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Mori Dream Spaces and blow-ups of weighted projective spaces

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ABSTRACT

For every $n \geq 3$, we find a sufficient condition for the blow-up of a weighted projective space $\mathbb{P}(a, b, c, d_1, \cdots, d_{n-2})$ at the identity point not to be a Mori Dream Space. We exhibit several infinite sequences of weights satisfying this condition in all dimensions $n \geq 3$.

1. Introduction

We study the question whether the blow-up of a projective, $\mathbb{Q}$-factorial toric variety over $\mathbb{C}$ of Picard number one, at the identity point p of the open torus, is a Mori Dream Space (MDS).

Mori Dream Spaces were introduced by Hu and Keel in [14]. By [1], log Fano varieties over $\mathbb{C}$ are Mori Dream Spaces. Projective, $\mathbb{Q}$-factorial toric varieties, being log Fano, are MDS. The property of being a MDS is nevertheless not a birational invariant. In fact, the blow-up of $\mathbb{P}^n$ at r very general points stops being a MDS if $r > 8$ for $\mathbb{P}^2$ and $\mathbb{P}^4$, $r > 7$ for $\mathbb{P}^3$, and $r > n + 3$ for $n \geq 5$ [16]. One of the motivations to study blow-ups of toric varieties at the identity point comes from the proof by Castravet and Tevelev [3] that the moduli spaces of stable rational curves $\overline{M}_{0,n}$ are not MDS when $n > 133$, which was later improved to $n > 12$ by González and Karu [8] and to $n > 9$ by Hausen, Keicher and Laface [12]. The proof of [3] used the examples of not MDS blow-ups of weighted projective planes (see 1.4 and 1.5) by Goto, Nishida and Watanabe [11].

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The discussion above prompts the question of searching for not MDS blow-ups of toric varieties of small Picard numbers, which was formulated in [2]. Historically, much research work was done for surfaces. For a weighted projective plane $S = \mathbb{P}(a, b, c)$, let p be the identity point of the open torus. If the anticanonical divisor $-K$ of the blow-up $\mathrm{Bl}_p S$ of S at p is big, then $\mathrm{Bl}_p S$ is a MDS [5]. If one of a, b, c is at most 4 or equals 6 then $\mathrm{Bl}_p S$ is a MDS [5][17]. The first examples where $\mathrm{Bl}_p S$ is not a MDS were given in [11]. A generalization was achieved by González and Karu [8] for toric varieties of Picard number one whose corresponding polytope Δ has specific numbers of lattice points in its columns. The question can be formulated as an interpolation problem on the lattice points in Δ and leads to 3 families of new nonexamples [13]. We note that for any weighted projective space X, $\mathrm{Bl}_p X$ is a MDS if and only if the Cox ring of $\mathrm{Bl}_p X$ is a finitely generated $\mathbb{C}$-algebra, which is also equivalent to the finite generation of the symbolic Rees algebra associated to X [5][11], which is of independent interest.

In higher dimensions not much was known until the recent work [9]. In [9] González and Karu constructed higher dimensional toric varieties X of Picard number one with $\mathrm{Bl}_p X$ not a MDS, by exhibiting a nef but not semiample divisor on $\mathrm{Bl}_p X$. Their examples include some weighted projective 3-spaces $X = \mathbb{P}(a, b, c, d)$ such that $\mathrm{Bl}_p X$ is not a MDS.

In this paper, we give a sufficient condition (Theorem 1.2) so that the blow-up of the weighted projective n-space $X = \mathbb{P}(a, b, c, d_1, d_2, \cdots, d_{n-2})$ at the identity p is not a MDS. We show new examples of such X in all dimensions $n \geq 3$.

We sum up our results below. We work over the complex numbers $\mathbb{C}$. Let $N = \mathbb{Z}^2$ and M be the dual lattice of N. Let S be a normal projective, $\mathbb{Q}$-factorial toric surface of Picard number 1, with fan Σ_S in $N \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^2$. Then a polarization $H = H_\Delta$ on S is determined by a rational triangle Δ in $M \otimes_{\mathbb{Z}} \mathbb{R}$ whose normal fan is Σ_S. Let the sides of Δ have rational slopes $s_1 < s_2 < s_3$. We choose Δ so that after translating one vertex of Δ to $(0, 0)$, the opposite side passes through $(0, 1)$. Then the width of this Δ equals $w := 1/(s_2 - s_1) + 1/(s_3 - s_2)$. This w is called the *width* of the polarized toric surface (S, H_Δ) (see [8, Thm. 1.2]).

A weighted projective plane $S = \mathbb{P}(a, b, c)$ is an example of normal $\mathbb{Q}$-factorial toric surfaces of Picard number 1. A triple $(e, f, -g)$ is called a *relation* between the weights (a, b, c) if $e, f, g \in \mathbb{Z}_{>0}$ and $ae + bf = cg$ [8, Thm. 1.5]. Then there exists a polarization H_Δ such that the width w of (S, H_Δ) is smaller than 1 if and only if there exists a relation $(e, f, -g)$ with $cg^2/ab = w < 1$. Such $(e, f, -g)$ is unique if it exists, even when permuting the weights a, b, c. Therefore for a relation $(e, f, -g)$ we define the *width* of $(e, f, -g)$ to be $cg^2/(ab)$.

Given $\xi = (e, f, -g)$ a relation with width $w < 1$, we can construct a fan Σ_ξ of S and the polytope Δ_ξ with width w as follows: By [13, Prop. 5.1], there exists a unique integer r such that $1 \leq r \leq g$, $g \mid er - b$ and $g \mid fr + a$. Then the following vectors are primitive and span $\mathbb{Z}^2$:

$$u_0 = \left(\frac{er - b}{g}, -e\right), \quad u_1 = \left(\frac{fr + a}{g}, -f\right), \quad u_2 = (-r, g). \tag{1}$$

Clearly $au_0 + bu_1 + cu_2 = 0$. Hence the fan Σ_ξ with ray generators u_0, u_1 and u_2 is a fan of $\mathbb{P}(a, b, c)$. The triangle Δ_ξ has vertices

$$(0, 0), \quad \left(-\frac{eg}{b}, -\frac{er - b}{b}\right), \quad \left(\frac{fg}{a}, \frac{fr + a}{a}\right), \tag{2}$$

which is normal to Σ_ξ and has width $w = cg^2/(ab)$ (see Fig. 1).

Throughout this paper, we always assume that the weights $q_0, q_1, \cdots, q_n$ of a weighted projective n-space $\mathbb{P}(q_0, q_1, \cdots, q_n)$ are well-formed, i.e., any n weights are relatively prime.

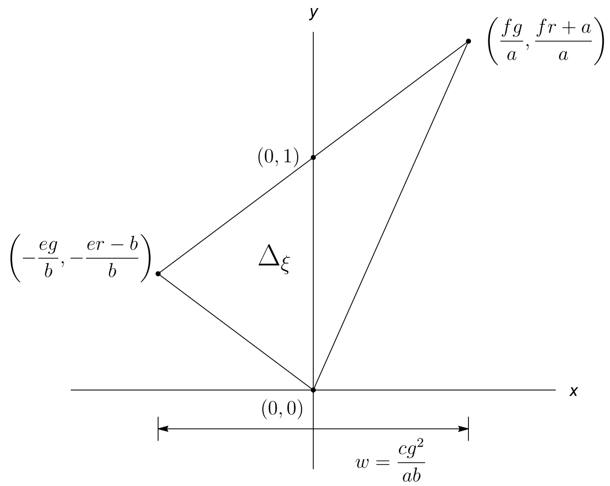

Fig. 1. The triangle Δ_ξ from the relation $\xi = (e, f, -g)$.

For any weighted projective space X, let p be the identity point of the open torus in X. For $S = \mathbb{P}(a, b, c)$, let B be the pseudo-effective divisor on S generating $\mathrm{Cl}(S) \cong \mathbb{Z}$. Let e be the exceptional divisor of the blow-up $\pi : \mathrm{Bl}_p S \to S$. Our main result is:

Theorem 1.1. *Let $X = \mathbb{P}(a, b, c, d_1, d_2, \cdots, d_{n-2})$ where a, b, c are pairwise coprime. Let $S = \mathbb{P}(a, b, c)$. Suppose there is a negative curve C on $\mathrm{Bl}_p S$, different from e, with $C \sim_{\mathbb{Q}} \lambda\pi^* B - \mu e$ for some $\lambda, \mu \in \mathbb{Q}$. Suppose all the following hold:*

(i) *every d_i lies in the semigroup generated by a, b and c (i.e., d_i is a linear combination of a, b, c with non-negative integer coefficients),*
(ii) *$d_i < \dfrac{abc\mu}{\lambda}$ for every i,*
(iii) *$\mathrm{Bl}_p \mathbb{P}(a, b, c)$ is not a MDS.*

Then $\mathrm{Bl}_p X$ is not a MDS.

We show a special case of Theorem 1.1 when there is a relation $(e, f, -g)$ between the weights (a, b, c) with $w < 1$. In this case, there exists a negative curve $C \sim cg\pi^* B - e$ on $\mathrm{Bl}_p S$, and we have:

Theorem 1.2. *Let $X = \mathbb{P}(a, b, c, d_1, d_2, \cdots, d_{n-2})$ be a weighted projective n-space where a, b, c are pairwise coprime. Let p be the identity point of the open torus in X. Suppose all the following hold:*

(i) *there is a relation between the weights (a, b, c) such that the width satisfies $w < 1$,*
(ii) *every d_i lies in the semigroup generated by a, b and c,*
(iii) *$d_i^2 w < abc$ for every i,*
(iv) *$\mathrm{Bl}_p \mathbb{P}(a, b, c)$ is not a MDS.*

Then $\mathrm{Bl}_p X$ is not a MDS.

In particular, if all $d_i = a$ and $a < b < c$ with $w < 1$, then $d_i^2 w = a^2 w < a^2 < abc$. Thus we have the following corollary:

Corollary 1.3. *Assume that $a < b < c$ are pairwise coprime. Suppose $\mathrm{Bl}_p \mathbb{P}(a, b, c)$ is not a MDS, and there is a relation between the weights (a, b, c) such that the width satisfies $w < 1$. Then $\mathrm{Bl}_p \mathbb{P}(a, b, c, a, \cdots, a)$ is not a MDS.*

Example 1.4. By [11], the Cox ring of the blow-up of $\mathbb{P}(a, b, c)$ at the identity point is not finitely generated as a $\mathbb{C}$-algebra when $(a, b, c) = (7m - 3, 8m - 3, (5m - 2)m)$ for $m \geq 4$ and $3 \nmid m$. Equivalently, the blow-up at p is not a MDS. The sequence of weights has relation $(e, f, -g) = (m, m, -3)$ so that $w < 1$.

By Theorem 1.2, we conclude that $\mathrm{Bl}_p \mathbb{P}(7m - 3, 8m - 3, (5m - 2)m, d_1, \cdots, d_{n-2})$ is not a MDS when

(i) $m \geq 4$ and $3 \nmid m$,
(ii) every d_i lies in the semigroup generated by $7m - 3, 8m - 3$ and $(5m - 2)m$, and
(iii) every $d_i < (7m - 3)(8m - 3)/3$.

By Corollary 1.3, $\mathrm{Bl}_p \mathbb{P}(7m - 3, 8m - 3, (5m - 2)m, 7m - 3, \cdots, 7m - 3)$ is not a MDS for $m \geq 4$ and $3 \nmid m$.

Example 1.5. Another infinite sequence given by [11] where the blow-ups at p are not MDS is $(a, b, c) = (7m - 10, 8m - 3, 5m^2 - 7m + 1)$ for any $m \geq 5$ such that $3 \nmid 7m - 10$ and $m \not\equiv -7 \pmod{59}$ (By [8] the blow-up at p is also not a MDS when $m = 3$). The sequence of weights has relation $(e, f, -g) = (m, m-1, -3)$ so that $w < 1$.

We conclude by Theorem 1.2 that $\mathrm{Bl}_p \mathbb{P}(7m-10, 8m-3, 5m^2-7m+1, d_1, \cdots, d_{n-2})$ is not a MDS when

(i) $m \geq 3$, $3 \nmid 7m - 10$ and $m \not\equiv -7 \pmod{59}$,
(ii) every d_i lies in the semigroup generated by $7m - 10, 8m - 3$ and $5m^2 - 7m + 1$, and
(iii) every $d_i < (7m - 10)(8m - 3)/3$.

Example 1.6. The infinite sequence $(a, b, c) = (7, 15 + 2t, 26 + 3t)$ for $t \geq 0$ has the relation $(e, f, -g) = (1, 3, -2)$. The weights (a, b, c) are pairwise coprime if and only if $7 \nmid t - 3$. They all satisfy the criterion of [8, Thm. 1.5], so $\mathrm{Bl}_p \mathbb{P}(a, b, c)$ is not MDS for every $t \geq 0$, where the width

$$w = \frac{4(26 + 3t)}{7(15 + 2t)} = \frac{104 + 12t}{105 + 14t} < 1$$

for $t \geq 0$. Theorem 1.2 (3) then gives the upper bound

$$d < \sqrt{\frac{abc}{w}} = \frac{ab}{g} = \frac{7(15 + 2t)}{2}.$$

Note that when $t \geq 0$, $a + b = 2t + 22 < \frac{7(15 + 2t)}{2}$. Hence $d = a + b$ is on the list. As a result, $\mathrm{Bl}_p \mathbb{P}(7, 15 + 2t, 26 + 3t, d_1, \cdots, d_{n-2})$ is not a MDS when

(i) $t \geq 0$ and $7 \nmid t - 3$,
(ii) every d_i lies in the semigroup generated by $7, 15 + 2t$ and $26 + 3t$, and
(iii) every $d_i < 7(15 + 2t)/2$.

The paper is organized as follows. In Section 2, we give a sufficient condition (Theorem 2.1) for the blow-up $\mathrm{Bl}_p X$ of a normal projective variety X with Picard number 1 not to be a MDS, with p a smooth point on X. Such $\mathrm{Bl}_p X$ has a nef but not semiample divisor. Sections 3 and 4 consider weighted projective n-spaces

X with properties described in Theorem 1.1. We show that X contains a closed subvariety isomorphic to $S = \mathbb{P}(a, b, c)$. Section 5 verifies the conditions in Theorem 2.1 for X and S, applying a result of Fulton and Sturmfels [7, Lem. 3.4]. In particular, we prove that $\mathrm{Bl}_p X$ is not a MDS.

In Section 6, we compare our results with the examples in [9]. Proposition 6.6 describes the overlap of our list in dimension 3 with González and Karu's in [9]. The only common examples are $X = \mathbb{P}(a, b, c, cg)$ where $(e, f, -g)$ is a relation between (a, b, c), and $\mathrm{Bl}_p \mathbb{P}(a, b, c)$ is not a MDS and satisfies the assumptions in [9, Cor. 2.5]. Note that we give more examples beyond the overlap (Examples 1.4, 1.5 and 1.6).

In Section 7, we apply Theorem 1.1 to the case when $X = \mathbb{P}(a, b, c, d_1, d_2, \cdots, d_{n-2})$ where $S = \mathbb{P}(a, b, c)$ being of the form considered in [10, Ex. 1.4]. Hence $\mathrm{Bl}_p S$ is again not a MDS. This leads to new examples where $\mathrm{Bl}_p X$ is not MDS in Corollary 7.1.

2. Blow-ups of varieties of Picard number one

Let X be a normal, projective, $\mathbb{Q}$-factorial variety of Picard number 1 and dimension $n \geq 3$. Suppose $Y_1, \cdots, Y_{n-2}$ are prime Weil divisors of X (Y_i not necessary normal), such that the set-theoretic intersection $S := \cap_{i=1}^{n-2} Y_i$, with the reduced subscheme structure on S, is a normal, projective, $\mathbb{Q}$-factorial surface of Picard number 1. In addition, suppose both $\mathrm{Pic}(X)$ and $\mathrm{Pic}(S)$ are finitely generated.

Let us blow up S and X at a point $p \in S$ which is smooth in X, S and each Y_j. Let $f : \mathrm{Bl}_p S \to \mathrm{Bl}_p X$ be the natural inclusion. Let E be the exceptional divisor of the blow-up $\pi_X : \mathrm{Bl}_p X \to X$ and e be the exceptional divisor of $\pi : \mathrm{Bl}_p S \to S$.

Theorem 2.1. *Let X, Y_i, S and f be defined as above. Suppose there exists an irreducible curve C in $\mathrm{Bl}_p S$, different from the exceptional divisor e in $\mathrm{Bl}_p S$, with $C^2 < 0$, such that for every i, $(f_*C).\mathrm{Bl}_p Y_i < 0$ in $\mathrm{Bl}_p X$. Then if $\mathrm{Bl}_p S$ is not a Mori Dream Space (MDS), then $\mathrm{Bl}_p X$ is not a Mori Dream Space.*

Proof. Here both $\mathrm{Bl}_p S$ and $\mathrm{Bl}_p X$ have Picard number 2. Since $C^2 < 0$ in $\mathrm{Bl}_p S$, C spans an extremal ray of the Mori cone $\overline{\mathrm{NE}}(\mathrm{Bl}_p S)$ [15, Lem. 1.22]. Since e is numerically equivalent to a general line in the exceptional divisor E of $\mathrm{Bl}_p X$, $[e]$ spans an extremal ray in both $\overline{\mathrm{NE}}(\mathrm{Bl}_p X)$ and $\overline{\mathrm{NE}}(\mathrm{Bl}_p S)$.

Let C_1 be the image of C in $\mathrm{Bl}_p X$, and e_1 be the image of e in $\mathrm{Bl}_p X$. We show that $[C_1]$ spans the other extremal ray of $\overline{\mathrm{NE}}(\mathrm{Bl}_p X)$. Since C is irreducible, C_1 is irreducible. Suppose towards a contradiction that C_1 is not extremal in $\overline{\mathrm{NE}}(\mathrm{Bl}_p X)$. Then $C_1 \equiv r_1 F_1 + s_1 e_1$ for some effective curve F_1 and some rational numbers $r_1, s_1 > 0$. Then there exists an irreducible component F_2 of F_1 such that $F_1 \equiv r_2 F_2 + s_2 e_1$ for some rational numbers $r_2 > 0$ and $s_2 \geq 0$. Therefore we can assume at the beginning that F_1 is irreducible. By assumption, $C_1 \cdot \mathrm{Bl}_p Y_i < 0$ for every i. Since $\mathrm{Bl}_p Y_i$ is isomorphic to the proper transform of Y_i in X, and the class of e_1 is the class of a line in E, we have $e_1 \cdot \mathrm{Bl}_p Y_i \geq 0$. Therefore $F_1 \cdot \mathrm{Bl}_p Y_i < 0$. The irreducibility assumption of F_1 implies that $F_1 \subset \mathrm{Bl}_p Y_i$. Run this for every i, and we have $F_1 \subset \cap_i \mathrm{Bl}_p Y_i = \mathrm{Bl}_p S$. Consider the pushforward $f_* : \mathrm{N}_1(\mathrm{Bl}_p S) \to \mathrm{N}_1(\mathrm{Bl}_p X)$ and the pullback $f^* : \mathrm{N}^1(\mathrm{Bl}_p X) \to \mathrm{N}^1(\mathrm{Bl}_p S)$. Since $\mathrm{N}^1(\mathrm{Bl}_p S)$ is spanned by $[f^*H]$ and $[e]$ where $H = \pi_X^* H_0$ is the total transform of a very ample divisor H_0 on X, and $e \equiv f^*E$, we have f^* is surjective. The dual paring between $\mathrm{N}^1(\mathrm{Bl}_p X)$ and $\mathrm{N}_1(\mathrm{Bl}_p X)$ (respectively $\mathrm{N}^1(\mathrm{Bl}_p S)$ and $\mathrm{N}_1(\mathrm{Bl}_p S)$) is perfect. Hence f_* is injective by the projection formula. Now $f_*(C - r_1 F_1 - s_1 e) \equiv C_1 - r_1 F_1 - s_1 e_1 \equiv 0$. By injectivity, $C - r_1 F_1 - s_1 e \equiv 0$. Then the ray $\mathbb{R}_{\geq 0}[C]$ is not extremal in $\overline{\mathrm{NE}}(\mathrm{Bl}_p S)$, which is a contradiction. Hence the ray $\mathbb{R}_{\geq 0}[C_1]$ is extremal in $\overline{\mathrm{NE}}(\mathrm{Bl}_p X)$.

Finally, suppose $\mathrm{Bl}_p X$ is a MDS. Since X is $\mathbb{Q}$-factorial, and p is smooth in X. $\mathrm{Bl}_p X$ is also $\mathbb{Q}$-factorial. Then the nef cone of $\mathrm{Bl}_p X$ is generated by semiample divisors. In particular, there is a semiample divisor D such that $D.C_1 = 0$. Therefore $f^*D \cdot C = f_*(f^*D \cdot C) = D \cdot f_*C = D \cdot C_1 = 0$ by projection formula. Hence $[f^*D]$ spans an extremal ray of $\mathrm{Nef}(\mathrm{Bl}_p S)$. Now f^*D is also semiample. This shows that $\mathrm{Bl}_p S$ is a MDS. $\square$

3. Divisors on weighted projective spaces

In this section we construct the fan of the weighted projective n-space $X = \mathbb{P}(a, b, c, d_1, \cdots, d_{n-2})$ and define $n-2$ divisors Y_j on X for $j = 3, 4, \cdots, n$, under the assumption (i) of Theorem 1.1. Then we show that the set-theoretic intersection of those Y_j equals the Zariski closure of a 2-dimensional subtorus in X.

Notation 3.1. We list some notations and terminology for later use.

- For any integer $n \geq 3$, let $J := \{3, 4, \cdots, n\}$.
- Let $N \cong \mathbb{Z}^n$ $(n \geq 3)$ be a lattice. Let $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$. Then T_N is a torus of dimension n. Let $M = \operatorname{Hom}(N, \mathbb{Z})$ be the dual lattice of N. Then $M = \operatorname{Hom}(T_N, \mathbb{G}_m)$, so each $u \in M$ defines a character χ^u on T_N.
- If $e_1, e_2, \cdots, e_n$ form a basis of N, then $e_1^*, \cdots, e_n^*$ form the dual basis of M. Write $\chi_j := \chi^{e_j^*}$. Then $T_N = \operatorname{Spec} \mathbb{C}[\chi_1, \chi_1^{-1}, \cdots, \chi_n, \chi_n^{-1}]$.
- For any lattice L, define $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$.
- Let $N_1 := \mathbb{Z}\{e_1\}$ be the sublattice of N spanned by e_1. Let $N_{12} := \mathbb{Z}\{e_1, e_2\}$ be the sublattice spanned by e_1 and e_2. Let $T_1 := N_1 \otimes_{\mathbb{Z}} \mathbb{C}^*$ and $T_{12} := N_{12} \otimes_{\mathbb{Z}} \mathbb{C}^*$ be the corresponding subtori of T_N. Let $M_{12} := \operatorname{Hom}(N_{12}, \mathbb{Z})$.
- Let $L_j := \mathbb{Z}\{e_1, e_2, \cdots, \widehat{e_j}, \cdots, e_n\}$ for $j \in J$. Let $T_j := L_j \otimes_{\mathbb{Z}} \mathbb{C}^*$.
- Let Σ be a full dimensional fan in $N_{\mathbb{R}}$. If X is the toric variety corresponding to the fan Σ, then T_N is the open torus in X. For any full dimensional cone $\sigma \in \Sigma$, let $U_\sigma := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$. Then $\{U_\sigma \mid \sigma \in \Sigma \text{ is full dimensional}\}$ is an affine open cover of X.
- Write $\tau \prec \sigma$ if τ is a face of σ. For any cone $\tau \in \Sigma$, let $O(\tau)$ be the T_N-orbit associated to τ in X. Then for a full dimensional cone σ and any cone τ in Σ, $O(\tau) \subseteq U_\sigma$ if and only if $\tau \prec \sigma$ (see [4, 3.2.6c]).
- Let $V(\tau)$ be the Zariski closure of $O(\tau)$ in X. Then $V(\tau)$ is a torus-invariant closed subvariety of X.
- A fan Σ is simplicial if any cone $\sigma \in \Sigma$ is generated by linearly independent generators. Assume that Σ is a simplicial fan in $\mathbb{R}^n$ with $n+1$ rays $R_0, R_1, \cdots, R_n$, where every n of them are linearly independent. For every $I \subseteq \{0, 1, \cdots, n\}$, let $\sigma_I \in \Sigma$ be the cone spanned by $\{R_i \mid i \in I\}$. Every cone $\sigma \in \Sigma$ corresponds to a unique subset I in the way above. Let $\Sigma(k)$ be the k-dimensional cones in Σ. Then $\Sigma(k) = \{\sigma_I \mid |I| = k\}$. We write $V(\sigma_I)$ as V_I, and $O(\sigma_I)$ as O_I. Then O_I is a torus of dimension $n - |I|$. If $I = \{i\}$, then we write the torus-invariant divisor $V(\sigma_{\{i\}})$ as D_i:.

We start with the fan of the weighted projective plane $\mathbb{P}(a, b, c)$. The assumption and conclusion of Proposition 1.1 are symmetric about a, b and c. Hence up to a permutation on (a, b, c), we can choose a fan Σ_S of S with ray generators $u_i = (x_i, y_i)$ such that both $y_0, y_1 < 0$ and $y_2 > 0$. Note that we have $au_0 + bu_1 + cu_2 = 0$.

Consider $N = \mathbb{Z}^n$. Fix a basis $e_1, e_2, \cdots, e_n$ of N. By assumption (ii), there exist nonnegative integers $\{m_{ij}\}$ such that $d_{j-2} = am_{0,j} + bm_{1,j} + cm_{2,j}$ for every $j \in J$. Define the following vectors in N:

$$\begin{aligned} v_0 &= (x_0, y_0, -m_{0,3}, \cdots, -m_{0,n}), \\ v_1 &= (x_1, y_1, -m_{1,3}, \cdots, -m_{1,n}), \\ v_2 &= (x_2, y_2, -m_{2,3}, \cdots, -m_{2,n}), \\ v_j &= e_j, \ \text{for } j \in J = \{3, 4, \cdots, n\}. \end{aligned} \tag{3}$$

Note that for every $j \in J$, at least one of the integers m_{0j}, m_{1j}, m_{2j} is necessarily nonzero.

Those v_i satisfy the relation

$$av_0 + bv_1 + cv_2 + d_1 v_3 + \cdots + d_i v_{i+2} + \cdots + d_{n-2} v_n = 0.$$

Moreover, each v_i is primitive, and together they span the lattice N. As a result, if we let Σ_X be the fan in $N_{\mathbb{R}}$ spanned by the $n+1$ rays along v_i $(i = 0, 1, \cdots, n)$, then Σ_X is a fan of $X = \mathbb{P}(a, b, c, d_1, \cdots, d_{n-2})$.

Definition 3.2. Let the fan Σ_X of $X = \mathbb{P}(a, b, c, d_1, \cdots, d_{n-2})$ be defined as above. For every $j \in J$, let Y_j be the Zariski closure of the subtorus $T_j = L_j \otimes_{\mathbb{Z}} \mathbb{C}^*$ in X. Define S to be the set-theoretic intersection $\cap_{j=3}^{n} Y_j$. Let Z be the Zariski closure of the subtorus $T_{12} = N_{12} \otimes_{\mathbb{Z}} \mathbb{C}^*$ in X.

By definition, all the Y_j and Z are irreducible. We claim:

Proposition 3.3.

(i) *The set-theoretic intersection S equals Z.*
(ii) *With the reduced subscheme structure, S is isomorphic to $\mathbb{P}(a, b, c)$. In particular, S is normal.*

We prove (ii) of Proposition 3.3 in the next section. Here we prove (i) by showing that Z is the unique irreducible component of the intersection S. We will reduce the question to the affine case and apply the following lemma.

Lemma 3.4. *Let σ in $N_{\mathbb{R}}$ be a simplicial cone spanned by n linearly independent rays R_i, $i = 1, \cdots, n$. Let $U_\sigma := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$. For any $u \in M$ such that u is primitive and $u \neq 0$, let T_u be the subtorus of T_N defined by $\chi^u = 1$, and take the Zariski closure $\overline{T_u}$ in U_σ. Then we have:*

(i) *If $\tau \prec \sigma$ such that $u \in \tau^\vee \cup (-\tau^\vee)$ and $u \notin \tau^\perp$, then the set-theoretic intersection $\overline{T_u} \cap O(\tau) = \emptyset$. In particular:*
  (a) *For $\tau = R_i$, if $u \notin \tau^\perp$, then $\overline{T_u} \cap O(\tau) = \emptyset$.*
  (b) *If $u \in \sigma^\vee \cup (-\sigma^\vee)$, then $\overline{T_u} \cap O(\sigma) = \emptyset$.*
(ii) *If $u \in \tau^\perp$ and $u \in \sigma^\vee \cup (-\sigma^\vee)$, then $\overline{T_u} \cap O(\tau)$ has codimension at least 1 in $O(\tau)$.*

Proof. When $\tau = R_i$ is a ray, $\tau^\vee \cup (-\tau^\vee) = M$. When $\tau = \sigma$, $\tau^\perp = \sigma^\perp = \{0\}$. Therefore the two special cases (a) and (b) of (i) follow from the general result. Now let τ be a d-dimensional face of σ such that $u \in \tau^\vee \cup (-\tau^\vee)$, and $u \notin \tau^\perp$. Then $O(\tau) \cong \operatorname{Spec} \mathbb{C}[\tau^\perp \cap M]$ is a $(n-d)$-dimensional torus (see Notation 3.1). Let $V(\tau)$ be the closure of $O(\tau)$ in U_σ. Then $V(\tau) \cong \operatorname{Spec} \mathbb{C}[\tau^\perp \cap \sigma^\vee \cap M]$. Then the inclusions

$$O(\tau) \cong \operatorname{Spec} \mathbb{C}[\tau^\perp \cap M] \hookrightarrow V(\tau) \cong \operatorname{Spec} \mathbb{C}[\tau^\perp \cap \sigma^\vee \cap M] \hookrightarrow U_\sigma \cong \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$$

correspond to the maps of $\mathbb{C}$-algebras

$$\mathbb{C}[\sigma^\vee \cap M] \xrightarrow{\phi_\tau} \mathbb{C}[\tau^\perp \cap \sigma^\vee \cap M] \to \mathbb{C}[\tau^\perp \cap M],$$

where ϕ_τ sends χ^u to χ^u if $u \in \tau^\perp$, and 0 otherwise. To prove that $\overline{T_u}$ does not intersect $O(\tau)$, it suffices to show that there is a regular function f vanishing on T_u but not vanishing anywhere on $O(\tau)$. There are two cases.

Case I. $u \in \sigma^\vee \cup (-\sigma^\vee)$ and $u \notin \tau^\perp$. Note that $\sigma^\vee \subseteq \tau^\vee$ since $\tau \prec \sigma$. Suppose $u \in -\sigma^\vee$. Then $-u \in \sigma^\vee$. By definition, $T_u = T_{-u}$, so we can assume $u \in \sigma^\vee$. Now $f := \chi^u - 1 = \chi^u - \chi^0 \in \mathbb{C}[\sigma^\vee \cap M]$ is a regular function on U_σ. Since $u \notin \tau^\perp$, $\phi_\tau(\chi^u) = 0$. Since $0 \in \tau^\perp$, $\phi_\tau(\chi^0) = 1$. Therefore $\phi_\tau(f) = -1$ is a regular function on $V(\tau)$ which does not vanish on $O(\tau)$.

Case II. $\tau \neq \sigma$ is a proper face, $u \in \tau^\vee \cup (-\tau^\vee)$ and $u \notin \sigma^\vee \cup (-\sigma^\vee)$ and $u \notin \tau^\perp$. For each $i = 1, \cdots, n$, let r_i be the ray generator of the ray R_i. Without loss of generality, we can assume τ is the face spanned by $r_1, \cdots, r_d$, with $d < n$, and $u \in \tau^\vee$. Let $\langle \cdot, \cdot \rangle : N \times M \to \mathbb{Z}$ be the dual pairing. Then $\langle r_i, u \rangle \geq 0$ for

$i = 1, \ldots, d$, with $\langle r_i, u\rangle > 0$ for some $i \leq d$, and $\langle r_j, u\rangle < 0$ for some $j \in \{d+1, \cdots, n\}$. We claim there exist $p, q \in \sigma^\vee \cap M - \{0\}$ and $k \in \mathbb{Z}_{>0}$ such that $ku = p - q$ and $q \in \tau^\perp$. Indeed, since σ is simplicial, $r_1, \cdots, r_n$ form a basis of $N \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $r_1^*, \cdots, r_n^*$ be the dual basis of $M \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $u = u_1 r_1^* + \cdots + u_n r_n^*$ for rational numbers u_i, $i = 1, \cdots, n$. Define

$$p' := \sum_{u_i > 0} u_i r_i^*, \text{ and } q' := -\sum_{u_i < 0} u_i r_i^*.$$

Then $u = p' - q'$. Indeed both p' and q' are in $\sigma^\vee$. Since $\langle r_i, u\rangle > 0$ for some $i \leq d$, and $\langle r_j, u\rangle < 0$ for some $j \in \{d+1, \cdots, n\}$, we have $p' \neq 0$ and $q' \neq 0$. Take any $k \in \mathbb{Z}_{>0}$ such that kp' and kq' are both in M. Let $p := kp'$ and $q := kq'$, then $ku = p - q$ and $p, q \in \sigma^\vee \cap M - \{0\}$, which proves the claim. Now let $f = \chi^q - \chi^p$. Then $f \in \mathbb{C}[\sigma^\vee \cap M]$. We have $f = \chi^q - \chi^p = -\chi^q(\chi^{ku} - 1)$. Since $\chi^u - 1$ divides $\chi^{ku} - 1$, and χ^q has no poles on T_u, f must vanish everywhere T_u. On the other hand, since $u \notin \tau^\perp$ and $q \in \tau^\perp$, $p = ku + q \notin \tau^\perp$. Therefore $\phi_\tau(\chi^p) = 0$, and $\phi_\tau(f) = \phi_\tau(\chi^q) = \chi^q$. When restricted to $O(\tau)$, χ^q is a nonzero monomial in the coordinate functions on $O(\tau)$, therefore χ^q does not vanish anywhere on the torus $O(\tau)$. This proves (i).

By the symmetry between u and $-u$, to prove (ii), we need only prove for the case when $u \in \tau^\perp \cap \sigma^\vee$. In this case, $\phi_\tau(\chi^u) = \chi^u$, so $\chi^u - 1$ is a regular function of $O(\tau)$. Now $\overline{T_u}$ is contained in the zero locus of $\chi^u - 1$. By assumption, $u \neq 0$, so $\chi^u \neq 1$. Restricting to $O(\tau)$, $\chi^u \neq 1$ is a monomial of the coordinate functions on $O(\tau)$, so $\chi^u = 1$ defines a subtorus of codimension 1 in $O(\tau)$. This proves (ii). □

Proof of Proposition 3.3(i). By Definition 3.2, S is the set-theoretic intersection of Y_j, $j \in J$. Since each Y_j has codimension one in X, the codimension of each irreducible component of S in X is at most $n-2$. For every $j \in J$, since $T_{12} \subseteq T_j$, Z is contained in Y_j. Hence Z is contained in S. Therefore it suffices to prove that Z is the unique irreducible component of S of dimension at least 2.

Here the fan Σ_X is simplicial, spanned by ray generator v_i. By Notation 3.1, $\Sigma_X = \{\sigma_I \mid I \subseteq \{0, 1, \cdots, n\}\}$. To prove that Z is the unique irreducible component of S of dimension at least 2, we need only show that $S \cap O_I$ is contained in a curve for every $1 \leq |I| \leq n-2$. Indeed, suppose $S \cap O_I$ is contained in a curve for every $1 \leq |I| \leq n-2$. Then $X \backslash T_N$ is a disjoint union of T_N-orbits O_I for $1 \leq |I| \leq n-2$, with $\dim O_I = n - |I|$. Therefore, if we assume there is some irreducible component S' of S disjoint from Z, then S' is contained in $X \backslash T_N$, hence $\dim S' \leq 1$. This proves that Z is the unique irreducible component of S of dimension at least 2.

It remains to show $S \cap O_I$ is contained in a curve for every $1 \leq |I| \leq n-2$. By Notation 3.1, $\{U_\sigma \mid \sigma \in \Sigma_X(n)\}$ is a torus-invariant open affine cover of X. For every T_N-orbit O_I with $1 \leq |I| \leq n-2$, we choose some $\sigma' \in \Sigma_X(n)$ such that $\sigma_I \prec \sigma'$. Then $O_I \subseteq U_{\sigma'}$. By definition, Y_j is the Zariski closure of $T_{e_j^*}$ in X. Indeed, $T_{e_j^*} \subseteq T_N \subseteq U_{\sigma'}$. Let Y_j' be the restriction of Y_j to this $U_{\sigma'}$. Then Y_j' equals the Zariski closure of $T_{e_j^*}$ in $U_{\sigma'}$. We apply Lemma 3.4 to $\sigma = \sigma'$, $\tau = \sigma_I$ and $u = e_j^*$. Recall (3) that $-m_{ij} \leq 0$ is the j-th entry of v_i for $i = 0, 1, 2$, $j \geq 3$. Define the following index sets:

$$\begin{aligned} I_+ &:= I \cap \{0, 1, 2\}, \\ J_- &:= \{j \in J \backslash I \mid m_{ij} > 0 \text{ for some } i \in I_+\}, \\ I_0 &:= \{j \in I \cap J \mid m_{ij} = 0 \text{ for all } i \in I_+\}. \end{aligned}$$

There are 4 possible cases: (a) $I_+ = \emptyset$; (b) $I_+ \neq \emptyset$ and $J_- \neq \emptyset$; (c) $I_+ \neq \emptyset$ and $I_0 \neq \emptyset$; and (d) $I_+ \neq \emptyset$ and $J_- = I_0 = \emptyset$.

In Cases (a) (b) and (c), we apply Lemma 3.4 (i) to show that there exists $j \in J$ such that $Y_j' \cap O_I = \emptyset$ for some $j \in J$ and for every choice of $\sigma_I \prec \sigma'$. Hence $S \cap O_I = \emptyset$. For (d), we apply Lemma 3.4 (ii) to show that $S \cap O_I$ is contained in a curve by choosing a specific σ'.

(a) $I_+ = \emptyset$. Choose any $j \in I$. Then $e_j^* \notin \sigma_I^\perp$ and $e_j^* \in \sigma_I^\vee$. Apply Lemma 3.4 (i) to any full dimensional σ' such that $\sigma_I \prec \sigma'$, $\tau = \sigma_I$ and $u = e_j^*$. Then $Y_j' \cap O_I = \emptyset$.

(b) $I_+ \neq \emptyset$ and $J_- \neq \emptyset$. Then choose any $j \in J_-$. We have $\langle v_i, e_j^* \rangle = -m_{ij} < 0$ for some $i \in I_+$, and $\langle v_i, e_j^* \rangle \leq 0$ for all $i \in I$. Hence $e_j^* \in -\sigma_I^\vee$ and $e_j^* \notin -\sigma_I^\perp$. Therefore $Y_j' \cap O_I = \emptyset$.

(c) $I_+ \neq \emptyset$ and $I_0 \neq \emptyset$. Choose any $j \in I_0$. Then $\langle v_j, e_j^* \rangle = 1 > 0$. If $i \in I$ and $i \neq j$, then either $i \in J$ or $i \in I_+$. If $i \in J$, then $v_i = e_i$ and $i \neq j$, so $\langle v_i, e_j^* \rangle = 0$. If $i \in I_+$, then $\langle v_i, e_j^* \rangle = -m_{ij} = 0$ since $j \in I_0$. Hence $e_j^* \in \sigma_I^\vee$ and $e_j^* \notin \sigma_I^\perp$, so $Y_j' \cap O_I = \emptyset$.

(d) $I_+ \neq \emptyset$ and $J_- = I_0 = \emptyset$. Since $|I| \leq n-2$, and $I_+ \neq \emptyset$, it must be that $J \not\subseteq I$. Therefore $I_+ \neq \{0,1,2\}$ (otherwise for every $j \in J \backslash I$, there exists an $m_{ij} > 0$, so $j \in J_-$), so $|I_+| = 1$ or 2. Fix some $j \in J \backslash I$. Since $J_- = \emptyset$, $m_{ij} = 0$ for all $i \in I_+$. Therefore $e_j^* \in \sigma_I^\perp$. For this $j \in J \backslash I$, define $I' = \{0, 1, 2, \cdots, \widehat{j}, \cdots, n\}$ and let $\sigma' := \sigma_{I'}$. Define Y_j' to be the restriction of Y_j to $U_{\sigma'}$ as discussed above. Then $U_{\sigma'}$ contains O_I, with $e_j^* \in -(\sigma')^\vee$. In Lemma 3.4 (ii), let $\sigma = \sigma'$, $\tau = \sigma_I$ and $u = e_j^*$. Then $Y_j' \cap O_I$ is of codimension at least one in O_I and is contained in the zero locus of $\chi_j - 1$, regarded as a regular function on O_I. Now the number of such j equals $|J \backslash I| = n - 2 - |I \cap J| = n - 2 - (|I| - |I_+|)$. Since $n - |I| = \dim O_I$, we have $|J \backslash I| = \dim O_I - (2 - |I_+|)$. Recall that $M = \mathbb{Z}\{e_1^*, \cdots, e_n^*\}$ and $O_I = \operatorname{Spec} \mathbb{C}[\sigma_I^\perp \cap M]$. Since $|I_+| = 1$ or 2, the semigroup $\sigma_I^\perp \cap M$ is generated by $\{e_i^* \mid i \in J \backslash I\}$ if $|I_+| = 2$, or by $\{e_i^* \mid i \in J \backslash I\}$ together with some $\xi \in \mathbb{Z}\{e_1^*, e_2^*\}$ if $|I_+| = 1$. Therefore each χ_j, $j \in J \backslash I$ restricts to different coordinate functions on O_I. Hence, the intersection of the zero loci of all those $\chi_j - 1$ ($j \in J \backslash I$) has dimension exactly $2 - |I_+|$, which is either 1 or 0. Therefore $S \cap O_I$ is contained in a curve. This finishes Case (d) and the proof. $\square$

4. Normality of the closure of subtori

In this section we prove (ii) of Proposition 3.3, namely that the surface S is normal and isomorphic to the weighted projective plane $\mathbb{P}(a, b, c)$.

We recall the following construction in [4, §2.1] of a projective toric variety X_A out of a finite set of lattice points $A \subset M$. Let $N = \mathbb{Z}^n$ and $M = \operatorname{Hom}(N, \mathbb{Z})$. Then each $m \in M$ gives a character χ^m of the torus T_N. Any list of k lattice points $A = (m_1, \cdots, m_k) \subset M$ defines a morphism ϕ_A from T_N to $\mathbb{P}^{k-1}$:

$$\begin{aligned} \phi_A : \quad & T_N \to T_k \xrightarrow{\mu} \mathbb{P}^{k-1}, \\ & t \mapsto (\chi^{m_1}(t), \cdots, \chi^{m_k}(t)) \mapsto [\chi^{m_1}(t) : \cdots : \chi^{m_k}(t)], \end{aligned} \tag{4}$$

where $T_k \cong (\mathbb{C}^*)^k$ and $\mu : T_k \to \mathbb{P}^{k-1}$ maps T_k to the open torus $\{[x_0 : \cdots : x_{k-1}] \mid \text{ all } x_i \neq 0\}$ of $\mathbb{P}^{k-1}$.

Definition 4.1. [4, Definition 2.1.1] We denote by X_A the not necessarily normal toric variety given by the Zariski closure of the image $\phi_A(T_N)$ in $\mathbb{P}^{k-1}$.

Remark 4.2. Up to isomorphism, the definition of X_A only depends on the set of points appearing in A. So up to isomorphism we can ignore the order of the points in A, and can remove possible duplicates from A.

We note that by definition, X_A is projective. However X_A need not be normal. One of the ways to obtain normal toric varieties is from polytopes. Let P be a full dimension polytope in $M_\mathbb{R}$. Call P a lattice polytope if the vertices of P are in M. Now consider a semigroup $\mathcal{S} \subset M$, with the addition inherited from M. Recall that $\mathcal{S}$ is said to be *saturated* if for every $m \in M$, every $k \in \mathbb{Z} - \{0\}$, $km \in \mathcal{S}$ implies $m \in \mathcal{S}$.

Definition 4.3. [4, Definition 2.2.17] A lattice polytope M is very ample if for every vertex $m \in P$, the semigroup $\mathcal{S}_{P,m}$ generated by the set $P \cap M - m$ is saturated in M.

Lemma 4.4. *[4, Cor. 2.2.19] If P is a full dimensional lattice polytope, then kP is very ample if $k \geq \dim P - 1$. In particular, if P is a lattice polygon in $\mathbb{R}^2$ then P is very ample.*

Definition 4.5. [4, Definition 2.3.14] Suppose that $P \subset M_{\mathbb{R}}$ is a full dimensional lattice polytope. Then define the toric variety X_P to be X_A with $A = kP \cap M$, for any integer $k > 0$ such that kP is very ample.

The toric variety X_P is well defined since $X_{kP\cap M}$ and $X_{\ell P\cap M}$ are isomorphic when both kP and ℓP are very ample (see [4, §2.3]).

Lemma 4.6. *If P is a full dimensional very ample lattice polytope, then $X_{P\cap M}$ is a normal projective toric variety, whose fan in N is the normal fan Σ of P.*

Proof. This follows from [4, Thm. 2.3.1, Thm. 1.3.5]. □

Now we are ready to prove that S is normal and isomorphic to $\mathbb{P}(a, b, c)$.

Proof of Proposition 3.3(ii). Let $M_{12} = \mathbb{Z}\{e_1^*, e_2^*\}$. We first show that S is a normal projective variety. By Lemma 4.6, we need only show $S \cong X_{Q\cap M_{12}}$ for some full dimensional very ample lattice polytope Q in $(M_{12})_{\mathbb{R}}$. Consider $X = \mathbb{P}(a, b, c, d_1, \cdots, d_{n-2})$, with the fan Σ_X defined by generators v_i in (3). Choose any lattice polytope P in $M_{\mathbb{R}}$ whose normal fan is Σ_X. By replacing P with some multiple kP, we can assume P is very ample. By Lemma 4.6, we have $X = X_P = X_{P\cap M}$. Let $m_0, m_1, \cdots, m_u$ be the distinct lattice points of $P \cap M$. Let $\psi := \phi_{P\cap M}$ be the map defined in (4). Then

$$\begin{aligned}\psi = \phi_{P\cap M}:\ & T_N \to T_{u+1} \to \mathbb{P}^u,\\ & t \mapsto (\chi^{m_0}(t), \chi^{m_1}(t), \cdots, \chi^{m_u}(t)) \mapsto [\chi^{m_0}(t) : \chi^{m_1}(t) : \cdots : \chi^{m_u}(t)].\end{aligned}$$

Then X equals the Zariski closure of $\psi(T_N)$ in $\mathbb{P}^u$. Let $\rho : M \to M_{12}$ be the projection map. If $t \in T_{12}$, then $\chi^{m_i}(t) = \chi^{\rho(m_i)}(t)$ for every i. Therefore, the restriction of ψ on T_{12} equals

$$\begin{aligned}\psi_{|T_{12}}:\ & T_{12} \to T_{u+1} \to \mathbb{P}^u,\\ & t \mapsto (\chi^{\rho(m_0)}(t), \cdots, \chi^{\rho(m_u)}(t)) \mapsto [\chi^{\rho(m_0)}(t) : \cdots : \chi^{\rho(m_u)}(t)].\end{aligned}$$

By Proposition 3.3 (i), S equals to the Zariski closure of $\psi(T_{12})$ in X. Since X is closed in $\mathbb{P}^u$, we have S equals the Zariski closure of $\psi(T_{12})$ in $\mathbb{P}^u$.

Define $A := \rho(P \cap M)$. Then A is the set of distinct elements in the list $A' = (\rho(m_1), \cdots, \rho(m_u))$. By Remark 4.2, we can remove the duplicates in A', so that $S \cong X_A$.

Now we only need to show that $\rho(P \cap M) = \rho(P) \cap M_{12}$ and $Q := \rho(P)$ is a full dimensional very ample lattice polytope in M_{12}. We first show that Q is a lattice triangle in $(M_{12})_{\mathbb{R}}$. Recall that P has the following facet presentation:

$$P = \{z \in M_{\mathbb{R}} \mid \langle v_i, z\rangle \le a_i \text{ for } i = 0, 1, \cdots, n\} \tag{5}$$

for some $a_i \in \mathbb{Z}$ (see [6, p. 66], [4, 2.2.1]). Since the normal fan of P is Σ_X, P has exactly $n+1$ facets F_i whose outer normal vectors are v_i, $i = 0, \cdots, n$ respectively. The reason that $a_i \in \mathbb{Z}$ is as follows: Fix $i \in \{0, 1, \cdots, n\}$. Let m be a vertex of the facet F_i. Then m is a vertex of P, so $m \in M$. Since $m \in F_i$, we in fact have $\langle v_i, m\rangle = a_i$. Thus $a_i \in \mathbb{Z}$ since $v_i \in N$.

Let $z = (z_1, \cdots, z_n) \in M_{\mathbb{R}}$. Then $\rho(z) = (z_1, z_2)$. By definition of u_i and v_i in (3), we have $\langle v_i, z\rangle = \langle u_i, \rho(z)\rangle - (z_3 m_{i,3} + \cdots + z_n m_{i,n})$ for $i = 0, 1, 2$, and $\langle v_j, z\rangle = z_j$ for $j \in J = \{3, 4, \cdots, n\}$. Therefore $z \in P$ if and only if $\langle u_i, \rho(z)\rangle \le a_i + (z_3 m_{i,3} + \cdots + z_n m_{i,n})$ for $i = 0, 1, 2$ and $z_j \le a_j$ for $j \in J$. Recall that every $m_{i,j} \ge 0$. As a result, $y \in Q$ if and only if $\langle u_i, y\rangle \le a_i + (a_3 m_{i,3} + \cdots + a_n m_{i,n})$ for $i = 0, 1, 2$. Define $q_i := a_i + (a_3 m_{i,3} + \cdots + a_n m_{i,n})$ for $i = 0, 1, 2$. Then

$$Q = \{y \in (M_{12})_{\mathbb{R}} \mid \langle u_i, y\rangle \le q_i, \text{ for } i = 0, 1, 2\}. \tag{6}$$

Indeed (5) is a facet presentation of Q. Thus Q is a triangle in $(M_{12})_{\mathbb{R}}$.

It remains to show that Q is a lattice triangle. A point $z \in P$ (or Q) is a vertex of P (or Q) if and only if z lives in all but one facets. By the facet presentation (5) of P, m is a vertex of P if and only if $\langle v_i, m\rangle = a_i$ for all v_i but one. Suppose that ξ_0, ξ_1, ξ_2 are the vertices of P where ξ_i lives in the n facets except F_i. We claim that $\rho(\xi_0), \rho(\xi_1)$ and $\rho(\xi_2)$ are the three vertices of Q. Indeed, we need only to prove this for ξ_0. Let $\xi_0 = (z_1, \cdots, z_n)$. Then $a_j = \langle v_j, \xi_0\rangle = z_j$ for $j \in J$, and $a_k = \langle v_k, \xi_0\rangle = \langle u_k, \rho(\xi_0)\rangle - (z_3 m_{k,3} + \cdots + z_n m_{k,n})$ for $k = 1, 2$. By definition, this shows that $\langle u_k, \rho(\xi_0)\rangle = q_k$ for $k = 1, 2$. Let F'_i be the facet of Q normal to u_i, for $i = 0, 1, 2$ (see (6)). Then $\rho(\xi_0) = F'_1 \cap F'_2$ is a vertex of Q. Since P is a lattice polytope, $\xi_0 \in M$, so $\rho(\xi_0) \in M_{12}$. Repeat this argument for ξ_1 and ξ_2. Then $\rho(\xi_0), \rho(\xi_1)$ and $\rho(\xi_2)$ are distinct vertices of Q. Therefore Q is a lattice triangle. By Lemma 4.4, any lattice triangle in M_{12} is very ample, so Q is very ample. Hence we verified that Q is a full dimensional very ample lattice polytope.

It remains to show $\rho(P \cap M) = \rho(P) \cap M_{12}$. By definition, $\rho(P \cap M) \subseteq \rho(P) \cap M_{12}$. Conversely, suppose $y = (z_1, z_2) \in \rho(P) \cap M_{12}$. Then $y = \rho(z)$ where $z := (z_1, z_2, a_3, \cdots, a_n)$. By (6), we have $\langle u_i, y\rangle \le q_i$ for $i = 0, 1, 2$. Hence $\langle u_i, \rho(z)\rangle \le q_i = a_i + (a_3 m_{i,3} + \cdots + a_n m_{i,n})$ for $i = 0, 1, 2$. The argument preceding (6) shows that $z \in P$. Since z_1, z_2, all a_i and all $m_{i,j}$ are integers, we have $z \in M$. Thus $\rho(P) \cap M_{12} \subseteq \rho(P \cap M)$. We conclude that $\rho(P \cap M) = \rho(P) \cap M_{12}$. Therefore, $S = X_{\rho(P)\cap M_{12}}$ is normal. Furthermore, by Proposition 4.6, the fan of S in N_{12} is the normal fan of Q with respect to N_{12}, hence is spanned by u_0, u_1 and u_2. By (3), the fan spanned by u_0, u_1 and u_2 is a fan of $\mathbb{P}(a, b, c)$. As a conclusion, $S \cong \mathbb{P}(a, b, c)$. □

5. Intersection products on weighted projective spaces

We prove Theorem 1.1 and Theorem 1.2 in this section. In Section 3 we constructed a fan Σ_X for $X = \mathbb{P}(a, b, c, d_1, \cdots, d_{n-2})$, under the assumption (i) of Theorem 1.1. Recall that S is defined as the intersection of Y_j for $j \in J$, where $J = \{3, 4, \cdots, n\}$. By Lemma 3.3 (ii), S is isomorphic to $\mathbb{P}(a, b, c)$.

We start with a review of the intersection products of various torus-invariant divisors on X and S. Let $A_d(X)$ be the Chow group of d-dimensional cycles in X. Since X is a complete simplicial toric variety, by [4, Lem. 12.5.1], $A_d(X)$ is generated by the classes of torus-invariant subvarieties $[V_I]$ where $|I| = n - d$. In particular, $A_{n-1}(X)$ is generated by the classes of torus-invariant Weil divisors $\{[D_i] \mid i = 0, 1, 2, \cdots, n\}$. The divisor class group $\mathrm{Cl}(X)$ of X is isomorphic to $\mathbb{Z}$ by [4, Ex. 4.1.5]. Let A be a pseudo-effective Weil divisor on X which generates $\mathrm{Cl}(X)$. Then in $A_{n-1}(X) = \mathrm{Cl}(X)$ we have

$$[D_0] = a[A], \quad [D_1] = b[A], \quad [D_2] = c[A], \quad [D_j] = d_{j-2}[A], \text{ for } j \ge 3. \tag{7}$$

Now Σ_X is simplicial (Notation 3.1). By [4, Lem. 12.5.2], we have the following intersection products:

$$\begin{aligned}
&[A]^n = \frac{1}{abcd_1 \cdots d_{n-2}},\\
&[D_3] \cdot [D_4] \cdot \ldots \cdot [D_n] = [V_J],\\
&[V_J] \cdot [D_i] = [V_{J\cup\{i\}}], \text{ for } i = 0, 1, 2,\\
&[D_1] \cdot [D_2] \cdot [V_J] = \frac{1}{a}, \quad [D_0] \cdot [D_2] \cdot [V_J] = \frac{1}{b}, \quad [D_0] \cdot [D_1] \cdot [V_J] = \frac{1}{c}.
\end{aligned} \tag{8}$$

By Notation 3.1, $N_{12} = \mathbb{Z}\{e_1, e_2\}$. Let Σ_S in $(N_{12})_{\mathbb{R}}$ be the fan of S generated by ray generators u_0, u_1 and u_2 (see (3)). Define $B_i := V(\sigma_{\{i\}})$ to be the torus-invariant divisors of S corresponding to u_i. By [4, Ex. 4.1.5], $\mathrm{Cl}(S) \cong \mathbb{Z}$. Let B be a pseudo-effective Weil divisor on S which generates $\mathrm{Cl}(S)$. Then

$$[B_0] = a[B], \quad [B_1] = b[B], \quad [B_2] = c[B], \quad [B]^2 = \frac{1}{abc}. \tag{9}$$

Next we recall a result by Fulton and Sturmfels [7]. Let W be a toric variety of a fan $\Sigma \subset N = \mathbb{Z}^n$. As in [7], define N_σ as $\mathbb{Z}(N \cap \sigma)$, the sublattice spanned by σ in N. Let L be a saturated d-dimensional sublattice of N. Let Y be the Zariski closure of the subtorus $T_L = L \otimes_{\mathbb{Z}} \mathbb{C}^*$ in W. For every lattice point $w \in N$, define

$$\Sigma(w) := \{\sigma \in \Sigma : L_{\mathbb{R}} + w \text{ meets } \sigma \text{ in exactly one point}\}.$$

Here $L_{\mathbb{R}} + w := \{x + w \mid x \in L_{\mathbb{R}}\}$.

Definition 5.1. [7, §3] w is called *generic* (with respect to L) if $\dim \sigma = n - d$ for all $\sigma \in \Sigma(w)$.

Lemma 5.2. *[7, Lem. 3.4] Let W, L and Y be defined as above. If $w \in N$ is a generic point with respect to L, then*

$$[Y] = \sum_{\sigma \in \Sigma(w)} m_\sigma [V(\sigma)] \in A_d(W),$$

where $m_\sigma := [N : L + N_\sigma]$ is the index of the lattice sum $L + N_\sigma$ in N.

For simplicity, when there are no ambiguity of the choice of L, and when the toric variety W has a simplicial fan Σ spanned by rays $r_0, r_1, \cdots, r_n$, we write $m_{\sigma_I} = [N : L + N_\sigma]$ as m_I, for $I \subset \{0, 1, \cdots, n\}$. When $I = \{i\}$, we write m_{σ_I} as m_i.

Lemma 5.3. *Let X, Y_j and S be defined as in Definition 3.2. Then $[Y_j] = [D_j]$ for all $j \in J$, and $[S] = [V_J]$.*

Proof. Fix $j \in J$. By Notation 3.1, $L_j := \mathbb{Z}\{e_1, e_2, \cdots, \widehat{e_j}, \cdots, e_n\}$. By Definition 3.2, Y_j is the Zariski closure of $T_j = L_j \otimes_{\mathbb{Z}} \mathbb{C}^*$ in X. We apply Lemma 5.2 to $W = X$, $Y = Y_j$ and $L = L_j$. First, e_j is generic with respect to L_j. Indeed if $j \notin I$, then $(L_j)_{\mathbb{R}} + e_j$ does not meet σ_I. If $j \in I$, then σ_I intersects $(L_j)_{\mathbb{R}} + e_j$ at a single point if and only if $I = \{j\}$. Hence $\Sigma(e_j) = \{\sigma_{\{j\}}\}$. Since $\sigma_{\{j\}}$ is a 1-dimensional cone, e_j is generic. By Lemma 5.2, $[Y_j] = m_j[D_j]$, and m_j equals the index of $L_j + N_{\sigma_{\{j\}}}$ in N, which equals to 1, so $[Y_j] = [D_j]$.

Similarly, $N_{12} := \mathbb{Z}\{e_1, e_2\}$, and S is the Zariski closure of $T_{12} := N_{12} \otimes_{\mathbb{Z}} \mathbb{C}^*$. The same argument above shows that $\Sigma(\omega) = \{\sigma_J\}$, where $\omega = (0, 0, 1, \cdots, 1) \in N$ is generic with respect to N_{12}. Apply Lemma 5.2 to $W = X, Y = S$ and $L = N_{12}$. Then we have $[S] = m_J[V_J]$. Here $m_J = 1$ since $N_{12} + N_{\sigma_J} = N$. $\square$

Definition 5.4. Let $N_1 = \mathbb{Z}\{e_1\}$ and $T_1 := N_1 \otimes_{\mathbb{Z}} \mathbb{C}^*$. Let C_1 be the Zariski closure of the subtorus T_1 in S.

Lemma 5.5. *Let C_1 be defined as above. Then*

(i) *The irreducible curve C_1 equals the closure of the subtorus T_1 in X.*
(ii) *The class $[C_1] = -y_0[V_{J\cup\{0\}}] - y_1[V_{J\cup\{1\}}] \in A_1(X)$.*
(iii) *The class $[C_1] = y_2[V_J] \cdot [D_2] \in A_1(X)$.*
(iv) *The class $[C_1] = y_2[B_2] = cy_2[B] \in A_1(S)$.*

Proof. Let $\overline{T_1}$ be the closure of T_1 in X. By definition, T_1 is contained in S. Since S is closed in X, $\overline{T_1}$ is contained in S. Therefore $C_1 = \overline{T_1}$. Hence, both C_1 and C are irreducible. This proves (i). For (ii), we work in $N = \mathbb{Z}^n$. Define $w = (w_1, w_2, 1, \cdots, 1) \in N$ such that (w_1, w_2) lies in the interior of the cone spanned by u_0 and u_1. We claim that w is generic with respect to N_1. Indeed, by the definition of u_i (see (3)), the second coordinates of u_0 and u_1 are negative and the second coordinate of u_2 is positive. Hence $w_2 < 0$. Suppose the line $\ell := (N_1)_{\mathbb{R}} + w$ intersects σ_I. Then $J \subset I$. Since $w_2 < 0$, ℓ misses σ_J and $\sigma_{J\cup\{2\}}$, and meets

$\sigma_{J\cup\{0\}}$ and $\sigma_{J\cup\{1\}}$ at a unique point. In the remaining case, $I = J \cup \{i_1, i_2\}$ with distinct $i_1, i_2 \in \{0, 1, 2\}$, so ℓ intersects σ_I at infinitely many points. As a conclusion, $\Sigma(w) = \{\sigma_{J\cup\{0\}}, \sigma_{J\cup\{1\}}\}$, so w is generic.

Apply Lemma 5.2 to $W = X, Y = C_1$ and $L = N_1$. We have

$$[C_1] = m_{J\cup\{0\}}[V_{J\cup\{0\}}] + m_{J\cup\{1\}}[V_{J\cup\{1\}}].$$

By definition, $m_{J\cup\{0\}} = [N : N_1 + N_{\sigma_{J\cup\{0\}}}]$. Since $N_1 + N_{\sigma_{J\cup\{0\}}}$ is spanned by $e_1, e_3, \cdots, e_n$ together with v_0, the index equals to the absolute value of the second coordinate of v_0. That is, $m_{J\cup\{0\}} = |y_0|$. Recall our assumption in Section 3 that $y_0, y_1 < 0$ and $y_2 > 0$. Hence $m_{J\cup\{0\}} = -y_0$. Similarly we have $m_{J\cup\{1\}} = -y_1$. This proves (ii). Now use formulas (7) and (8):

$$\begin{aligned}[C_1] &= -y_0[V_{J\cup\{0\}}] - y_1[V_{J\cup\{1\}}] = -y_0[V_J]\cdot[D_0] - y_1[V_J]\cdot[D_1] \\ &= [V_J]\cdot[-y_0 a[A] - y_1 b[A]] = cy_2[V_J]\cdot[A] = y_2[V_J]\cdot[D_2].\end{aligned}$$

This proves (iii).

Finally consider C_1 as a curve on S. The fan Σ_S lives in $(N_{12})_{\mathbb{R}}$ (see Notation 3.1). We have $\Sigma(e_2) = \{B_2\}$. Therefore $e_2 = (0, 1)$ is generic with respect to N_1. Apply Lemma 5.2 to $W = S, Y = C_1$ and $L = N_1$. Then $[C_1] = m_2[B_2] \in A_1(S)$ where $m_2 = [\mathbb{Z}^2 : (N_1)_{\mathbb{R}} + \mathbb{Z}u_2] = |y_2| = y_2$. This proves (iv). □

Lemma 5.6. *Consider the class $[B] \in A_1(X)$. Then we have $[B].[Y_j] = \dfrac{d_{j-2}}{abc}$, for $j \in J$.*

Proof. By Lemma 5.5, $[C_1] = cy_2[V_J]\cdot[A] \in A_1(X)$, and $[C_1] = cy_2[B] \in A_1(S)$. Therefore $cy_2[B] = cy_2[V_J]\cdot[A]$ in $A_1(X)$, so $[B] = [V_J]\cdot[A] = \dfrac{1}{a}[V_J]\cdot[D_0]$ in $A_1(X)$. Then

$$[B].[Y_j] = \frac{1}{a}[V_J]\cdot[D_0]\cdot\frac{d_{j-2}}{b}[D_1] = \frac{d_{j-2}}{abc}. \quad \square$$

Now we prove Theorem 1.1.

Proof of Theorem 1.1. By definition $X = \mathbb{P}(a, b, c, d_1, \cdots, d_{n-2})$ is a weighted projective n-space. By Proposition 3.3, $S = \mathbb{P}(a, b, c)$ is a weighted projective plane. Hence both X and S are normal projective $\mathbb{Q}$-factorial varieties, with finitely generated Picard groups. By Proposition 3.3, $S = \cap_{j=3}^{n} Y_j$. By assumption, C is a negative curve on $\mathrm{Bl}_p S$ and $C \neq e$. To apply Theorem 2.1 to X, Y_j, S and C, we need only verify that $(f_*C)\cdot \mathrm{Bl}_p Y_j < 0$ for $j = 3, 4, \cdots, n$. Here $(f_*C)\cdot \mathrm{Bl}_p Y_j = f_*C\cdot(\pi_X^* Y_j - E)$, and $C \sim_{\mathbb{Q}} \lambda\pi^* B - \mu e$. Hence by Lemma 5.6 and projection formula:

$$\begin{aligned}f_*C\cdot(\pi_X^*Y_j - E) &= (\pi_X)_* f_*[C]\cdot[Y_j] - f_*[C]\cdot[E] \\ &= \lambda[B].[Y_j] - \mu = \frac{\lambda d_{j-2}}{abc} - \mu < 1.\end{aligned}$$

By Theorem 2.1, $\mathrm{Bl}_p X$ is not a MDS. This proves the theorem. □

Finally we prove Theorem 1.2.

Proof of Theorem 1.2. Suppose there is a relation $(e, f, -g)$ between the weights (a, b, c) such that the width $w = cg^2/(ab) < 1$.

We need only show that there exists a non-exceptional negative curve C on $\mathrm{Bl}_p S$ satisfying the assumption in Theorem 1.1 with $\lambda = cg$ and $\mu = 1$, and $d_i < abc\mu/\lambda = ab/g$ for all $i = 0, 1, \cdots n-2$. We first choose a specific fan Σ_S and use Σ_S to define Σ_X. Indeed, by [13, Prop. 5.1], there exists a unique integer r with $1 \le r \le g$, $g \mid er - b$ and $g \mid fr + a$. Let $u_i = (x_i, y_i)$ be given by (1):

$$u_0 = \left(\frac{er-b}{g}, -e\right), \quad u_1 = \left(\frac{fr+a}{g}, -f\right), \quad u_2 = (-r, g). \tag{10}$$

Then u_i span a fan of S. Let this fan be Σ_S. We check that $y_0 = -e < 0$, $y_1 = -f < 0$ and $y_2 = g > 0$, so all the assumptions in Section 3 are satisfied. Then we can use u_i to define v_i and the fan Σ_X as in (3). Consider the curve C_1 in Definition 5.4. Let C be the proper transform of C_1 in $\mathrm{Bl}_p S$. Then $C \sim \pi^* C_1 - e$ on $\mathrm{Bl}_p S$. By Lemma 5.5 (iv), $C \sim cg\pi^* B - e$. Hence $\lambda = cg$ and $\mu = 1$. By (9), $[B]^2 = 1/abc$. Hence $[C_1]^2 = g^2c^2/abc = cg^2/ab = w$, and $[C]^2 = [C_1]^2 - 1 = w - 1 < 0$. Since $\pi(C) = C_1$ is not a point, C is not e. As a result, C is a non-exceptional negative curve on $\mathrm{Bl}_p S$. Finally by assumption (ii) of Theorem 1.2, for every i, $d_i^2 w < abc$. Therefore $d_i^2 cg^2/(ab) < abc$. That is, $d_i < ab/g$. By Theorem 1.1, we conclude that $\mathrm{Bl}_p X$ is not a MDS. $\square$

6. Comparison with González and Karu's examples

We compare the 3-dimensional case of Theorem 1.2 with [9, Thm. 2.3, Cor. 2.5].

Definition 6.1. Consider a n-dimensional convex polytope Δ in $\mathbb{R}^n$ such that all its vertices have rational coordinates.

- (i) For $n = 3$, we say such a polytope is *of González–Karu type* if the vertices of Δ are $(0, 0, 1)$, $(0, 1, 0)$, P_L and P_R, with P_L and P_R and 0 collinear, and $x(P_L) < 0 < x(P_R) \le x(P_L) + 1$, where $x(P_R)$ and $x(P_L)$ are the x-coordinates (see [9, §2.2]).
- (ii) For $n = 2$, we say such a polytope is of González–Karu type if Δ is a triangle with vertices $(0, 0)$, P_L and P_R, with P_L and P_R and $(0, 1)$ collinear, and $x(P_L) < 0 < x(P_R) < x(P_L) + 1$.
- (iii) In both dimension 2 and 3, define the *width* of a polytope of González–Karu type to be $x(P_R) - x(P_L)$.

By definition, 3-dimensional polytope Δ of González–Karu type has some evident properties:

- (a) The cross sections of Δ at $x = i \in \mathbb{N}$ are isosceles right triangles.
- (b) Projecting $\Delta \in \mathbb{R}^3$ of González–Karu type and of width < 1 to xy-plane or xz-plane, and then translating by the vector $(0, -1)$ will give a triangle of González–Karu type with the same width.

We first recall the following numerical criteria from [8], [9] for the weights for $\mathbb{P}(a, b, c, d)$ or $\mathbb{P}(a, b, c)$ to have a polytope of González–Karu type. We rephrase the criteria as follows:

Lemma 6.2.

- (i) *Given $w \in \mathbb{Q} \cap (0, 1)$. Consider $\mathbb{P}(a, b, c)$ with a, b, c pairwise coprime. Then $\mathbb{P}(a, b, c)$ has a polytope Δ of González–Karu type of width w if and only if there exist a relation $(e, f, -g)$ with $ae + bf = cg$ (up to a permutation of the weights a, b, c) and $w = cg^2/ab$. Furthermore, up to switching a with b, and up to a shear transformation $(x, y) \mapsto (x, y + kx)$ for some $k \in \mathbb{Z}$, Δ has vertices given by (2), i.e.,*

$$(0, 0), \quad \left(-\frac{eg}{b}, -\frac{er-b}{b}\right), \quad \left(\frac{fg}{a}, \frac{fr+a}{a}\right), \tag{11}$$

where r is the unique integer such that $1 \le r \le g$, $g \mid er - b$ and $g \mid fr + a$ [13, Prop. 5.1], and Δ is normal to the fan with ray generators given in (1). In particular, when $w < 1$, the numbers of lattice points on slices of Δ are determined by a, b, c.

(ii) *Given $W \in \mathbb{Q} \cap (0,1)$. Consider $\mathbb{P}(a,b,c,d)$ with every 3 weights relatively prime. Then $\mathbb{P}(a,b,c,d)$ has a polytope Δ of González–Karu type of width W if and only if there exist positive integers e, f, g_1, g_2 such that up to a permutation of the weights a, b, c and d, we have*

$$ae + bf = cg_1 = dg_2, \quad W = (dg_2)^3/(abcd), \quad \gcd(e,f,g_1) = \gcd(e,f,g_2) = \gcd(g_1,g_2) = 1.$$

The following definition is from [9]:

Definition 6.3. [9, §2.2] Suppose Δ is a 2 or 3-dimensional polytope of González–Karu type. Suppose m is a positive integer such $m\Delta$ is a lattice polytope. For any integer i such that $m \cdot x(P_L) \le i \le m \cdot x(P_R)$, the *slice* at $x = i$ is the set of lattice points in $m\Delta$ with x-coordinates i. When $\dim \Delta = 2$, a slice of $m\Delta$ consists of consecutive lattice points on a line. When $\dim \Delta = 3$, a slice of $m\Delta$ forms a right triangle with the same number n of lattice points on each side. Then say the slice at $x = i$ has size n.

To avoid ambiguity, in the following we use Γ to represent a 2-dimensional polytope of González–Karu type. We recall the following criteria in [8] and [9] for $\mathrm{Bl}_p X$ to be not a MDS where X is a toric surface or toric 3-fold with a polytope of González–Karu type.

Theorem 6.4. *[8, Thm. 1.2] Suppose S is a toric surface with fan Σ in $\mathbb{R}^2$. Suppose $\Gamma \subset \mathbb{R}^2$ is a triangle of González–Karu type with width w and normal fan Σ. Let $m > 0$ be a sufficiently large and divisible integer so that $m\Gamma$ is a lattice triangle. Then $\mathrm{Bl}_p S$ is not a MDS if the following hold:*

(i) *Let the slice at $m \cdot x(P_L) + 1$ of $m\Gamma$ have exactly n elements. Then the slice at $m \cdot x(P_R) - n + 1$ of $m\Gamma$ has exactly n elements.*
(ii) *$ns_2 \notin \mathbb{Z}$, where $s_2 := (y(P_R) - y(P_L))/w$ is the slope of the line through P_L and P_R.*

Theorem 6.5. *[9, Cor. 2.5] Suppose X is a toric 3-fold with fan Σ in $\mathbb{R}^3$. Suppose $\Delta \subset \mathbb{R}^3$ is a polytope of González–Karu type with width W and normal fan Σ. Let $m > 0$ be a sufficiently large and divisible integer so that $m\Delta$ is a lattice polytope. Then $\mathrm{Bl}_p X$ is not a MDS if the following hold:*

(i) *Let the slice at $m \cdot x(P_L) + 1$ of $m\Delta$ have size n. Then the slice at $m \cdot x(P_R) - n + 1$ of $m\Delta$ has size n.*
(ii) *$n(s_y, s_z) \notin \mathbb{Z}^2$, where $s_y := (y(P_R) - y(P_L))/W$ and $s_z := (z(P_R) - z(P_L))/W$ are the y, z-slopes of the line through P_L and P_R.*

Now a natural question is that whether there are examples of $\mathbb{P}(a,b,c,d)$ meeting assumptions in Theorem 1.2 and [9, Cor. 2.5]. The following proposition provides a precise answer on the overlap:

Proposition 6.6. *Suppose $\mathbb{P}(a,b,c,d)$ has a polytope Δ of González–Karu type and satisfies the assumptions including (i)–(iv) of Theorem 1.2. Then $d = cg$, where $(e, f, -g)$ is the unique relation between (a,b,c) with $w < 1$.*

Conversely, every weighted projective 3-space $\mathbb{P}(a,b,c,cg)$ such that (a,b,c) has a relation $(e,f,-g)$ with $w < 1$, and $\mathbb{P}(a,b,c)$ has a polytope satisfying the conditions in [8, Thm. 1.2] with width w, will satisfy the assumptions in both Theorem 1.2 and [9, Cor. 2.5].

Remark 6.7. In the proof of Theorem 1.2, we in fact showed that weighted projective spaces $\mathbb{P}(a,b,c,d)$ meeting the conditions of the theorem must contain the weighted projective plane $S = \mathbb{P}(a,b,c)$ where $\mathrm{Bl}_p S$

is not a MDS. Recall Theorem 3.3 that S is the Zariski closure of the subtorus $T_{12} = L_{12} \otimes \mathbb{C}^*$, where $(L_{12})_{\mathbb{R}}$ is the xy-plane.

Question: *Is there any $\mathbb{P}(a,b,c,d)$ such that $\mathrm{Bl}_p\,\mathbb{P}(a,b,c,d)$ is not a MDS, but for any 2-dimensional subtorus T' of the open torus T_N, the blow-up $\mathrm{Bl}_p\,\overline{T'}$ of the Zariski closure of T' is a MDS?*

Note that the Zariski closure $\overline{T'}$ may have Picard number 1 or 2.

We first prove Lemma 6.2. We note the following fact:

Lemma 6.8 *(See [8, §1]). Suppose a,b,c are pairwise coprime positive integers. Then there exist at most one relation $(e,f,-g)$ of (a,b,c) with $cg^2 < ab$, even when permuting a,b,c.*

Proof of Lemma 6.2. First we prove (i). Suppose $\mathbb{P}(a,b,c)$ has a relation of weight $w<1$, then the polytope in (2) is of González–Karu type with width w. Conversely, suppose $S = \mathbb{P}(a,b,c)$ has a polytope Γ of González–Karu type with width $w<1$. Then S has a fan Σ_S normal to Γ. Say the ray generators of Σ_S is $r_i = (x_i, y_i)$, $i=1,2,3$. Then we can assume $y_1<0$, $y_2<0$, $y_3>0$, $ar_1+br_2+cr_3=0$, and $P_L = s(y_1,-x_1)$, $P_R = t(-y_2,x_2)$ for some $s,t\in\mathbb{Q}$. Since r_i span the fan of $\mathbb{P}(a,b,c)$, the absolute values of the 2×2 minors of the following matrix should equal to (c,b,a) respectively:

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

Now the collinearity of P_L, P_R and $(0,1)$ gives $w = P_R - P_L = stc$. The condition that $\overline{P_L P_R}$ being perpendicular to r_3 gives $bs = at = |y_3| = y_3$. Therefore $w = stc = y_3^2 c/ab < 1$. So $ay_1+by_2+cy_3 = 0$ and $y_3^2c/ab<1$. Now $\gcd(a,b,c)=1$, so $\gcd(y_1,y_2,y_3)=1$. Write $y_1=-e$, $y_2=-f$ and $y_3=g$. By Lemma 6.8, $(e,f,-g)$ is the unique relation. After a shear transformation of the form $(x,y)\mapsto(x,y+kx)$ for some $k\in\mathbb{Z}$, we can assume $1\le x_3\le g$. Then $gx_1 = ex_3 \pm b$ and $gx_2 = -fx_3 \mp a$. So up to switching a with b, x_3 is the unique integer r such that $1\le r\le g$, $g \mid er-b$ and $g \mid fr+a$. This shows that Γ is of the required form, up to a reflection about the y-axis and a shear transformation. The shear transformations add the same integer k to the slopes of sides of Γ. Hence the numbers of lattice points on the slices are unchanged.

Next we prove (ii). Suppose $\mathbb{P}(a,b,c,d)$ has a polytope Δ of González–Karu type, with $P_R = (x,y,z)$, $x>0$ and $P_L = \lambda(x,y,z)$ for some $\lambda<0$. The fan Σ is normal to Δ. Therefore the four rays $R_1,\cdots,R_4$ of Σ are the outer normal vectors of the four faces of Δ. Direct calculation shows that R_i is spanned by the vector r_i:

$$\begin{aligned} r_1 &= (1-y-z, x, x), & r_2 &= (\lambda y + \lambda z - 1, -\lambda x, -\lambda x), \\ r_3 &= (y, -x, 0), & r_4 &= (-\lambda z, 0, \lambda x). \end{aligned} \tag{12}$$

Now let r_i' be the first lattice point in the ray R_i. Because $x>0$ and $\lambda<0$, there must exist positive integers e,f,g_1,g_2 and integers R,S,T,U such that

$$r_1' = (R,e,e), \quad r_2' = (S,f,f), \quad r_3' = (T,-g_1,0), \quad r_4' = (U,0,-g_2).$$

Since Σ is the fan of $\mathbb{P}(a,b,c,d)$, up to a permutation of the weights, we have $ar_1'+br_2'+cr_3'+dr_4'=0$. Take the last two components, we have $ae+bf = cg_1 = dg_2$. Since Σ is a fan of $\mathbb{P}(a,b,c,d)$, the weights (a,b,c,d) equal to the 3×3 minors of the matrix with rows $r_1',\cdots,r_4'$. For any 3 vectors v_1,v_2 and v_3 in $\mathbb{R}^3$, we denote by $\det(v_1,v_2,v_3)$ the determinant of the square matrix with row vectors v_1,v_2 and v_3. Then we have

$$a = |\det(r_2', r_3', r_4')| = \frac{g_1g_2}{x}|Sx + fy + fz| = \frac{g_1g_2}{x}\left|\frac{(\lambda y + \lambda z - 1)f}{-\lambda} + fy + fz\right| = -\frac{fg_1g_2}{\lambda x},$$
$$b = |\det(r_1', r_3', r_4')| = \frac{g_1g_2}{x}|Rx + ey + ez| = \frac{g_1g_2}{x}|(1 - y - z)e + ey + ez| = \frac{eg_1g_2}{x},$$

where we used that each r_i' is a scalar multiple of r_i. Note that the other two equations of c and d do not give new algebraic relations. As a result,

$$x = \frac{eg_1g_2}{b}, \quad \lambda = -\frac{bf}{ae}, \tag{13}$$

$$W = x(P_R) - x(P_L) = x - \lambda x = \frac{eg_1g_2}{b}\left(1 + \frac{bf}{ae}\right) = \frac{eg_1g_2}{b} \cdot \frac{dg_2}{ae} = \frac{cg_1 \cdot dg_2 \cdot dg_2}{abcd} = \frac{(dg_2)^3}{abcd}. \tag{14}$$

At last, the coprime conditions follow from the assumption that every 3 of a, b, c, d are relatively prime, and the expression of a, b, c, d as the determinants of r_i' with R, S, T and U are integers. This proves the 'only if' direction. Conversely, suppose $ae + bf = cg_1 = dg_2$ and $W = (dg_2)^3/(abcd)$. We can always choose integers T and U such that $\gcd(T, g_1) = \gcd(U, g_2) = 1$. Let $y = Tx/g_1$ and $z = Ux/g_2$, with x and λ given above in (13). The parameters x, y, z, λ determine a fan Σ' with rays r_i from (12), and a polytope Δ' with $P_R = (x, y, z)$, $x > 0$ and $P_L = \lambda(x, y, z)$. Then it is straightforward that Σ' is a fan of $\mathbb{P}(a, b, c, d)$, and Δ' is of González–Karu type with width W, whose normal fan is Σ'. This proves the 'if' direction. $\square$

Finally we prove Proposition 6.6.

Proof of Proposition 6.6. Suppose $\mathbb{P}(a, b, c, d)$ has a polytope Δ of González–Karu type and meets the assumptions of Theorem 1.2. Then by Lemma 6.2, there exist $e, f, g_1, g_2 \in \mathbb{Z}_{>0}$ such that $ae + bf = cg_1 = dg_2$ (up to a permutation of the weights a, b, c and d), and the width W of Δ equals $(dg_2)^3/(abcd) \leq 1$. In this equation, a and b are symmetric. The weights c and d are also symmetric. Hence up to symmetry either $\mathrm{Bl}_p \mathbb{P}(a, b, c)$ is not a MDS or $\mathrm{Bl}_p \mathbb{P}(b, c, d)$ is not a MDS.

Case I. $\mathrm{Bl}_p \mathbb{P}(a, b, c)$ is not a MDS, with relations $(E, F, -G)$ such that the width $w < 1$. By the argument above,

$$1 \geq W = \frac{(dg_2)^3}{abcd} = \frac{cg_1^2g_2}{ab}.$$

We claim $W < 1$. Otherwise $W = 1$. Then $cg_1^2g_2 = ab$, so $c \mid ab$, which contradicts the assumption of Theorem 2.1 that a, b, c are pairwise coprime.

Hence $cg_1^2/ab < 1/g_2 \leq 1$. By Lemma 6.2, $\gcd(e, f, g_1) = 1$. Now $(e, f, -g_1)$ is a relation between (a, b, c) with $\gcd(e, f, -g_1) = 1$ and width $c(g_1)^2/(ab) = cg_1^2/(ab) < 1$. By Lemma 6.8, we must have $e = E$, $f = F$ and $g_1 = G$, $ae + bf = cg_1$, and the width of $(e, f, -g_1)$ is

$$w = \frac{cG^2}{ab} = \frac{cg_1^2}{ab} < \frac{1}{g_2} \leq 1.$$

Suppose $g_2 \geq 2$. Then $w \leq 1/2$. By Theorem 2.5 and 2.6 of [13], if $w \leq 1/2$, then $\mathrm{Bl}_p \mathbb{P}(a, b, c)$ is a MDS, which contradicts the assumption. Therefore $g_2 = 1$, and $d = cg_1$.

Case II. $\mathrm{Bl}_p \mathbb{P}(b, c, d)$ is not a MDS, and $\gcd(b, c, d) = 1$. This together with $cg_1 = dg_2$ implies that $g_1 = kd$ and $g_2 = kc$ for some $k \in \mathbb{Z}_{>0}$. Now

$$1 \geq W = \frac{cg_1^2g_2}{ab} = \frac{k^3c^2d^2}{ab}.$$

Hence $k^3c^2d^2 \leq ab$. On the other hand, $kcd = cg_1 = ae+bf \geq a+b \geq 2\sqrt{ab}$. Hence $k^3c^2d^2 \geq k\cdot(4ab) > ab$, so we reached a contradiction. This shows Case II does not happen and proves the first half of Proposition 6.6.

Next we prove the second half of Proposition 6.6. Consider any $S = \mathbb{P}(a,b,c)$ such that a,b,c are pairwise coprime, $(e,f,-g)$ is a relation between (a,b,c) of width $w < 1$ and S satisfies the assumptions in [8, Thm. 1.2]. Then $\mathrm{Bl}_p\,\mathbb{P}(a,b,c)$ is not a MDS.

Now $X := \mathbb{P}(a,b,c,cg)$ satisfies conditions (i), (ii) and (iv) of Theorem 1.2. Since $d = cg$, we have $d^2w/(abc) = cg^2w/(ab) = w^2 < 1$. This verifies condition (iii). Hence $X = \mathbb{P}(a,b,c,cg)$ is an example of Theorem 1.2.

It remains to show that $X = \mathbb{P}(a,b,c,cg)$ satisfies the two assumptions in [9, Cor. 2.5]. Indeed, here $ae + bf = cg = d\cdot 1$ with $cg^2/ab < 1$. By Lemma 6.2, X and $S = \mathbb{P}(a,b,c)$ have polytopes Δ and Γ of González–Karu type. Let r be the unique integer such that $1 \leq r \leq g$, $g \mid er - b$ and $g \mid fr + a$. Recall the proof of Lemma 6.2. By setting $T = -r$ and $U = 0$, we can determine the parameters x, y, z and λ to give

$$P_L = \left(-\frac{fg}{a}, \frac{fr}{a}, 0\right), \quad P_R = \left(\frac{eg}{b}, -\frac{er}{b}, 0\right).$$

This gives a polytope Δ of González–Karu type. The fan Σ of X can be chosen as the fan with ray generators

$$r_1' = \left(\frac{er-b}{g}, e, e\right), \quad r_2' = \left(\frac{fr+a}{g}, f, f\right), \quad r_3' = (-r,-g,0), \quad r_4' = (0,0,-1).$$

Define Γ to be the projection of Δ to the xy-plane, after translating $(0,1)$ to $(0,0)$ and a reflection about y-axis. Then Γ is the triangle given by (2), which is a polytope of $S = \mathbb{P}(a,b,c)$.

Now let Γ' be the reflection of Γ about the y-axis. By the hypothesis and Lemma 6.2 (i), either (S,Γ) or (S,Γ') meets the assumptions of [8, Thm. 1.2]. By symmetry we can assume the case (S,Γ). Then [8, Thm. 1.2] (i) says that for some $m > 0$, the slice at $m\cdot x(P_L)+1$ of $m\Gamma$ has exactly n elements, and the slice at $m\cdot x(P_R)-n+1$ of $m\Gamma$ has exactly n elements too. By Definition 6.3, every slice of Δ forms a right triangle with the same number of lattice points on each right side. Hence, both slices of $m\Delta$ at $m\cdot x(P_L)+1$ and $m\cdot x(P_R)-n+1$ of $m\Delta$ have size n. This shows that (i) of [9, Cor. 2.5] holds. For (ii) of [9, Cor. 2.5], we have s_y equals s_2 of the triangle Γ in xy-plane. If Γ meets the assumption (ii) of [8, Thm. 1.2], then $ns_y = ns_2 \notin \mathbb{Z}$, so Δ meets the assumption (ii) of [9, Cor. 2.5]. Therefore, X satisfies the two assumptions in [9, Cor. 2.5]. $\square$

Remark 6.9. Consider $X = \mathbb{P}(a,b,c,cg)$ in the overlap described in Proposition 6.6. A comparison with [9, Lem. 5.1, 5.2] shows that the curve $C \subset \mathrm{Bl}_p\,X$ we constructed in Definition 5.4, whose class is extremal in the Mori cone $\overline{\mathrm{NE}}(\mathrm{Bl}_p\,X)$ (by Theorem 2.1), is the same curve C constructed in [9, Lem. 5.1, 5.2].

Example 6.10. An example in such family of $\mathbb{P}(a,b,c,cg)$ is $\mathbb{P}(7,15,26,52)$. By [8], $\mathrm{Bl}_p\,\mathbb{P}(7,15,26)$ is not a MDS. The relation is $(e,f,-g) = (1,3,-2)$. Both Theorem 1.2 and [9, Cor. 2.5] apply to $\mathbb{P}(7,15,26,52)$, so $\mathrm{Bl}_p\,\mathbb{P}(7,15,26,52)$ is not a MDS.

7. Application

We apply Proposition 1.1 to the following examples in [10]. By [10, Ex. 1.4], the blow-up $\mathrm{Bl}_p\,S$ of the following $S = \mathbb{P}(a,b,c)$ at the identity point p is not a MDS:

$$(a,b,c) = ((m+2)^2, (m+2)^3+1, (m+2)^3(m^2+2m-1)+m^2+3m+1), \tag{15}$$

where m is a positive integer.

We briefly review the geometry on those $\mathrm{Bl}_p S$. By [10, Thm. 1.1], for every positive integer $m \geq 1$, there exists an irreducible polynomial $\xi_m \in \mathbb{C}[x, y]$ such that ξ_m has vanishing order m at $(1, 1)$ and the Newton polygon of ξ_m is a triangle with vertices $(0, 0), (m-1, 0)$ and $(m, m+1)$. Now the weighted projective plane S above satisfies the conditions of [10, Thm. 1.3]. Then by [10, Thm. 1.3] and its proof, the polynomial ξ_m above defines a curve H in S, passing through p with multiplicity m, such that the proper transform C of H in $\mathrm{Bl}_p S$ is a negative curve. Then $C \neq e$. The proof of [10, Thm. 1.3] in fact shows that H is the polarization given by the triangle Δ with vertices $(-\alpha, 0), (m-1+\beta, 0), (m, m+1)$, with

$$\alpha = \frac{1}{(m+2)^2}, \quad \beta = \frac{(m+2)^2+1}{(m+2)^3+1}.$$

Therefore on S we have

$$H^2 = 2\mathrm{Area}(\Delta) = \frac{(m+1)^2 c}{ab}.$$

Let B be the pseudo-effective divisor on S generating $\mathrm{Cl}(S) \cong \mathbb{Z}$. Then $H \sim rB$ for some $r \in \mathbb{Q}_{>0}$. Since $B^2 = 1/abc$ and $H^2 = r^2 B^2$, we have $r = c(m+1)$, so $[H] = c(m+1)[B] \in \mathrm{Cl}(S)$. Therefore $C \sim c(m+1)\pi^* B - me$.

When $m \geq 2$, those S above have width $w \geq 1$, so Theorem 1.2 does not apply to S. Nevertheless, by Proposition 1.1, we have the following examples:

Corollary 7.1. *Let $X = \mathbb{P}(a, b, c, d_1, d_2, \cdots, d_{n-2})$ where*

$$(a, b, c) = ((m+2)^2, (m+2)^3+1, (m+2)^3(m^2+2m-1)+m^2+3m+1),$$

such that $m \in \mathbb{Z}_{>0}$, every d_i lies in the semigroup generated by a, b and c, and that every $d_i < abm/(m+1)$. Let p be the identity point of the open torus in X. Then $\mathrm{Bl}_p X$ is not a MDS.

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