ANALYSIS OF A MODEL OF ELASTIC DISLOCATIONS IN GEOPHYSICS

ANDREA ASPRI, ELENA BERETTA, ANNA L. MAZZUCATO, AND MAARTEN V. DE HOOP

ABSTRACT. We analyze a mathematical model of *elastic dislocations* with applications to geophysics, where by an elastic dislocation we mean an open, oriented Lipschitz surface in the interior of an elastic solid, across which there is a discontinuity of the displacement. We model the Earth as an infinite, isotropic, inhomogeneous, elastic medium occupying a half space, and assume only Lipschitz continuity of the Lamé parameters. We study the well posedness of very weak solutions to the forward problem of determining the displacement by imposing traction-free boundary conditions at the surface, continuity of the traction and a given jump on the displacement across the fault. We employ suitable weighted Sobolev spaces for the analysis. We utilize the well posedness of the forward problem and unique-continuation arguments to establish uniqueness in the inverse problem of determining the dislocation surface and the displacement jump from measuring the displacement at the surface of the Earth. Uniqueness holds for tangential or normal jumps and under some geometric conditions on the surface.

1. INTRODUCTION

In this paper, we analyze a mathematical model of *elastic dislocations* with applications to geophysics, see for example [13, 15, 36, 37, 43]. An *elastic dislocation* is an open, oriented surface in the interior of an elastic solid, across which there is a discontinuity of the displacement. It describes a fault plane undergoing slip over a limited area, a thin intrusion such as a dyke, or a crack the faces of which slide over one another or separate by the action of an applied stress. An elastic dislocation for which the displacement discontinuity varies from point to point of the internal surface is called a *Somigliana dislocation*, while in the particular case of a constant displacement discontinuity it is known as a *Volterra dislocation* (see [15, 40]).

We model the dislocation by an open, oriented Lipschitz surface S with Lipschitz boundary ∂S such that $\overline{S} \subset \mathbb{R}^3_-$. In particular, we assume that the dislocation is at positive distance from the surface of the Earth, identified with the plane $\{x_3 = 0\}$. We orient S by choosing a unit normal vector $\boldsymbol{n}$. In geophysical applications, one can assume that the closure $\overline{S}$ is compact. We assume the Earth's interior to be an isotropic and inhomogeneous infinite elastic medium. In the regime of small-amplitude deformations, we are led to study a boundary-value/transmission problem in a half space for the system of

linearized elasticity:

(1)
$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \boldsymbol{0}, & \text{in } \mathbb{R}^3_- \setminus \overline{S}, \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{e}_3 = \boldsymbol{0}, & \text{on } \{x_3 = 0\}, \\ [\boldsymbol{u}]_S = \boldsymbol{g}, \\ [(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}]_S = \boldsymbol{0}, \end{cases}$$

where $\mathbb{C}$ is the Lamé tensor with non-constant coefficients of Lipschitz class, satisfying the usual strong convexity assumption, $\boldsymbol{u}$ is the displacement field, $\boldsymbol{e}_3 = (0, 0, 1)$ is the unit normal vector on $\{x_3 = 0\}$, and $\boldsymbol{g}$ the displacement jump across the dislocation S. As customary, we denote the jump of a function or tensor field $\boldsymbol{f}$ across S by $[\boldsymbol{f}]_s := \boldsymbol{f}_+ - \boldsymbol{f}_-$, where $\pm$ denotes a non-tangential limit to each side of the oriented surface S, S_+ and S_-, where S_+ is by convention the side determined by $\boldsymbol{n}$. The *direct* or *forward problem* consists, knowing $\mathbb{C}$, S, and $\boldsymbol{g}$, in finding $\boldsymbol{u}$ solution of (1). The *inverse problem* consists in determining S and $\boldsymbol{g}$ from measurements made on $\boldsymbol{u}$. In seismology and geophysics, the data is typically in the form of measurements taken at the surface of the Earth. For the dislocation problem, since the solution is traction-free at the boundary, these data consist in measurements of the displacement at the surface induced by the jump $\boldsymbol{g}$ at the dislocation.

To be more specific, we investigate the inverse problem of determining dislocations S caused by a tangential slip along the dislocation surface (the case of a purely normal jump across the surface can also be included in our analysis) from surface measurements of the displacement $\boldsymbol{u}$ on some bounded open portion Σ of $\{x_3 = 0\}$. One of the main results of this work is the unique determination of the dislocation surface S and the slip strength $\boldsymbol{g}$ from knowledge of $\boldsymbol{u}$ on Σ, under some geometric conditions on S. These conditions are satisfied, for example, by polyhedral surfaces. This inverse problem is of particular interest in applications. In fact, the analysis of coseismic deformation through the inverse slip and dislocation problem might enhance the understanding of failure at faults and microseismicity, see for example [16] and references therein.

In order to investigate the inverse problem, one needs to have a good well-posedness theory for the forward problem. This problem is less studied than more classical transmission problems in bounded domains, and there are some additional technical difficulties that need to be overcome. For starters, the solution is discontinuous across the interface and the quadratic variational form needs to be properly augmented to obtain coercivity for weak solutions. In addition, the problem is naturally posed in an unbounded domain, a half space, which leads us to employ suitable weighted Sobolev spaces. Here, and throughout the paper, we denote standard L^2-based Sobolev spaces with H^s, $s \in \mathbb{R}$. The notation for the weighted spaces is discussed in Section 2.

If $\boldsymbol{g}$ belongs to the space $H^{1/2}(S)$ and has compact support in S, then the transmission problem (1) admits a unique variational solution $\boldsymbol{u}$ on $\mathbb{R}^3_- \setminus \overline{S}$ in a suitable weighted Sobolev space that takes into account the conditions at infinity (see [39] for the case of constant coefficients). In fact, $\boldsymbol{u}$ can be expressed as a double layer potential on the dislocation S, i.e.,

$$\boldsymbol{u}(\boldsymbol{y}) = -\int_S \left[(\mathbb{C}(\boldsymbol{x})\widehat{\nabla}_{\boldsymbol{x}}\mathbf{N}(\mathbf{x}, \mathbf{y}))\mathbf{n}(\mathbf{x}) \right]^T \boldsymbol{g}(\boldsymbol{x}) \, d\sigma(\boldsymbol{x}),$$

where $\mathbf{N}$ is the matrix-valued Neumann function

$$\begin{cases} \operatorname{div}\left(\mathbb{C}(\boldsymbol{x})\widehat{\nabla}\mathbf{N}(\mathbf{x},\mathbf{y})\right) = \delta_{\mathbf{y}}(\mathbf{x})\mathbf{I}, & \text{in } \mathbb{R}^3_-, \\ (\mathbb{C}(\boldsymbol{x})\widehat{\nabla}\mathbf{N}(\mathbf{x},\mathbf{y}))\mathbf{e_3} = \mathbf{0}, & \text{on } \{x_3 = 0\}, \end{cases}$$

satisfying certain decay conditions at infinity. The solution has then locally H^1-regularity from standard regularity results for potential theory on Lipschitz surfaces (see e.g. [32] and references therein). For the application to the inverse problem, we cannot restrict the support of $\boldsymbol{g}$, which may coincide with $\overline{S}$. In this case, even when $\boldsymbol{g} \in H^{1/2}(S)$, the existence of a variational solution with H^1 regularity locally in the complement of the dislocation is not guaranteed anymore. For example, if S is a rectangular Volterra dislocation, the double layer potential blows up logarithmically at the vertices of the rectangle, as already observed in [34].

Hence, establishing the well-posedness of the problem (1) in full generality is rather delicate and not covered by results in the literature. We devote the first half of the paper to investigating this problem. We find it convenient to reformulate the transmission problem as an equivalent source problem in the whole half space $\mathbb{R}^3_-$:

$$(2) \qquad \begin{cases} \operatorname{div}\left(\mathbb{C}\widehat{\nabla}\boldsymbol{u}\right) = \boldsymbol{f}_S, & \text{in } \mathbb{R}^3_-, \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{e}_3 = \mathbf{0}, & \text{on } \{x_3 = 0\}, \end{cases}$$

with the source term given by

$$(3) \qquad\qquad \boldsymbol{f}_S = \operatorname{div}(\mathbb{C}(\boldsymbol{g} \otimes \boldsymbol{n})\delta_S),$$

where δ_S represents the Dirac measure concentrated on S. In the first part of the paper we show well-posedness of this problem in the weighted Sobolev space $H^{1/2-\varepsilon}_{-1/2-\varepsilon}(\mathbb{R}^3_-)$ with $\varepsilon > 0$, that is in the context of very weak solutions. For a definition of this weighted space, see Section 2. We stress that this elliptic regularity result is optimal as the source (3) is a distribution with compact support belonging to $H^{-3/2-\varepsilon}(\mathbb{R}^3_-)$. The above result leads to a mathematically rigorous analysis of the elastic dislocation model in the geophysical framework with minimal assumptions on the elastic coefficients. We follow the approach of [26] for the case of regular coefficients on bounded domains, which is based on duality arguments and interpolation, adapted to the framework of weighted spaces in a half space in [20] and [3] when the source terms are integrable on $\mathbb{R}^3_-$. We extend it to the case of more singular source terms for the system of linearized elasticity with Lipschitz coefficients. We also extend the representation formula of the solution as a double layer potential. From regularity results for potential theory on Lipschitz surfaces (see again [32]), it follows that the displacement lies in $H^s(\mathbb{R}^3_- \setminus S)$ locally for $s < 1$, although it fails to belong to H^1 even locally near the dislocation surface. The analysis of the forward problem and, in particular, the hypothesis that $\operatorname{supp}(\boldsymbol{g}) = \overline{S}$ are essential in the second part of the paper to investigate the inverse problem of determining dislocations S caused by a tangential or normal slip along the dislocation surface from surface measurements of the displacement $\boldsymbol{u}$ on some bounded open portion Σ of $\{x_3 = 0\}$. We prove, by using unique-continuation properties of solutions to the Lamé system with Lipschitz parameters (see [25]), that one surface measurement of the displacement field is sufficient to recover uniquely both the dislocation surface and the slip, assuming some geometric conditions on S — for instance S can be piece-wise linear — and assuming the slip field is either purely tangential (corresponding to the case of a fault, the two sides of which slide one over the other) or directed in

the normal direction (corresponding to a crack, the two sides of which separate from one another). It is important to note that in three space dimensions, the unique continuation property may not hold for a second-order elliptic operator, if its coefficients are only in Hölder classes $C^{0,\alpha}$, $\alpha < 1$, and not Lipschitz continuous (see [29, 35]). In the framework of the Lamé operator with Lipschitz continuous coefficients very recent results on unique continuation are in [21, 25]. Therefore, the regularity assumption on the Lamé parameters that we impose appears optimal in this context.

In this paper, we do not tackle the problem of determining quantitative stability estimates for the inverse problem, which we leave for future work. However, we expect to be able to prove Lipschitz stability estimates for piece-wise linear dislocations in terms of the boundary data, thanks to the presence of singularities at the corners of the dislocation and generalizing quantitative unique continuation estimates obtained in [33]. In fact, in [10] the authors are able to prove a Lipschitz stability estimate for linear cracks in a two dimensional elastic homogeneous medium, taking advantage of the presence of singularities at the endpoint of the crack. We refer to [11] for a reconstruction algorithm in this setting. Concerning the reconstruction of dislocations and tangential slips in three space dimensions, several algorithms have been proposed, mostly related to the case of rectangular dislocations and homogeneous Lamé parameters. In the mathematics literature, we refer to [39], where the authors implement an iterative method for detecting the plane containing a fault and the tangential slip, supposed to be unidirectional, in the presence of only a finite number of surface measurements, via a constrained minimization of a suitable misfit functional. In the geophysics literature, we mention [7], where a two-step algorithm is discussed, first assuming a uniform slip and using a nonlinear, quasi-Newton method to locate the dislocation surface, and then recovering a non-uniform slip along the surface. We mention also the works [19, 42], [16] (and references therein), where a Bayesian framework and a sparsity-promoting, state-vector regularization method are employed, respectively.

The paper is organized as follows; in Section 2 we introduce needed notation, and recall the definition of certain weighted Sobolev spaces and their relevant properties useful for our analysis. Section 3 is devoted to the study of the well posedness of problem (2) and to an analysis of the regularity of the solution in a neighborhood of the dislocation surface. Finally, in Section 5 we prove a uniqueness result for the inverse problem of determining a piece-wise linear dislocation surface and its slip. We present an explicit calculation of the singularities in the displacement for the Volterra dislocation in the Appendix.

2. Notation and Functional Setting

In this section we introduce the weighted Sobolev spaces used for the analysis of the forward problem. We begin by recalling some standard, but needed, notation.

We denote scalar quantities in italics, e.g. λ, μ, ν, points and vectors in bold italics, e.g. $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$, matrices and second-order tensors in boldface, e.g. $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and fourth-order tensors in blackboard face, e.g. $\mathbb{A}, \mathbb{B}, \mathbb{C}$.

We indicate the symmetric part of a second-order tensor $\mathbf{A}$ by

$$\widehat{\mathbf{A}} = \tfrac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right),$$

where $\mathbf{A}^T$ is its transpose. We use the standard notation for inner product between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, $\boldsymbol{u} \cdot \boldsymbol{v} = \sum_i u_i v_i$, and between second-order tensors $\mathbf{A} : \mathbf{B} = \sum_{i,j} a_{ij} b_{ij}$. The tensor product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is denoted by $[\boldsymbol{u} \otimes \boldsymbol{v}]_{ij} = u_i v_j$. With $|\mathbf{A}|$ we denote the norm induced by the inner product between second-order tensors, that is,

$$|\mathbf{A}| = \sqrt{\mathbf{A} : \mathbf{A}}.$$

The vectors $\boldsymbol{e}_1, \boldsymbol{e}_2$, and $\boldsymbol{e}_3$ represent the standard orthonormal basis of $\mathbb{R}^3$, and we denote the lower half space by:

$$\{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_3 < 0\} = \mathbb{R}^3_-.$$

We canonically identify its boundary $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_3 = 0\}$ with $\mathbb{R}^2$, and denote a point in $\mathbb{R}^2$ by $\boldsymbol{x}' = (x_1, x_2)$. The set $B_r(\boldsymbol{x})$ represents the ball of center $\boldsymbol{x}$ and radius r and $B_r^-(\boldsymbol{x})$, $B_r^+(\boldsymbol{x})$ the lower and upper half balls, respectively. With $B_r'(\boldsymbol{x}')$ we mean the disk of center $\boldsymbol{x}'$ and radius r, namely

$$B_r'(\boldsymbol{x}') = \{\boldsymbol{y}' \in \mathbb{R}^2 \ : \ (y_1 - x_1)^2 + (y_2 - x_2)^2 < r^2\}.$$

Finally, we follow the standard multi-index notation for partial derivatives.

Weighted Sobolev Spaces. We next define the weighted Sobolev spaces utilized in our work. We briefly recall their main properties, referring the reader for instance to [2, 3, 4, 5, 6, 20] and references therein for a more in-depth discussion.

As usual, we begin by defining Sobolev spaces of integer regularity index, and we deal only with L^2-based spaces to avoid unnecessary technicalities. Throughout, we denote $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ for any non-empty set Ω.

Definition 2.1 (Weighted Sobolev spaces). *Let*

(4)
$$\varrho(\boldsymbol{x}) = (1 + |\boldsymbol{x}|^2)^{1/2}.$$

Let Ω be a domain in $\mathbb{R}^3$. For $m \in \mathbb{Z}_+$, $\alpha \in \mathbb{R}$, we define

(5)
$$H_\alpha^m(\Omega) = \left\{ f \in \mathcal{D}'(\Omega); \ 0 \leq |\kappa| \leq m, \ \varrho^{\alpha - m + |\kappa|} \, \partial^\kappa f \in L^2(\Omega) \right\},$$

where $\mathcal{D}'(\Omega)$ denotes the dual space of $\mathcal{D}(\Omega)$, that is, the space of distributions on Ω.

These function spaces are Banach spaces equipped with their natural norm

$$\|f\|_{H_\alpha^m(\mathbb{R}^3_-)}^2 = \sum_{|\kappa|=0}^m \|\varrho^{\alpha - m + |\kappa|} \, \partial^\kappa f\|_{L^2(\mathbb{R}^3_-)}^2.$$

If $\boldsymbol{u}$ is a vector function, then we will say $\boldsymbol{u} \in H_\alpha^m$ if each component belongs to that space. We similarly define weighted spaces in $\mathbb{R}^2$, replacing $\boldsymbol{x}$ with $\boldsymbol{x}'$ and ϱ with $\varrho' := (1 + |\boldsymbol{x}'|^2)^{1/2}$. With slight abuse of notation, we will refer to ρ, ρ^α, or simply to α, as the "weight".

Remark 2.1. *If Ω is a bounded domain, then the weighted spaces coincide with the usual Sobolev spaces, so we implicitly assume that Ω is unbounded, but may have non-empty boundary. For our purposes, it will be sufficient to consider the cases $\Omega = \mathbb{R}^d$, $d = 2, 3$, $\Omega = \mathbb{R}_-^3$, or $\Omega = \mathbb{R}_-^3 \setminus K$, where K is a compact set .*

We also stress that the spaces H_α^m do not fall into the typical framework of weighted spaces for the analysis on singular spaces, such as conormal Sobolev spaces [28]. In particular, it does not seem possible to reduce to standard Sobolev spaces by conjugation with an appropriate power of the weight.

As with the standard Sobolev spaces, one can show that the space $\mathcal{D}(\mathbb{R}^d)$ is dense in $H_\alpha^m(\mathbb{R}^d)$, while $\mathcal{D}(\overline{\mathbb{R}_-^3})$ is dense in $H_\alpha^m(\mathbb{R}_-^3)$. We hence define the space

$$(6) \qquad \mathring{H}_\alpha^m(\mathbb{R}_-^3) = \overline{\mathcal{D}(\mathbb{R}_-^3)}^{\|\cdot\|_{H_\alpha^m(\mathbb{R}_-^3)}},$$

which is a proper subset of $H_\alpha^m(\mathbb{R}_-^3)$, while $H_\alpha^m(\mathbb{R}^d) = \mathring{H}_\alpha^m(\mathbb{R}^d)$.

We define $H_{-\alpha}^{-m}(\mathbb{R}^d)$ as the dual space to $H_\alpha^m(\mathbb{R}^d)$ with pivot L^2. Similarly, we define $H_{-\alpha}^{-m}(\mathbb{R}_-^3)$ as the dual space to $\mathring{H}_\alpha^m(\mathbb{R}_-^3)$. Both are spaces of distributions.

Before introducing fractional spaces and discuss trace results, we recall some basic properties of the spaces H_α^m. For simplicity, we state most of these properties, such as Poincaré's and Korn's inequality only for the spaces H_α^m that we actually employ in our work, which are $H_0^1(\mathbb{R}_-^3)$ and $H_1^2(\mathbb{R}_-^3)$, if m is a positive integer. The following weighted Poincaré-type inequality holds (see for example [5, 20]):

$$(7) \qquad \begin{aligned} \|f\|_{H_0^1(\mathbb{R}_-^3)} &\leq C_1 \|\nabla f\|_{L^2(\mathbb{R}_-^3)}, \\ \|f\|_{H_1^2(\mathbb{R}_-^3)} &\leq C_2 \|\varrho \partial^2 f\|_{L^2(\mathbb{R}_-^3)}. \end{aligned}$$

Similarly, if a vector field $\boldsymbol{u}$ has square-integrable deformation tensor $\widehat{\nabla}\boldsymbol{u}$ and belongs to the space $H_{-1}^0(\mathbb{R}_-^3)$, then $\boldsymbol{u} \in H_0^1(\mathbb{R}_-^3)$ by a weighted Korn-type inequality [3, Theorem 2.10] (see also [22]) and the Poincaré's inequality above:

$$(8) \qquad \|\boldsymbol{u}\|_{H_0^1(\mathbb{R}_-^3)} \leq C \|\nabla \boldsymbol{u}\|_{L^2(\mathbb{R}_-^3)} \leq C \|\widehat{\nabla}\boldsymbol{u}\|_{L^2(\mathbb{R}_-^3)}.$$

For any $\gamma \in \mathbb{R}$ such that $(3/2 + \alpha - \gamma) \notin \{1, \cdots, m\}$, with $m \in \mathbb{N}$, the mapping

$$(9) \qquad \boldsymbol{u} \in H_\alpha^m(\Omega) \to \varrho^\gamma \boldsymbol{u} \in H_{\alpha-\gamma}^m(\Omega)$$

is an isomorphism; for any $\ell \in \mathbb{N}^3$, the mapping

$$(10) \qquad \boldsymbol{u} \in H_\alpha^m(\Omega) \to \partial^\ell \boldsymbol{u} \in H_\alpha^{m-|\ell|}(\Omega)$$

is continuous. For more details see for example [6, 20].

We next define fractional spaces on $\mathbb{R}^d$, $d = 2, 3$. We give the definition in $\mathbb{R}^3$ for $0 < s < 2$, the same definition applies in $\mathbb{R}^2$ for $0 < s < 1$ with the obvious change of notation (e.g. ϱ is replaced by ϱ'). In both cases, we assume $\alpha \geq -1$ and $d/2 + \alpha \neq s$, $d = 2, 3$.

For $0 < s < 1$, we define the weighted fractional space $H_\alpha^s(\mathbb{R}^3)$ as:

$$(11) \qquad H_\alpha^s(\mathbb{R}^3) = \left\{ f \in \mathcal{D}'(\mathbb{R}^3); \ \varrho^{\alpha-s} f \in L^2(\mathbb{R}^3), \ |f|_s < \infty \right\},$$

where $| \cdot |_s$ denotes a weighted Gagliardo seminorm:

$$|f|_s := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\varrho^\alpha(\boldsymbol{x}) \, f(\boldsymbol{x}) - \varrho^\alpha(\boldsymbol{y}) \, f(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^{3+2s}} \, d\boldsymbol{x} \, d\boldsymbol{y}.$$

For $1 < s < 2$, we define $H_\alpha^s(\mathbb{R}^3)$ as:
$$(12)$$

$$H_\alpha^s(\mathbb{R}^3) = \left\{ f \in \mathcal{D}'(\mathbb{R}^3); \ 0 \le |\kappa| \le [s]-1, \ \varrho^{\alpha-s+|\kappa|} \partial^\kappa f \in L^2(\mathbb{R}^3), \ \partial^{[s]} f \in H_\alpha^{s-[s]}(\mathbb{R}^3) \right\},$$

with $[s]$ the greatest integer less than or equal to s.

Remark 2.2. *In general, a logarithm correction to the weight is needed to properly define the spaces unless some conditions on the indices s and α for a given dimension d are satisfied (see e.g [3]). These conditions are met if s and α are taken as assumed above.*

Fractional spaces on $\mathbb{R}_-^3$ can be defined by restriction, that is,

$$(13) \qquad H_\alpha^s(\mathbb{R}_-^3) := \left\{ f \in \mathcal{D}'(\overline{\mathbb{R}_-^3}); \ \exists g \in H_\alpha^s(\mathbb{R}^3), \ f = g\lfloor_{\mathbb{R}_-^3} \right\},$$

equipped with the norm:

$$\|f\|_{H_\alpha^s(\mathbb{R}_-^3)} := \inf\{\|g\|_{H_\alpha^s}; \ g \in H_\alpha^s(\mathbb{R}^3), \ f = g\lfloor_{\mathbb{R}_-^3} \}.$$

Then, there exists a continuous restriction operator R and extension operator E such that RE is the identity map on $H_\alpha^s(\mathbb{R}_-^3)$ (see e.g. [38, Lemma 7], [20, Theorem I.4]).

We also define the spaces $\mathring{H}_\alpha^s(\mathbb{R}_-^3)$, $0 < s < 2$, $0 \le \alpha$, by the analog of formula (6), where there is a canonical extension operator given by the extension by zero to the whole $\mathbb{R}^3$. Lastly, we denote by $H_{-\alpha}^{-s}(\mathbb{R}_-^3)$ the dual of $\mathring{H}_\alpha^s(\mathbb{R}_-^3)$.

Fractional spaces can also be obtained by means of real interpolation of integer-order spaces. This result will be needed in Section 3. We will exploit the interpolation results in [38]. (In that work, more general spaces $w_{\alpha,p}^s$ are studied; by Theorem 2 there, the spaces $H_\alpha^s(\mathbb{R}^3)$ agree with the space $w_{2,\alpha}^s(\mathbb{R}^3)$ for any $0 \le s \le 2$ and any $\alpha \in \mathbb{R}$). By formula (59) in [38], which is a particular case of Theorem 3 (a), using Formula (15) in that same work, the following result holds:

$$(14) \qquad \left[H_{\alpha_0}^{s_0}(\mathbb{R}^3), H_{\alpha_1}^{s_1}(\mathbb{R}^3) \right]_{\Theta,2} = H_\alpha^s(\mathbb{R}^3),$$

$$s = s_0(1 - \Theta) + s_1 \, \Theta, \quad \alpha = \alpha_0(1 - \Theta) + \alpha_1 \, \Theta, \quad \Theta \in (0,1),$$

for any $\alpha_0, \ \alpha_1 \in \mathbb{R}$, $s_0, s_1 \in [0,2]$, as long as $s_0 \ne s_1$ and s is not an integer. The identification (14) holds also for weighted spaces on $\mathbb{R}_-^3$. In fact, by the definition of interpolation space (see e.g. [12, Definition 2.4.1]), the restriction and extension operators map between corresponding interpolation spaces, so:

$$\left[R(H_{\alpha_0}^{s_0}(\mathbb{R}^3)), R(H_{\alpha_1}^{s_1}(\mathbb{R}^3)) \right]_{\Theta,2} = R(H_\alpha^s(\mathbb{R}^3)),$$

but the restriction operator $R : H^s_\alpha(\mathbb{R}^3) \to H^s_\alpha(\mathbb{R}^3_-)$, $s \geq 0$, acts surjectively (see also Section 3.3 in [38]), so that $R(H^s_\alpha(\mathbb{R}^3)) = H^s_\alpha(\mathbb{R}^3_-)$. We specialize the interpolation formula to two cases of interest:

$$(15\text{a}) \qquad\qquad \left[H^0_{-1}(\mathbb{R}^3_-), H^1_0(\mathbb{R}^3_-) \right]_{\Theta,2} = H^\Theta_{-1+\Theta}(\mathbb{R}^3_-),$$

$$(15\text{b}) \qquad\qquad \left[H^1_0(\mathbb{R}^3_-), H^2_1(\mathbb{R}^3_-) \right]_{\Theta,2} = H^{1+\Theta}_\Theta(\mathbb{R}^3_-).$$

Next, we extend the interpolation analysis to negative spaces using duality. It is clear that (15) holds for the spaces $\mathring{H}^s_\alpha(\mathbb{R}^3_-)$, which are (not necessarily proper) closed subspaces of $H^s_\alpha(\mathbb{R}^3_-)$. We then recall the following standard fact about real interpolation (see e.g. again [12, Definition 2.4.1]), which applies since the intersection of any of the spaces $\mathring{H}^s_\alpha(\mathbb{R}^3_-)$ contains the dense subspace $\mathcal{D}(\mathbb{R}^3_-)$, specialized to the weighted spaces:

$$\left[H^{-s_0}_{-\alpha_0}(\mathbb{R}^3_-), H^{-s_1}_{-\alpha_1}(\mathbb{R}^3_-) \right]_{\Theta,2} = H^{-s}_{-\alpha}(\mathbb{R}^3_-),$$

using that $\mathring{H}^s_\alpha(\mathbb{R}^3_-)' = H^{-s}_{-\alpha}(\mathbb{R}^3_-)$. Finally, using that

$$[A_0, A_1]_{\Theta,2} = [A_1, A_0]_{1-\Theta,2},$$

(see e.g. [12, Theorem 3.4.1]) we conclude that:

$$(16\text{a}) \qquad\qquad \left[H^{-1}_0(\mathbb{R}^3_-), H^0_1(\mathbb{R}^3_-) \right]_{1-\Theta,2} = H^{-\Theta}_{1-\Theta}(\mathbb{R}^3_-),$$

$$(16\text{b}) \qquad\qquad \left[H^{-2}_{-1}(\mathbb{R}^3_-), H^{-1}_0(\mathbb{R}^3_-) \right]_{1-\Theta,2} = H^{-1-\Theta}_{-\Theta}(\mathbb{R}^3_-).$$

We close this section by recalling a trace result which we present only for spaces $H^m_\alpha(\mathbb{R}^3_-)$, where m is a positive integer and $\alpha \in \mathbb{R}$.

Proposition 2.3 ([3], Lemma 1.1). *Let $m \geq 1$ and $\alpha \in \mathbb{R}$. Then, there exists a continuous linear mapping*

$$(17) \qquad\qquad \boldsymbol{\gamma} = (\gamma_0, \cdots, \gamma_{m-1}) : H^m_\alpha(\mathbb{R}^3_-) \longrightarrow \prod_{j=0}^{m-1} H^{m-j-1/2}_\alpha(\mathbb{R}^2).$$

In addition, $\boldsymbol{\gamma}$ is surjective and

$$\operatorname{Ker} \boldsymbol{\gamma} = \mathring{H}^m_\alpha(\mathbb{R}^3_-).$$

3. The direct problem: formulation and well-posedness

In this section, we formulate the direct problem and study its well posedness. We begin by discussing in more detail the assumptions we make on the geometry of the dislocation surface S, on the displacement jump $\boldsymbol{g}$ across S, and on the elasticity tensor $\mathbb{C}$, which form the data for the direct problem.

3.1. **Main assumptions and a priori information.** We recall that we model the dislocation surface S by an open, bounded, oriented Lipschitz surface such that

$$\overline{S} \subset \mathbb{R}^3_-. \tag{18}$$

We assume that the interior of the Earth is an isotropic, inhomogeneous elastic medium. The associated elasticity tensor $\mathbb{C} = \mathbb{C}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^3_-$, is then of the form:

$$\mathbb{C}(\boldsymbol{x}) = \lambda(\boldsymbol{x})\mathbf{I} \otimes \mathbf{I} + 2\mu(\boldsymbol{x})\mathbb{I}, \qquad \forall \boldsymbol{x} \in \mathbb{R}^3_-, \tag{19}$$

where $\lambda = \lambda(\boldsymbol{x})$ and $\mu = \mu(\boldsymbol{x})$ are the Lamé coefficients. We suppose the Lamé parameters to have Lipschitz regularity, that is, there exists $M > 0$ such that

$$\|\mu\|_{C^{0,1}(\overline{\mathbb{R}^3_-})} + \|\lambda\|_{C^{0,1}(\overline{\mathbb{R}^3_-})} \leq M, \tag{20}$$

with $\| \cdot \|_{C^{0,1}(\overline{\mathbb{R}^3_-})} = \| \cdot \|_{L^\infty(\mathbb{R}^3_-)} + \|\nabla \cdot \|_{L^\infty(\mathbb{R}^3_-)}$. Moreover, we require that there exists a constant $C > 0$ such that

$$|\nabla \lambda| \leq \frac{C}{\varrho}, \qquad |\nabla \mu| \leq \frac{C}{\varrho}, \qquad \text{a.e. in } \overline{\mathbb{R}^3_-}, \tag{21}$$

where ϱ is the weight defined in (4). Finally, we assume the strong convexity condition for the elasticity tensor, i.e., there exist two positive constants α_0, β_0 such that

$$\mu(\boldsymbol{x}) \geq \alpha_0 > 0, \qquad 3\lambda(\boldsymbol{x}) + 2\mu(\boldsymbol{x}) \geq \beta_0 > 0, \quad \forall \boldsymbol{x} \in \mathbb{R}^3_-. \tag{22}$$

This condition implies that $\mathbb{C}$ defines a positive-definite quadratic form on symmetric matrices:

$$\mathbb{C}(\boldsymbol{x})\widehat{\mathbf{A}} : \widehat{\mathbf{A}} \geq c|\widehat{\mathbf{A}}|, \qquad \forall \boldsymbol{x} \in \mathbb{R}^3_-, \tag{23}$$

for $c = c(\alpha_0, \beta_0) > 0$.

3.2. **The transmission problem as a source problem.** The presence of a non-trivial jump $\boldsymbol{g}$ for the displacement $\boldsymbol{u}$ across the dislocation leads to reformulate the transmission-boundary-value problem (1) as boundary-value problem with rough interior source. In this way, we are able to prove an existence and uniqueness result for the solution globally in $\mathbb{R}^3_-$. This approach has been proposed in [14] for a similar transmission problem, but in the case of a scalar elliptic equation.

We recall Problem (1) here for the reader's sake:

$$\begin{cases} \text{div}\,(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \mathbf{0}, & \text{in } \mathbb{R}^3_- \setminus \overline{S}, \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{e}_3 = \mathbf{0}, & \text{on } \{x_3 = 0\}, \\ [\boldsymbol{u}]_S = \boldsymbol{g}, \\ [(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}]_S = \mathbf{0}, \end{cases} \tag{24}$$

where $\boldsymbol{e}_3 = (0,0,1)$ is the outward unit normal vector on $\{x_3 = 0\}$, $\boldsymbol{n}$ is a unit normal vector on S, and $\boldsymbol{g}$ is a vector field on S such that

$$\boldsymbol{g} \in H^{1/2}(S). \tag{25}$$

We rewrite this problem in the form

$$\begin{cases} \text{div}\,(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \boldsymbol{f}_S, & \text{in } \mathbb{R}^3_-, \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{e}_3 = \mathbf{0}, & \text{on } \{x_3 = 0\}, \end{cases} \tag{26}$$

where

(27)
$$\boldsymbol{f}_S = \operatorname{div}(\mathbb{C}(\boldsymbol{g} \otimes \boldsymbol{n})\delta_S).$$

Above, δ_S is the distribution on $\mathbb{R}^3$ defined by:

(28)
$$\langle \delta_S, \boldsymbol{\phi} \rangle = \int_S \boldsymbol{\phi}(\boldsymbol{x}) \, d\sigma(\boldsymbol{x}), \qquad \forall \boldsymbol{\phi} \in D(\mathbb{R}^3),$$

and, if $h \in L^1(S)$, the distribution $h \, \delta_S$ is defined by

(29)
$$\langle h \, \delta_S, \boldsymbol{\phi} \rangle = \int_S h(\boldsymbol{x}) \, \boldsymbol{\phi}(\boldsymbol{x}) \, d\sigma(\boldsymbol{x}), \qquad \forall \boldsymbol{\phi} \in D(\mathbb{R}^3),$$

We show the equivalence between (24) and (26) later in Lemma 3.12. Next, we establish the validity of Formula (27) and investigate the regularity of the source term.

Proposition 3.1. *The source term* $\boldsymbol{f}_S \in H^{-3/2-\varepsilon}(\mathbb{R}^3_-)$.

Proof. It is sufficient to show that $\mathbb{C}(\boldsymbol{g} \otimes \boldsymbol{n})\delta_S$ is a distribution in the space $H^{-1/2-\varepsilon}(\mathbb{R}^3_-)$, for all $\varepsilon > 0$. To this end, we introduce the function $\boldsymbol{\Psi} := \mathbb{C}(\boldsymbol{g} \otimes \boldsymbol{n})$, and observe that it lies in $L^2(S)$, since $\boldsymbol{n} \in L^\infty(S)$, $\mathbb{C}$ is bounded and continuous on S by Assumption (20), and $\boldsymbol{g}$ is assumed to be in $H^{1/2}(S)$. Next, we fix a bounded Lipschitz domain Ω such that $\overline{\Omega} \subset \mathbb{R}^3_-$ and $S \subset \partial\Omega$. The existence of this domain is guaranteed by the assumed regularity on S and the hypothesis that S is at positive distance from $\{x_3 = 0\}$. For all $\boldsymbol{\varphi} \in H^{1/2+\varepsilon}(\mathbb{R}^3_-)$, we have

$$\left| \int_S \boldsymbol{\Psi}(\boldsymbol{x})\boldsymbol{\varphi}(\boldsymbol{x}) \, d\sigma(\boldsymbol{x}) \right| \leq \|\boldsymbol{\Psi}\|_{L^2(S)} \|\boldsymbol{\varphi}\|_{L^2(S)} \leq c \, \|\boldsymbol{\varphi}\|_{L^2(\partial\Omega)}$$

$$\leq c \|\boldsymbol{\varphi}\|_{H^\varepsilon(\partial\Omega)} \leq c \|\boldsymbol{\varphi}\|_{H^{1/2+\varepsilon}(\mathbb{R}^3_-)},$$

where the fourth inequality follows by the Trace Theorem. Therefore, by definition $\boldsymbol{\Psi}\delta_S \in H^{-1/2-\varepsilon}(\mathbb{R}^3_-)$, which implies that $\boldsymbol{f}_S \in H^{-3/2-\varepsilon}(\mathbb{R}^3_-)$. $\qquad\square$

The well-posedness of Problem (26), with source term given by (27), will be obtained by duality and interpolation, once we have regularity for the source problem with source term in spaces of positive regularity. To do so, we study the general source problem:

(30)
$$\begin{cases} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \boldsymbol{f}, & \text{in } \mathbb{R}^3_-, \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{e}_3 = \boldsymbol{0}, & \text{on } \{x_3 = 0\}. \end{cases}$$

We follow the well-known approach of Lions and Magenes [26] to establish regularity for this problem. This approach was already adapted to the framework of weighted spaces in a half space in [20] and [3] for the Dirichlet and Neumann problems. However, our source problem is more singular, and we assume lower regularity on the coefficients.

Our strategy involves strong, weak, and very weak solutions, duality and interpolation. To introduce the proper functional setting, we need to define some auxiliary spaces.

Definition 3.1. *Let* $V(\mathbb{R}^3_-)$ *be the closed subspace of* $H^2_1(\mathbb{R}^3_-)$ *given by*

(31)
$$V(\mathbb{R}^3_-) = \left\{ \boldsymbol{v} \in H^2_1(\mathbb{R}^3_-) \; / \; (\mathbb{C}\widehat{\nabla}\boldsymbol{v})\boldsymbol{e}_3 = \boldsymbol{0} \text{ on } \{x_3 = 0\} \right\},$$

with norm $\|\boldsymbol{v}\|_{V(\mathbb{R}^3_-)} = \|\boldsymbol{v}\|_{H^2_1(\mathbb{R}^3_-)}$. *We denote the dual space of* $V(\mathbb{R}^3_-)$ *with* $V'(\mathbb{R}^3_-)$.

Remark 3.2. *We note that if $\boldsymbol{v} \in V(\mathbb{R}^3_-)$ then in particular $\boldsymbol{v} \in H^0_{-1}(\mathbb{R}^3_-)$ which implies that no infinitesimal rigid motion $\boldsymbol{v} = \mathbf{A}\boldsymbol{x} + \boldsymbol{c}$, where $\mathbf{A} \in \mathbb{R}^{3\times3}$ is a skew matrix and $\boldsymbol{c} \in \mathbb{R}^3$, can be an element of $V(\mathbb{R}^3_-)$.*

Definition 3.2. *Let*

$$E_0(\mathbb{R}^3_-) = \left\{ \boldsymbol{u} \in H^0_{-1}(\mathbb{R}^3_-) \;/\; \mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) \in V'(\mathbb{R}^3_-) \right\}, \tag{32}$$

equipped with the norm

$$\|\boldsymbol{u}\|_{E_0(\mathbb{R}^3_-)} = \|\boldsymbol{u}\|_{H^0_{-1}(\mathbb{R}^3_-)} + \|\mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\|_{V'(\mathbb{R}^3_-)}.$$

For the reader's convenience we give here a sketch of the strategy we use to establish well-posedness of (30):

- (i) we prove that, for any $\boldsymbol{f} \in (H^1_0(\mathbb{R}^3_-))'$, there exists a unique weak solution $\boldsymbol{u} \in H^1_0(\mathbb{R}^3_-)$;
- (ii) if $\boldsymbol{f} \in H^0_1(\mathbb{R}^3_-)$, we show that the weak solution in (i) is, in fact, a strong solution $\boldsymbol{u} \in H^2_1(\mathbb{R}^3_-)$;
- (iii) from (ii) by a duality argument, we establish that, for any $\boldsymbol{f} \in V'(\mathbb{R}^3_-)$, there exists a unique solution $\boldsymbol{u} \in H^0_{-1}(\mathbb{R}^3_-)$, that is, $\boldsymbol{u} \in E_0(\mathbb{R}^3_-)$;
- (iv) the well-posedness of (26) with source term (27) follows by interpolating the results obtained in (i) and (iii).

3.3. **Well-posedness.** In this section we prove the well-posedness of Problem (26) with $\boldsymbol{f}_S$ given by (27). Following the strategy presented in the scheme of the previous section, we start by proving (i).

Theorem 3.3 (Weak solution)**.** *Under Assumptions (20), (23), for any $\boldsymbol{f} \in (H^1_0)'(\mathbb{R}^3_-)$ there exists a unique solution $\boldsymbol{u} \in H^1_0(\mathbb{R}^3_-)$ of Problem (30) such that*

$$\|\boldsymbol{u}\|_{H^1_0(\mathbb{R}^3_-)} \leq C\|\boldsymbol{f}\|_{(H^1_0)'(\mathbb{R}^3_-)}. \tag{33}$$

Remark 3.4. *Above and in Theorems 3.5, 3.8, and 3.11, the constants generally depend on the constants in Assumptions (20), (21), and (22), such as M and c, but are independent of S and $\boldsymbol{g}$.*
We also note that is possible to show that the dual of $H^1_0(\mathbb{R}^3_-)$ can be canonically identified with the direct sum:

$$(H^1_0(\mathbb{R}^3_-))' \cong H^{-1}_0(\mathbb{R}^3_-) \oplus H^{-1/2}_0(\mathbb{R}^2).$$

However, we will not need this explicit identification.

Proof. We introduce the bilinear form $a : H^1_0(\mathbb{R}^3_-) \times H^1_0(\mathbb{R}^3_-) \to \mathbb{R}$ associated to the Lamé operator:

$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_{\mathbb{R}^3_-} \mathbb{C}\widehat{\nabla}\boldsymbol{u} : \widehat{\nabla}\boldsymbol{v} \, d\boldsymbol{x}. \tag{34}$$

Since $\mathcal{D}(\overline{\mathbb{R}^3_-})$ is dense in $H^1_0(\mathbb{R}^3_-)$, we can assume in the calculations below that $\boldsymbol{u} \in \mathcal{D}(\overline{\mathbb{R}^3_-})$. We test the first equation in (30) with $\boldsymbol{v} \in H^1_0(\mathbb{R}^3_-)$, we then integrate by parts once in $\boldsymbol{u}$ over $\mathbb{R}^3_-$ on the left-hand side, using the traction-free boundary condition, to obtain:

$$\int_{\mathbb{R}^3_-} \mathbb{C}\widehat{\nabla}\boldsymbol{u} : \widehat{\nabla}\boldsymbol{v} \, d\boldsymbol{x} = -\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{((H^1_0)'(\mathbb{R}^3_-), H^1_0(\mathbb{R}^3_-))}, \qquad \forall \boldsymbol{v} \in H^1_0(\mathbb{R}^3_-).$$

We have hence the following variational formulation of Problem (30):
find $\boldsymbol{u} \in H_0^1(\mathbb{R}_-^3)$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = l(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in H_0^1(\mathbb{R}_-^3),$$

where $l : H_0^1(\mathbb{R}_-^3) \to \mathbb{R}$ is the linear functional given by

$$(35) \qquad\qquad l(\boldsymbol{v}) = -\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{((H_0^1)'(\mathbb{R}_-^3), H_0^1(\mathbb{R}_-^3))}.$$

Above and throughout the paper, we use the notation $\langle, \rangle_{(X',X)}$ to denote the duality pairing between a Banach space X and its dual X'.

The assertion of the theorem then follows by applying the Lax-Milgram theorem, once the continuity and coercivity of the bilinear form a and the continuity of the functional l are established .

Continuity and coercivity of (34): From (20), we have

$$|a(\boldsymbol{u}, \boldsymbol{v})| = \left| \int_{\mathbb{R}_-^3} \mathbb{C}\widehat{\nabla}\boldsymbol{u} : \widehat{\nabla}\boldsymbol{v} \, d\boldsymbol{x} \right| \le c\|\widehat{\nabla}\boldsymbol{u}\|_{L^2(\mathbb{R}_-^3)}\|\widehat{\nabla}\boldsymbol{v}\|_{L^2(\mathbb{R}_-^3)}$$

$$\le c\|\boldsymbol{u}\|_{H_0^1(\mathbb{R}_-^3)}\|\boldsymbol{v}\|_{H_0^1(\mathbb{R}_-^3)}.$$

Coercivity follows from the strong convexity of $\mathbb{C}$ (Assumptions (22) and (23)), by applying the weighted Korn's (8) inequality:

$$a(\boldsymbol{u}, \boldsymbol{u}) = \int_{\mathbb{R}_-^3} \mathbb{C}\widehat{\nabla}\boldsymbol{u} : \widehat{\nabla}\boldsymbol{u} \, d\boldsymbol{x} \ge c\|\widehat{\nabla}\boldsymbol{u}\|_{L^2(\mathbb{R}_-^3)}^2$$

$$\ge c\|\nabla\boldsymbol{u}\|_{L^2(\mathbb{R}_-^3)}^2 \ge c\|\boldsymbol{u}\|_{H_0^1(\mathbb{R}_-^3)}^2.$$

Continuity of (35)*:* l is by definition an element of the dual of $H_0^1(\mathbb{R}_-^3)$, hence it defines a continuous linear functional on H_0^1 via the duality form.

The conclusion now follows from the Lax-Milgram Theorem. $\qquad\qquad\square$

Next we establish (ii), i.e., the fact that the unique weak solution is actually a strong solution, if the source term in (30) belongs to $H_1^0(\mathbb{R}_-^3)$.

Theorem 3.5 (Strong solution). *Under the assumptions of Theorem 3.3 and Assumption (21) on the decay of the Lamé coefficients, if the source $\boldsymbol{f} \in H_1^0(\mathbb{R}_-^3)$, Problem (30) admits a unique solution $\boldsymbol{u} \in H_1^2(\mathbb{R}_-^3)$, satisfying*

$$(36) \qquad\qquad \|\boldsymbol{u}\|_{H_1^2(\mathbb{R}_-^3)} \le c\|\boldsymbol{f}\|_{H_1^0(\mathbb{R}_-^3)}.$$

Proof. We begin by observing that, if $\boldsymbol{f} \in H_1^0(\mathbb{R}_-^3)$, then $\boldsymbol{f} \in (H_0^1(\mathbb{R}_-^3))'$. In fact, using that $\varrho(x) \ge 1$ and that $\boldsymbol{f}$ is now locally an integrable function, we can estimate (35) as follows:

$$|l(\boldsymbol{v})| = \left| \int_{\mathbb{R}_-^3} \boldsymbol{f} \cdot \boldsymbol{v} \, d\boldsymbol{x} \right| \le \|\varrho\boldsymbol{f}\|_{L^2(\mathbb{R}_-^3)}\left\|\frac{\boldsymbol{v}}{\varrho}\right\|_{L^2(\mathbb{R}_-^3)} \le \|\boldsymbol{f}\|_{H_1^0(\mathbb{R}_-^3)}\|\boldsymbol{v}\|_{H_0^1(\mathbb{R}_-^3)}.$$

By uniqueness of weak solutions, it is enough to show that we can bootstrap regularity for the source problem, if $\boldsymbol{f}$ is more regular. To bootstrap, we will prove that suitable weighted derivatives of the solutions satisfy a similar source problem. To be specific, we study the source-boundary-value problem formally satisfied by $\varrho\partial_i\boldsymbol{u}$, for $i = 1, 2$. Since we take only derivatives that are tangent to the boundary, we can show that this problem

is in a form similar to the original problem (30) (see (39)). Theorem 3.3 then gives, again by uniqueness of solutions, that $\varrho\partial_i\boldsymbol{u} \in H_0^1(\mathbb{R}^3_-)$, which in turn means by (9) that $\partial_i\boldsymbol{u} \in H_1^1(\mathbb{R}^3_-)$ or, equivalently, that $\varrho\nabla(\partial_i\boldsymbol{u}) \in L^2(\mathbb{R}^3_-)$. Lastly, by using the specific form of the Lamé system, we are able to prove that the regularity of the tangential derivatives implies $\varrho\partial_3^2\boldsymbol{u} \in L^2(\mathbb{R}^3_-)$ as well. Therefore, we can conclude that $\varrho\partial^2\boldsymbol{u} \in L^2(\mathbb{R}^3_-)$.

First Step: We seek to find the problem satisfied by $\varrho\partial_i\boldsymbol{u}$, for $i = 1, 2$, knowing that $\boldsymbol{u}$ satisfies (30). We proceed formally first. The manipulations below are justified *a posteriori*, given the regularity on $\boldsymbol{u}$, ϱ, and the data. We have that

$$\text{div}(\mathbb{C}\widehat{\nabla}(\varrho\partial_i\boldsymbol{u})) = \varrho\partial_i\boldsymbol{f} + \text{div}(\mathbb{C}(\widehat{\nabla\varrho \otimes \partial_i\boldsymbol{u}})) + (\mathbb{C}\widehat{\nabla}(\partial_i\boldsymbol{u}))\nabla\varrho - \varrho\,\text{div}\,(\partial_i\mathbb{C}\widehat{\nabla}\boldsymbol{u}),$$

and, since

$$\varrho\,\text{div}(\partial_i\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \text{div}(\varrho\partial_i\mathbb{C}\widehat{\nabla}\boldsymbol{u}) - \partial_i\mathbb{C}\widehat{\nabla}\boldsymbol{u}\nabla\varrho,$$

we find that

$$(37) \qquad \text{div}(\mathbb{C}\widehat{\nabla}(\varrho\partial_i\boldsymbol{u}) - \mathbf{G}_i) = \boldsymbol{F}_i, \qquad \text{in } \mathbb{R}^3_-,$$

where

$$\boldsymbol{F}_i := \varrho\partial_i\boldsymbol{f} + (\mathbb{C}\widehat{\nabla}(\partial_i\boldsymbol{u}))\nabla\varrho + \partial_i\mathbb{C}\widehat{\nabla}\boldsymbol{u}\nabla\varrho, \quad \text{and} \quad \mathbf{G}_i := \mathbb{C}(\widehat{\nabla\varrho \otimes \partial_i\boldsymbol{u}}) - \varrho\partial_i\mathbb{C}\widehat{\nabla}\boldsymbol{u}.$$

Next, from the Neumann boundary condition on $\boldsymbol{u}$ in (30), it follows that

$$\mathbb{C}\widehat{\nabla}(\varrho'\partial_i\boldsymbol{u})\boldsymbol{e}_3 = \mathbb{C}(\widehat{\nabla\varrho' \otimes \partial_i\boldsymbol{u}})\boldsymbol{e}_3 - \varrho'(\partial_i\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{e}_3, \qquad \text{for } x_3 = 0,$$

where $\varrho' = \varrho\lfloor_{\mathbb{R}^2}$. That is,

$$(38) \qquad (\mathbb{C}\widehat{\nabla}(\varrho'\partial_i\boldsymbol{u}) - \mathbf{G}_i)\boldsymbol{e}_3 = \mathbf{0}, \qquad \text{on } \{x_3 = 0\}.$$

Hence, by combining (37) and (38), we obtain the problem

$$(39) \qquad \begin{cases} \text{div}(\mathbb{C}\widehat{\nabla}(\varrho\partial_i\boldsymbol{u}) - \mathbf{G}_i) = \boldsymbol{F}_i, & \text{in } \mathbb{R}^3_-, \\ (\mathbb{C}\widehat{\nabla}(\varrho'\partial_i\boldsymbol{u}) - \mathbf{G}_i)\boldsymbol{e}_3 = \mathbf{0}, & \text{on } \{x_3 = 0\}. \end{cases}$$

We write the weak formulation of this problem. The quadratic form associated to the left-hand side of the equation above is:

$$(40) \qquad \int_{\mathbb{R}^3_-} (\mathbb{C}\widehat{\nabla}(\varrho\partial_i\boldsymbol{u}) - \mathbf{G}_i) \cdot \nabla\boldsymbol{\varphi}\,d\boldsymbol{x},$$

for $\boldsymbol{\varphi} \in H_0^1(\mathbb{R}^3_-)$. This expression is justified by the fact that $\boldsymbol{u} \in H_0^1(\mathbb{R}^3_-)$ and $\mathbf{G}_i \in L^2(\mathbb{R}^3_-)$ by the regularity and decay conditions on $\mathbb{C}$ (in particular, the fact that $\varrho\nabla\mathbb{C} \in L^\infty(\mathbb{R}^3_-)$ from (21)).

We further observe that $\boldsymbol{F}_i \in (H_0^1(\mathbb{R}^3_-))'$. In fact, taking again $\boldsymbol{\varphi} \in H_0^1(\mathbb{R}^3_-)$, by the hypothesis $\boldsymbol{f} \in H_1^0(\mathbb{R}^3_-)$ and Theorem 3.3, we have

$$(41) \qquad \begin{aligned} \left|\langle\varrho\partial_i\boldsymbol{f}, \boldsymbol{\varphi}\rangle_{(H_0^1(\mathbb{R}^3_-))', H_0^1(\mathbb{R}^3_-)}\right| &= \left|\langle\partial_i(\varrho\boldsymbol{f}) - \boldsymbol{f}\partial_i\varrho, \boldsymbol{\varphi}\rangle_{(H_0^1(\mathbb{R}^3_-))', H_0^1(\mathbb{R}^3_-)}\right| \\ &\leq C\left(\|\boldsymbol{f}\|_{H_1^0(\mathbb{R}^3_-)}\|\nabla\boldsymbol{\varphi}\|_{L^2(\mathbb{R}^3_-)} + \|\boldsymbol{f}\|_{H_1^0(\mathbb{R}^3_-)}\left\|\frac{\boldsymbol{\varphi}}{\varrho}\right\|_{L^2(\mathbb{R}^3_-)}\right) \\ &\leq C\,\|\boldsymbol{f}\|_{H_1^0(\mathbb{R}^3_-)}\|\boldsymbol{\varphi}\|_{H_0^1(\mathbb{R}^3_-)}. \end{aligned}$$

We point out that, in the first inequality above, there are no boundary terms, because we take tangential derivatives. Next, from (21) it follows that

$$
(42) \quad \left| \langle (\mathbb{C}\widehat{\nabla}\partial_i \boldsymbol{u})\nabla\varrho, \boldsymbol{\varphi} \rangle_{(H_0^1(\mathbb{R}_-^3))', H_0^1(\mathbb{R}_-^3)} \right| = \left| \langle \widehat{\nabla}\boldsymbol{u}, \partial_i(\mathbb{C}(\nabla\varrho \otimes \boldsymbol{\varphi})) \rangle \right|
$$
$$
\leq C \, \|\widehat{\nabla}\boldsymbol{u}\|_{L^2(\mathbb{R}_-^3)} \|\boldsymbol{\varphi}\|_{H_0^1(\mathbb{R}_-^3)},
$$

where the last inequality comes from the fact that $\varrho\,\partial_i\nabla\varrho$ are bounded in $\mathbb{R}_-^3$, for $i = 1, 2$. Again we stress that there are not boundary elements because we take tangential derivatives. Finally, using again the fact that $\nabla\varrho$ and $\varrho\,\partial_i\mathbb{C}$ are bounded, we get

$$
(43) \quad \left| \langle (\nabla\varrho^T\partial_i\mathbb{C}\widehat{\nabla}\boldsymbol{u}), \boldsymbol{\varphi} \rangle_{(H_0^1(\mathbb{R}_-^3))', H_0^1(\mathbb{R}_-^3)} \right| \leq C\|\widehat{\nabla}\boldsymbol{u}\|_{L^2(\mathbb{R}_-^3)} \left\| \frac{\boldsymbol{\varphi}}{\varrho} \right\|_{L^2(\mathbb{R}_-^3)}
$$
$$
\leq C \, \|\widehat{\nabla}\boldsymbol{u}\|_{L^2(\mathbb{R}_-^3)} \|\boldsymbol{\varphi}\|_{H_0^1(\mathbb{R}_-^3)}.
$$

By (40) then, we can write Problem (39) in weak form as:

$$
(44) \quad \int_{\mathbb{R}_-^3} (\mathbb{C}\widehat{\nabla}(\varrho\partial_i\boldsymbol{u})) \cdot \nabla\boldsymbol{\varphi}\,d\boldsymbol{x} = \int_{\mathbb{R}_-^3} \mathbf{G}_i \cdot \nabla\boldsymbol{\varphi}\,d\boldsymbol{x} - \langle \boldsymbol{F}_i, \boldsymbol{\varphi} \rangle_{((H_0^1(\mathbb{R}_-^3))', H_0^1(\mathbb{R}_-^3))},
$$

for $\boldsymbol{\varphi} \in H_0^1(\mathbb{R}_-^3)$. Since $\mathbf{G}_i \in L^2(\mathbb{R}_-^3)$, it is clear that the right-hand side of the equality above defines a continuous functional, which we call h_i, $i = 1, 2$, on $H_0^1(\mathbb{R}_-^3)$. In fact, we have already explicitly estimated the terms containing $\boldsymbol{F}_i$. For the term containing $\mathbf{G}_i$, we note that, using (20), we obtain

$$
(45) \quad \left| \int_{\mathbb{R}_-^3} \mathbb{C}(\widehat{\nabla\varrho \otimes \partial_i\boldsymbol{u}}) : \widehat{\nabla}\boldsymbol{\varphi}\,d\boldsymbol{x} \right| \leq C\|\partial_i\boldsymbol{u}\|_{L^2(\mathbb{R}_-^3)} \|\nabla\boldsymbol{\varphi}\|_{L^2(\mathbb{R}_-^3)}
$$
$$
\leq C\|\partial_i\boldsymbol{u}\|_{L^2(\mathbb{R}_-^3)} \|\boldsymbol{\varphi}\|_{H_0^1(\mathbb{R}_-^3)},
$$

and by (21),

$$
\left| \int_{\mathbb{R}_-^3} \varrho\partial_i\mathbb{C}\widehat{\nabla}\boldsymbol{u} : \nabla\boldsymbol{\varphi}\,d\boldsymbol{x} \right| \leq C\|\nabla\boldsymbol{u}\|_{L^2(\mathbb{R}_-^3)} \|\nabla\boldsymbol{\varphi}\|_{L^2(\mathbb{R}_-^3)} \leq C\|\nabla\boldsymbol{u}\|_{L^2(\mathbb{R}_-^3)} \|\boldsymbol{\varphi}\|_{H_0^1(\mathbb{R}_-^3)}.
$$

We therefore study the variational problem:
find $\boldsymbol{w}_i \in H_0^1(\mathbb{R}_-^3)$, *for* $i = 1, 2$, *such that*

$$
(46) \quad a(\boldsymbol{w}_i, \boldsymbol{\varphi}) = h_i(\boldsymbol{\varphi}), \qquad \forall \boldsymbol{\varphi} \in H_0^1(\mathbb{R}_-^3),
$$

where a is the bilinear form defined in (34), *and h_i, $i = 1, 2$, is the linear operator on* $H_0^1(\mathbb{R}_-^3)$ *defined by the right-hand side in* (44) *as a function of $\boldsymbol{\varphi}$.*
If this problem has a unique solution, then necessarily $\boldsymbol{w}_i = \varrho\partial_i\boldsymbol{u}$, $i = 1, 2$.

As proved in Theorem 3.3, the bilinear form a is continuous and coercive, and we have shown above h_i is continuous. Hence, the Lax-Milgram Theorem then ensures the existence of a unique solution $\boldsymbol{w}_i = \varrho\partial_i\boldsymbol{u}$ of (39) in $H_0^1(\mathbb{R}_-^3)$, satisfying

$$
\|\varrho\partial_i\boldsymbol{u}\|_{H_0^1(\mathbb{R}_-^3)} \leq C\|\boldsymbol{f}\|_{H_1^0(\mathbb{R}_-^3)}.
$$

Second Step: From the previous step, by the isomorphism (9) we have that, for $i = 1, 2$, $\partial_i \boldsymbol{u} \in H_1^1(\mathbb{R}_-^3)$ and satisfies the estimate:

$$\|\partial_i \boldsymbol{u}\|_{H_1^1(\mathbb{R}_-^3)} \leq C \|\varrho \partial_i \boldsymbol{u}\|_{H_0^1(\mathbb{R}_-^3)} \leq C \|\boldsymbol{f}\|_{H_1^0(\mathbb{R}_-^3)}. \tag{47}$$

If $\partial_3 \boldsymbol{u} \in H_1^1(\mathbb{R}_-^3)$, then $\nabla \boldsymbol{u} \in H_1^1(\mathbb{R}_-^3)$, that is, $\boldsymbol{u} \in H_1^2(\mathbb{R}_-^3)$, given that $\boldsymbol{u} \in H_0^1(\mathbb{R}_-^3)$ by Theorem 3.3. In fact, it is enough to prove that $\varrho \partial_3^2 \boldsymbol{u} \in L^2(\mathbb{R}_-^3)$. To this end, we use equation (30), written in the form:

$$\mu \Delta \boldsymbol{u} + (\lambda + \mu) \nabla \operatorname{div} \boldsymbol{u} + \nabla \lambda \operatorname{div} \boldsymbol{u} + 2 \widehat{\nabla} \boldsymbol{u} \nabla \mu = \boldsymbol{f}. \tag{48}$$

From this equation, it follows, in particular, that:

$$\begin{aligned}
\mu \partial_3^2 \boldsymbol{u}' &= -\mu \Delta' \boldsymbol{u}' - (\lambda + \mu) \nabla' \operatorname{div} \boldsymbol{u} - \nabla' \lambda \operatorname{div} \boldsymbol{u} - 2 (\widehat{\nabla} \boldsymbol{u})' \nabla \mu + \boldsymbol{f}', \\
(\lambda + 2\mu) \partial_3^2 u_3 &= -\mu \Delta' u_3 - (\lambda + \mu) \partial_3 \operatorname{div} \boldsymbol{u}' - \partial_3 \lambda \operatorname{div} \boldsymbol{u} - 2 (\widehat{\nabla} \boldsymbol{u})_3 \cdot \nabla \mu + f_3,
\end{aligned} \tag{49}$$

where we used the prime notation to denote projection onto the first two variables. From (49), since the tangential derivatives are in $H_1^1(\mathbb{R}_-^3)$, we have that

$$\varrho \mu \partial_3^2 \boldsymbol{u}' \in L^2(\mathbb{R}_-^3), \qquad \text{and} \qquad \varrho (\lambda + 2\mu) \partial_3^2 u_3 \in L^2(\mathbb{R}_-^3). \tag{50}$$

From (50), the bounds on the Lamé parameters (see (20)), and the strong convexity condition (22), we obtain that

$$\varrho \partial_3^2 \boldsymbol{u} \in L^2(\mathbb{R}_-^3). \tag{51}$$

The regularity estimate (36) then follows from (47), the assumed regularity of the source and the coefficients, and (49). □

We now tackle (iii), that is, the existence of a very weak solution for Problem (30). More precisely, we will prove the existence of a unique solution $\boldsymbol{u} \in E_0(\mathbb{R}_-^3)$ for $\boldsymbol{f} \in V'$, where, we recall, the space V is defined in (31) and $E_0(\mathbb{R}_-^3)$ is given in (32). To do so, we first establish the validity of integration by parts, when $\operatorname{div}(\mathbb{C} \widehat{\nabla} \boldsymbol{u}) \in V'$. This result will be a direct consequence of standard integration by parts provided $\mathcal{D}(\overline{\mathbb{R}_-^3})$ is dense into $E_0(\mathbb{R}_-^3)$. The density follows from an extension to the case of weighted Sobolev spaces of results in [26] and from some ideas contained in [3]. We include the proof of this density result for completeness.

Lemma 3.6. *The space $\mathcal{D}(\overline{\mathbb{R}_-^3})$ is dense in $E_0(\mathbb{R}_-^3)$.*

Proof. By the Riesz Representation Theorem, for every functional $T \in (E_0(\mathbb{R}_-^3))'$ there exist $\boldsymbol{u}_1 \in H_1^0(\mathbb{R}_-^3)$ and $\boldsymbol{u}_2 \in V(\mathbb{R}_-^3)$ such that, for every $\boldsymbol{v} \in E_0(\mathbb{R}_-^3)$,

$$\langle T, \boldsymbol{v} \rangle = \int_{\mathbb{R}_-^3} \boldsymbol{u}_1 \cdot \boldsymbol{v} \, d\boldsymbol{x} + \langle \operatorname{div} \mathbb{C} \widehat{\nabla} \boldsymbol{v}, \boldsymbol{u}_2 \rangle_{(V'(\mathbb{R}_-^3), V(\mathbb{R}_-^3))}. \tag{52}$$

Next, we suppose that T is the zero functional when restricted to $\mathcal{D}(\overline{\mathbb{R}_-^3})$:

$$\langle T, \boldsymbol{\varphi} \rangle = 0, \qquad \forall \boldsymbol{\varphi} \in \mathcal{D}(\overline{\mathbb{R}_-^3}). \tag{53}$$

We will show that T is then the zero functional on E_0:

$$\langle T, \boldsymbol{v} \rangle = 0, \qquad \forall \boldsymbol{v} \in E_0(\mathbb{R}_-^3).$$

Since $\boldsymbol{\varphi} \in \mathcal{D}(\overline{\mathbb{R}^3_-})$, there exists $\boldsymbol{\psi} \in \mathcal{D}(\mathbb{R}^3)$ such that

$$\boldsymbol{\psi}_{|\overline{\mathbb{R}^3_-}} = \boldsymbol{\varphi}.$$

Following the approach of Lions-Magenes (see [26] p. 173), we let $\widetilde{\boldsymbol{u}}_1$ and $\widetilde{\boldsymbol{u}}_2$ denote the extension by zero of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ to $\mathbb{R}^3$, respectively. We also extend the isotropic elastic tensor $\mathbb{C}$ to an isotropic tensor $\widetilde{\mathbb{C}}$ on $\mathbb{R}^3$, satisfying

$$\widetilde{\mathbb{C}} \in C^{0,1}(\mathbb{R}^3) \qquad \text{and} \qquad \widetilde{\mathbb{C}}_{|\overline{\mathbb{R}^3_-}} = \mathbb{C}.$$

Such as extension can be done by even reflection, and, hence, we can assume that there exist $\widetilde{\alpha}_0 > 0$ and $\widetilde{\beta}_0 > 0$ such that

$$\widetilde{\mu}(\boldsymbol{x}) \geq \widetilde{\alpha}_0 > 0, \qquad \text{and} \qquad 3\widetilde{\lambda}(\boldsymbol{x}) + 2\widetilde{\mu}(\boldsymbol{x}) \geq \widetilde{\beta}_0 > 0, \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^3,$$

and we can also assume that

$$(54) \qquad\qquad\qquad |\nabla\widetilde{\lambda}| \leq \frac{C}{\varrho}, \qquad |\nabla\widetilde{\mu}| \leq \frac{C}{\varrho},$$

almost everywhere in $\mathbb{R}^3$.

Therefore it follows that

$$\langle T, \boldsymbol{\psi} \rangle = \int_{\mathbb{R}^3} \left[\widetilde{\boldsymbol{u}_1} \cdot \boldsymbol{\psi} + \mathrm{div}(\widetilde{\mathbb{C}}\widehat{\nabla}\boldsymbol{\psi}) \cdot \widetilde{\boldsymbol{u}}_2 \right] d\boldsymbol{x}$$

$$= \int_{\mathbb{R}^3_-} \left[\boldsymbol{u}_1 \cdot \boldsymbol{\varphi} + \mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi}) \cdot \boldsymbol{u}_2 \right] d\boldsymbol{x} = 0,$$

where we used that $\mathrm{div}(\widetilde{\mathbb{C}}\widehat{\nabla}\boldsymbol{\psi})$ is bounded function with compact support and that $\widetilde{\boldsymbol{u}}_2$ is a locally L^2 function. Consequently, for any $\boldsymbol{\psi} \in \mathcal{D}(\mathbb{R}^3)$,

$$\langle \widetilde{\boldsymbol{u}}_1, \boldsymbol{\psi} \rangle + \langle \mathrm{div}\,\widetilde{\mathbb{C}}\widehat{\nabla}\widetilde{\boldsymbol{u}}_2, \boldsymbol{\psi} \rangle = 0,$$

from which it follows that

$$(55) \qquad\qquad\qquad \mathrm{div}(\widetilde{\mathbb{C}}\widehat{\nabla}\widetilde{\boldsymbol{u}}_2) = -\widetilde{\boldsymbol{u}}_1 \quad \text{in } \mathcal{D}'(\mathbb{R}^3).$$

We next show that $\widetilde{\boldsymbol{u}}_2 \in H^2_1(\mathbb{R}^3)$ by using the well posedness and regularity for the equation

$$\mathrm{div}(\widetilde{\mathbb{C}}\widehat{\nabla}\widetilde{\boldsymbol{u}}) = \widetilde{\boldsymbol{f}}$$

on all of $\mathbb{R}^3$. The decay condition at infinity imposed on $\widetilde{\boldsymbol{u}}$ as an element of $H^2_1(\mathbb{R}^3)$ (or just $H^1_0(\mathbb{R}^3)$) ensures the global coercivity of the quadratic form from the strong convexity of the Lamé tensor. Therefore, since $\widetilde{\boldsymbol{f}} = -\widetilde{\boldsymbol{u}}_1 \in H^0_1(\mathbb{R}^3)$ we can first prove that the solution belongs to $H^1_0(\mathbb{R}^3)$ proceeding as in the proof of Theorem 3.3. Then following a similar approach as that in the proof of Theorem 3.5, we are able to establish that the unique solution, which must agree with $\widetilde{\boldsymbol{u}}_2$, is in $H^2_1(\mathbb{R}^3)$. Now, since $\widetilde{\boldsymbol{u}}_2 \in H^2_1(\mathbb{R}^3)$ is an extension by zero of $\boldsymbol{u}_2$ in $\mathbb{R}^3$, $\boldsymbol{u}_2$ and $\nabla\boldsymbol{u}_2$ must have trace zero on $\{x_3 = 0\}$, that is,

$$\boldsymbol{u}_2 = \widetilde{\boldsymbol{u}}_{2|\mathbb{R}^3_-} \in \mathring{H}^2_1(\mathbb{R}^3_-).$$

Exploiting identity (55), we can rewrite (52) as:

$$\langle T, \boldsymbol{v} \rangle = -\int_{\mathbb{R}^3_-} \mathrm{div}\,(\mathbb{C}\widehat{\nabla}\boldsymbol{u}_2) \cdot \boldsymbol{v}\, d\boldsymbol{x} + \langle \mathrm{div}\,(\mathbb{C}\widehat{\nabla}\boldsymbol{v}), \boldsymbol{u}_2 \rangle_{(V'(\mathbb{R}^3_-), V(\mathbb{R}^3_-))}.$$

Since $\mathcal{D}(\mathbb{R}^3_-)$ is dense in $\mathring{H}^2_1(\mathbb{R}^3_-)$ by definition, there exists

(56) $$\{\boldsymbol{\varphi}_k\} \in \mathcal{D}(\mathbb{R}^3_-) \quad \text{such that} \quad \boldsymbol{\varphi}_k \overset{k\to\infty}{\longrightarrow} \boldsymbol{u}_2 \quad \text{in } \mathring{H}^2_1(\mathbb{R}^3_-).$$

Hence we find that, for every $\boldsymbol{v} \in E_0(\mathbb{R}^3_-)$,

$$\langle T, \boldsymbol{v}\rangle = \lim_{k\to\infty} \left[-\int_{\mathbb{R}^3_-} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi}_k) \cdot \boldsymbol{v}\, d\boldsymbol{x} + \langle \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{v}), \boldsymbol{\varphi}_k\rangle \right] = 0.$$

We conclude that $\mathcal{D}(\overline{\mathbb{R}^3_-})$ is dense in $E_0(\mathbb{R}^3_-)$ from the Hahn-Banach Theorem. In fact, setting for notational convenience $E_1 := \mathcal{D}(\overline{\mathbb{R}^3_-})$, we suppose by contradiction that $\overline{E}_1$ is a proper closed subset of $E_0(\mathbb{R}^3_-)$. Therefore, there exists $\boldsymbol{f}_0 \in E_0(\mathbb{R}^3_-)$ such that $\boldsymbol{f}_0 \notin \overline{E}_1$, and we can define a continuous functional $\widehat{T}$ on $\overline{E}_1 \cup \{\boldsymbol{f}_0\}$) by:

$$\widehat{T}(\boldsymbol{f}) = 0 \qquad \forall \boldsymbol{f} \in \overline{E}_1,$$
$$\widehat{T}(\boldsymbol{f}_0) = 1,$$

Then by the Hahn-Banach theorem the operator $\widehat{T}$ can be extended to a non-zero functional $\widehat{\widehat{T}} \in (E_0(\mathbb{R}^3_-))'$, a contradiction, since we proved that any functional that is zero on E_1 is zero on E_0. $\qquad\square$

By the previous lemma and results in [3], we can prove the following Green's formula.

Proposition 3.7. *Let $\boldsymbol{u} \in E_0(\mathbb{R}^3_-)$. Then for any $\boldsymbol{\phi} \in V(\mathbb{R}^3_-)$, we have*
(57)
$$\langle \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}), \boldsymbol{\phi}\rangle_{(V'(\mathbb{R}^3_-), V(\mathbb{R}^3_-))} = \int_{\mathbb{R}^3_-} \boldsymbol{u}\cdot\operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\phi})\, d\boldsymbol{x} + \langle (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{e}_3, \boldsymbol{\phi}\rangle_{(H^{-3/2}_{-1}(\mathbb{R}^2), H^{3/2}_1(\mathbb{R}^2))}$$

Proof. From Lemma 3.6, it is sufficient to prove (57) for $\boldsymbol{u} \in \mathcal{D}(\overline{\mathbb{R}^3_-})$. Therefore, for any $\boldsymbol{u} \in \mathcal{D}(\overline{\mathbb{R}^3_-})$ and $\boldsymbol{\phi} \in V(\mathbb{R}^3_-)$ we have

(58) $$\int_{\mathbb{R}^3_-} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\cdot\boldsymbol{\phi}\, d\boldsymbol{x} = \int_{\mathbb{R}^3_-} \boldsymbol{u}\cdot\operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\phi})\, d\boldsymbol{x} + \int_{\mathbb{R}^2} (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{e}_3)\cdot\boldsymbol{\phi}\, d\boldsymbol{x}'.$$

Next we prove that, if $\boldsymbol{u} \in E_0(\mathbb{R}^3_-)$, then

(59)
$$\gamma_N : E_0(\mathbb{R}^3_-) \longrightarrow H^{-3/2}_{-1}(\mathbb{R}^2),$$
$$\boldsymbol{u} \mapsto (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{e}_3,$$

is a linear and continuous functional. Let $\boldsymbol{\zeta} \in H^{3/2}_1(\mathbb{R}^2)$. By the Trace Theorem, there exists a lifting function $\boldsymbol{\phi} \in H^2_1(\mathbb{R}^3_-)$ such that $\boldsymbol{\phi} = \boldsymbol{\zeta}$ and $(\mathbb{C}\widehat{\nabla}\boldsymbol{\phi})\boldsymbol{e}_3 = 0$ on $\mathbb{R}^2$ (see [3, Lemma 2.2]); hence $\boldsymbol{\phi} \in V(\mathbb{R}^3_-)$. From (58), we obtain

$$\int_{\mathbb{R}^2} ((\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{e}_3)\cdot\boldsymbol{\zeta}\, d\sigma(\boldsymbol{x}) = \langle \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}), \boldsymbol{\phi}\rangle_{(V'(\mathbb{R}^3_-), V(\mathbb{R}^3_-))} - \langle \boldsymbol{u}, \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\phi})\rangle_{(H^0_{-1}(\mathbb{R}^3_-), H^0_1(\mathbb{R}^3_-))}.$$

The functional on the left-hand side is, therefore, well-defined on $H^{3/2}_1(\mathbb{R}^2)$ as $\boldsymbol{u} \in \mathcal{D}(\overline{\mathbb{R}^3_-})$. Moreover, the lifting function $\boldsymbol{\phi}$ satisfies

$$\|\boldsymbol{\phi}\|_{H^2_1(\mathbb{R}^3_-)} \le c\,\|\boldsymbol{\zeta}\|_{H^{3/2}_1(\mathbb{R}^2)},$$

so that

$$|\langle \gamma_N(\boldsymbol{u}), \boldsymbol{\zeta} \rangle| \leq C \left[\|\mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\|_{V'(\mathbb{R}^3_-)}\|\boldsymbol{\phi}\|_{V(\mathbb{R}^3_-)} + \|\boldsymbol{u}\|_{H^0_{-1}(\mathbb{R}^3_-)}\|\mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\phi})\|_{H^0_1(\mathbb{R}^3_-)} \right]$$

$$\leq C\|\boldsymbol{u}\|_{E_0(\mathbb{R}^3_-)}\|\boldsymbol{\phi}\|_{H^2_1(\mathbb{R}^3_-)} \leq C\|\boldsymbol{u}\|_{E_0(\mathbb{R}^3_-)}\|\boldsymbol{\zeta}\|_{H^{3/2}_1(\mathbb{R}^2)}.$$

That is, (59) holds and the statement of the proposition follows. $\qquad\square$

Next, we prove existence and uniqueness of a very weak solution.

Theorem 3.8 (Very weak solution). *For any $\boldsymbol{f} \in V'(\mathbb{R}^3_-)$, there exists a unique solution $\boldsymbol{u} \in H^0_{-1}(\mathbb{R}^3_-)$ to Problem* (30) *such that*

$$\|\boldsymbol{u}\|_{H^0_{-1}(\mathbb{R}^3_-)} \leq C \, \|\boldsymbol{f}\|_{V'(\mathbb{R}^3_-)}.$$

Proof. Thanks to the Green's formula (57), for any $\boldsymbol{f} \in V'(\mathbb{R}^3_-)$ Problem (30) is equivalent to the following variational formulation:
find $\boldsymbol{u} \in E_0(\mathbb{R}^3_-)$ such that

$$(60) \qquad \int_{\mathbb{R}^3_-} \boldsymbol{u} \cdot \mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{v})\, d\boldsymbol{x} = \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{V'(\mathbb{R}^3_-),V(\mathbb{R}^3_-)}, \qquad \forall \boldsymbol{v} \in V(\mathbb{R}^3_-).$$

We first note that, from the well-posedness of (30) in $H^2_1(\mathbb{R}^3_-)$ (Theorem 3.5), for any $\overline{\boldsymbol{f}} \in H^0_1(\mathbb{R}^3_-)$ there exists $\overline{\boldsymbol{v}} \in V(\mathbb{R}^3_-)$ satisfying (30) with $\boldsymbol{f}$ replaced by $\overline{\boldsymbol{f}}$ such that

$$\|\overline{\boldsymbol{v}}\|_{V(\mathbb{R}^3_-)} = \|\overline{\boldsymbol{v}}\|_{H^2_1(\mathbb{R}^3_-)} \leq c\,\|\overline{\boldsymbol{f}}\|_{H^0_1(\mathbb{R}^3_-)}.$$

The bilinear form

$$\Phi_{\boldsymbol{f}}(\overline{\boldsymbol{f}}) = \langle \boldsymbol{f}, \overline{\boldsymbol{v}} \rangle_{(V'(\mathbb{R}^3_-), V(\mathbb{R}^3_-))}$$

is then continuous, as

$$|\Phi_{\boldsymbol{f}}(\overline{\boldsymbol{f}})| \leq C\,\|\boldsymbol{f}\|_{V'(\mathbb{R}^3_-)}\|\overline{\boldsymbol{v}}\|_{V(\mathbb{R}^3_-)} \leq C\|\boldsymbol{f}\|_{V'(\mathbb{R}^3_-)}\|\overline{\boldsymbol{f}}\|_{H^0_1(\mathbb{R}^3_-)}.$$

Consequently, from the Riesz Representation Theorem there exists a unique $\boldsymbol{u} \in H^0_{-1}(\mathbb{R}^3_-)$ such that

$$\Phi_{\boldsymbol{f}}(\overline{\boldsymbol{f}}) = \langle \boldsymbol{u}, \overline{\boldsymbol{f}} \rangle_{(H^0_{-1}(\mathbb{R}^3_-), H^0_1(\mathbb{R}^3_-))}, \qquad \forall \overline{\boldsymbol{f}} \in H^0_1(\mathbb{R}^3_-).$$

Since the solution operator of Problem (26) for strong solutions

$$\Phi : H^0_1(\mathbb{R}^3_-) \longrightarrow V(\mathbb{R}^3_-)$$
$$\overline{\boldsymbol{f}} \longrightarrow \overline{\boldsymbol{v}}$$

is an isomorphism, the assertion of the theorem follows. $\qquad\square$

We lastly address the well-posedness of the source problem, Problem (26), by means of interpolation. We recall Formula (16b),

$$\left[H^{-2}_{-1}(\mathbb{R}^3_-), H^{-1}_0(\mathbb{R}^3_-) \right]_{1-\theta,2} = H^{-1-\Theta}_{-\Theta}(\mathbb{R}^3_-).$$

The above result is relevant in view of the following auxiliary result.

Proposition 3.9. *If $\boldsymbol{f} \in H^{-2}_{-1}(\mathbb{R}^3_-)$ has compact support, then $\boldsymbol{f} \in V'(\mathbb{R}^3_-)$.*

Proof. We define ψ, a regular cut-off function in $\mathbb{R}^3_-$, such that $\psi = 1$ on a compact neighborhood of the support of $\boldsymbol{f}$. Then

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle_{(V'(\mathbb{R}^3_-), V(\mathbb{R}^3_-))} = \langle \boldsymbol{f}\psi, \boldsymbol{v} \rangle_{(V'(\mathbb{R}^3_-), V(\mathbb{R}^3_-))} = \langle \boldsymbol{f}, \psi\boldsymbol{v} \rangle_{(H^{-2}_{-1}(\mathbb{R}^3_-), \mathring{H}^2_1(\mathbb{R}^3_-))},$$

is well defined and satisfies

$$\left| \langle \boldsymbol{f}, \boldsymbol{v} \rangle_{(V'(\mathbb{R}^3_-), V(\mathbb{R}^3_-))} \right| \leq \|\boldsymbol{f}\|_{H^{-2}_{-1}(\mathbb{R}^3_-)} \|\psi\boldsymbol{v}\|_{\mathring{H}^2_1(\mathbb{R}^3_-)} \leq c\|\boldsymbol{f}\|_{H^{-2}_{-1}(\mathbb{R}^3_-)} \|\boldsymbol{v}\|_{H^2_1(\mathbb{R}^3_-)}.$$

The assertion follows. $\qquad\square$

Remark 3.10. *By Proposition 3.9, Theorem 3.8 also holds for any $\boldsymbol{f} \in H^{-2}_{-1}(\mathbb{R}^3_-)$ with compact support in $\mathbb{R}^3_-$.*

We are now in the position to establish the well-posedness of (26).

Theorem 3.11. *Problem (26) with source term given in (27) has a unique distributional solution $\boldsymbol{u} \in H^{1/2-\varepsilon}_{-1/2-\varepsilon}(\mathbb{R}^3_-)$, for all $\varepsilon > 0$.*

Proof. We specialize the interpolation Formula (16b) to the case $\Theta = 1/2 + \varepsilon$ to obtain:

$$(61) \qquad H^{-3/2-\varepsilon}_{-1/2-\varepsilon}(\mathbb{R}^3_-) = [H^{-2}_{-1}(\mathbb{R}^3_-), H^{-1}_0(\mathbb{R}^3_-)]_{\frac{1}{2}-\varepsilon,2},$$

and similarly Formula (15a) to the case $\Theta = 1/2 - \varepsilon$ to obtain:

$$(62) \qquad H^{1/2-\varepsilon}_{-1/2-\varepsilon}(\mathbb{R}^3_-) = [H^0_{-1}(\mathbb{R}^3_-), H^1_0(\mathbb{R}^3_-)]_{\frac{1}{2}-\varepsilon,2}.$$

From Proposition (3.1), $\boldsymbol{f}_S \in H^{-\frac{3}{2}-\varepsilon}$ and, since it has compact support in $\mathbb{R}^3_-$ by (27), we deduce that

$$\boldsymbol{f}_S \in H^{-3/2-\varepsilon}_{-1/2-\varepsilon}(\mathbb{R}^3_-).$$

By Theorems 3.3 and 3.8, and Remark 3.10, we have a bounded solution operator Φ, where $\Phi(\boldsymbol{f}) = \boldsymbol{u}$, mapping

$$\Phi : V'(\mathbb{R}^3_-) \to H^0_{-1}(\mathbb{R}^3_-), \qquad \Phi : (H^1_0)'(\mathbb{R}^3_-) \to H^1_0(\mathbb{R}^3_-).$$

From the hypotheses on the dislocation surface, we can assume that $S \subset \Omega$, where Ω is a bounded open set such that $\overline{\Omega} \subset \mathbb{R}^3_-$. We then restricts all source terms $\boldsymbol{f}$ to have compact support in $\overline{\Omega}$, and we denote by $H^{-s}_{-\alpha,\Omega}(\mathbb{R}^3_-)$ the closed subspace of $H^{-s}_{-\alpha}(\mathbb{R}^3_-)$ of distributions with support in $\overline{\Omega}$. Then $\boldsymbol{f}_S \in H^{-3/2-\varepsilon}_{-1/2-\varepsilon,\Omega}(\mathbb{R}^3_-)$. Furthermore, since $V'(\mathbb{R}^3_-) \supset H^{-2}_{-1,\Omega}(\mathbb{R}^3_-)$ and $(H^1_0)'(\mathbb{R}^3_-) \supset H^{-1}_{0,\Omega}(\mathbb{R}^3_-)$, from (61) we have:

$$H^{-3/2-\varepsilon}_{-1/2-\varepsilon,\Omega}(\mathbb{R}^3_-) = [H^{-2}_{-1,\Omega}(\mathbb{R}^3_-), H^{-1}_{0,\Omega}(\mathbb{R}^3_-)]_{\frac{1}{2}-\varepsilon,2} \subset [V'(\mathbb{R}^3_-), (H^1_0)'(\mathbb{R}^3_-)]_{\frac{1}{2}-\varepsilon,2}.$$

By interpolation, Φ extends as a continuous solution operator

$$\Phi : [V'(\mathbb{R}^3_-), (H^1_0(\mathbb{R}^3_-))']_{\frac{1}{2}-\varepsilon,2} \to [H^0_{-1}(\mathbb{R}^3_-), H^1_0(\mathbb{R}^3_-)]_{\frac{1}{2}-\varepsilon,2},$$

so that, using (62), we also have:

$$\Phi : H^{-3/2-\varepsilon}_{-1/2-\varepsilon,\Omega}(\mathbb{R}^3_-) \to H^{1/2-\varepsilon}_{-1/2-\varepsilon}(\mathbb{R}^3_-),$$

as a continuous operator, which gives the conclusion of the theorem. $\qquad\square$

We close the discussion of well-posedness of the direct problem by showing the equivalence of the transmission problem to the source problem.

Lemma 3.12. *Problem* (24) *and Problem* (26) *are equivalent.*

Proof. We first observe that both solutions of (24) and (26) satisfy $\mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \boldsymbol{0}$, for all $\boldsymbol{x} \in \mathbb{R}^3_- \setminus \overline{S}$, and $(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{e}_3 = \boldsymbol{0}$ on $\{x_3 = 0\}$. Hence we have only to verify that the jump relations on the dislocation surface S are satisfied.

We take a point $\overline{\boldsymbol{x}} \in S$ and a ball $B_\eta(\overline{\boldsymbol{x}})$, with η sufficiently small, such that $(B_\eta(\overline{\boldsymbol{x}})\cap S) \subset S$, since S is an open surface. We indicate with $B_\eta^+(\overline{\boldsymbol{x}})$ $(B_\eta^-(\overline{\boldsymbol{x}}))$ the half ball on the same (opposite) side of the unit normal vector $\boldsymbol{n}$ on the boundary of S.

To simplify notation, we define $D_\eta^+ := S^+ \cap B_\eta^+(\overline{\boldsymbol{x}})$, $D_\eta^- := S^- \cap B_\eta^-(\overline{\boldsymbol{x}})$ and $D_\eta := S \cap B_\eta(\overline{\boldsymbol{x}})$.

Let $\boldsymbol{\varphi} \in \mathcal{D}(B_\eta(\overline{\boldsymbol{x}}))$ and let $\boldsymbol{u}$ be the solution to (26). We recall that $\boldsymbol{u} \in H^{1/2-\varepsilon}$ in a neighborhood of S and that its trace can be defined in $H^{-\varepsilon}(S)$ in a weak sense. Then, since $\boldsymbol{u}$ is a solution to $\mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}) = \boldsymbol{0}$ in $B_\eta^+(\overline{\boldsymbol{x}})$ and $B_\eta^-(\overline{\boldsymbol{x}})$, we have

(63)
$$
\int\limits_{B_\eta^+(\overline{\boldsymbol{x}})} \mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi}) \cdot \boldsymbol{u}\, d\boldsymbol{x} = -\langle \boldsymbol{u}^+, (\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi})\boldsymbol{n}\rangle_{(H^{-\varepsilon}(D_\eta^+),\, H^\varepsilon(D_\eta^+))} - \int\limits_{B_\eta^+(\overline{\boldsymbol{x}})} \mathbb{C}\widehat{\nabla}\boldsymbol{\varphi} : \widehat{\nabla}\boldsymbol{u}\, d\boldsymbol{x}
$$
$$
= -\langle \boldsymbol{u}^+, (\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi})\boldsymbol{n}\rangle_{(H^{-\varepsilon}(D_\eta^+),\, H^\varepsilon(D_\eta^+))} + \langle ((\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n})^+, \boldsymbol{\varphi}\rangle_{(H^{-1-\varepsilon}(D_\eta^+),\, H^{1+\varepsilon}(D_\eta^+))}.
$$

Analogously,

(64)
$$
\int\limits_{B_\eta^-(\overline{\boldsymbol{x}})} \mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi}) \cdot \boldsymbol{u}\, d\boldsymbol{x} = \langle \boldsymbol{u}^-, (\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi})\boldsymbol{n}\rangle_{(H^{-\varepsilon}(D_\eta^-),\, H^\varepsilon(D_\eta^-))} - \int\limits_{B_\eta^-(\overline{\boldsymbol{x}})} \mathbb{C}\widehat{\nabla}\boldsymbol{\varphi} : \widehat{\nabla}\boldsymbol{u}\, d\boldsymbol{x}
$$
$$
= \langle \boldsymbol{u}^-, (\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi})\boldsymbol{n}\rangle_{(H^{-\varepsilon}(D_\eta^-),\, H^\varepsilon(D_\eta^-))} - \langle ((\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n})^-, \boldsymbol{\varphi}\rangle_{(H^{-1-\varepsilon}(D_\eta^-),\, H^{1+\varepsilon}(D_\eta^-))}.
$$

Summing (63) and (64) gives:

(65)
$$
\int\limits_{B_\eta(\overline{\boldsymbol{x}})} \mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi}) \cdot \boldsymbol{u}\, d\boldsymbol{x} = -\langle [\boldsymbol{u}]_S, (\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi})\boldsymbol{n}\rangle_{(H^{-\varepsilon}(D_\eta),\, H^\varepsilon(D_\eta))} + \langle [(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}], \boldsymbol{\varphi}\rangle_{(H^{-1-\varepsilon}(D_\eta),\, H^{1+\varepsilon}(D_\eta))},
$$

where $[\cdot]_S$ denotes the jump on S. Since $\boldsymbol{u}$ is a solution of (26) with source term (27), it follows that

(66)
$$
\int\limits_{B_\eta^+(\overline{\boldsymbol{x}})} \mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi}) \cdot \boldsymbol{u}\, d\boldsymbol{x} + \int\limits_{B_\eta^-(\overline{\boldsymbol{x}})} \mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi}) \cdot \boldsymbol{u}\, d\boldsymbol{x} = \int\limits_{B_\eta(\overline{\boldsymbol{x}})} \mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi}) \cdot \boldsymbol{u}\, d\boldsymbol{x}
$$
$$
= \langle \mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{u}), \boldsymbol{\varphi}\rangle_{(H^{-3/2-\varepsilon}(B_\eta(\overline{\boldsymbol{x}})),\, H^{3/2+\varepsilon}(B_\eta(\overline{\boldsymbol{x}})))}
$$
$$
= -\langle \mathbb{C}(\boldsymbol{g} \otimes \boldsymbol{n})\delta_S, \nabla\boldsymbol{\varphi}\rangle_{(H^{-1/2-\varepsilon}(B_\eta(\overline{\boldsymbol{x}})),\, H^{1/2+\varepsilon}(B_\eta(\overline{\boldsymbol{x}})))}
$$
$$
= -\int\limits_{D_\eta} \nabla\boldsymbol{\varphi} : \mathbb{C}(\boldsymbol{g} \otimes \boldsymbol{n})\, d\sigma(\boldsymbol{x}).
$$

By the symmetries the tensor $\mathbb{C}$ satisfies, we also have that

$$
\nabla\boldsymbol{\varphi} : \mathbb{C}(\boldsymbol{g} \otimes \boldsymbol{n}) = \boldsymbol{g} \cdot (\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi})\boldsymbol{n}.
$$

Hence, equation (66) becomes

$$(67) \qquad \int_{B_\eta(\overline{\boldsymbol{x}})} \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi}) \cdot \boldsymbol{u} \, d\boldsymbol{x} = -\int_{D_\eta} \boldsymbol{g} \cdot (\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi})\boldsymbol{n} \, d\sigma(\boldsymbol{x}).$$

Comparing (65) and (67) we have that

$$(68)$$
$$-\int_{D_\eta} \boldsymbol{g} \cdot (\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi})\boldsymbol{n} \, d\sigma(\boldsymbol{x}) = -\big\langle [\boldsymbol{u}], \, (\mathbb{C}\widehat{\nabla}\boldsymbol{\varphi})\boldsymbol{n} \big\rangle_{(H^{-\varepsilon}(D_\eta), H^{\varepsilon}(D_\eta))} + \big\langle [(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}], \, \boldsymbol{\varphi} \big\rangle_{(H^{-1-\varepsilon}(D_\eta), H^{1+\varepsilon}(D_\eta))},$$

for any $\boldsymbol{\varphi} \in \mathcal{D}(B_\eta(\bar{\boldsymbol{x}}))$. By taking $\boldsymbol{\varphi}$ constant near $\bar{\boldsymbol{x}}$, we conclude from the identity above that $[(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}] = \boldsymbol{0}$ in D_η, and hence it follows $[(\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n}] = \boldsymbol{0}$ in S. Then, $[\boldsymbol{u}] = \boldsymbol{g}$ in S. We have shown that, if $\boldsymbol{u}$ is a solution of (26) , it is also a very weak solution of (24). The converse implication follows by simply reversing all arguments in the proof. $\qquad\square$

From Theorem 3.11 and the previous lemma, we finally have the well-posedness of the direct problem in $\mathbb{R}^3_-$.

Corollary 3.13. *There exists a unique very weak solution* $\boldsymbol{u} \in H^{1/2-\varepsilon}_{-1/2-\varepsilon}(\mathbb{R}^3_-)$ *of the boundary-value/transmission problem* (24).

4. The solution as a double layer potential

In this section, we prove the existence of a Neumann function in the half-space, for an isotropic, non-homogeneous, elastic tensor satisfying (20) and (23). The Neumann function is utilized to give a representation of the solution $\boldsymbol{u}$ to Problem (24) as a double layer potential.

4.1. **The Neumann function.** In this subsection, we prove the existence of a distributional solution to the problem:

$$(69) \qquad \begin{cases} \operatorname{div}\left(\mathbb{C}(\boldsymbol{x})\widehat{\nabla}\mathbf{N}(\mathbf{x},\mathbf{y})\right) = \delta_{\mathbf{y}}(\mathbf{x})\mathbf{I}, & \text{in } \mathbb{R}^3_-, \\ \left(\mathbb{C}(\boldsymbol{x})\widehat{\nabla}\mathbf{N}(\mathbf{x},\mathbf{y})\right)\mathbf{e_3} = \mathbf{0}, & \text{on } \{x_3 = 0\}, \end{cases}$$

where $\delta_{\boldsymbol{y}}(\cdot)$ is the Dirac distribution supported at $\boldsymbol{y} \in \mathbb{R}^3_-$. To prove the existence of the Neumann function, we work column-wise and consider the system

$$(70) \qquad \operatorname{div}\left(\mathbb{C}(\boldsymbol{x})\widehat{\nabla}\boldsymbol{N}^{(k)}(\boldsymbol{x},\boldsymbol{y})\right) = \boldsymbol{e}_k \delta_{\boldsymbol{y}}(\boldsymbol{x}), \qquad \text{in } \mathbb{R}^3_-,$$

for $k = 1, 2, 3$. We observe that, by freezing the coefficients of the elastic tensor at $\boldsymbol{x} = \boldsymbol{y}$, we formally have $\boldsymbol{e}_k \delta_{\boldsymbol{y}}(\boldsymbol{x}) = \operatorname{div}\left(\mathbb{C}(\boldsymbol{y})\widehat{\nabla}\widetilde{\boldsymbol{N}}^{(k)}(\boldsymbol{x},\boldsymbol{y})\right)$, where $\widetilde{\boldsymbol{N}}^{(k)}$ is the k-th column vector of the Neumann function for the Lamé system with constant coefficients. Therefore, subtracting $\operatorname{div}(\mathbb{C}(\boldsymbol{x})\widehat{\nabla}\widetilde{\boldsymbol{N}}^{(k)})$ from both sides in Equation (70) gives:

$$(71) \qquad \operatorname{div}\left(\mathbb{C}(\boldsymbol{x})\widehat{\nabla}\boldsymbol{M}^{(k)}(\boldsymbol{x},\boldsymbol{y})\right) = -\operatorname{div}\left[(\mathbb{C}(\boldsymbol{x}) - \mathbb{C}(\boldsymbol{y}))\widehat{\nabla}\widetilde{\boldsymbol{N}}^{(k)}(\boldsymbol{x},\boldsymbol{y})\right], \qquad \text{in } \mathbb{R}^3_-,$$

where $\boldsymbol{M}^{(k)} := \boldsymbol{N}^{(k)} - \widetilde{\boldsymbol{N}}^{(k)}$, for $k = 1, 2, 3$. We next recall the decay estimates satisfied by the Neumann function $\widetilde{\mathbf{N}}$ in the case of constant coefficients. In particular, it is not difficult

to see from Theorem 4.9 in [9] that there exists a positive constant $C = C(\alpha_0, \beta_0, M)$, such that for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3_-$ with $\boldsymbol{x} \neq \boldsymbol{y}$,

$$(72) \qquad \begin{aligned} |\widetilde{\mathbf{N}}(\boldsymbol{x}, \boldsymbol{y})| &\leq C|\boldsymbol{x} - \boldsymbol{y}|^{-1}, \\ |\nabla_{\boldsymbol{x}} \widetilde{\mathbf{N}}(\boldsymbol{x}, \boldsymbol{y})| &\leq C|\boldsymbol{x} - \boldsymbol{y}|^{-2}. \end{aligned}$$

We recall that α_0, β_0, M are the constants appearing in the assumptions on the elasticity tensor $\mathbb{C}$ in Subsection 3.1.

Next, we establish rigorously the existence of the Neumann function $\mathbf{N}$, by showing that there exists a unique variational solution $\mathbf{M}$ for the vector problem (71) in $H^1_0(\mathbb{R}^3_-)$. This result also implies that, as expected, the singularities of $\mathbf{N}(\boldsymbol{x}, \boldsymbol{y})$ near $\boldsymbol{y}$ are those of the constant-coefficient Neumann function obtained by freezing the coefficient at $\boldsymbol{y}$. This fact will be used in Subsection 4.2.

Proposition 4.1. *Assume that* (20) *holds, and let*

$$(73) \qquad \mathbf{F}_{\boldsymbol{y},k}(\boldsymbol{x}) := [(\mathbb{C}(\boldsymbol{x}) - \mathbb{C}(\boldsymbol{y}))\widehat{\nabla}\widetilde{\boldsymbol{N}}^{(k)}(\boldsymbol{x}, \boldsymbol{y})], \quad k = 1, 2, 3.$$

Then $\mathbf{F}_{\boldsymbol{y},k} \in L^2(\mathbb{R}^3_-)$ *for any* $\boldsymbol{y} \in \mathbb{R}^3_-$. *Moreover, for any* $\boldsymbol{y} \in \mathbb{R}^3_-$ *the boundary value problem*

$$(74) \qquad \begin{cases} \operatorname{div}(\mathbb{C}(\boldsymbol{x})\widehat{\nabla}\boldsymbol{M}^{(k)}(\boldsymbol{x}, \boldsymbol{y})) = -\operatorname{div}\mathbf{F}_{\boldsymbol{y},k}, & \text{in } \mathbb{R}^3, \\ \mathbb{C}(\boldsymbol{x})\widehat{\nabla}\boldsymbol{M}^{(k)}\boldsymbol{e_3} = -\mathbb{C}(\boldsymbol{x})\widehat{\nabla}\widetilde{\boldsymbol{N}}^{(k)}\boldsymbol{e_3}, & \text{on } \{x_3 = 0\}, \end{cases}$$

admits a unique solution, satisfying

$$(75) \qquad \|\boldsymbol{M}^{(k)}\|_{H^1_0(\mathbb{R}^3_-)} \leq C\|\mathbf{F}_{\boldsymbol{y},k}\|_{L^2(\mathbb{R}^3_-)}.$$

In particular the matrix $\mathbf{M} = [\boldsymbol{M}^{(1)} \, \boldsymbol{M}^{(2)} \, \boldsymbol{M}^{(3)}]$ *belongs to* $H^1_0(\mathbb{R}^3_-)$.

Proof. We start by showing that $\mathbf{F}_{\boldsymbol{y},k} \in L^2(\mathbb{R}^3_-)$. We choose r sufficiently small so that $B_r(\boldsymbol{y}) \subset \mathbb{R}^3_-$. From the regularity assumption on the elasticity tensor (20), it follows that there exists a positive constant C independent of $\boldsymbol{y}$ and r such that, for $\boldsymbol{x} \in B_r(\boldsymbol{y})$,

$$|\mathbb{C}(\boldsymbol{x}) - \mathbb{C}(\boldsymbol{y})| \leq C|\boldsymbol{x} - \boldsymbol{y}|.$$

By (72) then,

$$\int_{B_r(\boldsymbol{y})} |[\mathbb{C}(\boldsymbol{x}) - \mathbb{C}(\boldsymbol{y})]\widehat{\nabla}\widetilde{\boldsymbol{N}}^{(k)}|^2 \, d\boldsymbol{x} \leq C \int_{B_r(\boldsymbol{y})} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|^2} \, d\boldsymbol{x} < \infty.$$

On the other hand, (20) also implies the elastic parameters are uniformly bounded and, again by (72), we find that

$$\int_{B_r^C(\boldsymbol{y}) \cap \mathbb{R}^3_-} |[\mathbb{C}(\boldsymbol{x}) - \mathbb{C}(\boldsymbol{y})]\widehat{\nabla}\widetilde{\boldsymbol{N}}^{(k)}|^2 \, d\boldsymbol{x} \leq C \int_{B_r^C(\boldsymbol{y}) \cap \mathbb{R}^3_-} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|^4} \, d\boldsymbol{x} < \infty,$$

where $B_r^C(\boldsymbol{y})$ is the complementary set of $B_r(\boldsymbol{y})$. Combining these two results gives that $\mathbf{F}_{\boldsymbol{y},k}(\boldsymbol{x}) \in L^2(\mathbb{R}^3_-)$ for any $\boldsymbol{y} \in \mathbb{R}^3_-$.

Recalling that

$$(\mathbb{C}(\boldsymbol{y})\widehat{\nabla}\widetilde{\mathbf{N}}(\mathbf{x}, \mathbf{y}))\mathbf{e_3} = \mathbf{0}, \quad \text{on } \{\mathbf{x_3} = \mathbf{0}\},$$

we can rewrite (74) in the equivalent form

(76)
$$\begin{cases} \operatorname{div}\left(\mathbb{C}(\boldsymbol{x})\widehat{\nabla}\boldsymbol{M}^{(k)}(\boldsymbol{x},\boldsymbol{y})+\mathbf{F}_{\boldsymbol{y},k}\right)=\mathbf{0} & \text{in } \mathbb{R}^3_-, \\ \left(\mathbb{C}(\boldsymbol{x})\widehat{\nabla}\boldsymbol{M}^{(k)}+\mathbf{F}_{\boldsymbol{y},k}\right)\boldsymbol{e_3}=\mathbf{0} & \text{on } \{x_3=0\}. \end{cases}$$

Proceeding as in the first step of the proof of Theorem 3.5 (cf. Problem 39), the existence and regularity of $\boldsymbol{M}^{(k)}$ as the unique solution of (74) follows from the well-posedness of the variational formulation of (76), i.e., find $\boldsymbol{w} \in H^1_0(\mathbb{R}^3_-)$ such that

$$a(\boldsymbol{w},\boldsymbol{v})=G_k(\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in H^1_0(\mathbb{R}^3_-),$$

where the functional

(77)
$$G_k(\boldsymbol{v})=-\int_{\mathbb{R}^3_-}(\mathbb{C}(\boldsymbol{x})-\mathbb{C}(\boldsymbol{y}))\widehat{\nabla}\widetilde{\boldsymbol{N}}^{(k)}(\boldsymbol{x},\boldsymbol{y}):\widehat{\nabla}\boldsymbol{v}(\boldsymbol{x})\,d\boldsymbol{x}.$$

By Lax-Milgram, it is enough to show that G_k is continuous on $H^1_0(\mathbb{R}^3_-)$. Since $\mathbf{F}_{\boldsymbol{y},k}$ was shown to belong to $L^2(\mathbb{R}^3_-)$, we have that

$$|G_k(\boldsymbol{v})| \le \int_{\mathbb{R}^3_-}|\mathbf{F}_{\boldsymbol{y},k}:\widehat{\nabla}\boldsymbol{v}|\,d\boldsymbol{x} \le \|\mathbf{F}_{\boldsymbol{y},k}\|_{L^2(\mathbb{R}^3_-)}\|\widehat{\nabla}\boldsymbol{v}\|_{L^2(\mathbb{R}^3_-)} \le \|\mathbf{F}_{\boldsymbol{y},k}\|_{L^2(\mathbb{R}^3_-)}\|\boldsymbol{v}\|_{H^1_0(\mathbb{R}^3_-)}.$$

$\square$

Remark 4.2. If $\operatorname{dist}(\boldsymbol{y},\{x_3=0\}) \ge d_0 > 0$, then from (75) it follows that

$$\|\boldsymbol{M}^{(k)}\|_{H^1_0(\mathbb{R}^3_-)} \le C$$

where C does not depend on $\boldsymbol{y} \in \mathbb{R}^3_-$.

In the next section, to provide an integral representation formula of the solution $\boldsymbol{u}$ of Problem (24) as a double layer potential, we need to prove higher regularity than H^1_0 on the Neumann function once we are sufficiently far from the singularity $\boldsymbol{y}$.

Proposition 4.3. For any $r > 0$ such that $B_r(\boldsymbol{y}) \subset \mathbb{R}^3_-$, we have that

$$\mathbf{N} \in H^2_1(\mathbb{R}^3_- \setminus \overline{B_r(\boldsymbol{y})}).$$

Proof. We fix $r > 0$ such that $B_r(\boldsymbol{y}) \subset \mathbb{R}^3_-$, and define a cut-off function $\varphi \in \mathcal{D}(\mathbb{R}^3_-)$ with the property that

$$\varphi = \begin{cases} 1, & \text{in } \overline{B_{r/2}(\boldsymbol{y})}, \\ 0, & \text{in } \mathbb{R}^3_- \setminus B_r(\boldsymbol{y}). \end{cases}$$

We also let

(78)
$$\boldsymbol{N}^{(k)}_\dagger := (1-\varphi)\boldsymbol{N}^{(k)} = \begin{cases} \mathbf{0}, & \text{in } \overline{B_{r/2}(\boldsymbol{y})}, \\ \boldsymbol{N}^{(k)}, & \text{in } \mathbb{R}^3_- \setminus B_r(\boldsymbol{y}), \end{cases}$$

for $k = 1, 2, 3$. From the definition of φ and the fact that $\boldsymbol{N}^{(k)}$ solves the homogeneous equation $\operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{N}^{(k)}) = \mathbf{0}$ for $\boldsymbol{x} \ne \boldsymbol{y}$, it is straightforward to find the equation solved by $\boldsymbol{N}^{(k)}_\dagger$, that is

$$\operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{N}^{(k)}_\dagger) = -\operatorname{div}(\mathbb{C}(\widehat{\nabla\varphi \otimes \boldsymbol{N}^{(k)}})) - (\mathbb{C}\widehat{\nabla}\boldsymbol{N}^{(k)})\nabla\varphi,$$

with homogeneous Neumann boundary conditions. We observe that the source term

$$\boldsymbol{h} := -\mathrm{div}(\mathbb{C}(\widehat{\nabla\varphi \otimes \boldsymbol{N}^{(k)}})) - (\mathbb{C}\widehat{\nabla}\boldsymbol{N}^{(k)})\nabla\varphi$$

has compact support in $\overline{B_r(\boldsymbol{y})}\setminus B_{r/2}(\boldsymbol{y})$ and, moreover, $\boldsymbol{N}^{(k)} \in H^1(B_r(\boldsymbol{y})\setminus\overline{B_{r/2}(\boldsymbol{y})})$, which follows from the result in Proposition 4.1 and the representation $\boldsymbol{N}^{(k)} = \boldsymbol{M}^{(k)} - \widetilde{\boldsymbol{N}}^{(k)}$. Therefore, $\boldsymbol{h} \in L^2(B_r(\boldsymbol{y})\setminus\overline{B_{r/2}(\boldsymbol{y})})$ and, since $\boldsymbol{h}$ has compact support, $\boldsymbol{h} \in H^0_1(\mathbb{R}^3_-)$ as well. We then consider the problem:

$$\begin{cases} \mathrm{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{N}^{(k)}_\dagger) = \boldsymbol{h}, & \text{in } \mathbb{R}^3_-, \\ (\mathbb{C}\widehat{\nabla}\boldsymbol{N}^{(k)}_\dagger)\boldsymbol{e}_3 = \boldsymbol{0}, & \text{on } \{x_3 = 0\} \end{cases}$$

for given $\boldsymbol{h} \in H^0_1(\mathbb{R}^3_-)$. Following the steps in the proof of Theorem 3.3 and Theorem 3.5, one can prove that there exists a unique $\boldsymbol{N}^{(k)}_\dagger \in H^2_1(\mathbb{R}^3_-)$, that is, $\boldsymbol{N}^{(k)} \in H^2_1(\mathbb{R}^3_- \setminus \overline{B_r(\boldsymbol{y})})$ from (78), for $k = 1, 2, 3$. $\qquad\square$

4.2. **A representation formula for solutions to** (24)**.** In this subsection, using the Neumann function defined in (69), we give an integral representation formula for the solution to Problem (24). Then we take advantage of this integral representation to study the regularity of the solution in the complement of the dislocation surface S in $\mathbb{R}^3_-$. In fact, we will determine the singularities of the solution, when S is a rectangular dislocation surface parallel to the plane $\{x_3 = 0\}$ and $\boldsymbol{g}$ is a constant vector, in the special case that the medium is homogeneous.

This explicit example shows that, if $\boldsymbol{g} \in H^{1/2}(S)$, but without assuming that $\boldsymbol{g}$ has compact support in S, generically solutions to (24) are not in $H^1_0(\mathbb{R}^3_- \setminus \overline{S})$. We begin with a preliminary result proved by following an approach similar to that in [14] and using also results in [17] on growth properties of Neumann functions in a neighborhood of the singularity.

Proposition 4.4. *The unique solution to* (24) *can be represented as a double layer potential on S, that is,*

$$(79) \qquad \boldsymbol{u}(\boldsymbol{y}) = -\int\limits_S \left[(\mathbb{C}(\boldsymbol{x})\widehat{\nabla}_{\boldsymbol{x}}\mathbf{N}(\mathbf{x},\mathbf{y}))\mathbf{n}(\mathbf{x}) \right]^T \boldsymbol{g}(\boldsymbol{x})\, d\sigma(\boldsymbol{x}),$$

where $\mathbf{N}$ is the Neumann function satisfying (69)*.*

Proof. We recall that the transmission problem (24) is equivalent to the source problem (26), so we provide the integral representation formula starting from (26). From regularity results for elliptic systems (see e.g. [24]), it is immediate that the solution $\boldsymbol{u}$ is regular in $\mathbb{R}^3_- \setminus \overline{S}$ and has traction zero on $\{x_3 = 0\}$.

We fix $\boldsymbol{y} \in \mathbb{R}^3_- \setminus\overline{S}$ and we consider a ball $B_r(\boldsymbol{y})$ such that $\overline{B_r(\boldsymbol{y})} \subset \mathbb{R}^3_-$ with $B_r(\boldsymbol{y})\cap\overline{S} = \emptyset$.

From Proposition 3.1, $\boldsymbol{f}_S \in H^{-3/2-\varepsilon}$ and has compact support in $\mathbb{R}^3_-$. Without loss of generality, we assume the support of $\boldsymbol{f}_S$ lies in an open set $\Omega \subset \mathbb{R}^3_-$, the closure of which does not meet $\mathbb{R}^2$ nor the boundary of the ball $B_r(\boldsymbol{y})$. We then have have from Proposition 3.9 that

$$\boldsymbol{f}_S \in H^{-3/2-\varepsilon}_{-1/2-\varepsilon,\Omega} \subset H^{-2}_{-1,\Omega} \subset (H^2_1)' \subset V',$$

where $H^{-s}_{-\alpha,\Omega}$ are the spaces of distributions with compact support in $\overline{\Omega}$. Then, $\boldsymbol{u}$ also solves the source problem in V', that is, $\boldsymbol{u} \in E_0(\mathbb{R}^3)$. Moreover, for $k = 1, 2, 3$, $\boldsymbol{N}^{(k)} \in H^2_1(\mathbb{R}^3_- \setminus \overline{B_r(\boldsymbol{y})})$ from Proposition 4.3, and $(\mathbb{C}\widehat{\nabla}\boldsymbol{N}^{(k)})\boldsymbol{e}_3 = \boldsymbol{0}$ on $\{x_3 = 0\}$ by hypothesis. We observe that we can then apply Green's formula (57) in $\mathbb{R}^3_- \setminus \overline{B_r(\boldsymbol{y})}$ with $\boldsymbol{N}^{(k)}$ as test function. (That formula is derived in $\mathbb{R}^3_-$, but it can be extended to $\mathbb{R}^3_- \setminus \overline{B_r(\boldsymbol{y})})$ in this case, since both $\boldsymbol{u}$ and $\mathbf{N}$ are regular near $\partial B_r(\boldsymbol{y})$.)

As a result, we obtain that

(80)
$$\int_{\partial B_r(\boldsymbol{y})} (\mathbb{C}(\boldsymbol{x})\widehat{\nabla}\boldsymbol{N}^{(k)}(\boldsymbol{x}, \boldsymbol{y}))\boldsymbol{n} \cdot \boldsymbol{u}(\boldsymbol{x}) \, d\sigma(\boldsymbol{x}) - \int_{\partial B_r(\boldsymbol{y})} (\mathbb{C}(\boldsymbol{x})\widehat{\nabla}\boldsymbol{u}(\boldsymbol{x}))\boldsymbol{n} \cdot \boldsymbol{N}^{(k)}(\boldsymbol{x}, \boldsymbol{y}) \, d\sigma(\boldsymbol{x})$$
$$= \langle \boldsymbol{f}_S, \boldsymbol{N}^{(k)}(\cdot, \boldsymbol{y}) \rangle_{(H^2_1)'(\mathbb{R}^3_- \setminus \overline{B_r(\boldsymbol{y})}), H^2_1(\mathbb{R}^3_- \setminus \overline{B_r(\boldsymbol{y})})}$$
$$= -\langle \mathbb{C}(\boldsymbol{g} \otimes \boldsymbol{n})\delta_S, \nabla\boldsymbol{N}^{(k)}(\cdot, \boldsymbol{y}) \rangle_{(H^1_1)'(\mathbb{R}^3_- \setminus \overline{B_r(\boldsymbol{y})}), H^1_1(\mathbb{R}^3_- \setminus \overline{B_r(\boldsymbol{y})})},$$

where we used that $\text{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{N}^{(k)}(\boldsymbol{x}, \boldsymbol{y})) = \boldsymbol{0}$ in $\mathbb{R}^3_- \setminus \overline{B_r(\boldsymbol{y})}$ and the traction-free boundary condition. The last equality above follows from the fact that

$$\mathbb{C}(\boldsymbol{g} \otimes \boldsymbol{n})\delta_S \in H^{-1/2-\varepsilon}_{-1/2-\varepsilon,\Omega}(\mathbb{R}^3_- \setminus \overline{B_r(\boldsymbol{y})}) \subset (H^{1/2+\varepsilon}_{1/2+\varepsilon})'(\mathbb{R}^3_- \setminus \overline{B_r(\boldsymbol{y})}) \subset (H^1_1)'(\mathbb{R}^3_- \setminus \overline{B_r(\boldsymbol{y})}).$$

As discussed in [17, 18], the Neumann function admits the decomposition:

$$\mathbf{N}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{\Gamma}(\boldsymbol{x} - \boldsymbol{y}) + \mathbf{H}^{(k)}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}),$$

where $\boldsymbol{\Gamma}$ denotes the fundamental solution for the Lamé operator with constant coefficients (we freeze the coefficients at $\boldsymbol{y}$) and $\boldsymbol{H_y}$ is a more regular remainder. Using the Lipschitz continuity of the Lamé coefficients, it is possible to prove that there exist C_1, C_2, C_3, $C_4 > 0$, which depend on the constants in the a priori assumptions (20) and (23) on $\mathbb{C}$, and on the distance of the fixed point $\boldsymbol{y}$ from $\{x_3 = 0\}$, such that

(81)
$$|\boldsymbol{N}^{(k)}(\boldsymbol{x}, \boldsymbol{y})| \leq C_1 |\boldsymbol{x} - \boldsymbol{y}|^{-1} + C_2$$
$$|\nabla_{\boldsymbol{x}} \boldsymbol{N}^{(k)}(\boldsymbol{x}, \boldsymbol{y})| \leq C_3 |\boldsymbol{x} - \boldsymbol{y}|^{-2} + C_4 |\boldsymbol{x} - \boldsymbol{y}|^{-1}.$$

Following the same calculations as, for example, in [8, Theorem 3.3] and employing the local estimates (81), it is straightforward to prove that

(82)
$$\left| \int_{\partial B_r(\boldsymbol{y})} (\mathbb{C}\widehat{\nabla}\boldsymbol{u})\boldsymbol{n} \cdot \boldsymbol{N}^{(k)} \, d\sigma(\boldsymbol{x}) \right| \to 0, \text{ as } r \to 0,$$

and

(83)
$$\int_{\partial B_r(\boldsymbol{y})} (\mathbb{C}\widehat{\nabla}\boldsymbol{N}^{(k)})\boldsymbol{n} \cdot \boldsymbol{u} \, d\sigma(\boldsymbol{x}) \to u_k(\boldsymbol{y}), \text{ as } r \to 0,$$

where u_k is k-th component of the displacement vector $\boldsymbol{u}$. To handle the last term in (80), we use the density of the space $\mathcal{D}(\overline{\mathbb{R}^3_-} \setminus B_r(\boldsymbol{y}))$ in $H^1_1(\mathbb{R}^3_- \setminus \overline{B_r(\boldsymbol{y})})$. By density, there exist

$$\{\boldsymbol{\eta}^{(k)}_j\} \in \mathcal{D}(\overline{\mathbb{R}^3_-} \setminus B_r(\boldsymbol{y})) \text{ such that } \boldsymbol{\eta}^{(k)}_j \xrightarrow{j \to \infty} \boldsymbol{N}^{(k)}(\cdot, \boldsymbol{y}) \text{ in } H^2_1(\mathbb{R}^3_- \setminus \overline{B_r(\boldsymbol{y})}),$$

hence $\nabla \boldsymbol{\eta}_j^{(k)} \to \nabla \boldsymbol{N}^{(k)}(\cdot, \boldsymbol{y})$ in $H^{1/2}(S)$, for $k = 1, 2, 3$. Therefore,

(84)
$$\lim_{j \to \infty} \int_S \mathbb{C}(\boldsymbol{x})(\boldsymbol{g}(\boldsymbol{x}) \otimes \boldsymbol{n}(\boldsymbol{x})) : \nabla \boldsymbol{\eta}_j^{(k)}(\boldsymbol{x}) \, d\sigma(\boldsymbol{x}) = \int_S \mathbb{C}(\boldsymbol{x})(\boldsymbol{g}(\boldsymbol{x}) \otimes \boldsymbol{n}(\boldsymbol{x})) : \nabla \boldsymbol{N}^{(k)}(\boldsymbol{x}, \boldsymbol{y}) \, d\sigma(\boldsymbol{x}),$$

where we used that, by (29),

$$\langle \mathbb{C}(\boldsymbol{g} \otimes \boldsymbol{n}) \delta_S, \nabla \boldsymbol{\eta} \rangle = \int_S (\mathbb{C}(\boldsymbol{g} \otimes \boldsymbol{n})) : \nabla \boldsymbol{\eta} \, d\sigma(\boldsymbol{x}).$$

From (84), (83), (82) and (80), taking the limit $r \to 0$, we obtain the representation formula (79) exploiting the symmetries of the tensor $\mathbb{C}$. $\qquad\square$

Remark 4.5. *It is possible to show that $\boldsymbol{u} \in H_{loc}^\sigma(\mathbb{R}_-^3 \setminus \overline{S})$, with $\sigma < 1$, when $\boldsymbol{g} \in H^{1/2}(S)$. In fact, we can assume that S is part of the boundary of a compact Lipschitz domain D with $\overline{D} \subset \mathbb{R}_-^3$. Since ∂S is assumed also Lipschitz, we can extend $\boldsymbol{g}$ by zero to the complement of S in ∂D and have that $\boldsymbol{g} \in H^s(\partial D)$ for $s < 1/2$. Then, from regularity results for layer potentials in the case of a Lipschitz surface (see [32, Theorem 8.7]), it follows that $\boldsymbol{u} \in H^{1/2+s}(D)$, and $\boldsymbol{u} \in H^{1/2+s}(\widetilde{D})$, where $\widetilde{D}$ is a compact Lipschitz domain, the boundary of which contains ∂D and which is contained in the complement of $\overline{D}$ in $\mathbb{R}_-^3$.*

An explicit example. We consider now the particular case of the an isotropic homogeneous half space. We denote the constant elasticity tensor with $\mathbb{C}_0$ and its Lamé coefficients with μ_0 and λ_0. These satisfy the strong convexity condition $\mu_0 > 0$ and $3\lambda_0 + 2\mu_0 > 0$. In this setting, taking S to be a rectangular Volterra dislocation (which, we recall, means a constant displacement jump distribution on S), in [34] Okada gives an explicit expression of the solution to Problem (24), highlighting the presence of singularities on the vertices of the rectangular dislocation surface. This solution is well known and applied in the geophysical literature (see for example [37, 43] and references therein).

We denote the Neumann function for a homogeneous and isotropic half space by $\mathbf{N}_0(\boldsymbol{x}, \boldsymbol{y})$. Its explicit expression can be found, for instance, in [30, 31, 8, 27]. Moreover, we assume S is a rectangle parallel to the plane $\{x_3 = 0\}$, that is,

(85) $\qquad S = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : a \le y_1 \le b, \ c \le y_2 \le d, \ y_3 = -|\alpha|\},$

with $a, b, c, d, \alpha \in \mathbb{R}$, and we assume that $\boldsymbol{g} := \boldsymbol{g}_c = (k_1, k_2, k_3)^T$ on S, with $k_i \in \mathbb{R}$, $i = 1, 2, 3$. This choice for S and $\boldsymbol{g}_c$ can be seen as a particular case of the rectangular dislocation surface considered by Okada in [34], and it is the simplest case in which the source has support on the whole of $\overline{S}$.

In this setting, we show that the solution $\boldsymbol{u}$ of (24) is not in $H_0^1(\mathbb{R}_-^3 \setminus \overline{S})$. The representation formula (79) becomes

(86) $$\boldsymbol{u}(\boldsymbol{x}) = -\int_S \left(\mathbb{C}_0 \widehat{\nabla}_{\boldsymbol{y}}(\mathbf{N}_0(\boldsymbol{y}, \boldsymbol{x})) \boldsymbol{e}_3 \right)^T \boldsymbol{g}_c \, d\sigma(\boldsymbol{y}).$$

Again, we use that $\mathbf{N}_0 = \boldsymbol{\Gamma} + \mathbf{R}$, where $\boldsymbol{\Gamma}$ is the fundamental solution of the constant-coefficient Lamé operator, also known as the *Kelvin fundamental solution* (see [23]), and $\mathbf{R}$ is a regular function in $\mathbb{R}_-^3$ (see e.g. [8]). The singularities of $\boldsymbol{u}$ are then contained in the term

$$\boldsymbol{u}_\Gamma := \int_S (\mathbb{C}_0 \widehat{\nabla}_{\boldsymbol{y}} \boldsymbol{\Gamma}(\boldsymbol{x} - \boldsymbol{y}) \boldsymbol{e}_3)^T \boldsymbol{g}_c \, d\sigma(\boldsymbol{y}).$$

From straightforward calculations we find that

$$\boldsymbol{\Xi} := (\mathbb{C}_0 \widehat{\nabla}\boldsymbol{\Gamma}\boldsymbol{e}_3)^T = \begin{bmatrix} \mu\left(\frac{\partial\Gamma_{11}}{\partial y_3} + \frac{\partial\Gamma_{31}}{\partial y_1}\right) & \mu\left(\frac{\partial\Gamma_{21}}{\partial y_3} + \frac{\partial\Gamma_{31}}{\partial y_2}\right) & \lambda\mathrm{div}\,\boldsymbol{\Gamma}^{(1)} + 2\mu\frac{\partial\Gamma_{31}}{\partial y_3} \\ \mu\left(\frac{\partial\Gamma_{12}}{\partial y_3} + \frac{\partial\Gamma_{32}}{\partial y_1}\right) & \mu\left(\frac{\partial\Gamma_{22}}{\partial y_3} + \frac{\partial\Gamma_{32}}{\partial y_2}\right) & \lambda\mathrm{div}\,\boldsymbol{\Gamma}^{(2)} + 2\mu\frac{\partial\Gamma_{32}}{\partial y_3} \\ \mu\left(\frac{\partial\Gamma_{13}}{\partial y_3} + \frac{\partial\Gamma_{33}}{\partial y_1}\right) & \mu\left(\frac{\partial\Gamma_{23}}{\partial y_3} + \frac{\partial\Gamma_{33}}{\partial y_2}\right) & \lambda\mathrm{div}\,\boldsymbol{\Gamma}^{(3)} + 2\mu\frac{\partial\Gamma_{33}}{\partial y_3} \end{bmatrix},$$

where $\boldsymbol{\Gamma}^{(i)}$, for $i = 1, 2, 3$, represents the i-th column vector of the matrix $\boldsymbol{\Gamma}$. Therefore

$$(87) \qquad \boldsymbol{u}_\Gamma = \int_a^b\int_c^d \boldsymbol{\Xi}(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{g}_c\,d\sigma(\boldsymbol{y}) = \sum_{j=1}^3 \int_a^b\int_c^d \Xi_{ij}(\boldsymbol{x}-\boldsymbol{y})k_j\,d\sigma(\boldsymbol{y}), \qquad i = 1, 2, 3.$$

We find that $\boldsymbol{u}_\Gamma(\boldsymbol{x})$ has logarithmic singularities as $\boldsymbol{x}$ approaches the vertices of the rectangle S, which comes exclusively from the entries Ξ_{31}, Ξ_{32} (and Ξ_{13}, Ξ_{23} due to the symmetries of Γ) of the matrix $\boldsymbol{\Xi}$. (See Appendix A for explicit formulas of these terms.)

Remark 4.6. *Because of the nature of the entries in $\boldsymbol{\Xi}$ from which the singularities originates, we stress that these singularities are present both for tangential as well as normal (and hence oblique) jumps $\boldsymbol{g}$.*

5. Formulation of the inverse problem: a uniqueness result

In this section, we formulate and study an inverse dislocation problem. To be more precise, we are interested in the following boundary inverse problem:
Let $\boldsymbol{u}$ be solution to Problem (24) with unknown dislocation surface S and jump distribution $\boldsymbol{g}$ on S. Assume that $\boldsymbol{u} = \boldsymbol{u}_\Sigma$ is known in a bounded open region Σ of the boundary $x_3 = 0$, then determine both S and $\boldsymbol{g}$.

In particular, we want to establish under which assumptions the displacement $\boldsymbol{u}_\Sigma$ determines S and $\boldsymbol{g}$ uniquely. To prove uniqueness, we will need an extra geometrical assumption on S and supplementary assumptions on $\boldsymbol{g}$, but we allow the elastic coefficients to be only Lipschitz continuous by adapting the proofs in [1] (valid for scalar equations) to the Lamé system. Generically, we expect this regularity to be optimal for the inverse problem, unless further assumptions on the form of the coefficients is made, such as assuming the Lamé parameters to be piece-wise constant on a given mesh. We reserve to address this point in future work. We remark again that the unique continuation property may not hold for a second-order elliptic operator, if its coefficients are only in Hölder classes $C^{0,\alpha}$, $\alpha < 1$, and not Lipschitz continuous (see [29, 35]). Our uniqueness result is the following theorem.

Theorem 5.1. *Let S_1, S_2 be two open, bounded, piece-wise linear surfaces such that $\overline{S}_1, \overline{S}_2 \subset \mathbb{R}^3_-$, and let $\boldsymbol{g}_i$ be bounded tangential fields in $H^{1/2}(S_i)$ with $\mathrm{supp}\,\boldsymbol{g}_i = \overline{S}_i$, for $i = 1, 2$. Let $\boldsymbol{u}_i$, for $i = 1, 2$, be the unique solution of (24) in $H^{1/2-\varepsilon}_{-1/2-\varepsilon}(\mathbb{R}^3_-)$ corresponding to $\boldsymbol{g} = \boldsymbol{g}_i$ and $S = S_i$. Let Σ be an open set in $\{x_3 = 0\}$. If $\boldsymbol{u}_1\big|_\Sigma = \boldsymbol{u}_2\big|_\Sigma$, then $S_1 = S_2$ and $\boldsymbol{g}_1 = \boldsymbol{g}_2$.*

Remark 5.2. *We first note that by local regularity results, since $\mathbb{C} \in C^{0,1}(\overline{\mathbb{R}^3_-})$, the displacements $\boldsymbol{u}_i$, for $i = 1, 2$, are regular in a neighborhood $\mathcal{U}$ of $\{x_3 = 0\}$, more precisely $u_i \in C^{1,\alpha}(\mathcal{U})$ with $0 \le \alpha < 1$ (see e.g. [24]), so that $\boldsymbol{u}_i\big|_\Sigma$ can be interpreted as the*

trace of $\boldsymbol{u}_i$ pointwise. The same conclusion holds for the traction-free boundary condition $(\mathbb{C}\widehat{\nabla}\boldsymbol{u}_i)\boldsymbol{e}_3 = 0$ *on* $\mathbb{R}^2$.

Proof of Theorem 5.1. We let $\boldsymbol{w} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, defined on $\mathbb{R}^3_- \setminus \overline{S_1 \cup S_2}$. Then

$$\boldsymbol{w}_{|\Sigma} = (\mathbb{C}\widehat{\nabla}\boldsymbol{w})\boldsymbol{e}_{3|\Sigma} = \boldsymbol{0}, \qquad \text{and} \qquad \text{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{w}) = \boldsymbol{0} \ \text{ in } \ \mathbb{R}^3_- \setminus \overline{S_1 \cup S_2}.$$

From the homogeneous boundary conditions, it follows that also the Cauchy data are zero. Hence,

$$(88) \qquad \boldsymbol{w}_{|\Sigma} = \frac{\partial \boldsymbol{w}}{\partial x_3}\bigg|_{\Sigma} = \boldsymbol{0}.$$

Without loss of generality, we assume that $\Sigma = B'_R(\boldsymbol{0})$ (see Section 2 for notation). We define $\widetilde{\boldsymbol{w}}$, extension of $\boldsymbol{w}$, in $B^R(\boldsymbol{0})$ by

$$(89) \qquad \widetilde{\boldsymbol{w}} = \begin{cases} \boldsymbol{w}, & \text{in } B^-_R(\boldsymbol{0}), \\ \boldsymbol{0}, & \text{in } B^+_R(\boldsymbol{0}). \end{cases}$$

We also extend the elasticity tensor $\mathbb{C}$ to $B^+_R(\boldsymbol{0})$ in such a way that the resulting tensor is Lipschitz continuous and strongly convex in $B_R(\boldsymbol{0})$ (for example, by even reflection). We will show that $\widetilde{\boldsymbol{w}}$ is a weak solution to the Lamé system in $B_R(\boldsymbol{0})$. We let $\boldsymbol{\varphi} \in \mathcal{D}(B_R(\boldsymbol{0}))$, and test the equation with φ:

$$\int_{B_R(\boldsymbol{0})} \text{div}(\mathbb{C}\widehat{\nabla}\widetilde{\boldsymbol{w}}) \cdot \boldsymbol{\varphi} \, d\boldsymbol{x} = -\int_{B^-_R(\boldsymbol{0})} (\mathbb{C}\widehat{\nabla}\boldsymbol{w}) : \widehat{\nabla}\boldsymbol{\varphi} \, d\boldsymbol{x} = 0,$$

where in the last equality we have integrated by parts, and used the fact that $\boldsymbol{w}$ satisfies the Lamé system in $B^-_R(\boldsymbol{0})$. Consequently, $\widetilde{\boldsymbol{w}} \in H^1(B_R(\boldsymbol{0}))$ is a weak solution to

$$\text{div}(\mathbb{C}\widehat{\nabla}\widetilde{\boldsymbol{w}}) = \boldsymbol{0}, \qquad \text{in } B_R(\boldsymbol{0}).$$

We can then apply Theorem 1.3 in [25] to obtain the strong unique continuation property in $B_R(\boldsymbol{0})$, which implies the weak continuation property. Since $\widetilde{\boldsymbol{w}} = \boldsymbol{0}$ in $B^+_R(\boldsymbol{0})$ and the weak continuation property holds in $B_R(\boldsymbol{0})$, we have that $\widetilde{\boldsymbol{w}} = \boldsymbol{0}$ in $B_R(\boldsymbol{0})$. That is, $\widetilde{\boldsymbol{w}} = \boldsymbol{w} = \boldsymbol{0}$ in $B^-_R(\boldsymbol{0})$. Using the three-spheres inequality (see Theorem 1.1 in [25]) in the connected component G of $\mathbb{R}^3_- \setminus \overline{S_1 \cup S_2}$ containing Σ gives that $\boldsymbol{w} = \boldsymbol{0}$ in G. We now distinguish two cases:

   (i) the complement of $\overline{S_1 \cup S_2}$ in $\mathbb{R}^3_-$ is path connected;
   (ii) the complement of $\overline{S_1 \cup S_2}$ in $\mathbb{R}^3_-$ is not path connected.

We start from Case (i). We assume that $\overline{S}_1 \neq \overline{S}_2$, and we fix $\boldsymbol{y} \in \overline{S}_1$ such that $\boldsymbol{y} \notin \overline{S}_2$. Then there exists a sphere $B_r(\boldsymbol{y})$ that does not intersect $\overline{S}_2$. Hence,

$$\boldsymbol{0} = [\boldsymbol{w}]_{B_r(\boldsymbol{y}) \cap S_1} = [\boldsymbol{u}_1]_{B_r(\boldsymbol{y}) \cap S_1} = \boldsymbol{g}_1,$$

and this equality is a contradiction, since $\text{supp}\,\boldsymbol{g}_1 = \overline{S}_1$. We can repeat the same argument, switching the role of S_1 and S_2, to conclude that $\overline{S}_1 = \overline{S}_2$. Therefore,

$$\boldsymbol{0} = [\boldsymbol{w}]_{S_1} = [\boldsymbol{w}]_{S_2} \Rightarrow [\boldsymbol{u}_1]_{S_1} = [\boldsymbol{u}_2]_{S_2} \Rightarrow \boldsymbol{g}_1 = \boldsymbol{g}_2.$$

We now turn to Case (ii). Then there exists a connected component D, the boundary of which is contained in $\overline{S_1 \cup S_2}$. Since S_1 and S_2 are assumed to be open Lipschitz surfaces, ∂D is exactly the union of a connected component of S_1 and of a connected component

of S_2 that share the same boundary. On the connected components of S_1 and S_2 that do not have a common boundary, we can apply the argument for Case (i) (as, by unique continuation, we can always find a neighborhood of these open components where $\boldsymbol{w} = \boldsymbol{0}$), and so we can always assume, without loss of generality, that $\partial D = \overline{S_1 \cup S_2}$. Given that $\boldsymbol{w} = \boldsymbol{0}$ in $D^C \cap \mathbb{R}^3_-$ and $(\mathbb{C}\widehat{\nabla}\boldsymbol{u}_1)\boldsymbol{n}$ and $(\mathbb{C}\widehat{\nabla}\boldsymbol{u}_2)\boldsymbol{n}$ are continuous in the trace sense on S_1 and S_2 respectively, we have that

$$(90) \qquad (\mathbb{C}\widehat{\nabla}\boldsymbol{w}^+)\boldsymbol{n} = \boldsymbol{0} \quad \text{a.e. on } \partial D,$$

where $\boldsymbol{w}^+$ indicates the function $\boldsymbol{w}$ restricted to D and $\boldsymbol{n}$ the outward unit normal to D. Moreover, $\boldsymbol{w}^+$ satisfies

$$(91) \qquad \operatorname{div}(\mathbb{C}\widehat{\nabla}\boldsymbol{w}^+) = \boldsymbol{0} \quad \text{in } D.$$

We conclude from (91) and (90) that $\boldsymbol{w}^+$ is a rigid motion, i.e.,

$$\boldsymbol{w}^+ = \mathbf{A}\boldsymbol{x} + \boldsymbol{c},$$

where $\boldsymbol{c} \in \mathbb{R}^3$ and $\mathbf{A} \in \mathbb{R}^{3\times 3}$ is a skew matrix.

On the other hand, since $\boldsymbol{w}^- = \boldsymbol{0}$ on ∂D, we have

$$[\boldsymbol{w}]_{\partial D} = \boldsymbol{w}^+_{|\partial D},$$

but on S_1 we have $[\boldsymbol{w}]_{S_1} = [\boldsymbol{u}_1]_{S_1}$, which is tangential by assumption; the same argument can be repeated on S_2. Therefore,

$$(92) \qquad \boldsymbol{w}^+ \cdot \boldsymbol{n} = \boldsymbol{0} \qquad \text{on } \partial D.$$

We will now show that (92) implies that $\mathbf{A}$ and $\boldsymbol{c}$ are zero, which means that $\boldsymbol{w} = \boldsymbol{0}$ in $\mathbb{R}^3_- \setminus \overline{S_1 \cup S_2}$. Then, reasoning as for Case (i), $\overline{S}_1 = \overline{S}_2$ and $\boldsymbol{g}_1 = \boldsymbol{g}_2$. Let $\boldsymbol{x}_0$ denote a vertex of $\overline{S_1 \cup S_2}$. This vertex is contained in at least three faces F_1, F_2, F_3 having independent normals $\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3$. We pick three points $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ on the faces F_1, F_2, F_3. Then

$$(\mathbf{A}\boldsymbol{x}_i + \boldsymbol{c}) \cdot \boldsymbol{n}_i = 0, \qquad \forall\, i = 1, 2, 3.$$

By continuity, there is a well-defined limit of $(\mathbf{A}\boldsymbol{x} + \boldsymbol{c}) \cdot \boldsymbol{n}$ when $\boldsymbol{x}$ approaches $\boldsymbol{x}_0$. It follows that

$$(\mathbf{A}\boldsymbol{x}_0 + \boldsymbol{c}) \cdot \boldsymbol{n}_i = 0, \qquad \forall\, i = 1, 2, 3,$$

which implies by linear independence that

$$(93) \qquad \mathbf{A}\boldsymbol{x}_0 + \boldsymbol{c} = \boldsymbol{0}.$$

An analogous argument can be applied to obtain a similar relation for any other vertex in $\overline{S_1 \cup S_2}$. Now, observe that there exist at least three vertices that are not co-planar. We denote them by $\boldsymbol{x}_0^1, \boldsymbol{x}_0^2, \boldsymbol{x}_0^3$. We assume that $\boldsymbol{x}_0^1$ does not lie in the plane spanned by $\boldsymbol{x}_0^2, \boldsymbol{x}_0^3$. Then the vectors $\boldsymbol{x}_0^1 - \boldsymbol{x}_0^2$ and $\boldsymbol{x}_0^1 - \boldsymbol{x}_0^3$ are independent. Combining by linearity all the relations of the form (93) for the three vertices gives:

$$\begin{cases} \mathbf{A}(\boldsymbol{x}_0^1 - \boldsymbol{x}_0^2) = \boldsymbol{0} \\ \mathbf{A}(\boldsymbol{x}_0^1 - \boldsymbol{x}_0^3) = \boldsymbol{0} \end{cases}$$

The above relations give six independent scalar linear equations for the entries of A. Since $\mathbf{A}$ is a skew-symmetric matrix, it follows that $\mathbf{A} = \boldsymbol{0}$ and, therefore, $\boldsymbol{c} = \boldsymbol{0}$. $\qquad\square$

Remark 5.3. *We observe that the statement of the previous theorem is true also in the case of slip fields directed in the normal direction. Indeed, instead of* (92), *we have* $\boldsymbol{w}^+ \cdot \boldsymbol{\tau} = \boldsymbol{0}$ *on* ∂D, *where* $\boldsymbol{\tau}$ *is any tangent vector to* ∂D, *so that we can still derive three independent conditions at each vertex and repeat the same arguments of the last part of the proof. In the oblique case, this condition cannot be guaranteed.*

The condition that the surface S is piece-wise linear is restrictive. However, it includes the important case in applications of surfaces that are exactly triangularized by polyhedral meshes. In addition, we only use the fact that piece-wise linear, closed surfaces have at least three vertices that are not co-planar and each vertex is adjoined by three faces with normal vectors that are independent of each other. Consequently, we can extend our result to more general surfaces that satisfy these conditions. For example, we can consider surfaces that are exactly triangularized by curvilinear meshes, as long as each element of the mesh is not co-planar with any other element, so that vertices of adjoining elements form real corners. Our method of proof cannot handle smooth surfaces, and non-uniqueness may in fact hold in this class.

The significance of effects originating from heterogeneity around and across the fault (including damage zones) and the fault geometric complexity has been recognized since the mid 1980s. These are still subject to active research; for a recent study, see [41]. *Fault geometric complexities have been shown to strongly affect a fault's seismic behavior at the corresponding spatial scales, while affecting the dynamic rupture process.*

Appendix A. Singularities for a rectangular dislocation surface parallel to $\{x_3 = 0\}$

In this section we give the explicit expression for $\boldsymbol{u}_\Gamma$ defined in (87). To this end, we define the following functions:

$$h_0(x, y, z) = \sqrt{x^2 + y^2 + z^2}, \qquad h_1(x, y, z) = \arctan\left(\frac{xy}{zh_0(x, y, z)}\right),$$

$$h_2(x, y, z) = \frac{xyz}{(x^2 + z^2)h_0(x, y, z)}, \qquad h_3(x, y, z) = \ln(y + h_0(x, y, z)),$$

$$h_4(x, y, z) = \frac{yz^2}{(x^2 + z^2)h_0(x, y, z)}, \qquad h_5(x, y, z) = \frac{xyzh_0(x, y, z)}{(x^2 + z^2)(y^2 + z^2)},$$

$$h_6(x, y, z) = \frac{xyz^3}{(x^2 + z^2)(y^2 + z^2)h_0(x, y, z)}$$

We denote $x_a^1 := x_1 - a$, $x_b^1 := x_1 - b$, $x_c^2 := x_2 - c$, $x_d^2 := x_2 - d$, $x_\alpha^3 := x_3 + |\alpha|$, and define the constant $C_\nu = -1/(8\pi(1 - \nu))$. We calculate the integrals in (87) (some of the calculations were performed using the computer algebra software Mathematica$^\copyright$), and we

find that

$$C_\nu^{-1}(u_\Gamma)_1 = \frac{2k_1(\lambda + 2\mu)}{\lambda + \mu} \left[h_1(x_a^1, x_c^2, x_\alpha^3) - h_1(x_a^1, x_d^2, x_\alpha^3) + h_1(x_b^1, x_d^2, x_\alpha^3) - h_1(x_b^1, x_c^2, x_\alpha^3) \right]$$

$$+ k_1 \left[h_2(x_a^1, x_d^2, x_\alpha^3) - h_2(x_a^1, x_c^2, x_\alpha^3) + h_2(x_b^1, x_c^2, x_\alpha^3) - h_2(x_b^1, x_d^2, x_\alpha^3) \right]$$

$$+ k_2 x_\alpha^3 \left[\frac{1}{h_0(x_a^1, x_c^2, x_\alpha^3)} - \frac{1}{h_0(x_b^1, x_c^2, x_\alpha^3)} + \frac{1}{h_0(x_b^1, x_d^2, x_\alpha^3)} - \frac{1}{h_0(x_a^1, x_d^2, x_\alpha^3)} \right]$$

$$+ \frac{k_3 \mu}{\lambda + \mu} \left[h_3(x_a^1, x_c^2, x_\alpha^3) - h_3(x_b^1, x_c^2, x_\alpha^3) + h_3(x_b^1, x_d^2, x_\alpha^3) - h_3(x_a^1, x_d^2, x_\alpha^3) \right]$$

$$+ k_3 \left[h_4(x_a^1, x_d^2, x_\alpha^3) - h_4(x_a^1, x_c^2, x_\alpha^3) + h_4(x_b^1, x_c^2, x_\alpha^3) - h_4(x_b^1, x_d^2, x_\alpha^3) \right],$$

$$C_\nu^{-1}(u_\Gamma)_2 = k_1 x_\alpha^3 \left[\frac{1}{h_0(x_a^1, x_c^2, x_\alpha^3)} - \frac{1}{h_0(x_b^1, x_c^2, x_\alpha^3)} + \frac{1}{h_0(x_b^1, x_d^2, x_\alpha^3)} - \frac{1}{h_0(x_a^1, x_d^2, x_\alpha^3)} \right]$$

$$+ \frac{2k_2(\lambda + 2\mu)}{\lambda + \mu} \left[h_1(x_a^1, x_c^2, x_\alpha^3) - h_1(x_a^1, x_d^2, x_\alpha^3) + h_1(x_b^1, x_d^2, x_\alpha^3) - h_1(x_b^1, x_c^2, x_\alpha^3) \right]$$

$$+ k_2 \left[h_2(x_d^2, x_a^1, x_\alpha^3) - h_2(x_c^2, x_a^1, x_\alpha^3) + h_2(x_c^2, x_b^1, x_\alpha^3) - h_2(x_d^2, x_b^1, x_\alpha^3) \right]$$

$$+ \frac{k_3 \mu}{\lambda + \mu} \left[h_3(x_c^2, x_a^1, x_\alpha^3) - h_3(x_d^2, x_a^1, x_\alpha^3) + h_3(x_d^2, x_b^1, x_\alpha^3) - h_3(x_c^2, x_b^1, x_\alpha^3) \right]$$

$$+ k_3 \left[h_4(x_d^2, x_a^1, x_\alpha^3) - h_4(x_c^2, x_a^1, x_\alpha^3) + h_4(x_c^2, x_b^1, x_\alpha^3) - h_4(x_d^2, x_b^1, x_\alpha^3) \right],$$

$$C_\nu^{-1}(u_\Gamma)_3 = \frac{k_1 \mu}{\lambda + \mu} \left[h_3(x_b^1, x_c^2, x_\alpha^3) - h_3(x_a^1, x_c^2, x_\alpha^3) + h_3(x_a^1, x_d^2, x_\alpha^3) - h_3(x_b^1, x_d^2, x_\alpha^3) \right]$$

$$+ k_1 \left[h_4(x_a^1, x_d^2, x_\alpha^3) - h_4(x_a^1, x_c^2, x_\alpha^3) + h_4(x_b^1, x_c^2, x_\alpha^3) - h_4(x_b^1, x_d^2, x_\alpha^3) \right]$$

$$+ \frac{k_2 \mu}{(\lambda + \mu)} \left[h_3(x_d^2, x_a^1, x_\alpha^3) - h_3(x_c^2, x_a^1, x_\alpha^3) + h_3(x_c^2, x_b^1, x_\alpha^3) - h_3(x_d^2, x_b^1, x_\alpha^3) \right]$$

$$+ k_2 \left[h_4(x_d^2, x_a^1, x_\alpha^3) - h_4(x_c^2, x_a^1, x_\alpha^3) + h_4(x_c^2, x_b^1, x_\alpha^3) - h_4(x_d^2, x_b^1, x_\alpha^3) \right]$$

$$+ \frac{k_3(\lambda + 2\mu)}{\lambda + \mu} \left[h_1(x_a^1, x_c^2, x_\alpha^3) - h_1(x_a^1, x_d^2, x_\alpha^3) + h_1(x_b^1, x_d^2, x_\alpha^3) - h_1(x_b^1, x_c^2, x_\alpha^3) \right]$$

$$+ k_3 \left[h_5(x_a^1, x_c^2, x_\alpha^3) - h_5(x_a^1, x_d^2, x_\alpha^3) + h_5(x_b^1, x_d^2, x_\alpha^3) - h_5(x_b^1, x_c^2, x_\alpha^3) \right]$$

$$+ k_3 \left[h_6(x_a^1, x_c^2, x_\alpha^3) - h_6(x_a^1, x_d^2, x_\alpha^3) + h_6(x_b^1, x_d^2, x_\alpha^3) - h_6(x_b^1, x_c^2, x_\alpha^3) \right].$$

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Johann Radon Institute for Computational and Applied Mathematics (RICAM)
E-mail address: `andrea.aspri@ricam.oeaw.ac.at`

Dipartimento di Matematica, Politecnico di Milano and Department of Mathematics, NYU-Abu Dhabi
E-mail address: `elena.beretta@polimi.it`

Department of Mathematics, Penn State University
E-mail address: `alm24@psu.edu`

Department of Computational and Applied Mathematics and Department of Earth, Environmental, and Planetary Sciences, Rice University
E-mail address: `mdehoop@rice.edu`