NPM-BUNDLE: Non-Preemptive Multitask Scheduling for Jobs with BUNDLE-Based Thread-Level Scheduling

Corey Tessler
Wayne State University, Detroit, Michigan, USA
corey.tessler@wayne.edu

Nathan Fisher
Wayne State University, Detroit, Michigan, USA
fishern@wayne.edu

Abstract

The BUNDLE and BUNDLEP scheduling algorithms are cache-cognizant thread-level scheduling algorithms and associated worst case execution time and cache overhead (WCETO) techniques for hard real-time multi-threaded tasks. The BUNDLE-based approaches utilize the inter-thread cache benefit to reduce WCETO values for jobs. Currently, the BUNDLE-based approaches are limited to scheduling a single task. This work aims to expand the applicability of BUNDLE-based scheduling to multiple task multi-threaded task sets.

BUNDLE-based scheduling leverages knowledge of potential cache conflicts to selectively preempt one thread in favor of another from the same job. This thread-level preemption is a requirement for the run-time behavior and WCETO calculation to receive the benefit of BUNDLE-based approaches. This work proposes scheduling BUNDLE-based jobs non-preemptively according to the earliest deadline first (EDF) policy. Jobs are forbidden from preempting one another, while threads within a job are allowed to preempt other threads.

An accompanying schedulability test is provided, named Threads Per Job (TPJ). TPJ is a novel schedulability test, input is a task set specification which may be transformed (under certain restrictions); dividing threads among tasks in an effort to find a feasible task set. Enhanced by the flexibility to transform task sets and taking advantage of the inter-thread cache benefit, the evaluation shows TPJ scheduling task sets fully preemptive EDF cannot.

2012 ACM Subject Classification Computer systems organization → Real-time systems; Software and its engineering → Real-time schedulability

Keywords and phrases Scheduling algorithms, Cache Memory, Multi-threading, Static Analysis

Digital Object Identifier 10.4230/LIPIcs.ECRTS.2019.15

Supplement Material ECRTS 2019 Artifact Evaluation approved artifact available at https://dx.doi.org/10.4230/DARTS.5.1.2

Funding The research presented in this paper was supported by the National Science Foundation under Grant Nos. CNS-1618185 and IIS-1724227.

1 Introduction

Hard real-time multi-threaded task systems which incorporate cache memory, must account for the variation in execution time and cache related preemption delays found in single-threaded task systems. For multi-threaded task systems, the complexity of cache interactions is increased due to thread-level cache interference and preemptions. Worst-case execution time (WCET) and schedulability analysis of hard real-time multi-threaded tasks commonly treat threads independently [21] or utilize cache management techniques [33] to limit the cache interference.

Analysis techniques focusing on independent treatment or limiting of cache interference exclude the possible benefit of caches. Multi-threaded tasks may benefit from caches. By virtue of sharing the same address space one thread of a task may cache values on behalf of

31st Euromicro Conference on Real-Time Systems (ECRTS 2019).
Editor: Sophie Quinton; Article No. 15; pp. 15:1–15:23


Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

another reducing the total execution time to complete both. This positive effect is referred to as the *inter-thread cache benefit* [26].

Currently, only the BUNDLE [26] and BUNDLEP [27] analysis techniques and cache congnizant thread-level scheduling algorithms incorporate the inter-thread cache benefit into WCET and schedulability analysis. These BUNDLE-based approaches are currently limited to a single multi-threaded task. The primary focus of this work is to provide a scheduling algorithm and schedulability test for multi-threaded task sets with multiple tasks, where individual jobs utilize BUNDLE-based scheduling. As the first scheduling algorithm to incorporate BUNDLE-based thread-level scheduling, a non-preemptive algorithm was chosen to avoid necessary modifications to BUNDLE and BUNDLEP. Non-preemptive EDF was selected as the task-level scheduler, as the proposed schedulability test presented in Section 4 is based upon Baruah's limited-preemption for EDF [6] algorithm.

An additional consideration is made for alternative approaches and the unforeseen benefits to schedulability of thread-level schedulers of non-preemptive multi-threaded jobs. If the WCET of jobs can be expressed as a strictly increasing discrete concave function of the number of threads per job, the schedulability test developed for this work applies without modification to the BUNDLE-based approaches or non-preemptive EDF scheduling.

In the following sections, the key contributions are:

1. A model of hard real-time multi-threaded tasks which is compatible with existing single-threaded models, where tasks sets may be transformed through division of threads.
2. A schedulability test named Threads Per Job (TPJ) that provides a schedulability result and transformed feasible task set if the specified task set could not be scheduled non-preemptively.
3. Proof of TPJ's optimality with respect to non-preemptive multi-threaded feasibility.
4. An improvement to Baruah's [6] non-preemptive chunk algorithm, increasing chunk sizes.
5. An evaluation of over 500,000 task sets, comparing the schedulability ratio of TPJ to those of non-preemptive and (limited) preemptive EDF, with an accompanying implementation available for download [28].

These contributions are presented following the related research of Section 2. Section 3 introduces the proposed model, application of non-preemptive EDF scheduling for thread-level schedulers, and the requirements of task transformation. Section 4 introduces then improves upon the non-preemptive chunk algorithm [6], followed by the TPJ schedulability algorithm and proof of feasibility. Section 5 compares the schedulability ratio of TPJ to other non-preemptive and preemptive scheduling algorithms, before concluding with Section 6.

2 Discussion of Related Research

Single-Threaded Tasks. The challenge of dealing with the non-uniformity of execution times in real-time systems due to cache misses or hits has received considerable attention [34, 30]. In particular, much of the prior real-time systems work on understanding caches vis-à-vis scheduling has focused upon the contention in the cache due to tasks preempting each other. Roughly speaking, a large majority of this research can be classified into two categories: *cache-related preemption delay (CRPD) analysis* and *deferred/limited-preemption scheduling*. The goal of CRPD analysis is to bound the number of cache blocks of a task that need to be reloaded due to evictions caused by a preempting task. The foundation of CRPD analysis is the development of techniques for counting and bounding the number of blocks affected by preemption; this is achieved by categorizing a task's cache blocks into sets of useful cache blocks (UCBs) or evicting cache block (ECBs) [17, 31]. The size of these sets can be used as

an upper bound on the cache cost of a preemption. Subsequent research based upon this UCB and ECB categorization has refined these sets and incorporated the CRPD analysis into schedulability analysis [1, 2, 3, 20, 24, 25]. However, please note that these CRPD approaches only quantify the cache effect of preemption into existing scheduling approaches and do not change any scheduling decision based upon the knowledge of preemption.

In limited/deferred-preemption scheduling, a higher-priority task may preempt a lower-priority task only when some condition is satisfied. The overall effect of deferring or limiting preemptions is to reduce the number of times a task may be preempted during its execution. The hope is that by limiting the number of preemptions this will lead to a decrease in the execution time of job due to the cache overhead of preemption. Different conditions for deferring preemptions have been considered. The fixed preemption-point approach [11] selects specific locations in a task code that are most appropriate for the program but preserve the system schedulability. The preemption-threshold scheduling approach [32] sets a threshold that only task with higher-priority than this threshold may preempt a currently-executing lower priority task. The floating preemption-point model [6, 19] computes the maximum time duration that a lower-priority task may delay the preemption of a higher-priority task. Each of the deferred preemption approaches have been shown to limit the number of preemptions but do not incorporate the CRPD overhead cost in its decision on how to defer preemption.

More recently, a line of research has emerged to combine the aspects of CRPD analysis and limited/deferred preemption scheduling by explicitly placing preemption points in the code to minimize CRPD effects. Early heuristics were proposed by Simonson and Patel [23] and Lee et al. [17]. Bril et al. [10] integrated CRPD analysis into preemption-threshold scheduling. Bertogna et al. [8] provide a more formal approach for optimally determining preemptions in programs that can be represented by linear control flowgraphs given the CRPD overhead of each preemption and a bound on the maximum non-preemption region [6]. Later work, extended this to more general control flowgraphs [22] or more precise CRPD characterizations of the preemption costs [14]. However, all of this aforementioned research assumes each task is single-threaded. The techniques proposed in this paper extend the CRPD and limited preemption concepts to scheduling multi-threaded tasks by combining and extending the limited-preemption scheduling results of Baruah [6] to the cache-cognizant thread-level scheduling algorithms that minimize cache contention between threads called `BUNDLE` [26] and `BUNDLEP` [27].

Multi-Threaded Tasks. Cache interference amplifies the variation in execution times of multi-threaded task sets. Threads of the same task share cache locations, with the potential to increase misses and hits depending on the order of execution of threads. This variability is an addition to the variation already present when considering CRPD with other tasks.

There are few works we are aware which directly address the inter-thread variability due to caches in multi-threaded task sets. The approaches focus on isolating execution or managing cache behavior. Memory-Centric Scheduling [4] isolates contentious execution by scheduling tasks according to their cache behavior. To create such isolation, tasks must be PREM-compliant [21], with distinct load and execution phases. Cache management utilize techniques that limit the contention in the cache, such as coloring and blocking found in [33]. These approaches come at a cost of modified or restricted executable objects, reduced cache sizes, or additional cache misses of blocked lines. Yet, with these limitations, the inter-thread variability is not accounted for within multi-threaded tasks.

`BUNDLE` [26] and `BUNDLEP` [27] address inter-thread variability due to cache interactions. These `BUNDLE`-based approaches analyze executable object coupled with a cache-cognizant thread-level scheduling algorithm without the added detriment of modified (or restricted)

objects, or cache management penalties. We are not aware of any other technique that addresses inter-thread variability, with the execption of Calandrino's [13] limited cache spread. However, the results of [13] are strictly empirical.

3 Model

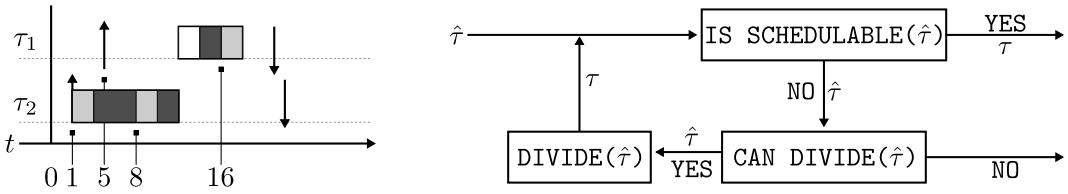
To permit non-preemptive jobs that utilize thread-level schedulers, a new model is proposed in this work. The set of n multi-threaded tasks is given by $\tau = \{\tau_0, \tau_1, ..., \tau_{n-1}\}$. Each job of a task $\tau_i = (p_i, d_i, m_i, c_i : \mathbb{N} \mapsto \mathbb{R}^+)$ has a minimum inter-arrival time of p_i and relative deadline d_i. For every job release of τ_i, a positive integer m_i identical threads are released. Each thread of τ_i executes over the same object o_i on the shared processor. An object is a set of executable machine instructions, mapping to one set of in memory addresses, such that all threads execute the same instruction from the same address. All threads share the same deadline as their job. The worst-case execution time (WCET) of τ_i is a function of the number of threads per job, $c_i(m_i)$.

Scheduling and schedulability analysis proposed in this work relies upon a relationship between the number of threads scheduled per multi-threaded job and the WCET of the job executed non-preemptively. To clarify, the scheduling mechanism proposed in this work precludes preemptions between jobs of different tasks. For threads within a job of a task, a thread-level scheduler may execute threads preemptively. Figure 1a illustrates the scheduling behavior.

In Figure 1a, at $t = 1$ a job of τ_2 is released. The job of τ_2 cannot be preempted by the job of τ_1 released at $t = 5$. During the execution of τ_2, the two threads (given distinct colors) may preempt one another according to the thread-level scheduler, at $t = 8$ for instance. Thread-level scheduling and preemption decisions are not prescribed by this work. The thread-level scheduling policies of τ_1 and τ_2 are independent of the non-preemptive task-level scheduling of non-preemptive EDF used in this work.

Thread-level scheduling algorithms must be characterized by a WCET function $c_i(m_i)$ for m_i threads per job and $c_i(m_i)$ must be strictly increasing discrete and concave (detailed in Subsection 3.2). Thread-level schedulers that produce concave $c_i(m_i)$ functions establish a relationship between the execution requirements of a task and the number of threads, where the requirement for one job of m_i threads is less than m_i jobs of one thread. For `BUNDLE`-based schedulers, concavity is the result of the inter-thread cache benefit, where $c_i(m) - c_i(m-1) \geq c_i(m+1) - c_i(m)$; it is this relationship the proposed scheduling behavior and analysis seek to exploit.

Not all tasks and thread-level schedulers will produce concave WCET functions. For a task τ_i with a convex WCET function (where there is no benefit in grouping threads together), the m_i threads of τ_i may be replaced with m_i single-threaded tasks. These single-threaded have vacuously concave WCET functions by virtue of executing no more than one thread.



(a) Scheduling Behavior.

(b) Schedulability and Transformable Task Sets.

Figure 1 Scheduling and Schedulability of the Proposed Model.

The task set τ provided by the system designer to schedulability analysis is referred to as the task set *specification*. Commonly [5, 18, 6, 8, 11, 12], task set specifications are immutable in hard-real time models. The number of tasks, their WCET time, period, and deadline are provided by the system designer, not to be changed. Schedulability analysis determines if the task set specification is feasible. In this work, task sets are transformable (obeying some restrictions).

Transformation of a task set exploits the concavity of execution requirements, redistributing the threads of individual tasks to multiple tasks. A greater number of threads per job reduces the WCET of a task but increases the non-preemptive execution requirement. Conversely, a fewer number of threads per task increases the total WCET for all tasks while decreasing the non-preemptive execution requirement. Schedulability analysis in this non-preemptive setting encompasses the search for a distribution of the fixed number threads from the task set specification to a variable number of tasks, resolving the tension between a greater number of tasks and a greater number of threads per task to find a feasible task set.

Under the proposed model, schedulability analysis is a process that begins by considering the current task set named the *anterior* task set $\hat{\tau}$. If the set is schedulable, the set is unmodified and processing ceases with a positive result. If the task set $\hat{\tau}$ cannot be scheduled as described, the task set is transformed into a *posterior* task set τ, and processed again as an anterior set. Processing ceases with a negative result when there are no available transformations of $\hat{\tau}$.

Figure 1b illustrates the schedulability analysis process. Division is the transformative operation of the process and is described in Subsection 3.1. The figure highlights the ability of a single task set to be both anterior and posterior to different sets during processing. To aid in explanation, properties of a task may be referred to in terms of the set the task was transformed from and to. By example, if the number of threads assigned to τ_i in the anterior set $\hat{\tau}$ is reduced by one in the posterior task set τ, the posterior threads of τ_i may be written as $m_i = \hat{m}_i - 1$.

As a process, schedulability analysis of the specified task set serves two purposes under this model. The first, is to determine if there exists a posterior task set which is feasible. Second, to produce the feasible posterior task set if one exists. It is the feasible posterior task set τ found by schedulability analysis that is then deployed on the target architecture. From the system designer's perspective, each task $\tau_i \in \tau$ of the specified task set is a request to execute m_i threads of the object o_i with shared periods p_i and deadlines d_i for **any** posterior task set τ. A task set specification is flexible, for one object there may be multiple tasks with variable numbers of threads per job. However, the specified m_i of a task is a ceiling on any m_i of a posterior task.

3.1 Dividing and Task Parts

A task set may be transformed by *dividing* tasks of the set. Dividing a task reduces the number of threads executed by each job, splitting the anterior task into two or more tasks in the posterior set.

▶ **Definition 1** (Task Division). *In the anterior task set $\hat{\tau}$, a task $\tau_i = (p_i, d_i, c_i(m_i))$ may be* divided *into two (or more) posterior tasks τ_j and τ_k with three restrictions: 1.) the periods of τ_j and τ_k are equal to the period of τ_i 2.) the relative deadlines of τ_j and τ_k are equal to the deadline of τ_i 3.) the sum of threads of τ_j and τ_k are equal to τ_i 4.) the objects of τ_i, τ_j, and τ_k are equal. Enumerated, the restrictions are:*

1. $p_i = p_j = p_k$
2. $d_i = d_j = p_k$
3. $m_i = m_j + m_k$
4. $o_i = o_j = o_k$

▶ **Definition 2** (Partial Tasks). *When an anterior task τ_i is divided into τ_j and τ_k posterior tasks, τ_j and τ_k are referred to as* partial tasks *or* parts *of τ_i.*

▶ **Definition 3** (Partial Task Set). *For convenience, the set of posterior tasks of τ_i is denoted Φ_i and called the* partial task set *of τ_i, where $m_i = \sum_{\tau_k \in \Phi_i} m_k$.*

3.2 Worst-Case Execution Time Function Growth

Motivation for the task model and schedulability analysis process proposed in this work stems from the inter-thread cache benefit of `BUNDLE`-based scheduling [26, 27]. The previous works [26, 27] are limited to a single task; this work extends the method (non-preemptively) to multiple tasks. Schedulability analysis for `BUNDLE`-based scheduling algorithms produce, for each task τ_i, a worst-case execution time combined with cache overhead (WCETO) function $c_i(m)$ in terms of m the number of threads per job scheduled in a cache-cognizant manner. For tasks that benefit from `BUNDLE`-based scheduling and analysis, $c_i(m)$ is a strictly increasing discrete concave function. Tasks that do not are made vacuously concave by restricting jobs to release one thread.

In the WCETO analysis of `BUNDLE` and `BUNDLEP`, threads are assigned to paths through the conflict-free region graph of the executable object which maximize their contribution to $c_i(m_i)$. When considering the addition of a thread $m_i + 1$, only the greatest increase in $c_i(m_i)$ is permitted. Subsequently, the addition of thread $m_i + 2$ must increase $c_i(m_i)$ by less than or equal to the increase from $m_i + 1$ or the increase of $m_i + 1$ would not have been maximal. Therefore, for any $m_a < m_b < m_c$ the point $(m_b, c_i(m_b))$ lies above the straight line described by $(m_a, c_i(m_a))$ and $(m_c, c_i(m_c))$ – subsequently, $c_i(m_i)$ is concave.

A consequence of $c_i(m)$'s strictly increasing discrete concavity is a limit on the increase of the WCET as the number of threads increases. This property is referred to as the *concave restricted growth* (*concave growth* for brevity) of $c_i(m)$ and is leveraged in Sections 4 and 5.

▶ **Property 1** (Concavity Restriction on WCET Growth). *For a strictly increasing discrete concave WCET function $c_i(m)$:*

$$\forall m \in \mathbb{N}^+ \mid c_i(m) - c_i(m-1) \geq c_i(m+1) - c_i(m) \tag{1}$$

It then follows for $m_x \geq m_y > 0$

$$\begin{aligned} c_i(m_x + 1) - c_i(m_x) &\leq c_i(m_x) - c_i(m_x - 1) \\ &\leq c_i(m_x - 1) - c_i(m_x - 2) \\ &\ldots \\ &\leq c_i(m_y) - c_i(m_y + 1) \\ &\leq c_i(m_y) - c_i(m_y - 1) \end{aligned}$$

A WCET function $c_i(m)$ that obeys Property 1, will produce a value for $c_i(m + 1)$ threads which is greater than $c_i(m)$. The difference between $c_i(m + 1)$ and $c_i(m)$ must be less than or equal to the difference of $c_i(m)$ and $c_i(m - 1)$. As the number of threads increase, $c_i(m)$ increases at a decreasing (or stable) rate.

For the purposes of comparison and evaluation in Section 5, an upper bound on the growth of $c_i(m)$ is called the *growth factor* $\mathbb{F}_i$ of τ_i. Growth factors relate the WCET of one thread $c_i(1)$ to the WCET of an arbitrary number of threads $c_i(m)$ for $m > 0$. A growth factor $\mathbb{F}_i \in (0, 1]$, for a task τ_i, is a real number that satisfies Equation 2.

▶ **Definition 4** (Growth Factor for τ_i).

$$\forall m \mid c_i(m) \leq c_i(1) + (m - 1) \cdot \mathbb{F} \cdot c_i(1) \tag{2}$$

For an $\mathbb{F}$ satisfying Equation 2, the pessimistic upper bound provides a linear function that can be rearranged to find an upper bound on the WCET of one thread in terms of m threads. The result is Equation 3, which will be used in the evaluation Section 5 when constructing task sets. Note, since $m \in \mathbb{N}$ each increase of m increases $c_i(m)$ by $\mathbb{F} \cdot c_i(1)$.

$$c_i(m) = c_i(1) + (m-1) \cdot \mathbb{F} \cdot c_i(1) \tag{3}$$

4 Non-Preemptive EDF Schedulability

Preemptive earliest deadline first (EDF) schedulability analysis for sporadic task sets has been well studied [18, 5, 15]. In the fully preemptive setting for which the algorithm is optimal, the overhead of a large number of preemptions may be a detriment to schedulability. Baruah [6] addresses this concern with an algorithm for calculating the non-preemptive chunk size q_i of each task $\tau_i \in \tau$. The non-preemptive chunk size q_i guarantees that task τ_i may execute up to q_i time units non-preemptively without introducing a deadline miss for any task in τ scheduled by preemptive EDF.

Section 4.3 introduces the non-preemptive feasibility algorithm Thread Per Job (TPJ) based upon the non-preemptive chunks algorithm from [6]. TPJ differs from the non-preemptive chunks algorithm by requiring the non-preemptive chunk size q_i of each task τ_i to be greater than or equal to its WCET: $c_i(m_i) \leq q_i$. As such, all jobs can be scheduled non-preemptively without fear of a deadline miss. To clearly convey TPJ, a description of the non-preemptive chunks algorithm and its dependencies is provided in the immediate subsection. Subsection 4.2 describes, by example, the available improvements to the non-preemptive chunks algorithm [6]. Subsection 4.4 defines and proves TPJ's optimality.

4.1 Non-Preemptive Chunks

The non-preemptive chunks algorithm depends on the demand bound function, EDF feasibility, ordering of absolute deadlines, and slack for the task set τ. Ordered absolute deadlines are given by $\{D_1, D_2, ...\}$ with $D_n < D_{n+1}$ for all n, where each task $\tau_i \in \tau$ contributes deadlines $D = k \cdot p_i + d_i$ for $k \in \mathbb{Z}^+$.

For a sporadic task τ_i the demand bound function for a task $\text{DBF}(\tau_i,t)$ is an upper bound on the amount of execution requirement generated from jobs released by τ_i over t units of time. The demand bound function is presented as Equation 4 as $\text{DBF}(\tau_i,t)$ modified from [5] to suit the task set model used in this work.

▶ **Definition 5** (Demand Bound Function for a Task τ_i and Interval t)**.**

$$\text{DBF}(\tau_i,t) = \max\left(0, \left(\left\lfloor \frac{t-d_i}{p_i} \right\rfloor + 1\right) \cdot c_i(m_i)\right) \tag{4}$$

When necessary for brevity, Equation 5 will be used to represent the sum of demand of all tasks over an interval of length t.

▶ **Definition 6** (Demand Bound Function for the Task Set τ and Interval t)**.**

$$\text{DBF}(\tau,t) = \sum_{\tau_i \in \tau} \text{DBF}(\tau_i,t) \tag{5}$$

Slack of the task set τ at deadline D_k is given by Equation 6. Intuitively, slack is the minimum time the processor will be idle over an interval. It is the difference between the demand over the interval and the length of the interval.

▶ **Definition 7** (Slack at Deadline D_k)**.**

$$\text{SLACK}(D_k) = \min_{j \leq k} \left(D_j - \sum_{\tau_i \in \tau} \text{DBF}(\tau_i, D_k) \right) \tag{6}$$

For EDF, feasibility is determined by examining increasing time intervals and calculating the demand and supply. If demand exceeds supply, the system is infeasible. Equation 7 provides a formal definition of feasibility for the task set τ.

▶ **Definition 8** (EDF Feasibility Demand Bound Test)**.**

$$\forall t \geq 0, \left(\sum_{\tau_i \in \tau} \text{DBF}(\tau_i, t) \right) \leq t \tag{7}$$

In [6], the number of time instants tested by Equation 7 is limited to the values of the ordered set of absolute deadlines $\{D_1, D_2, ...\}$. The ordered set of absolute deadlines is an infinite set, impractical for feasibility test. There is an upper bound on the value of all time instants (absolute deadlines) that must be tested and is denoted $T^*(\tau)$. Taken from [15], $T^*(\tau)$ is given by Equation 8 below. Among all tasks the largest deadline is $d_{\max} = \max_{\tau_j \in \tau}(d_j)$. Utilization of τ_j is defined as $U_j = \frac{c_j(m_j)}{p_j}$. Among all tasks, the greatest difference of period and deadline is given by $\Delta_{max} = \max_{\tau_i \in \tau}(p_i - d_i)$. The hyper-period of all tasks (the least common multiple of all relative deadlines) is given by P.

▶ **Definition 9** (Feasibility Test Bound t for τ)**.**

$$T^*(\tau) = \min \left(P, \max \left(d_{max}, \frac{1}{1-U} \cdot \Delta_{max} \cdot \sum_{i=0}^{n-1} U_i \right) \right) \tag{8}$$

The non-preemptive chunks algorithm from [6] is presented (with additional details) as pseudocode in Algorithm 1 and named NP-CHUNKS. In addition to determining if the task set is schedulable under EDF, the algorithm produces a non-preemptive chunk size q_j for each task $\tau_j \in \tau$. Jobs of τ_j may execute up to q_j time units non-preemptively without negatively impacting schedulability. This setting, where a task τ_j may execute non-preemptively for some period of time q_j is referred to as *limited-preemption*.

Algorithm 1 Non-Preemptive Chunks (NP-CHUNKS).

```
1: SLACK(D_1) ← D_1 − Σ_{τ_i∈τ} DBF(τ_i, D_1)
2: for τ_j ∈ {τ_i ∈ τ | (d_i = D_1)} do
3:     q_j ← c_j(m_j)
4: end for
5: for k ∈ {D_2, D_3, ..., } do
6:     if D_k > T*(τ) then
7:         return feasible
8:     end if
9:     SLACK(D_k) ← min(SLACK(D_{k−1}), D_k − Σ_{τ_i∈τ} DBF(τ_i, D_k))
10:    if SLACK(D_k) < 0 then
11:        return infeasible
12:    end if
13:    for τ_j ∈ {τ_i ∈ τ | (d_i = D_k)} do
14:        q_j ← SLACK(D_k)
15:    end for
16: end for
```

For a detailed description of NP-CHUNKS refer to [6]. To summarize, NP-CHUNKS begins by seeding the slack of the smallest interval D_1 and the non-preemptive chunk size of tasks with the smallest relative deadline equal to their WCET. During each iteration of $D_k \in \{D_2, D_3, ..., \}$, the slack for the interval D_k is calculated as the minimum of the current slack and the previous slack value. If there is less than zero slack, the system is infeasible. If the slack is zero or greater, each task with relative deadline equal to the current interval size is assigned the available slack as the task's non-preemptive chunk size. A task τ_j is assigned a non-preemptive chunk once, before assignment $q_j = \varnothing$ afterwards $q_j \neq \varnothing$. If the interval being examined D_k exceeds $T^*(\tau)$, the task set must be schedulable.

4.2 Improving the Non-Preemptive Chunk Size

From the description of NP-CHUNKS in [6], there is an opportunity to improve the available slack for each of the k deadlines considered. Alogrithm 1 is pessimistic in the amount of available slack at any deadline D_k. To illustrate, consider the task set and intermediate values described by Table 1.

Table 1 Example Task Set $\tau = \{\tau_0, \tau_1, \tau_2\}$.

i	p_i	d_i	m_i	$c_i(m_i)$
τ_0	4	2	1	1
τ_1	3	3	1	1
τ_2	3	3	1	1

P	D_k	$\tau_j : d_j = D_k$	DBF(τ,D_i)	SLACK(D_i)	q_j
12	$D_1 = 2$	τ_0	1	1	1
	$D_2 = 3$	τ_1	3	0	**0**
		τ_2	3	0	**0**

There are three tasks in the task set of Table 1, with utilization of approximately 0.92. For τ_0, initialization assigns a non-preemptive chunk of $q_0 = 1$ time units. By observation, after release τ_0 may be delayed from execution by at most one time unit or it will miss its deadline. Consequently, the non-preemptive chunk size available to τ_1 and τ_2 is 1. As such NP-CHUNKS would be expected to find $q_0 = 1, q_1 = 1, q_2 = 1$.

Note, it is not possible for τ_0 to be blocked for 1 or more time units if both τ_1 and τ_2 execute non-preemptively for 1 time unit each. If τ_0 is blocked for less than 1 time unit by τ_1, then τ_0 will be the highest priority task when τ_1 completes (similarly for τ_2). It is impossible for τ_0 to be blocked 1 time unit or more by τ_1 or τ_2, τ_0 would have to be released at the same time instant as τ_1 or τ_2 and τ_1 or τ_2 would have to execute before τ_0, since the relative deadline of τ_0 is less than the other two, limited-preemption EDF executes τ_0: the task with earliest absolute deadline.

For τ_0, q_0 is calculated as expected $q_0 = c_0(m_0) = 1$, by Lines 2-4 of Algorithm 1. However, τ_1 has a non-preemptive chunk size of $q_1 = 0$. The reason is Line 9, where SLACK(D_2) is calculated which includes the execution demand of τ_1 and τ_2. Slack is an upper bound on the non-preemptive chunk size assigned to a task (in this case τ_1). Giving a task the available slack permits the task to execute longer, delaying higher priority jobs from executing in the interval by delaying them for as much time as there is slack.

By example in Table 1, the available slack for τ_1 is determined from the interval of length $D_2 = 3$. The execution requirement of τ_1 and τ_2 is included in DBF(τ,3) because $d_1 = d_2 = 3$. Thus SLACK(D_2) is zero. Since τ_1's execution requirement is already included, it cannot further interfere over the interval D_2. Furthermore, τ_1 must have executed some portion without being preempted or the system would not be schedulable. Inclusion of τ_1's execution requirement within the interval over which slack is calculated for is pessimistic with respect to the non-preemptive chunk q_1 in this specific example, and q_j in general.

In the pseudocode implementation of NP-CHUNKS adopted from [6], Line 9 calculates the non-preemptive chunk size according Equation 9 (Equation 7 of Theorem 1 in [6]).

Comparing Line 9 of Algorithm 1 to Equation 9 a mismatch between the algorithm and the infeasibility test is illuminated.

▶ **Definition 10** (Infeasibility Test, Equation 7, from [6]).

$$\exists \tau_j \in \tau, t \in [0, d_j) \mid t < q_j + \sum_{\substack{i=0 \\ i \neq j}}^{n-1} \text{DBF}(\tau_i, t) \tag{9}$$

If the condition of Equation 9 is satisfied for a task set τ, the task set is unschedulable given a limited-preemption task set with assigned non-preemptive chunks q. The interval considered in the demand of Equation 9 is over $[0, d_j)$. The demand used in Algorithm 1 to calculate q_j is over the interval $[0, d_j]$. Extending the interval to include d_j introduces the pessimism identified by the example and is not required by Equation 9.

Table 1 illustrates the pessimism of NP-CHUNKS found in [6]. The example uses the notation of assigning non-preemptive chunks to individual tasks from [6]. A later work [7] uses a different notation, assigning non-preemptive chunks to interval lengths for the remaining execution of a job. The conceptual pessimism of including demand for tasks with deadline equal to the current interval (described by Table 1) is also found in [7].

4.3 Threads per Job (TPJ) Scheduling Algorithm

In this work, the NP-CHUNKS algorithm is modified for several purposes. First, the unnecessary pessimism is removed from chunk calculations. Second, the schedulability test is adapted to the model used herein. Lastly, when a given assignment of tasks and threads are infeasible, tasks are divided (when possible) to fit into their chunks. The division process is repeated until the task set is feasible, or no possible divisions remain and the task set is reported as infeasible. The algorithm is named the *Threads Per Job* (TPJ) scheduling algorithm.

A full description of TPJ is presented at the end of this subsection. To reach the complete description, an intermediate algorithm named *Bigger Non-Preemptive Chunks* (BNC) is presented as pseudocode in Algorithm 2. BNC removes the pessimism described in Section 4.2. The algorithm takes advantage of a property of the demand function $\text{DBF}(\tau, t)$ noted in [6].

▶ **Property 2** (Demand Change). *Demand for a task does not change for values of t that do not equal an absolute deadline. In terms of the set of ordered absolute deadlines,* $\text{DBF}(\tau, D_{i-1}) = \text{DBF}(\tau, D_i - \epsilon)$*, for* $0 < \epsilon \leq (D_i - D_{i-1})$.

Algorithm 2 Bigger Non-Preemptive Chunks (BNC).

1: $\text{SLACK}(D_0) \leftarrow \infty$
2: **for** $k \in \{D_1, D_2, D_3, ..., \}$ **do**
3: **if** $D_k > T^*(\tau)$ **then**
4: return *feasible*
5: **end if**
6: $\text{SLACK}(D_k) \leftarrow \min\left(\text{SLACK}(D_{k-1}), D_k - \sum_{\tau_i \in \tau} \text{DBF}(\tau_i, D_k)\right)$
7: **if** $\text{SLACK}(D_k) < 0$ **then**
8: return *infeasible*
9: **end if**
10: **for** $\tau_j \in \{\tau_i \in \tau \mid (d_i = D_k)\}$ **do**
11: $q_j \leftarrow \min(c_j(m_j), \text{SLACK}(D_{k-1}))$
12: **end for**
13: **end for**

Line 11 of Algorithm 2 implements the improvement of BNC over NP-CHUNKS. The non-preemptive chunk q_j of task τ_j is taken from the slack of the previous interval D_{k-1} or the task's WCET $c_j(m_j)$, whichever is smaller. The algorithm verifies the condition set by Equation 9, selecting the correct interval length by Property 2, which precludes the inclusion of τ_j's execution requirement in the interval (and other tasks with deadline D_k).

Algorithm 3 Threads-Per-Job (TPJ).

```
1:  SLACK(D_0) ← ∞
2:  for k ∈ {D_1, D_2, D_3, ...,} do
3:      if D_k > T*(τ) then
4:          return feasible
5:      end if
6:      for τ̂_j ∈ {τ_i ∈ τ | (d_i = D_k)} do
7:          if SLACK(D_{k-1}) < ĉ_j(1) then
8:              return infeasible
9:          end if
10:         Φ_j ← {τ̂_j}
11:         if SLACK(D_{k-1}) < ĉ_j(m̂_j) then                ▷ Jobs must be divided
12:             Φ_j ← DIVIDE(τ̂_j, SLACK(D_{k-1}))
13:             τ ← τ \ τ̂_j                                 ▷ Anterior task τ̂_j is represented by Φ_j
14:             τ ← τ ∪ Φ_j                                  ▷ Partial tasks include all threads of τ̂_j
15:         end if
16:         for τ_j ∈ Φ_j do
17:             q_j ← c_j(m_j)
18:         end for
19:     end for
20:     SLACK(D_k) ← min (SLACK(D_{k-1}), D_k − Σ_{τ_i∈τ} DBF(τ_i, D_k))
21:     if SLACK(D_k) < 0 then
22:         return infeasible
23:     end if
24: end for
```

The Threads per Job scheduling Algorithm 3, is a modification of BNC from limited-preemption EDF (EDF-LP) scheduling to non-preemptive EDF (EDF-NP). Input to the schedulability test is a task set specification τ, if TPJ returns a *feasibile* result there exists a posterior task set which can be scheduled by non-preemptive EDF and the posterior task set is returned as τ. An *infeasible* result indicates that TPJ could not guarantee τ would be schedulable by EDF-NP for any posterior task set. Since non-preemptive EDF is not optimal with respect to feasibility [12], TPJ is a sufficient test but cannot be necessary.

Algorithm 3 (TPJ) modifies BNC, the modifications are limited to Lines 6-19. An additional benefit of BNC removing the pessimism of each q_j, is that each q_j can be calculated without consideration of the current task τ_j and the demand at D_k. Chunk values depend on the demand of D_{k-1} instead. This permits an efficient implementation of TPJ by moving the slack calculation of the current interval to the end of each iteration. Otherwise, if slack were calculated earlier in each iteration, the changes to demand resulting from Lines 6-19 would force the demand and slack of D_k to be recalculated.

The first notable change to BNC is introduced on Line 7, comparing the available slack to the WCET of a single thread of $\hat{\tau}_j$. If there is insufficient slack to execute just one thread of $\hat{\tau}_j$ to completion, the task cannot be executed non-preemptively for any number of threads and the task set is infeasible non-preemptively.

Lines 11-15 introduce several subtle changes. For clarity, it is simpler to discuss the negative case ($\text{SLACK}(D_{k-1}) \geq \hat{c}_j(\hat{m}_j)$) before the positive. When there is sufficient slack for $\hat{m}_j$ threads to execute without preemption, $\hat{\tau}_j$ is given its full WCET ($\hat{c}_j(\hat{m}_j)$) as its non-preemptive chunk. In other words, no division of $\hat{\tau}_j$ is required and the posterior task set τ is unchanged (with respect to $\hat{\tau}_j$). Lines 11-15 are avoided, the algorithm progresses to the next task such that $d_i = D_k$.

However, in the positive case on Line 11 (when $\text{SLACK}(D_{k-1}) < \hat{c}_j(\hat{m}_j)$), $\hat{m}_j$ threads of $\hat{\tau}_j$ cannot feasibly execute without being preempted. Therefore, $\hat{\tau}_j$ must be divided. The DIVIDE procedure creates a partial task Φ_j set of $\hat{\tau}_j$, such that all tasks $\tau_p \in \Phi_j$ will complete within the available slack $c_p(m_p) \leq D_{k-1}$. The posterior task set τ has $\hat{\tau}_j$ removed, and is replaced by the partial set Φ_j maintaining the specified number of threads for $\hat{\tau}_j$.

For any task $\hat{\tau}_j$, the task is transformed into a partial task set Φ_j and assigned a non-preemptive chunk only once in the iteration where the absolute deadline D_k is equal to the relative deadline of the task: $D_k = \hat{d}_j$. Since tasks of τ are evaluated in strictly increasing absolute deadline order, the impact on demand and non-preemptive chunk sizes of processing $\hat{\tau}_j$ exclusively impacts demand for larger intervals $D_\ell > D_k$ and non-preemptive chunk sizes for tasks $\tau_\ell \in \tau$ with greater relative deadlines $d_\ell > \hat{d}_k$.

▶ **Property 3** (Divisions of $\hat{\tau}_j$ Exclusively Impacts Interval of Length $t \geq \hat{d}_j$)**.** *Division of $\hat{\tau}_j$ into the partial set Φ_j, and replacing $\hat{\tau}_j$ in τ with Φ_j will impact demand exclusively for intervals of length $D_k \geq \hat{d}_j$, slack of absolute deadlines $D_k > \hat{d}_j$ and therefore non-preemptive chunk values q_ℓ for tasks $\tau_\ell \in \tau$ with relative deadlines $d_\ell \geq D_k$*

By definition of $\text{DBF}(\hat{\tau}_j,t)$, no task of Φ_j or $\hat{\tau}_j$ can impact the task set τ demand $\text{DBF}(\tau,t)$ when $t < d_j$. Thus replacing $\hat{\tau}_j$ in τ, only affects the demand of intervals with length $\hat{d}_j$ or greater. Slack over the interval D_k is calculated from exclusively shorter intervals. Since the demand of the current interval D_k does not influence the slack at D_k, replacing $\hat{\tau}_j$ in τ only affects the slack of intervals with length greater than D_k. Non-preemptive chunk sizes are assigned based on the available slack, and only those assigned for an interval of length greater than D_k can be affected by replacing $\hat{\tau}_j$ in τ.

Algorithm 4 DIVIDE.

```
1:  procedure DIVIDE(τ̂_j, q)
2:      Φ_j ← {}
3:      m ← argmax_{m∈ℤ⁺} (ĉ_i(m) ≤ q)
4:      r ← m̂_j
5:      while r > 0 do
6:          m_p ← min(r, m)
7:          τ_p ← (p̂_j, d̂_j, m_p, ĉ_j)      ▷ Posterior task, same period, deadline, WCET function.
8:          Φ_j ← Φ_j ∪ τ_p
9:          r ← r − m_p
10:     end while
11:     return Φ_j
12: end procedure
```

On Line 12 of the TPJ Algorithm 3, the task $\hat{\tau}_j$ is divided into Φ_j by the DIVIDE procedure. Pseudocode of DIVIDE is given by Algorithm 4. The number of tasks in Φ_j are determined by the maximum number of threads m of $\hat{\tau}_j$ that can execute non-preemptively within q time units. Each task $\tau_k \in \Phi_j$ is assigned m threads of $\hat{\tau}_j$ or however many remain, whichever is less. The result is that each task set has the following properties.

▶ **Property 4** (Partial Task Sets Returned from DIVIDE). *The partial task set Φ_j of an anterior task $\hat{\tau}$ for a specific q value (and related maximum threads assigned per job m such that $c_j(m) \leq q$) contains posterior tasks where:*

1. *The exact number of posterior tasks is $|\Phi_j| = \lceil \frac{\hat{m}_j}{m} \rceil$*
2. *Exactly $\lfloor \frac{\hat{m}_j}{m} \rfloor$ tasks of Φ_j are assigned m threads per job.*
3. *There is at most one task $\tau_g \in \Phi_j$ with exactly $m_g = \hat{m}_j \mod m$ threads.*

4.4 Non-Preemptive Feasibility of TPJ and DIVIDE

The DIVIDE Algorithm 4 creates a partial task set Φ_j for an anterior task $\hat{\tau}_j$, assigning as many threads to each task in Φ_j as possible. Upon returning Φ_j to TPJ, $\hat{\tau}_j$ is replaced in the task set τ. Algorithm 4 is one method of dividing of $\hat{\tau}_j$ which TPJ could employ when creating the posterior task set τ. This section justifies DIVIDE's method by demonstrating the effect on schedulability and optimality of TPJ.

This section's ultimate objective is to clearly convey Theorem 5; concluding that TPJ is optimal with respect to task-level non-preemptive multi-threaded feasibility. The theorems that precede Theorem 5 establish minimal demand and WCET sums for partial sets created by DIVIDE necessary to illustrate TPJ's optimality.

Non-preemptive EDF scheduling of jobs of multiple threads ordered by a thread-level scheduler (such as `BUNDLE` or `BUNDLEP`) allows preemptions between threads of the same job but precludes preemptions between jobs. Each task benefits from the advantages of thread-level scheduling by the exclusive use of the processor and shared resources. Since task set specifications may be divided, a specification is feasible when threads of the specification $\hat{\tau}$ may be assigned to tasks such that the posterior task set τ is feasible by EDF-NP.

▶ **Definition 11** (npm-feasible). *A task set specification $\hat{\tau}$ is task-level non-preemptive multi-threaded feasible (*npm-feasible*) if there exists a posterior task set τ of $\hat{\tau}$ such that all multi-threaded jobs scheduled by EDF-NP will always meet their deadlines.*

For the theorems that follow, unless necessary to discriminate between anterior and posterior tasks, the anterior task $\hat{\tau}_i$ will be written τ_i. The sum of the demand of the partial tasks of τ_i for an interval of length t is $\sum_{\tau_k \in \Phi_i} \text{DBF}(\tau_k, t)$.

▶ **Theorem 1** (Minimal Demand of Partial Task Sets Over All Intervals). *For a partial task set Φ_i of an anterior task τ_i with m_i threads, minimizing $\sum_{\tau_k \in \Phi_i} \text{DBF}(\tau_k, d_i)$ minimizes $\sum_{\tau_k \in \Phi_i} \text{DBF}(\tau_k, t)$ for all $t \geq 0$.*

Proof. Provided into two parts, when $t < d_i$ and $t \geq d_i$. The first portion is a simple direct argument. The second portion is by contradiction.

Part 1: When $t < d_i$, $0 = \sum_{\tau_k \in \Phi_i} \text{DBF}(\tau_k, t)$. By definition of the demand bound function (Equation 4) the execution requirement of a task is zero before the first possible deadline. All tasks $\tau_k \in \Phi_i$ share the same relative deadlines $d_k = d_i$ and absolute deadlines because $p_k = p_i$. These follow from the definition of division (Definition 1) and partial tasks (Definition 2). Since $t < d_i$, $\text{DBF}(\tau_k, t) = 0$ for all $\tau_k \in \Phi_i$. Therefore, $\sum_{\tau_k \in \Phi_i} \text{DBF}(\tau_k, t)$ will be minimal (exactly zero) when $t < d_i$, regardless of $\sum_{\tau_k \in \Phi_i} \text{DBF}(\tau_k, d_i)$.

Part 2: When $t \geq d_i$, assume $\sum_{\tau_k \in \Phi_i} \text{DBF}(\tau_k, d_i)$ is minimal and $\sum_{\tau_k \in \Phi_i} \text{DBF}(\tau_k, t)$ is not minimal. Since all partial tasks $\tau_k \in \Phi_i$ share absolute deadlines (as described in Part 1), demand for each task $\text{DBF}(\tau_k, t)$ increases only for values of t that equal absolute deadlines. Furthermore, the execution requirement of every τ_k increases exactly by $c_k(m_k)$ for each

absolute deadline of $\tau_i = \{D_1, D_2, ...\}$:

$$\begin{aligned}
\text{DBF}(\tau_k,D_1) &= 1 \cdot c_k(m_k) \\
\text{DBF}(\tau_k,D_2) &= 2 \cdot c_k(m_k) \\
&... \\
\text{DBF}(\tau_k,D_z) &= z \cdot c_k(m_k)
\end{aligned}$$

Utilizing Property 2, for $t \geq d_i$ and D_z, where D_z is the greatest absolute deadline of τ_i less than or equal to t ($D_z \leq t$):

$$\sum_{\tau_k \in \Phi_i} \text{DBF}(\tau_k,t) = \sum_{\tau_k \in \Phi_i} z \cdot \text{DBF}(\tau_k,d_i) \quad = z \cdot \sum_{\tau_k \in \Phi_i} \text{DBF}(\tau_k,d_i)$$

Because z depends on t (and is completely independent of the division of the partial task set), if $\sum_{\tau_k \in \Phi_i} \text{DBF}(\tau_k,t)$ were not minimal then $\sum_{\tau_k \in \Phi_i} \text{DBF}(\tau_k,d_i)$ could not be minimal, contradicting the assumption.

Combining Parts 1 and 2, when the demand for the partial tasks of τ_i is minimized for the interval d_i, the demand of partial tasks of τ_i is minimized for all intervals of length $t \geq 0$. ◀

▶ **Corollary 2** (Minimal WCET Sum of Φ_i Minimizes Demand Over the Interval d_i)**.** *The demand of Φ_i over the interval d_i is minimized when the sum of WCET of Φ_i is minimized.*

Proof. Following directly from Theorem 1, where the demand over the interval d_i of each task $\tau_k \in \Phi_i$ is given by $\text{DBF}(\tau_k,d_i) = 1 \cdot c_k(m_k) = c_k(m_k)$. Then,

$$\sum_{\tau_k \in \Phi_i} \text{DBF}(\tau_k,d_i) = \sum_{\tau_k \in \Phi_i} c_k(m_k)$$

Thus, minimizing $\sum_{\tau_k \in \Phi_i} c_k(m_k)$ minimizes $\sum_{\tau_k \in \Phi_i} \text{DBF}(\tau_k,d_i)$ ◀

▶ **Corollary 3** (Minimal WCET Sum of Φ_i Minimizes Demand Over all Intervals $t \geq 0$)**.** *The demand of Φ_i over alls interval $t \geq 0$ is minimized when the sum of WCET of Φ_i is minimized.*

Proof. Following directly from Theorem 1 and Corollary 2. ◀

▶ **Definition 12** (Assumptions of Theorem 4)**.** *For the following theorem, there are several assumptions that must be upheld for the result to be valid. These assumptions are consequences of the non-preemptive setting and requirements of the task set specification.*

1. *All tasks τ_i must be characterized by strictly increasing discrete concave WCET function $c_i(m_i)$.*
2. *Any task $\tau_i \in \tau$ where $c_i(m_i) > q_i$ is not schedulable non-preemptively. Consequently, no assignment of m_i may cause $c_i(m_i) > q_i$ or the task set is infeasible.*
3. *The greatest number of threads assigned to a task τ_i such that $c_i(m_i) \leq q_i$ is named $m = \underset{m \in \mathbb{Z}^+}{\text{argmax}} \, (c_i(m) \leq q_i)$.*

▶ **Theorem 4** (Minimal Sum of WCET of Φ_i for any q by DIVIDE)**.** *For an anterior task $\hat{\tau}_i$ and non-preemptive chunk size q,* DIVIDE *will produce a partial task set Φ_i with minimum WCET sum among all possible partial task sets of $\hat{\tau}_i$.*

Proof. To illustrate a contradiction, assume Φ_i returned from DIVIDE does not have the minimal WCET sum for a specific q and task $\hat{\tau}_i$. There must exist a partial task set Φ_k of $\hat{\tau}_i$ that differs, ie. $\Phi_i \neq \Phi_k$ and

$$\sum_{\tau_k \in \Phi_k} c_k(m_k) < \sum_{\tau_j \in \Phi_i} c_j(m_j)$$

By Property 4 of partial tasks created by DIVIDE, Φ_i will have at most one task with less than m threads assigned to it. For Φ_k to differ, it must have at least two tasks with less than m threads assigned to them. Call these two tasks with less than m threads $\tau_x, \tau_y \in \Phi_k$. Select τ_x as the task with the greater number of threads $m_x \geq m_y$.

Consider the impact on $\sum_{\tau_k \in \Phi_k} c_k(m_k)$ of moving one thread of τ_y to τ_x, as the operation of adding the difference of WCET values for $c_x(m_x + 1)$ and $c_y(m_y - 1)$ to the sum.

$$\left(\sum_{\tau_k \in \Phi_k} c_k(m_k) \right) - c_x(m_x) + c_x(m_x + 1) - c_y(m_y) + c_y(m_y - 1)$$

$$= \left(\sum_{\tau_k \in \Phi_k} c_k(m_k) \right) + (c_x(m_x + 1) - c_x(m_x)) - (c_y(m_y) - c_y(m_y - 1))$$

By the concave growth Property 1 and virtue of $m_y \leq m_x$, the quantity $(c_x(m_x + 1) - c_x(m_x))$ is less than or equal to $(c_y(m_y) - c_y(m_y - 1))$ so the difference must be less than or equal to zero. Therefore:

$$\left(\sum_{\tau_k \in \Phi_k} c_k(m_k) \right) + (c_x(m_x + 1) - c_x(m_x)) - (c_y(m_y) - c_y(m_y - 1)) \leq \sum_{\tau_k \in \Phi_k} c_k(m_k)$$

The WCET sum of Φ_k can be reduced by moving one thread of τ_y to τ_x. When $m_x = m$ no more threads may be assigned to τ_x or the system will be infeasible by Definition 12. While there are two (or more) tasks of $\tau_x, \tau_y \in \Phi_k$ with fewer than m threads assigned, moving one thread from τ_y to τ_x will reduce the WCET sum. By repeatedly moving tasks to reduce the WCET sum, Φ_k will satisfy all aspects of Property 4 of partial task sets created by DIVIDE, ie. $\Phi_i = \Phi_k$ after all moves have been completed. This contradicts the assumption that $\Phi_i \neq \Phi_k$ and the relationship of their WCET sums, therefor Φ_i is minimal. ◀

▶ **Theorem 5** (TPJ is Optimal with Respect to npm-feasibility)**.** *For a task set specification $\hat{\tau}$,* TPJ *returns feasible if and only if there exists an npm-feasible posterior task set τ of $\hat{\tau}$.*

Proof. *Forward Direction* (TPJ returns feasible for $\hat{\tau} \implies \exists$ a posterior task set $\tau \mid \tau$ is npm-feasible): The TPJ algorithm returned a posterior task set τ where the infeasibility condition (Equation 9) is never satisfied across intervals of length $0 \leq t \leq T^*(\tau)$ and every job of $\tau_i \in \tau$ executes non-preemptively for $c_i(m_i) \leq q_i$ time units. Therefore, τ is npm-feasible.

Reverse Direction ($\exists$ a posterior task set $\tau \mid \tau$ is npm-feasible $\implies$ TPJ returns feasible for $\hat{\tau}$): For the purpose of demonstrating a contradiction, assume TPJ returns infeasible for an npm-feasible task set $\hat{\tau}$. Name the absolute deadline which TPJ returned infeasibility for D_x from the set ordered deadlines $\{D_1, D_2, ...\}$ and the task which generated D_x, $\hat{\tau}_x$. Name the set of tasks with relative deadlines smaller than $\hat{d}_x$, $\bar{\tau}$.

For any task $\tau_k \in \bar{\tau}$ and partial task set Φ_k of τ_k included in the posterior set τ, the number of tasks and threads assigned to each Φ_k cannot be affected by $\hat{\tau}_x$ due to $\hat{d}_x > d_k$ and Property 3. The combined set of posterior tasks of $\bar{\tau}$ in τ is denoted $\dot{\tau} = \cup_{\tau_k \in \bar{\tau}} \Phi_k$.

There are two cases where TPJ will return infeasible for $\hat{\tau}$, on Line 8 and Line 22. Both illustrate a contradiction with the respect to demand.

Line 8: If TPJ returns infeasible for $\hat{\tau}$ on Line 8 there is insufficient slack q_x to execute any one-thread job of $\hat{\tau}_x$ non-preemptively. Since slack is inversely related to demand, the demand of $\dot{\tau}$ is too great to allow any thread of τ_x as part of a feasible task set.

Line 22: If TPJ returns infeasible for $\hat{\tau}$ on Line 22, there is insufficient supply for Φ_x (the set of partial tasks of $\hat{\tau}_x$). By Corollary 2 and Theorem 4 the demand of Φ_x is minimal over all intervals for the available slack q_x. Due to Property 3 only tasks with shorter relative deadlines i.e. $\dot{\tau}$, can impact the demand of Φ_x by affecting q_x. In this case, the demand of $\dot{\tau}$ is too great for the demand of Φ_x to be included as part of a feasible task set.

By assumption $\hat{\tau}$ is npm-feasible, the infeasibility conditions on Lines 8 and 22 of TPJ indicate the demand of $\dot{\tau}$ is too great. However, TPJ adds each partial set Φ_k to $\dot{\tau}$ in increasing deadline order. By Property 3, every Φ_k added to $\dot{\tau}$ exclusively impacts the demand of larger deadlines. Every Φ_k increases the demand of $\dot{\tau}$ minimally starting with D_1, maximizing the slack available for partial task sets with greater deadlines; thus the demand of $\dot{\tau}$ is minimal and cannot be reduced. For $\hat{\tau}$ to be npm-feasible, there must be another partial task set that reduces $\dot{\tau}$'s demand, which is a direct contradiction. Therefore, TPJ must return feasible. ◀

5 Evaluation

Evaluation [28] of TPJ and the non-preemptive multi-threaded task model presented in this work focuses on the schedulability ratio of synthetic task sets and a case study based upon the evaluation of `BUNDLEP` [29]. The ratio of task set specifications deemed schedulable by TPJ for EDF-NP will be compared to NP-CHUNKS in both limited and fully preemptive settings for EDF. What follows is a description of the parameters to task set specification generation, the prescribed evaluation metrics, and analysis of the results.

5.1 Generating Task Sets

A specified task set τ is generated with four parameters, M the total number of threads of execution, U the target utilization, a maximum growth factor $\mathbb{F}$, and m the maximum number of threads per task. The number of threads M may be one of $\{3, 5, 7, 10, 25, 50, 100\}$ with dependent m values of $\{2, 2, 3, 4, 8, 16, 32\}$. Utilization varies from [0.1, 0.9] and the growth factor varies from [0.1, 0.9] independently by increments of 0.1.

Each task $\tau_i \in \tau$ is assigned m_i threads from a random uniform integer distribution over $[1, m]$, such that the sum of all threads is equal to $M = \sum_{\tau_i \in \tau} m_i$. A task's period p_i is from a uniform integer distribution over [10, 1000]. Utilization u_i of each task τ_i is calculated using the UUniFast(n, U) [9] algorithm, where $n = |\tau|$.

A task's WCET is assigned for m_i threads, $c_i(m_i) = \lceil p_i \cdot U_i \rceil$. Tasks are given a growth factor $\mathbb{F}_i$ in a uniform real distribution over $[0.1, \mathbb{F}]$. The remaining $m_i - 1$ WCET values are determined by substituting $\mathbb{F}_i$ into Equation 3. The relative deadline of τ_i, d_i is taken from a uniform integer distribution over $[\max(c_i(m_i), p_i/2), 1000]$.

For each combination of $(M, m, U, \mathbb{F})$, 1000 task sets specifications are generated. Table 2 summarizes the parameters of task set generation. The smaller values of M are taken from [7] and the dependent m values were selected to avoid one task consuming more than half of the threads in the task set specification (where possible).

Table 2 Task Set Generation Parameters.

U	[0.1, 0.9]	M	{3, 5, 7, 10, 25, 50, 100}
$\mathbb{F}$	[0.1, 0.9]	m	{2, 2, 3, 4, 8, 16, 32}

5.1.1 Applicability of Parameters

To avoid favoring TPJ, the task set generation parameters m and $\mathbb{F}$ were carefully selected. For the threads per task m, a large m favors TPJ. Therefore, no single task my be assigned more than half the total threads: $m \leq \lfloor \frac{M}{2} \rfloor$ (except for $M = 3$).

The growth factor $\mathbb{F}$ is informed by previous results for `BUNDLEP` [29]. In [29], multi-threaded tasks are constructed from the Mälardalen WCET benchmarks [16]. Task analysis in [29] yields growth factors below 0.1 for several benchmarks. A lower bound (0.1) on $\mathbb{F}$ greater than observed values is pessimistic, resulting in less favorable results for TPJ.

5.2 Case Study

`BUNDLEP`'s evaluation covers 18 benchmarks for distinct architecture configurations. An architecture configuration includes the block reload time (BRT), cycles per instruction (CPI), and number of cache lines. One of the least favorable in terms of the analytical benefit of `BUNDLEP` is a BRT of 100, CPI of one, and 32 cache lines. From this configuration, the WCET values and growth factors were extracted, growth factors ranging in the range $[0.08, 3.02]$.[1]

From these results of `BUNDLEP` 1000 task sets with 18 tasks (one per benchmark) and a total 100 threads were generated per utilization target. The utilization target ranged from 0.1 to 1.0 increments of 0.1. Threads were assigned to each task τ_i from a distribution over $m_i \in [2, 8]$. Each tasks utilization, period, and deadline, $c_i(m_i)$ were assigned using the same method as synthetic tasks. The WCET values for fewer threads $1 \leq k < m_i$, were scaled such that the value of $c_i(k)/c_i(m_i)$ remained constant after the $c_i(m_i) = \lceil p_i \cdot U_i \rceil$ assignment.

5.3 Evaluation Metrics

TPJ is compared with the NP-CHUNKS schedulability test in non-preemptive (EDF-NP) and preemptive (EDF-P) settings. The focus of the evaluation is on the non-preemptive setting. The preemptive setting serves as a comparison to alternative scheduling strategies and the theoretical best case. For EDF-P, preemptions incur **no** penalty, CRPD or otherwise. In this highly advantageous setting for EDF-P, TPJ can still produce feasible non-preemptive task sets NP-CHUNKS deems infeasible in a preemptive setting!

To compare schedulability tests, each task set specification $\hat{\tau}$ is provided to TPJ without modification under EDF-NP scheduling. TPJ will transform the task set producing a posterior task set τ if a feasible one exists. A task set specification $\hat{\tau}$ cannot be provided directly to NP-CHUNKS, since NP-CHUNKS has no concept of threads per job.

To be suitable for analysis by NP-CHUNKS, a task set specification $\hat{\tau}$ is transformed into two posterior task sets. The first task set, τ^1 represents single-threaded tasks by including all threads of $\hat{\tau}$ as individual tasks. The second task set, τ^m represents the tasks of $\hat{\tau}$ as indivisible, executing all specified threads without preemption per job. Each task in τ^m benefits from the thread-level scheduler but does not expose the threaded nature of the task to the scheduling algorithm. This is achieved by modifying an anterior task $\hat{\tau}_j$ with $\hat{m}_j > 1$ and $\hat{c}_j(\hat{m}_j)$ to a posterior task τ_j with $m_j = 1$ and $c_j(1) = \hat{c}_j(\hat{m}_j)$.

The NP-CHUNKS schedulability test will produce results for τ^1 and τ^m in both preemptive and non-preemptive settings. For non-preemptive schedulability analysis, each task $\tau_i \in \tau^1$ or τ^m must have a non-preemptive chunk size $q_i \geq c_i(m_i)$. When evaluating preemptive EDF schedulability for τ^1 and τ^m, the results are labeled EDF-P:1 and EDF-P:M respectively. When evaluating non-preemptive EDF schedulability, the results are labeled EDF-NP:1 and

[1] Due to length restrictions the full listing of WECT and growth factors are omitted.

Table 3 Schedulability Test Combinations.

Test	Task Set	EDF-NP	EDF-P
TPJ	$\hat{\tau}$	EDF-TPJ	-
NP-CHUNKS	τ^1	EDF-NP:1	EDF-P:1
	τ^m	EDF-NP:M	EDF-P:M

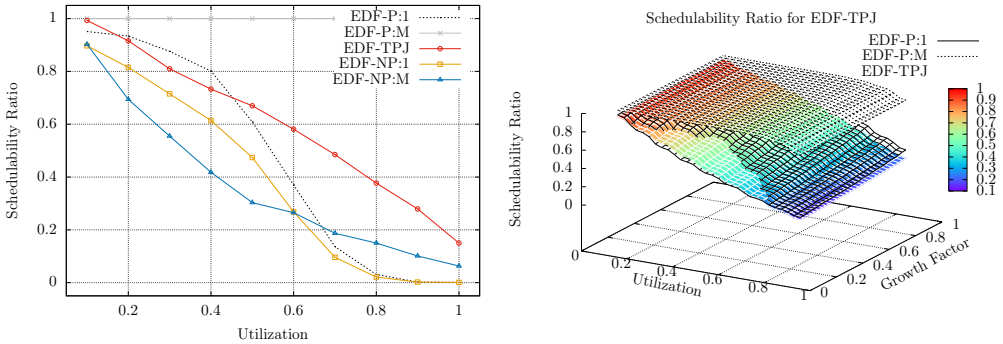
EDF-NP:M. Schedulability results for TPJ under EDF-NP scheduling are labeled EDF-TPJ. Table 3 gives a synopsis of the schedulability tests. Schedulability ratios for each of the combinations are calculated for every $(M, m, U, \mathbb{F})$ configuration.

It must be noted that EDF-P:M is an unrealistic schedulability test. It serves only as a theoretical limit to the benefits of concave growth. Concave growth is a result of scheduling threads of the same job without preemption by another job with a `BUNDLE`-based thread-level scheduler. However, current `BUNDLE` implementations require that an executing task cannot be preempted by a different task. Such a preemption would destroy the cache benefits and analysis of `BUNDLE` scheduling. Analysis of EDF-P:M assumes preemptions between jobs are allowed and have zero cost. It is included as a reference for TPJ's performance, as a ceiling for what is theoretically possible given ideal (but likely impossible) conditions.

As a consequence of transforming multi-threaded task set specifications $\hat{\tau}$ to single-threaded task sets τ^1, some single threaded task sets may not be feasible. One reason for a task set τ^1 to become infeasible is the utilization exceeding one, while τ^m and $\hat{\tau}$ have utilization less than one. In this setting, EDF-TPJ is capable of scheduling task sets that preemptive EDF cannot.

For a task set specification configuration $(M, m, U, \mathbb{F})$, call S the set of all task set specifications $\hat{\tau}$ generated for the configuration. Call s the set of τ^1 task sets transformed from $\hat{\tau} \in S$ such that τ^1 has utilization greater than one. The set s^{TPJ} is the subset of s deemed feasible by the TPJ schedulability test. That is, s^{TPJ} is the set of all tasks TPJ could schedule, yet EDF-P:1 could not (even) when CRPD values are zero.

5.4 Results



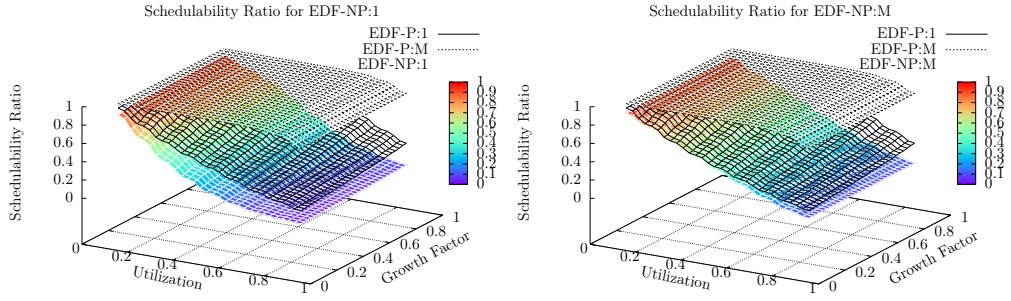
(a) `BUNDLEP` Case Study. **(b)** EDF-TPJ Summary.

Figure 2 Case Study and EDF-TPJ Summary Results.

Schedulability ratios from the `BUNDLEP` case study are given in Figure 2a. For the target architecture and 18 benchmarks, EDF-TPJ consistently outperforms the other non-preemptive algorithms. For preemptive EDF-P:1 (with zero cost preemptions), EDF-TPJ has higher schedulability ratios for the majority of target utilization values. EDF-TPJ's

comparative performance increases with the target utilization. This case study demonstrates the benefit of TPJ to non-preemptive and (potentially) preemptive approaches.

Figures 2b, 3a, and 3b, summarize the results for the synthetic task sets varied by the utilization and growth factor. Within each graph, the schedulability ratios provided by EDF-P:1 and EDF-P:M serve as references. The difference between EDF-P:1 and the subject of the graph illustrate the benefit of preemptive scheduling. Inclusion of EDF-PM highlights the theoretical limit of concave growth to schedulability.



(a) EDF-NP:1 Summary.

(b) EDF-NP:M Summary.

Figure 3 EDF-NP:1 and EDF-NP:M Summary.

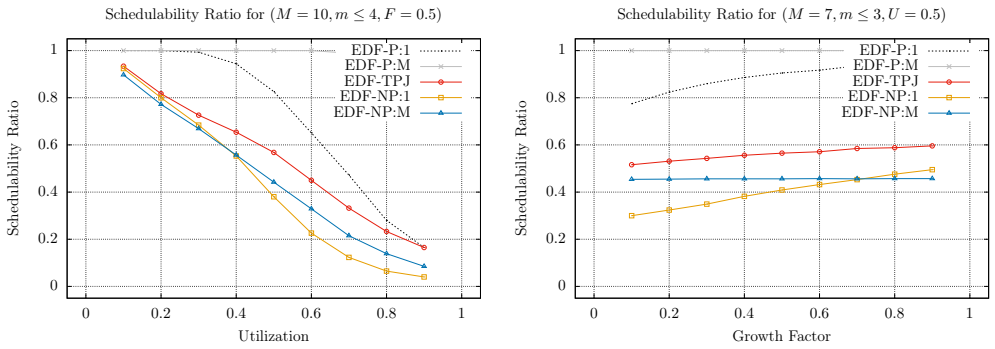
Including EDF-P:1 and EDF-P:M in each of the summary graphs eases the comparison between EDF-NP:1, EDF-NP:M, and EDF-TPJ. Comparing EDF-NP:1 (3a) to EDF-NP:M (3b), illustrates the benefits of the model and scheduling mechanism. EDF-NP:M has a consistently higher schedulability ratio for all utilizations and growth factors. EDF-TPJ (2b) outperforms EDF-NP:M, with higher schedulability ratios for all utilizations and growth factors due to the ability to transform task sets. EDF-TPJ performs best among the non-preemptive tests across all configurations. Additionally, EDF-TPJ is able to schedule task sets deemed infeasible for EDF-P:1.

Table 4 summarizes the infeasible utilization findings for the synthetic tasks. For moderate and larger values of $M(\geq 25)$, the number of infeasible by utilization task sets dominate the specifications. For 25, 50, and 100 total threads, the infeasible by utilization comprise 44, 59, and 74 percent of the task sets respectively, with EDF-TPJ finding 25, 34, and 45 percent feasible. This illustrates the large potential of the proposed model, in conjunction with concave growth WCET functions of thread-level schedulers (e.g. `BUNDLE` and `BUNDLEP`).

Table 4 $U > 1$ Feasibility.

(M, m)	$(3, 2)$	$(5, 2)$	$(7, 3)$	$(10, 4)$	$(25, 8)$	$(50, 16)$	$(100, 32)$	Total
$\lvert S \rvert$	81000	81000	81000	81000	81000	81000	81000	567000
$\lvert s \rvert$	3131	4973	11744	18689	36565	49147	59412	183661
$\lvert s^{\text{TPJ}} \rvert$	465	291	1437	3065	9426	16912	25832	57428

There are two noteworthy trends within the schedulability results. The simpler of the two is the relationship between utilization and schedulability ratio for a fixed growth factor. Figure 4a illustrates the trend common among $M \leq 10$ total threads. The trend for preemptive and non-preemptive schedulability tests when utilization increases is for the schedulability ratio to decrease. However, EDF-TPJ always outperforms the other non-preemptive tests.



(a) $(M, m, U, \mathbb{F}) = (10, 4, *, 0.5)$.

(b) $(M, m, U, \mathbb{F}) = (7, 3, 0.5, *)$.

Figure 4 $M \leq 10$ Performance.

The second trend is slightly more complex. Figure 4b was selected for the smallest M and U values with visually distinct plots per schedulability test. The growth factor and the schedulability ratio are correlated. As the growth factor increases, so does the schedulability ratio. This is due to the utilization being held constant. When the growth factor is small, the WCET of the first thread of each task is larger. Larger WCET values are harder to schedule non-preemptively.

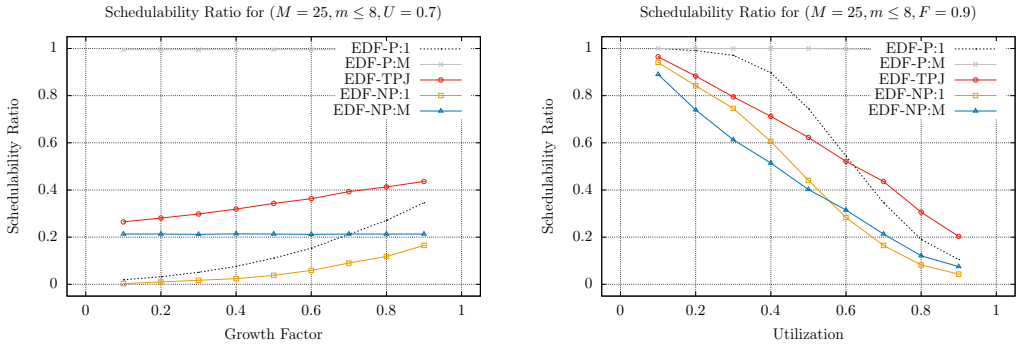


Figure 5 $M > 10$ EDF-TPJ Performance Above EDF-P:1.

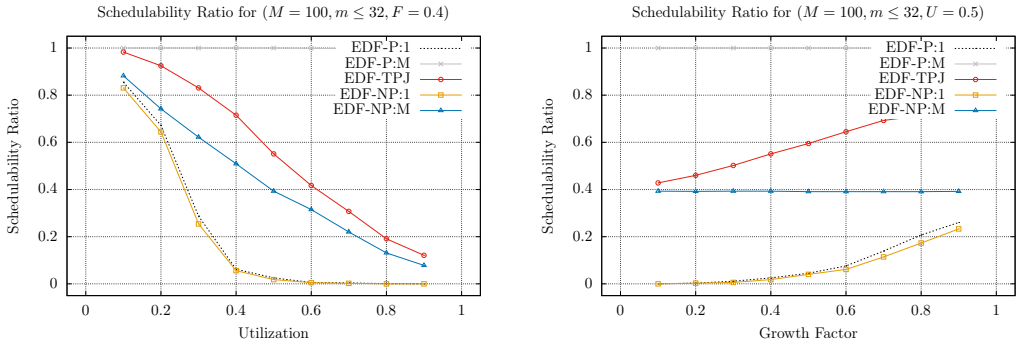


Figure 6 $M = 100$ EDF-TPJ Performance.

As M increases beyond 10 total threads, the number of infeasible by utilization task sets s grows. This contributes to the schedulability ratio of EDF-TPJ surpassing EDF-P:1 for threshold utilization and growth factor values. For $M = 25$, the threshold of utilization is between $[0.6, 0.7]$ shown in Figure 5.

For $M = 100$ and $\mathbb{F} \leq 0.4$, EDF-TPJ outperforms EDF-P:1. Figure 6 highlights the advantage of EDF-TPJ compared to EDF-P:1 by virtue of concave growth. It also highlights the benefit of dividing tasks, as the performance of EDF-NP:M is always below EDF-TPJ.

The comparative performance of EDF-TPJ is at its lowest for $M < 10$ threads and $U > .4$ utilization. In these ranges EDF-TPJ maintains the highest schedulability ratio among the non-preemptive methods, but the ratio is closer to EDF-NP:M or EDF-NP:1 than EDF-P:1. This suggests, the decrease in EDF-TPJ's performance is more likely due to the non-preemptive setting combined with larger WCET values for individual threads.

6 Conclusion

Motivation for this work stemmed from BUNDLE-based thread-level schedulers limitation of a single task and single job. The primary goal was to create a multi-task scheduling technique and schedulability test for those BUNDLE-based thread-level schedulers which leverages without decreasing the inter-thread cache benefit.

In addition to achieving the primary goal, the scheduling technique and schedulability test developed for the multi-task BUNDLE-based scheduler can be applied to any thread level scheduler with strictly increasing discrete concave WCET functions. This allows any compatible thread-level scheduling technique to benefit from the TPJ approach developed in this work. As a non-preemptive multi-threaded schedulability test TPJ is optimal with respect to npm-feasibility, always producing a feasible task set if one is schedulable by EDF-NP.

For future work, the primary focus is upon a fully or limited preemption scheduling algorithm that permits the inter-thread cache benefit of BUNDLE-based schedulers and other schedulers characterized by concave growth to retain their thread-level scheduling benefits.

References

1. S. Altmeyer, R. Davis, and C. Maiza. Improved cache related pre-emption delay aware response time analysis for fixed priority pre-emptive systems. *Real Time Systems*, 48(5), 2012.
2. S. Altmeyer, R. I. Davis, and C. Maiza. Cache Related Pre-emption Delay Aware Response Time Analysis for Fixed Priority Pre-emptive Systems. In *IEEE Real-Time Systems Symposium*, pages 261–271, November 2011. doi:10.1109/RTSS.2011.31.
3. Sebastian Altmeyer and Claire Maiza Burguière. Cache-related Preemption Delay via Useful Cache Blocks: Survey and Redefinition. *Journal of Systems Architecture*, 57(7):707–719, August 2011. doi:10.1016/j.sysarc.2010.08.006.
4. S. Bak, G. Yao, R. Pellizzoni, and M. Caccamo. Memory-Aware Scheduling of Multicore Task Sets for Real-Time Systems. In *IEEE International Conference on Embedded and Real-Time Computing Systems and Applications*, pages 300–309, August 2012. doi:10.1109/RTCSA.2012.48.
5. S. K. Baruah, A. K. Mok, and L. E. Rosier. Preemptively scheduling hard-real-time sporadic tasks on one processor. In *[1990] Proceedings 11th Real-Time Systems Symposium*, pages 182–190, December 1990. doi:10.1109/REAL.1990.128746.
6. Sanjoy Baruah. The limited-preemption uniprocessor scheduling of sporadic task systems. In *17th Euromicro Conference on Real-Time Systems (ECRTS'05)*, pages 137–144, July 2005. doi:10.1109/ECRTS.2005.32.
7. M. Bertogna and S. Baruah. Limited Preemption EDF Scheduling of Sporadic Task Systems. *IEEE Transactions on Industrial Informatics*, 6(4):579–591, November 2010. doi:10.1109/TII.2010.2049654.
8. M. Bertogna, O. Xhani, M. Marinoni, F. Esposito, and G. Buttazzo. Optimal Selection of Preemption Points to Minimize Preemption Overhead. In *Proceedings of the Euromicro Conference on Real-Time Systems*, pages 217–227, July 2011. doi:10.1109/ECRTS.2011.28.

9 E. Bini and G. C. Buttazzo. Biasing effects in schedulability measures. In *Proceedings. 16th Euromicro Conference on Real-Time Systems, 2004. ECRTS 2004.*, pages 196–203, July 2004. doi:10.1109/EMRTS.2004.1311021.

10 R. Bril, S. Altmeyer, M. van den Heuvel, R. Davis, and M. Behnam. Fixed priority scheduling with pre-emption thresholds and cache-related pre-emption delays: integrated analysis and evaluation. *Real-Time Systems*, 53(4):403–466, July 2017.

11 A. Burns. *Advances in Real-Time Systems*, chapter Preemptive priority-based scheduling: an appropriate engineering approach, pages 225–248. Prentice Hall, Inc., 1995.

12 Giorgio C. Buttazzo. *Hard Real-Time Computing Systems: Predictable Scheduling Algorithms and Applications*. Springer Publishing Company, Incorporated, 3rd edition, 2011.

13 John Michael Calandrino. *On the Design and Implementation of a Cache-aware Soft Real-time Scheduler for Multicore Platforms*. PhD thesis, University of North Carolina at Chapel Hill, Chapel Hill, NC, USA, 2009.

14 J. Cavicchio, C. Tessler, and N. Fisher. Minimizing Cache Overhead via Loaded Cache Blocks and Preemption Placement. In *Proceedings of the Euromicro Conference on Real-Time Systems*, 2015.

15 Laurent George, Nicolas Rivierre, and Marco Spuri. Preemptive and Non-Preemptive Real-Time UniProcessor Scheduling. Research Report RR-2966, INRIA, 1996. Projet REFLECS. URL: https://hal.inria.fr/inria-00073732.

16 Jan Gustafsson, Adam Betts, Andreas Ermedahl, and Björn Lisper. The Mälardalen WCET Benchmarks: Past, Present And Future. In *International Workshop on Worst-Case Execution Time Analysis*, volume 15, pages 136–146, Dagstuhl, Germany, 2010. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

17 C.-G. Lee, J. Hahn, S.L. Min, R. Ha, S. Hong, C.Y. Park, M. Lee, and C.S. Kim. Analysis of cache-related preemption delay in fixed-priority preemptive scheduling. *IEEE Transactions on Computers*, 47(6):700–713, 1998.

18 C. L. Liu and James W. Layland. Scheduling Algorithms for Multiprogramming in a Hard-Real-Time Environment. *J. ACM*, 20(1):46–61, January 1973. doi:10.1145/321738.321743.

19 J. M. Marinho, V. Nelis, S.M. Petters, and I. Puaut. An Improved Preemption Delay Upper Bound for Floating Non-preemptive Region. In *Proceedings of IEEE International Symposium on Industrial Embedded Systems*, 2012.

20 H. S. Negi, T. Mitra, and A. Roychoudhury. Accurate Estimation of Cache Related Preemption Delay. In *Proceedings of IEEE/ACM/IFIP International Conference on Hardware/ Software Codesign and Systems Synthesis (CODES)*, 2003.

21 R. Pellizzoni, E. Betti, S. Bak, G. Yao, J. Criswell, M. Caccamo, and R. Kegley. A Predictable Execution Model for COTS-Based Embedded Systems. In *IEEE Real-Time and Embedded Technology and Applications Symposium*, pages 269–279, April 2011. doi:10.1109/RTAS.2011.33.

22 B. Peng, N. Fisher, and M. Bertogna. Explicit Preemption Placement for Real-Time Conditional Code. In *Proceedings of Euromicro Conference on Real-Time Systems*, 2014.

23 J. Simonson and J.H. Patel. Use of preferred preemption points in cache based real-time systems. In *Proceedings of IEEE International Computer Performance and Dependability Symposium*, 1995.

24 J. Staschulat and R. Ernst. Scalable Precision Cache Analysis for Real-Time Software. *ACM Transactions on Embedded Computing Systems (TECS)*, 6(4), September 2005.

25 Y. Tan and V. Mooney. Integrated intra- and inter-task cache analysis for preemptive multi-tasking real-time systems. In *Proceedings of International Workshop on Software and Compilers for Embedded Systems (SCOPES)*, 2004.

26 C. Tessler and N. Fisher. BUNDLE: Real-Time Multi-threaded Scheduling to Reduce Cache Contention. In *2016 IEEE Real-Time Systems Symposium (RTSS)*, pages 279–290, November 2016. doi:10.1109/RTSS.2016.035.

27 C. Tessler and N. Fisher. BUNDLEP: Prioritizing Conflict Free Regions in Multi-threaded Programs to Improve Cache Reuse. In *2018 IEEE Real-Time Systems Symposium (RTSS)*, pages 325–337, December 2018. doi:10.1109/RTSS.2018.00048.

28 Corey Tessler. NPM-BUNDLE Artifacts, 2019. URL: http://www.cs.wayne.edu/~fh3227/npm-bundle/.

29 Corey Tessler and Nathan Fisher. BUNDLEP: prioritizing conflict free regions in multi-threaded programs to improve cache reuse - extended results and technical report. *CoRR*, abs/1805.12041, 2018. arXiv:1805.12041.

30 Henrik Theiling, Christian Ferdinand, and Reinhard Wilhelm. Fast and Precise WCET Prediction by Separated Cache and Path Analyses. *Real-Time Systems*, 18(2):157–179, May 2000. doi:10.1023/A:1008141130870.

31 H. Tomiyama and N. D. Dutt. Program Path Analysis to Bound Cache-Related Preemption Delay in Preemptive Real-Time Systems. In *Proceedings of the Eighth International Workshop on Hardware/Software Codesign (CODES)*, 2000.

32 Y. Wang and M. Saksena. Scheduling fixed-priority tasks with preemption threshold. In *Proceedings of the International Conference on Real Time Computing Systems and Applications*, 1999.

33 B. C. Ward, J. L. Herman, C. J. Kenna, and J. H. Anderson. Making Shared Caches More Predictable on Multicore Platforms. In *Euromicro Conference on Real-Time Systems*, pages 157–167, July 2013. doi:10.1109/ECRTS.2013.26.

34 Reinhard Wilhelm, Jakob Engblom, Andreas Ermedahl, Niklas Holsti, Stephan Thesing, David Whalley, Guillem Bernat, Christian Ferdinand, Reinhold Heckmann, Tulika Mitra, Frank Mueller, Isabelle Puaut, Peter Puschner, Jan Staschulat, and Per Stenström. The Worst-case Execution-time Problem – Overview of Methods and Survey of Tools. *ACM Trans. Embed. Comput. Syst.*, 7(3):36:1–36:53, May 2008. doi:10.1145/1347375.1347389.