Algebraic Optimization of Binary Spatially Coupled Measurement Matrices for Interval Passing

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***Abstract*—We consider binary spatially coupled (SC) low density measurement matrices for low complexity reconstruction of sparse signals via the interval passing algorithm (IPA). The IPA is known to fail due to the presence of harmful sub-structures in the Tanner graph of a binary sparse measurement matrix, so called termatiko sets. In this work we construct array-based (AB) SC sparse measurement matrices via algebraic lifts of graphs, such that the number of termatiko sets in the Tanner graph is minimized. To this end, we show for the column-weight-three case that the most critical termatiko sets can be removed by eliminating all length-12 cycles associated with the Tanner graph, via algebraic lifting. As a consequence, IPA-based reconstruction with SC measurement matrices is able to provide an almost error free reconstruction for significantly denser signal vectors compared to uncoupled AB LDPC measurement matrices.**

I. Introduction

Compressed sensing [1], [2] is a tool for estimating a sparse signal $\boldsymbol{x} \in \mathbb{R}^n$ of sparsity order k from a compressed version of the signal $\boldsymbol{y} \in \mathbb{R}^m$, where $k \ll n$ and $m \ll n$. The compressed signal can be obtained by taking m random linear projections of the original signal via the operation $\boldsymbol{y} = A\boldsymbol{x}$, where $A \in \mathbb{R}^{m \times n}$ is an $m \times n$ measurement matrix.

A straightforward way for reconstructing the signal is to find a vector $\hat{\boldsymbol{x}}$ with the smallest l_0 norm. However, as its complexity is NP-hard, this approach is rendered infeasible for most practical applications [2]. A more efficient approach based on linear programming (LP), called Basis Pursuit, has been proposed in [3], which however is still too complex for applications that require fast reconstruction. To overcome these complexity issues, message passing schemes such as verification decoding and iterative thresholding algorithms have been proposed for reconstructing compressed signals [4], [5]. An improved messaging passing algorithm known as Approximate Message Passing (AMP) is proposed in [6], which has an identical sparsity to sampling ratio trade-off as LP, albeit at a much lower computational complexity.

The interval-passing algorithm (IPA) was first proposed in [7] for both binary and non-negative real measurement matrices. For measurement matrices derived from parity check matrices of LDPC codes, the IPA is known to fail due to the presence of stopping sets. In particular, in [8] it is shown that if the Tanner graph associated with the support of a signal $\boldsymbol{x}$ contains a non-empty stopping set, then the IPA fails to fully recover $\boldsymbol{x}$, but some of the samples inside these sets can be recovered. In [9] a complete graphical description of harmful substructures causing a recovery failure, the so called termatiko sets, is provided. In particular, if the Tanner graph associated with the support of $\boldsymbol{x}$ contains a termatiko set, then the IPA completely fails to recover the signal.

In this work we are mainly interested in the reconstruction performance of array-based (AB) spatially coupled (SC) measurement matrices, obtained by coupling regular AB LDPC code-based measurement matrices. Note that AB SC LDPC codes can be constructed via an edge-spreading process applied to a base Tanner graph of the LDPC block code (BC), yielding an SC protograph. Recently, general edge-spreading schemes [10] have been proposed as an extension of the widely used cutting vector approach [11] for constructing SC codes. Additionally, [10] considers the design of generalized cutting vectors with the objective of maximizing the minimum distance of the corresponding SC protograph, thus also maximizing the size of the smallest stopping set in the Tanner graph of the code [12]. In [13] we have proposed a new algebraic lifting strategy for constructing AB SC LDPC codes, which outperforms existing schemes in terms of reducing critical substructures in the Tanner graph of the AB SC code.

Also, it is known that AB block measurement matrices are able to outperform Gaussian measurement matrices under AMP decoding [14]. Further, in [15] it is shown that SC LDPC measurement matrices obtained from randomly generated regular LDPC BCs outperform uncoupled measurement matrices under verification decoding. However, to the best of our knowledge, the use of binary AB SC LDPC code-based measurement matrices under IPA reconstruction has not been studied so far. In particular, we propose to construct binary SC AB measurement matrices such that the number of both size-three and size-six termatiko sets in the underlying Tanner graph is minimized. As one of our main results we show that for the column-weight-three AB case, these termatiko sets can be removed efficiently by eliminating length-12 cycles in the Tanner graph. As a consequence, IPA-based reconstruction in conjunction with binary SC LDPC code based measurement matrices is able to provide a low complexity, almost error free reconstruction for significantly denser signal vectors compared to uncoupled AB LDPC based measurement matrices.

II. Preliminaries

A. Algebraic lifting

Let the Tanner graph associated to a $m \times n$ binary matrix A be represented by $G = (V \cup C, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ is a set of variable nodes (VNs), $C = \{c_1, c_2, \ldots, c_m\}$ is a set of check nodes (CNs), and $E = \{(v_i, c_j) | v_i \in V, c_j \in C, A(j, i) = 1\}$ is the set of edges connecting v_i to c_j, for $i = \{1, \ldots, m\}$ and $j = \{1, \ldots, n\}$. We also denote the set of neighbors for each node v_i and c_j as $\mathcal{N}(v_i) = \{c_j \in C | (v_i, c_j) \in E\}$ and $\mathcal{N}(c_j) = \{v_i \in V | (v_i, c_j) \in E\}$, respectively. In general, a *degree J lift* of G is a graph $\hat{G}$ with VN set $\hat{V} = \{v_{1_1}, \ldots, v_{1_J}, \ldots, v_{n_1}, \ldots, v_{n_J}\}$ of size nJ and CN set $\hat{C} = \{c_{1_1}, \ldots, c_{1_J}, \ldots, c_{m_1}, \ldots, c_{m_J}\}$ of size mJ and for each $e \in E$, if $e = (v_i, c_j)$ in G, then there are J edges from $\{v_{i_1}, \ldots, v_{i_J}\}$ to $\{c_{j_1}, \ldots, c_{j_J}\}$ in $\hat{G}$ in a one-to-one mapping. The graph $\hat{G}$ can be obtained algebraically by assigning permutations to each of the edges in G so that if $e = (v_i, c_j)$ is assigned a permutation $\tau(k) \in \{1, \ldots, J\}$, the corresponding edges in $\hat{G}$ are $(v_{i_k}, c_{j_{\tau(k)}})$ for $1 \leq k \leq J$.

The protograph approach to the construction of SC LDPC codes involves the base Tanner graph with a parity check matrix represented as $H(\gamma, p)$, where p is odd and $p \geq \gamma$. In case of AB codes, this matrix is given as

$$H(\gamma, p) = \begin{bmatrix} I & I & I & \cdots & I \\ I & \sigma & \sigma^2 & \cdots & \sigma^{p-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ I & \sigma^{\gamma-1} & \sigma^{2(\gamma-1)} & \cdots & \sigma^{(\gamma-1)(p-1)} \end{bmatrix},$$

where I and σ^z for $z \in \{1, \ldots, 2(p-1)\}$ are identity and permutation matrices, resp., of dimension $p \times p$. This matrix can also be considered as a 2-D array of submatrices where each row (column) of matrices denotes a row (column) *group* with p column groups and γ row groups in total. A SC protograph is then obtained from $H(\gamma, p)$ via *edge-spreading*. The idea is to split $H(\gamma, p)$ into a sum of $m+1$ matrices of the same dimension as $H(\gamma, p) = H_0 + H_1 +, \ldots, H_m$, where m represents the memory of the code. These matrices are arranged as

$$H(\gamma, p, L) = \begin{bmatrix} H_0 & & & & \\ H_1 & H_0 & & & \\ & \ddots & \ddots & & \\ & & H_m & \cdots & H_0 \\ & & & \ddots & \vdots \\ & & & & H_m \end{bmatrix}$$

to form the parity-check matrix of a terminated SC protograph $H(\gamma, p, L) \in \mathbb{F}_2^{\gamma(L+1)p \times Lp^2}$, where L is the coupling length and $\mathbb{F}_2$ is the binary field. The final SC LDPC measurement matrix $H(\gamma, p, L, J) \in \mathbb{F}_2^{\gamma(L+1)Jp \times LJp^2}$ is then obtained by a *terminal lift* of $H(\gamma, p, L)$, where J is the terminal lifting parameter.

For algebraic lifting, let τ_L^κ be a $L \times L$ permutation matrix obtained by left shifting the identity permutation by an amount of κ, where $0 \leq \kappa \leq m$. Then, the SC protograph corresponding to $H(\gamma, p, L)$ can alternatively be constructed by lifting each of the edges of the base protograph by a τ_L^κ matrix; this is equivalent to replacing the non-zero entires of $H(\gamma, p)$ by τ_L^κ, and the zero entries by all-zero matrices of the same size, respectively. In the same way, the check matrix $H(\gamma, p, L, J)$ of the final SC LDPC code can be obtained by lifting each of the edges of the SC protograph by any $J \times J$ permutation matrix [13].

B. Compressed sensing and the IPA

Let $\boldsymbol{x} \in \mathbb{R}^n$ be an n dimensional k-sparse signal (which means it has at most k nonzero entries). We consider the recovery of $\boldsymbol{x}$ from measurements $\boldsymbol{y} = A\boldsymbol{x} \in \mathbb{R}^m$, where $m \ll n$ and $k \ll n$, and A is the $m \times n$ measurement matrix. The IPA denotes an iterative algorithm IPA(y, A) to reconstruct a nonnegative real signal $\boldsymbol{x} \in \mathbb{R}^n$ from a measurement vector $\boldsymbol{y}$. It has been stated in [9] that the IPA reconstruction performance is independent of whether binary or non-negative measurement matrices A and signal vectors $\boldsymbol{x}$ are used. Therefore, without loss of generality we consider $A \in \mathbb{F}_2^{\times n}$ and $\boldsymbol{x} \in \mathbb{F}_2^n$. We also denote the elements of $\boldsymbol{x}$ and $\boldsymbol{y}$ as $\boldsymbol{x} = [x(v_1), \ldots, x(v_n)]^T$ and $\boldsymbol{y} = [y(c_1), \ldots, y(c_m)]^T$, respectively. Recovery takes place by iteratively exchanging messages on the Tanner graph of A, where the measurement nodes will be denoted as VNs and the function nodes as CNs in the following.

C. Stopping sets in AB measurement matrices

Stopping sets are harmful structures in the Tanner graph of A that can cause the IPA to fail. In this work, we first analyze the structure of minimum stopping sets in AB SC LDPC measurement matrices which are the most harmful to the IPA decoder.

Definition 1 ([16])**.** *A stopping set $S(M) = \{v_1, \ldots, v_M\} \subset V$ is a non-empty subset of the set of M variable nodes V such that all neighbors of $S(M)$ are connected to it at least twice.*

In the following, we focus on AB parity check matrix with a column weight of $\gamma = 3$ for the sake of simplicity[1]. For $\gamma = 3$, each VN has 3 neighbors, so there must be $3M$ edges connected to $S(M)$. The Tanner graph of an AB code also consists of *cycles* with the following structural properties.

Remark 1. *For $\gamma = 3$, a cycle of length ℓ in an AB code consists of ℓ edges that are connected to $\ell/2$ CNs (resp. VNs) of degree 2 with respect to the VNs (resp., CNs) of the cycle. By the pattern consistency condition [17] each CN associated to the cycle is connected to a distinct pair of VNs of the cycle.*

We now address the structure of some *small* stopping sets of size ≤ 12. Let $\mathcal{N}(S(M))$ be the set of all neighboring CNs of $S(M)$, and let $e(S(M))$ be the set of all the edges connecting $S(M)$ to $\mathcal{N}(S(M))$.

Remark 2. *For $\gamma = 3$, the minimum stopping set $S(6)$ in an AB code consists of 6 degree 3 VNs that are connected via 18 edges to 9 CNs of degree 2 with respect to the VNs in $S(6)$.*

Since there are no 4-cycles in AB codes [17], a pair of neighbors of $S(6)$ cannot be connected to the same pair of VNs in $S(6)$. In $e(S(6))$, nine CNs are connected to nine VN pairs via 18 edges. There are $\binom{6}{2} = 15$ possible pairs of VNs in $S(6)$. However, we only consider nine pairs such that each VN of $S(6)$ appears in exactly three pairs out of those nine (because the VN degree is 3). This also implies that there exists two pairs out of those nine that have a common VN, and that is true for all VNs in $S(6)$. Thus, we obtain the following lemma.

Lemma 1. *For $\gamma = 3$, the minimum stopping set $S(6)$ in an AB code consists of six VNs of degree 3 that are connected via 18 edges to nine CNs of degree 2 with respect to the VNs in $S(6)$. Here, a pair of neighboring CNs of $S(6)$, denoted by*

[1]Note that in the following "AB codes" refers to AB codes with $\gamma = 3$.

c and c', respectively, that have a common neighbor v_k, must be connected to three VNs $\{v_i, v_k, v_q\} \in S(6)$ via four edges (c, v_i), (c, v_k), (c', v_k) and (c', v_q), where $i \neq q \neq k$.

Corollary 1. *There are two sets of VNs $V' \subset S(6)$ and $\hat{V} = S(6) \setminus V'$, where $|V'| = |\hat{V}| = 3$, that are connected to all CNs in $\mathcal{N}(S(6))$.*

III. Termatiko Sets in AB Measurement Matrices

A. Preliminaries

In [9] it is shown that stopping sets may not cause a total failure of the IPA. Under some conditions, some of the non-zero values of the signal can be recovered even if the VNs in the Tanner graph of the measurement matrix corresponding to the non-zero values are associated with a stopping set. However, there are sets of VNs inside a stopping set, termed termatiko sets, that cause a total failure of the IPA if the support of $\boldsymbol{x}$, $\mathrm{supp}(\boldsymbol{x}) = \{v \in V : x(v) \in \boldsymbol{x}, x(v) \neq 0\}$, is a termatiko set.

Definition 2 ([9])**.** *A subset $T_{w,M} \subseteq S(M)$ is a termatiko set of size $w \leq M$ if and only if the function IPA$(A\boldsymbol{x}_{T_{w,M}}, A)$ returns $\hat{\boldsymbol{x}} = \boldsymbol{0}$, where $\boldsymbol{x}_{T_{w,M}}$ is a binary vector with $\mathrm{supp}(\boldsymbol{x}_{T_{w,M}}) = T_{w,M}$.*

We denote by N the set of CNs connected to $T_{w,M}$. Moreover, we denote by $\hat{S} = \{v \in V \setminus T_{w,M} : \mathcal{N}_N(v) = \mathcal{N}(v)\}$ the set of remaining VNs outside $T_{w,M}$ connected only to N, where $\mathcal{N}_N(v)$ is the set of neighbors of v in N. $T_{w,M}$ exists only if for each $c \in N$ one of the following conditions is true [9]:

(i) A CN $c \in N$ is connected to $\hat{S}$.

(ii) If $c \in N$ is not connected to $\hat{S}$, then it must have at least two neighbors belonging to set $T_{w,M}$ satisfying the following constraint: all CNs $c' \in N$ connected to these neighbors must have at least two neighbors in $T_{w,M}$.

B. Minimum termatiko sets

In the following we analyze the structure of minimum termatiko sets, residing in minimum stopping sets, which have the smallest possible value of $w > 0$. For AB measurement matrices with $\gamma = 3$, this minimum termatiko set of this type is denoted as $T_{3,6}$.

Proposition 1 (see also [18])**.** *A set of three VNs in $S(6)$ constitutes a $T_{3,6}$ set if it is connected to all nine CNs in $\mathcal{N}(S(6))$. Also, a $S(6)$ stopping set consists of two $T_{3,6}$ sets.*

Proof: As in [9] we assume without loss of generality that $V = T_{w,M} \cup \hat{S}$. Recall from Corollary 1 that there exists two sets of VNs $V' \subset S(6)$ and $\hat{V} = S(6) \setminus V'$, where $|V'| = |\hat{V}| = 3$, and each of them are connected to all nine CNs in $\mathcal{N}(S(6))$. Thus, according to Condition (i) above we obtain that $T_{3,6} = V', \hat{S} = \hat{V}$, and $T_{3,6} = \hat{V}, \hat{S} = V'$, respectively, and $N = \mathcal{N}(S(6))$.

On the other hand, assume now that a set of three VNs $\tilde{V} \subset S(6)$ is not equal to V' or $\hat{V}$. Then, according to the structure of $e(S(6))$, the total number of CNs connected to $\tilde{V}$ is less than nine. If we assume for a moment that $\tilde{V} = T_{3,6}$, then this would imply that $|N| < |\mathcal{N}(S(6))| = 9$; in other words $N \neq \mathcal{N}(S(6))$. Then, due to the properties of $e(S(6))$, there would be less than nine CNs in the set $\mathcal{N}(S(6)) \setminus N$ that has neighbors in the set $\tilde{S} = S(6) \setminus \tilde{V} = S(6) \setminus T_{3,6}$. However, this also implies that $\tilde{S} \neq \hat{S}$. Consequently, $\tilde{V} \neq T_{3,6}$. ■

Fig. 1 shows that a set of VNs $\{v_2, v_3, v_5\}$ is not connected to all the neighbors of $S(6)$, hence it cannot form a $T_{3,6}$ set, since if it did, it would imply that $\hat{S} = \{v_1, v_4, v_6\}$. This contradicts the definition of $\hat{S}$ as the set $\{v_1, v_4, v_6\}$ is not connected to all green CNs (which are neighbors of the candidate termatiko set $\{v_2, v_3, v_5\}$ in this example).

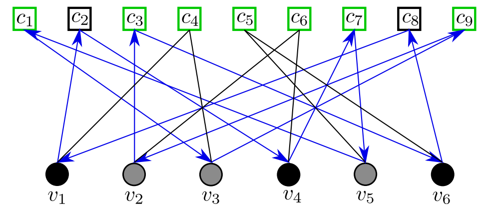

Fig. 1: Example of a case where a set of VNs $\{v_2, v_3, v_5\}$ cannot form a termatiko set in $S(6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. The underlying 12-cycle is shown in blue. The VNs $\{v_1, v_2, v_5\}$ are connected to all neighbors of $S(6)$ and therefore represent a $T_{3,6}$ termatiko set with $\hat{S} = \{v_3, v_4, v_6\}$.

Remark 3. *From the proof of Proposition 1 we have seen that a $T_{3,6}$ set can exist in $S(6)$ in two possible configurations: $T_{3,6} = V', \hat{S} = \tilde{V}$, and $T_{3,6} = \tilde{V}, \hat{S} = V'$. In other words, both V' and $\tilde{V}$ are termatiko sets. The fact that $V' \cup \hat{V} = S(6)$ satisfies Condition (ii) above with the set $\hat{S}$ being the empty set implies that $S(6)$ is also a $T_{6,6}$ termatiko set.*

Lemma 2. *A $T_{6,6}$ set contains at least two 12-cycles.*

Proof: Consider the Tanner graph of an AB code with m CNs and n VNs. Let $V_1 : \{v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_5}, v_{i_6}\} \subset V$ and $\hat{C} : \{c_{j_1}, c_{j_2}, \ldots, c_{j_9}\} \subset C$, where $j_k \in \{1, 2, \ldots, m\}$, $k \in \{1, 2, \ldots, 9\}$ and $i_\ell \in \{1, 2, \ldots, n\}$, $\ell \in \{1, 2, \ldots, 6\}$. We split $\hat{C}$ into three subsets, C_1, C_2 and C_3, respectively, where $C_1 : \{c_{j_1}, c_{j_2}, c_{j_3}\}$, $C_2 : \{c_{j_4}, c_{j_5}, c_{j_6}\}$, and $C_3 : \{c_{j_7}, c_{j_8}, c_{j_9}\}$. We now establish a condition under which the VNs in V_1 connected to CNs in set $C_1 \cup C_2$ are associated to a 12-cycle. The six edges connecting V_1 to C_1 and C_2, respectively, are denoted as e_1 and e_2, respectively. Next, we establish a condition under which the VNs in V_1 connected to CNs in set $C_1 \cup C_3$ are associated to another 12-cycle, where the six edges connecting V_1 to C_3 are denoted as e_3. Finally, we show that under these conditions $e(S(6)) = e(T_{6,6}) = e_1 \cup e_2 \cup e_3$ by invoking Lemma 1. Further details are omitted in the interest of space. ■

C. Other termatiko sets associated to 12-cycles

Remark 4. *In the same way as above we can show that an $S(8)$ stopping set contains a 12-cycle whose VNs form a $T_{6,8}$, and that an $S(12)$ stopping set contains a 12-cycle whose VNs form a $T_{6,12}$, respectively. Details are omitted due to space constraints. Since $|\mathcal{N}(T_{6,8})| \neq |\mathcal{N}(S(6)|$ we can conclude that $T_{6,8} \neq S(6)$. Likewise, $T_{6,12} \neq S(6)$.*

D. Eliminating small termatiko sets via algebraic lifting

In the algebraic lifting process described in Section II-A, a ℓ-cycle can be broken by the lift if we ensure that the net permutation, which is the product of the oriented edge labels, assigned to its edges is not identical to the identity permutation. Let the assignments to the edges of a ℓ-cycle be $\tau_L^{\kappa_1}, \ldots, \tau_L^{\kappa_\ell}$, where τ_L^{κ} is a permutation matrix as discussed in Section II-A. Without loss of generality, then, the net permutation of the cycle is given by $\tau_L^{\sum_{i=1}^{\ell}(-1)^{i+1}k_i}$. This becomes the identity permutation only when

$$\sum_{i=1}^{\ell}(-1)^{i+1}k_i = 0, \tag{1}$$

where $0 \le k_i \le m$. For example, for $\ell = 12$, a 12-cycle will be eliminated by the algebraic lifting process if (1) is non-zero.

IV. Optimization of AB SC Measurement Matrices

In our previous work [13] we have shown that all harmful $(3,3)$ absorbing sets can be removed from an AB SC protograph by eliminating all 6-cycles due to the fact that each $(3,3)$ absorbing set contains a 6-cycle. In the same fashion we can see from Lemma 2 and Remark 4, that if we remove all 12-cycles via a properly chosen algebraic lifting, we can eliminate all $T_{6,\{6,8,12\}}$ termatiko sets. Since $T_{6,6}$ sets include two $T_{3,6}$ sets, by this method we can also remove all $T_{3,6}$ termatiko sets. In the following, we focus on two lifting schemes for constructing the SC protograph, namely cutting vector based [19] and algebraic lifting schemes [13].

A. Enumeration of termatiko sets of size 6

Let $C(12) \subset V$, $|C(12)| = 6$, represent the six VNs of a 12-cycle in G. From Lemma 2 and Remark 4 it is evident that $T_{6,\{6,8,12\}}$ sets are in fact $C(12)$ sets. Let $\mathscr{C}_{12}$ denote the set of all unique $C(12)$ sets, i.e., all 12-cycles with a different set of VNs. In order to find the VN index i associated to an edge (v_i, c_j) of a 12-cycle in G, we employ a cycle detection algorithm, such as the improved message passing algorithm proposed in [20]. Such an algorithm has polynomial complexity and for AB codes, the complexity can be further reduced by factor p. We then obtain the set $\mathscr{C}_{12}$ by employing an efficient (binary) search algorithm to detect duplicate cycles associated with the same set of VNs. This search algorithm has a complexity of $O(\log \mu)$, where μ is the number of all detected (non-unique) 12 cycles, which potentially can be very large. Algorithm 1 proposes a simple enumeration algorithm for all $T_{6,M}$ sets, $\mu_{T_{6,M}}$ with $M \in \{6,7,\ldots,Lp^2\}$, associated with a 12-cycle. Note that all $C(12)$ sets in G are not necessarily associated to a termatiko set of size 6. In order to determine whether or not a $C(12)$ set is a $T_{6,M}$ set, we adopt the following rule in Algorithm 1: If and only if the IPA outputs a vector $\hat{\boldsymbol{x}} = \boldsymbol{0}$ corresponding to an input data vector $\boldsymbol{x}$ with support $C(12)$, represented as $\boldsymbol{x}_{C(12)}$, then $C(12) = T_{6,M}$. Note that $\mu_{T_{6,6}} + \mu_{T_{6,8}} + \mu_{T_{6,12}} \le \mu_{T_{6,M}} \forall M \in \{6,7,\ldots,Lp^2\}$, and equality holds if $T_{6,\{6,8,12\}}$ are the only size 6 termatiko sets associated to 12-cycles.

Algorithm 1: Enumeration of all $T_{6,M}$ sets with $M \in \{6,7,\ldots,Lp^2\}$ in an AB measurement matrix A ($\gamma = 3$)

```
Input  : 𝒞_12, A
Output: μ_{T_{6,M}}
Initialization: μ_{T_{6,M}} = 0
foreach C(12) ∈ 𝒞_12 do
    Fix a binary x_{C(12)} with supp(x_{C(12)}) = C(12)
    Compute y_{C(12)} = A x^T_{C(12)}
    Run IPA(y_{C(12)}, A)
    if x̂ = 0 then
        μ_{T_{6,M}} = μ_{T_{6,M}} + 1
    end
end
```

B. Optimization of the SC protograph

Let us define the permutation indicator matrix $B_1 \in \{0,1\}^{\gamma \times p}$, where a 1 (resp., 0) in position (i,j) of this matrix indicates that all the non-zero elements of block (i,j) of $H(\gamma,p)$ will be lifted by τ_L^{κ} (resp., I), for $\kappa \in \{1,2,\ldots,m\}$, resulting in the $H(\gamma,p,L)$ SC protograph matrix. The process of obtaining optimized SC protographs by using both cutting vector and algebraic lifting approaches is described as follows:

(i) We first choose an $H(3,p)$ AB block matrix.

(ii) For the cutting vector approach based on the $H(3,p)$ AB block matrix, we construct SC protograph matrices by choosing a cutting vector $\boldsymbol{\xi}^*$ from [12, Table III] that provides a maximal minimum distance of 8 for the AB SC protograph. For such a code the minimum distance is equivalent to the stopping distance [12], and therefore it follows from Proposition 1 and Remark 3 that the AB SC protograph does not contain any $T_{3,6}$ and $T_{6,6}$ termatiko sets.

(iii) In case of algebraic lifting, we minimize the number of 12-cycles in the Tanner graph of the SC protograph obtained from $H(3,p)$. We numerically optimize the B_1 permutation matrix by using the approach in [13], and the cycle counting algorithm of [20] is utilized to count the number of 12-cycles in each optimization step. This leads to an optimized SC protograph matrix $H(3,p,L)$ that contains a smaller number of $T_{6,M}$ sets compared to the non-optimized protograph.

(iv) Finally, for both SC protographs discussed previously, we apply a degree J lift to $H(3,p,\boldsymbol{\xi}^*,L)$ and $H(3,p,L)$, resp., and obtain the corresponding optimized AB SC measurement matrix A, whose Tanner graph is used for reconstruction by the IPA.

Proposition 2. *Let $\hat{G}$ be a Tanner graph obtained by applying a degree J lift to the Tanner graph G. Let $\mu_{C(12)}$ (resp. $\hat{\mu}_{C(12)}$) represent the total number of 12-cycles in the graph G (resp., $\hat{G}$). Also, let $\mu_{T_{3,6}}$, $\mu_{T_{6,M}}$ (resp. $\hat{\mu}_{T_{3,6}}$, $\hat{\mu}_{T_{6,M}}$) represent the total number of $T_{3,6}$, $T_{6,M}$ sets in the graph G (resp., $\hat{G}$). We then have $\hat{\mu}_{C(12)} \le J\mu_{C(12)}$ and $\hat{\mu}_{T_{3,6}} \le J\mu_{T_{3,6}}$, $\hat{\mu}_{T_{6,M}} \le J\mu_{T_{6,M}}$.*

The proof is a simple consequence of the properties of graph lifting.

V. Simulation Results

We now provide results for the IPA reconstruction performance for different constructions of measurement matrices via Monte Carlo simulations.

- A_1 is obtained as a block diagonal matrix where each block is obtained from a $H(3,7)$ AB base matrix of size $3p \times p^2$ and then individually uplifted by factor J.
- A_2 represents a non AB SC LDPC matrix obtained by coupling L copies of a $(3,7)$ *random* regular LDPC matrix of size $3p \times p^2$, uplifted by a factor J.
- A_3 represents a $H(3,7,\boldsymbol{\xi}^*,L,J)$ matrix obtained by applying a degree J lift to the protograph of the $H(3,7,\boldsymbol{\xi}^*,L)$ SC protograph matrix from a cutting vector approach.
- A_4 represents a $H(3,7,L,J)$ matrix obtained by applying a degree J lift to the protograph of the optimized $H(3,7,L)$ SC protograph matrix based on algebraic lifting.

- A_5 represents a Gaussian matrix with same dimension as A_4 whose elements are $\mathcal{N}(0, \sigma^2)$ Gaussian random variables. Without loss of generality, $\sigma^2 = 1$.

The matrices A_1 to A_4 have the same constraint length of Jp^2, and all matrices have dimension $3(L+1)Jp \times LJp^2$. As parameters we select $\gamma = 3$, $p = 7$, $m = 1$, $J = 5$, $L = 10$, which leads to a blocklength of $n = 2450$ for all matrices. For these parameters Table I shows the total number of 12-cycles and $T_{6,M}$ sets, $M \in \{6, 7, \dots Lp^2\}$, for the corresponding protograph matrices of A_1, A_3 and A_4 [2]. We observe that spatial coupling is able to provide a significant reduction of both $T_{3,6}$ and $T_{6,\{6,8,12\}}$ termatiko sets. Also, Table I verifies that for the cutting vector approach with $\boldsymbol{\xi}^*$ all $T_{3,6}$ sets are eliminated. We also see that by optimizing the AB SC measurement matrix via algebraic lifting, $T_{6,M}$ sets can be completely removed from the protograph, which also implies the elimination of $T_{3,6}$ and $T_{6,\{6,8,12\}}$ sets. By invoking Proposition 2, these results also hold for the terminally lifted Tanner graph of A_4.

Number of	protograph of A_1	protograph of A_3	protograph of A_4
12-cycles	2409050	661311	227150
$T_{3,6}$ sets	4900	0	0
$T_{6,M}$ sets	9800	63	0

TABLE I: Total number of 12-cycles and $T_{6,M}$ sets, $M \in \{6, 7, \dots, Lp^2\}$, in the corresponding protograph matrices for A_1, A_3, A_4 with the parameters $m = 1$, $p = 7$, $L = 10$.

Fig. 2 displays the IPA reconstruction performance of matrices A_1 to A_4, and the LP reconstruction performances of A_4 and A_5. For the IPA, the probability of reconstruction is defined as $\Pr(\hat{\boldsymbol{x}} = \boldsymbol{x})$. For the LP, the probability of reconstruction is given as $\Pr(\max_{i \in \{1,2,\dots,n\}} |\hat{x}_i - x_i| \leq 10^{-3})$. All data points on the performance curve are averaged over 1000 realizations of the binary vector $\boldsymbol{x}$.

From Fig. 2 we can observe a behavior similar to the results shown in Table I, i.e., that spatially coupling leads to a significant increase in IPA reconstruction performance: for the same probability of reconstruction the density of the signal can be much higher. We also observe that LP based reconstruction for A_4 outperforms IPA decoding, albeit at a significantly higher reconstruction complexity. Whereas the IPA has a complexity of only $O(n(\log(n/k))^2 \log(k))$ [7], LP-based reconstruction has a complexity which is polynomial in time. Therefore, IPA based reconstruction with algebraically lifted SC measurement matrices serves as a good compromise between complexity and performance, in particular for larger block lengths.

[2]The enumeration result for A_2 has been excluded as termatiko set enumeration for non-AB matrices is beyond the scope of this paper.

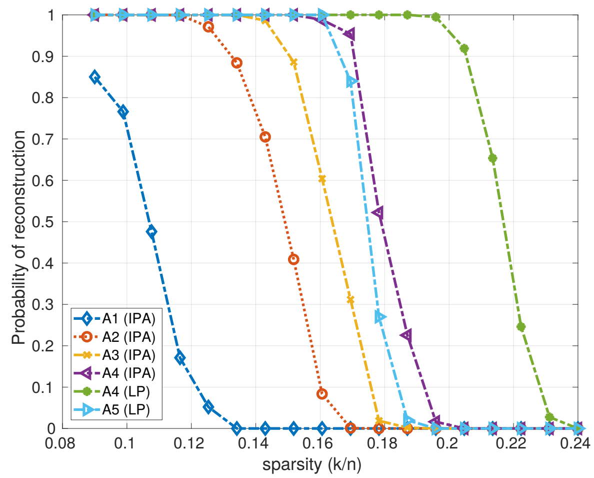

Fig. 2: Reconstruction probability versus sparsity of the data vector $\boldsymbol{x}$ for sparse measurement matrices A_1 to A_4 with $J = 5$, $L = 10$, $p = 7$, $m = 1$, and for a Gaussian matrix A_5. All matrices have dimension $m \times n$ with $m = 1155$ and $n = 2450$.

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