \end{pmatrix}_{\rightarrow\times 3} & \\
& & & \begin{pmatrix} \xi - \mathrm{i}\mathsf{F} & \iota_\eta \end{pmatrix}
\end{pmatrix}.$$

Then $\nabla_{\mathsf{F}}^* \nabla_{\mathsf{F}}$ has symbol

$$\begin{pmatrix}
\xi^2 + \mathsf{F}^2 + |\eta|^2 & 0 & 0 & 0 \\
0 & \xi^2 + \mathsf{F}^2 + |\eta|^2 & 0 & 0 \\
0 & 0 & \xi^2 + \mathsf{F}^2 + |\eta|^2 & 0 \\
0 & 0 & 0 & \xi^2 + \mathsf{F}^2 + |\eta|^2
\end{pmatrix}. \tag{3.2}$$

Similar to our calculation in the proof of Lemma 2.2, we get the principal symbol of $\mathrm{d}^s_{\mathsf{F}}\delta^s_{\mathsf{F}}$ on symmetric 3-tensors,

$$\begin{pmatrix} \xi^2+\mathsf{F}^2 & (\xi+\mathrm{i}\mathsf{F})\iota_\eta & 0 & 0 \\ \frac{1}{3}(\xi-\mathrm{i}\mathsf{F})\eta\otimes & \frac{2}{3}(\xi^2+\mathsf{F}^2)+\frac{1}{3}\eta\otimes\iota_\eta & \frac{2}{3}(\xi+\mathrm{i}\mathsf{F})\iota^s_\eta & 0 \\ 0 & \frac{2}{3}(\xi-\mathrm{i}\mathsf{F})\eta\otimes_s & \frac{1}{3}(\xi+\mathsf{F}^2)+\frac{2}{3}\eta\otimes_s\iota_\eta & \frac{1}{3}(\xi+\mathsf{F})\iota^s_\eta \\ 0 & 0 & (\xi-\mathrm{i}\mathsf{F})\eta\otimes_s & \eta\otimes_s\iota_\eta \end{pmatrix}$$
$$+\begin{pmatrix} 0 & 0 & 3(\xi+\mathrm{i}\mathsf{F})\langle d^\flat,\cdot\rangle & 0 \\ 0 & 0 & \eta\langle d^\flat,\cdot\rangle & \frac{1}{3}(\xi+\mathrm{i}\mathsf{F})\langle e^\flat,\cdot\rangle \\ (\xi-\mathrm{i}\mathsf{F})d^\flat & d^\flat\iota_\eta & 3d^\flat\langle d^\flat,\cdot\rangle & \frac{1}{3}\eta\otimes_s\langle e^\flat,\cdot\rangle \\ 0 & (\xi+\mathrm{i}\mathsf{F})e^\flat & e^\flat\iota^s_\eta & \frac{1}{2}e^\flat\langle e^\flat,\cdot\rangle \end{pmatrix}.$$

Here, $\eta\otimes\iota_\eta$ at the $(2,2)$-block has (i'_1,i'_2)-entry

$$\eta_{i'_1,i'_2},$$

$\eta\otimes_s\iota_\eta$ at the $(3,3)$-block has (i'_1,j'_1,i'_2,j'_2)-entry

$$\frac{1}{4}(\eta_{i'_1}\eta_{i'_2}\delta_{j'_1j'_2}+\eta_{i'_1}\eta_{j'_2}\delta_{i'_1j'_2}+\eta_{i'_2}\eta_{j'_1}\delta_{i'_1j'_2}+\eta_{j'_1}\eta_{j'_2}\delta_{i'_1i'_2})$$

and $\eta\otimes_s\iota_\eta$ at the $(4,4)$-block has $(i'_1,j'_1,k'_1,i'_2,j'_2,k'_2)$-entry

$$\begin{aligned}\frac{1}{18}\Big(&\eta_{i'_1}\eta_{i'_2}(\delta_{j'_1j'_2}\delta_{k'_1k'_2}+\delta_{j'_1k'_2}\delta_{k'_1j'_2})+\eta_{i'_1}\eta_{j'_2}(\delta_{j'_1i'_2}\delta_{k'_1k'_2}+\delta_{j'_1k'_2}\delta_{i'_1k'_2})+\eta_{i'_1}\eta_{k'_2}(\delta_{j'_1j'_2}\delta_{k'_1i'_2}+\delta_{j'_1i'_2}\delta_{k'_1j'_2})\\ &+\eta_{j'_1}\eta_{k'_2}(\delta_{i'_1j'_2}\delta_{k'_1i'_2}+\delta_{i'_1i'_2}\delta_{k'_1j'_2})+\eta_{j'_1}\eta_{j'_2}(\delta_{i'_1i'_2}\delta_{k'_1k'_2}+\delta_{i'_1k'_2}\delta_{i'_1k'_2})+\eta_{k'_1}\eta_{k'_2}(\delta_{i'_1i'_2}\delta_{j'_1j'_2}+\delta_{i'_1j'_2}\delta_{j'_1i'_2})\\ &+\eta_{j'_1}\eta_{i'_2}(\delta_{i'_1j'_2}\delta_{k'_1k'_2}+\delta_{i'_1k'_2}\delta_{j'_1k'_2})+\eta_{k'_1}\eta_{i'_2}(\delta_{j'_1j'_2}\delta_{i'_1k'_2}+\delta_{j'_1k'_2}\delta_{i'_1j'_2})+\eta_{j'_1}\eta_{j'_2}(\delta_{i'_1k'_2}\delta_{j'_1i'_2}+\delta_{i'_1i'_2}\delta_{j'_1k'_2})\Big).\end{aligned}$$

We note that the principal symbol of $\delta^s_{\mathsf{F}}\mathrm{d}^s_{\mathsf{F}}$ is the same as the one of $\frac{1}{4}\nabla^*_{\mathsf{F}}\nabla_{\mathsf{F}}+\frac{3}{4}\mathrm{d}^s_{\mathsf{F}}\delta^s_{\mathsf{F}}$, which is positive definite with a lower bound $\frac{1}{4}(\xi^2+\mathsf{F}^2+|\eta|^2)$. Suppose $a^\flat,b^\flat,c^\flat,d^\flat,d^\flat$ have a common bound C, then A has a bound $C^2+C|\eta|+C\mathsf{F}+C\xi\le C'(1+\epsilon^{-1})+\epsilon(\xi^2+\mathsf{F}^2+|\eta|^2)$. Then we can choose $\mathsf{F}>0$ large enough, and complete the proof. ☐

Let Ω_j be a domain in M with boundary $\partial\Omega_j$ transversal to ∂X. Let $\dot{H}^{m,l}_{sc}(\Omega_j)$ be the subspace of $H^{m,l}_{sc}(X)$ consisting of distributions supported in $\overline{\Omega_j}$, and let $\bar{H}^{m,l}_{sc}(\Omega_j)$ be the space of restrictions of elements of $H^{m,l}_{sc}(X)$ to Ω_j. Thus, $\dot{H}^{m,l}_{sc}(\Omega_j)^*=\bar{H}^{-m,-l}_{sc}(\Omega_j)$.

Lemma 3.2. *There exists $\mathsf{F}_0>0$ such that for $\mathsf{F}\ge\mathsf{F}_0$, the operator $\Delta_{\mathsf{F},s}=\delta^s_{\mathsf{F}}\mathrm{d}^s_{\mathsf{F}}$, considered as a map $\dot{H}^{1,0}_{sc}\to(\dot{H}^{1,0}_{sc})^*=\bar{H}^{-1,0}_{sc}$, is invertible.*

Proof. Since δ^s_{F} is defined as the adjoint of $\mathrm{d}^s_{\mathsf{F}}$ relative to the scattering metric, we have

$$\begin{aligned}\|\mathrm{d}^s_{\mathsf{F}}u\|^2_{L^2_{sc}}&=\langle\mathrm{d}^s_{\mathsf{F}}u,\mathrm{d}^s_{\mathsf{F}}u\rangle=\langle\Delta_{\mathsf{F},s}u,u\rangle\\ &\le\|\Delta_{\mathsf{F},s}u\|_{\bar{H}^{-1,0}_{sc}}\|u\|_{\dot{H}^{1,0}_{sc}}\le\epsilon^{-1}\|\Delta_{\mathsf{F},s}u\|^2_{\bar{H}^{-1,0}_{sc}}+\epsilon\|u\|^2_{\dot{H}^{1,0}_{sc}}.\end{aligned}\tag{3.3}$$

By (3.1) and (3.2), we have

$$\delta^s_{\mathsf{F}}\mathrm{d}^s_{\mathsf{F}}=\frac{1}{4}\nabla^*\nabla+\frac{1}{4}\mathsf{F}^2+\frac{3}{4}\mathrm{d}^s_{\mathsf{F}}\delta^s_{\mathsf{F}}+A+\tilde{R},\tag{3.4}$$

where $A\in\mathrm{Diff}^1_{sc}(X)$ with

$$|\langle Au,u\rangle|\le C\|u\|_{\dot{H}^{1,0}_{sc}}\|u\|_{L^2_{sc}}+C\mathsf{F}\|u\|^2_{L^2_{sc}},$$

and $\tilde{R}\in x\mathrm{Diff}^1_{sc}(X)$. This follows by rewriting $\nabla^*_{\mathsf{F}}\nabla_{\mathsf{F}}$ using (3.2), which modifies R in (3.1). Thus, we have

$$\|\mathrm{d}^s_{\mathsf{F}}u\|^2_{L^2_{sc}}=\frac{1}{4}\|\nabla u\|^2_{L^2_{sc}}+\frac{1}{4}\mathsf{F}^2\|u\|^2_{L^2_{sc}}+\frac{3}{4}\|\delta^s_{\mathsf{F}}u\|^2_{L^2_{sc}}+\langle Au,u\rangle+\langle\tilde{R}u,u\rangle.$$

This gives us

$$\|\nabla u\|^2_{L^2_{sc}} + \mathsf{F}^2\|u\|^2_{L^2_{sc}} \le C\|\mathrm{d}^s_{\mathsf{F}} u\|^2_{L^2_{sc}} + C\|x^{1/2}u\|^2_{L^2_{sc}} + C\|u\|_{\dot H^{1,0}_{sc}}\|u\|_{L^2_{sc}} + C\mathsf{F}\|u\|^2_{L^2_{sc}}. \tag{3.5}$$

Then for sufficiently large F,

$$\|\nabla u\|^2_{L^2_{sc}} + \mathsf{F}^2\|u\|^2_{L^2_{sc}} \le C\|\mathrm{d}^s_{\mathsf{F}} u\|^2_{L^2_{sc}} + C\|x^{1/2}u\|^2_{L^2_{sc}}, \tag{3.6}$$

where C is a constant depending on F, and thus

$$\|\nabla u\|^2_{L^2_{sc}} + \langle (1-Cx)u, u\rangle \le C\|\mathrm{d}^s_{\mathsf{F}} u\|^2_{L^2_{sc}}.$$

Now suppose that Ω_j is contained in $\{x \le x_0\}$. If x_0 is sufficiently small, this gives

$$\|\nabla u\|_{L^2_{sc}} + \|u\|_{L^2_{sc}} \le C\|\mathrm{d}^s_{\mathsf{F}} u\|_{L^2_{sc}}. \tag{3.7}$$

If x_0 is larger, we can still have

$$\|\nabla u\|_{L^2_{sc}} + \|u\|_{L^2_{sc}} \le C\|\mathrm{d}^s_{\mathsf{F}} u\|_{L^2_{sc}} + C\|u\|_{L^2_{sc}(\{x_1 \le x \le x_0\})},$$

with x_1 small, and thus have (3.7) by the standard Poincaré inequality (See [15, Equation (28)] for one forms). Then, with (3.3), and choosing $\epsilon > 0$ small, we find that

$$\|u\|_{\dot H^{1,0}_{sc}} \le C\|\Delta_{\mathsf{F},s} u\|_{\bar H^{-1,0}_{sc}}.$$

Therefore, we have proved the invertibility of $\Delta_{\mathsf{F},s}$. ☐

Using Lemma 4.4 in [18], in parallel to the above lemmas, we obtain

Lemma 3.3. *There exists $\mathsf{F}_0 > 0$ such that for $\mathsf{F} > \mathsf{F}_0$, the operator $\Delta_{\mathsf{F},s} = \delta^s_{\mathsf{F}}\mathrm{d}^s_{\mathsf{F}}$ on symmetric 3-tensors is invertible as a map $\dot H^{1,r}_{sc} \to \bar H^{-1,r}_{sc}$ for all $r \in \mathbb{R}$.*

Lemma 3.4. *Let Ω_j be a domain contained in X as above. For $\mathsf{F} > 0$ and $r \in \mathbb{R}$,*

$$\|u\|_{\bar H^{1,r}_{sc}(\Omega_j)} \le C(\|x^{-r}\mathrm{d}^s_{\mathsf{F}} u\|_{L^2_{sc}(\Omega_j)} + \|u\|_{x^{-r}L^2_{sc}(\Omega_j)}),$$

for symmetric 3-tensors $u \in \bar H^{1,r}_{sc}(\Omega_j)$.

Proof. By the proof of Lemma 4.5 in [18], we only need to consider the case $r = 0$. Let $\tilde\Omega_j$ be a domain in X with C^∞ boundary, transversal to ∂X, containing $\overline{\Omega_j}$. We show that there exist a continuous extension map $E : \bar H^{1,2}_{sc}(\Omega_j) \to \dot H^{1,2}_{sc}(\tilde\Omega_j)$ such that

$$\|\mathrm{d}^s_{\mathsf{F}} Eu\|_{L^2_{sc}(\tilde\Omega_j)} + \|Eu\|_{L^2_{sc}(\tilde\Omega_j)} \le C(\|\mathrm{d}^s_{\mathsf{F}} u\|_{L^2_{sc}(\Omega_j)} + \|u\|_{L^2_{sc}(\Omega_j)}), \quad u \in \bar H^{1,0}_{sc}(\Omega_j). \tag{3.8}$$

Once (3.8) is proved, by Lemma 3.1, with $v = Eu$, we have

$$\begin{aligned}\|\nabla v\|^2_{L^2_{sc}(\tilde\Omega_j)} + \|v\|^2_{L^2_{sc}(\tilde\Omega_j)} &\le C(\|\mathrm{d}^s_{\mathsf{F}} v\|^2_{L^2_{sc}(\tilde\Omega_j)} + \|v\|^2_{L^2_{sc}(\tilde\Omega_j)}) \\ &\le C(\|\mathrm{d}^s_{\mathsf{F}} u\|^2_{L^2_{sc}(\Omega_j)} + \|v\|^2_{L^2_{sc}(\Omega_j)}).\end{aligned}$$

This finally gives

$$\|u\|_{\bar H^{1,0}_{sc}(\Omega_j)} \le C(\|\mathrm{d}^s_{\mathsf{F}} u\|^2_{L^2_{sc}(\Omega_j)} + \|v\|^2_{L^2_{sc}(\Omega_j)}).$$

The only thing remaining is to construct E. By a partition of unity, this can be reduced to the situation where locally $X = \overline{\mathbb{R}^n}$, $\overline{\Omega_j} = \overline{\mathbb{R}^n_+}$; see the proof of Lemma 4.5 in [18]. We only need to analyze the extension of a symmetric 3-tensor on $\overline{\mathbb{R}^n_+}$ to $\overline{\mathbb{R}^n}$.

We let $\Phi_q(x', x'_n) = (x', -qx_n)$ for $x_n < 0$ be a diffeomorphism from $\{x_n < 0\}$ to $\{x_n > 0\}$. For $f_{ijk}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k$ on $\{x_0 \ge 0\}$, we define E_1 to be the extension to R^n,

$$E_1(f_{ijk}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k)(x', x_n) = \sum_{q=1}^{5} C_q\Phi_q^*(f_{ijk}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k), \quad x_n < 0$$

and

$$E_1(f_{ijk}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k)(x', x_n) = f_{ijk}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k, \quad x_n \geq 0,$$

with C_q chosen so that $E_1 : C^1(\overline{\mathbb{R}^n_+}) \to C^1(\overline{\mathbb{R}^n})$. By calculation

$$\begin{aligned}
&\Phi_q^* f_{ijk}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k = f_{ijk}(x', -qx_n)\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k, \quad i,j,k \neq n,\\
&\Phi_q^* f_{ijn}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^n = -qf_{ijn}(x', -qx_n)\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^n, \quad i,j \neq n,\\
&\Phi_q^* f_{inn}\mathrm{d}x^i \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n = q^2 f_{inn}(x', -qx_n)\mathrm{d}x^i \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n, \quad i \neq n,\\
&\Phi_q^* f_{nnn}\mathrm{d}x^n \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n = -q^3 f_{nnn}(x', -qx_n)\mathrm{d}x^n \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n,\\
&\partial_l\Phi_q^* f_{ijk}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k = \partial_l f_{ijk}(x', -qx_n)\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k, \quad i,j,k,l \neq n,\\
&\partial_l\Phi_q^* f_{ijn}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^n = -q\partial_l f_{ijn}(x', -qx_n)\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^n, \quad i,j,l \neq n,\\
&\partial_l\Phi_q^* f_{inn}\mathrm{d}x^i \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n = q^2\partial_l f_{inn}(x', -qx_n)\mathrm{d}x^i \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n, \quad i,l \neq n,\\
&\partial_l\Phi_q^* f_{nnn}\mathrm{d}x^n \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n = -q^3\partial_l f_{nnn}(x', -qx_n)\mathrm{d}x^n \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n, \quad l \neq 0\\
&\partial_n\Phi_q^* f_{ijk}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k = -q\partial_n f_{ijk}(x', -qx_n)\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^k, \quad i,j,k \neq n,\\
&\partial_n\Phi_q^* f_{ijn}\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^n = q^2\partial_n f_{ijn}(x', -qx_n)\mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \mathrm{d}x^n, \quad i,j \neq n,\\
&\partial_n\Phi_q^* f_{inn}\mathrm{d}x^i \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n = -q^3\partial_n f_{inn}(x', -qx_n)\mathrm{d}x^i \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n, \quad i \neq n,\\
&\partial_n\Phi_q^* f_{nnn}\mathrm{d}x^n \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n = q^4\partial_n f_{nnn}(x', -qx_n)\mathrm{d}x^n \otimes \mathrm{d}x^n \otimes \mathrm{d}x^n.
\end{aligned}$$

The matching of the derivatives at $x_n = 0$, which gives the C^1 property, yields

$$\begin{aligned}
&C_1 + C_2 + C_3 + C_4 + C_5 = 1,\\
&C_1 + 2C_2 + 3C_3 + 4C_4 + 5C_5 = -1,\\
&C_1 + 4C_2 + 9C_3 + 16C_4 + 25C_5 = 1,\\
&C_1 + 8C_2 + 27C_3 + 64C_4 + 125C_5 = -1,\\
&C_1 + 16C_2 + 81C_3 + 256C_4 + 625C_5 = 1.
\end{aligned}$$

The linear system, with a Vandermonde matrix, is solvable. With the $C_q, q = 1, 2, \cdots, 5$, satisfying the linear system above, we obtain the property $E_1 : C^1_c(\overline{\mathbb{R}^n_+}) \to C^1_c(\overline{\mathbb{R}^n})$ and

$$\|E_1 u\|_{H^1(\mathbb{R}^n)} \leq C\|u\|_{H^1(\mathbb{R}^n_+)}.$$

With Φ_q^* acting on 4-tensors as usual, we have

$$\mathrm{d}^s\Phi_q^* = \Phi_q^*\mathrm{d}^s,$$

and thus

$$\|\mathrm{d}^s\Phi_q^* u\|_{L^2(\mathbb{R}^n_-)} \leq C\|\mathrm{d}^s u\|_{L^2(\mathbb{R}^n_+)}.$$

Then

$$\|\mathrm{d}^s E_1 u\|_{L^2(\mathbb{R}^n)} \leq C\|\mathrm{d}^s u\|_{L^2(\mathbb{R}^n_+)}$$

which completes the proof. □

Now we define

$$\begin{aligned}
\mathcal{S}_{\mathsf{F},\Omega_j} &= \mathrm{Id} - \mathrm{d}^s_{\mathsf{F}}\Delta^{-1}_{\mathsf{F},s}\delta^s_{\mathsf{F}},\\
\mathcal{P}_{\mathsf{F},\Omega_j} &= \mathrm{d}^s_{\mathsf{F}}Q_{\mathsf{F},\Omega_j}, \quad Q_{\mathsf{F},\Omega_j} = \Delta^{-1}_{\mathsf{F},s}\delta^s_{\mathsf{F}}.
\end{aligned}$$

In parallel to Corollaries 4.6, 4.7, 4.8 in [18], we have obtain for the Dirichlet Laplacian $\Delta_{\mathsf{F},s}$

Corollary 3.5. *Let ϕ on $C_c^\infty(\overline{\Omega_j} \setminus \partial_{\mathrm{int}}\Omega_j)$. Then on symmetric 3-tensors, there exists $\mathsf{F}_0 > 0$ such that for any $\mathsf{F} \geq \mathsf{F}_0$, $\phi\Delta_{\mathsf{F},s}^{-1}\phi : \bar{H}_{sc}^{-1,k} \to \dot{H}_{sc}^{1,k}$ is in $\Psi_{sc}^{-2,0}(X)$.*

Corollary 3.6. *Let $\phi \in C_c^\infty(\overline{\Omega_j} \setminus \partial_{\mathrm{int}}\Omega_j)$, $\chi \in C^\infty(\overline{\Omega_j})$ with disjoint support and with χ constant near $\partial_{\mathrm{int}}\Omega$. Let F, F_0 as in Corollary 3.5. Then the operator $\chi\Delta_{\mathsf{F},s}^{-1}\phi : \bar{H}_{sc}^{-1,k}(\Omega_j) \to \dot{H}_{sc}^{1,k}(\Omega_j)$ in fact maps $H_{sc}^{s,r}(X) \to \dot{H}_{sc}^{1,k}(\Omega_j)$ for all s, r, k.*

Similarly, $\phi\Delta_{\mathsf{F},s}^{-1}\chi : \bar{H}_{sc}^{-1,k}(\Omega_j) \to \dot{H}_{sc}^{1,k}(\Omega_j)$ in fact maps $\bar{H}_{sc}^{-1,k}(\Omega_j) \to H_{sc}^{s,r}(X)$ for all s, r, k.

Corollary 3.7. *Let $\phi \in C_c^\infty(\overline{\Omega_j} \setminus \partial_{\mathrm{int}}\Omega_j)$, $\chi \in C^\infty(\overline{\Omega_j})$ with disjoint support and with χ constant near $\partial_{\mathrm{int}}\Omega_j$. Let F, F_0 as in Corollary 3.5.*

Then $\phi\mathcal{S}_{\mathsf{F},\Omega_j}\phi \in \Psi_{sc}^{0,0}(X)$, while $\chi\mathcal{S}_{\mathsf{F},\Omega_j}\phi : H_{sc}^{s,r}(X) \to x^k L_{sc}^2(\Omega_j)$ and $\phi\mathcal{S}_{\mathsf{F},\Omega_j}\chi : x^k L_{sc}^2(\Omega_j) \to H_{sc}^{s,r}(X)$ for all s, r, k.

We also have properties in parallel to Lemmas 4.9 to 4.13, and then arrive at the main, local, result

Theorem 3.8. *For $\Omega = \Omega_c$, $c > 0$ small, there exists $\mathsf{F}_0 > 0$ large enough, such that for $\mathsf{F} > \mathsf{F}_0$, the geodesic ray transform on symmetric 4-tensors $f \in e^{\mathsf{F}/x}L_{sc}^2(\Omega)$ satisfying $\delta^s(e^{-2\mathsf{F}/x}f) = 0$, is injective.*

The above local theorem leads to the global result, Theorem 1.2 similar to [18, Theorem 4.19].

Acknowledgments. M. V. de Hoop gratefully acknowledges support from the Simons Foundation under the MATH + X program, the National Science Foundation under grant DMS-1815143, and the corporate members of the Geo-Mathematical Group at Rice University. G. Uhlmann was partly supported by NSF, a Walker Family Endowed Professorship at UW and a Si-Yuan Professorship at IAS, HKUST. J. Zhai acknowledges the great hospitality of IAS, HKUST, where this work was initiated.

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