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Inverse Problem of Travel Time Difference Functions on a Compact Riemannian Manifold with Boundary

Maarten V. de Hoop[1] · Teemu Saksala[1]

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Abstract
We show that the travel time difference functions, between common interior points and pairs of points on the boundary, determine a compact Riemannian manifold with smooth boundary up to Riemannian isometry if the boundary satisfies a certain visibility condition. This corresponds with the inverse microseismicity problem. In the proof of this result, we also construct an explicit smooth atlas from the travel time difference functions.

Keywords Inverse problems · Differential geometry · Geodesics · Distance functions

Mathematics Subject Classification 35R30 · 53C22

1 Introduction

Let (N, g) be a complete, connected smooth Riemannian manifold of dimension two or higher. We split the manifold into two parts that are a closed set M, with non-empty interior M^{int}, and the closure of the exterior $F := N \setminus M^{\text{int}}$. We assume that the boundary ∂M of M is a smooth co-dimension one manifold. The set F is the known observation domain, and M is the object of interest, for instance, the Earth. The Riemannian metric g can be seen as a proxy of the material parameters of M.

For any $p, q \in N$, we denote by $d_N(p, q)$ the length of a distance minimizing geodesic of (N, g) that connects p to q. If the wave speed in F is much smaller than in M and if ∂M is strictly convex, we may assume that distance minimizing geodesics of (N, g) connecting p to q stay inside M if $p, q \in M$. This implies

$$d_M(p, q) = d_N(p, q), \quad p, q \in M, \tag{1}$$

✉ Teemu Saksala
teemu.saksala@rice.edu

[1] Department of Computational Applied Mathematics, Rice University, Houston, USA

where $d_M(p,q)$ is the distance from p to q in M, that is given as the infimum of lengths of curves from p to q that stay in M. For the present, we assume that (1) holds and we denote $d_M = d_g$.

Suppose that there exists a Dirac point source $(p,s) \in M \times \mathbb{R}$ of a Riemannian wave equation, with zero Cauchy data. It follows from [4,6,7] that the singularities emitted from (p,s) propagate along the co-geodesic flow of (N,g) (see for instance [12] for more details). When this flow is projected to M, we obtain the family of unit speed geodesics emitted from p.

For every $z \in \partial M$, we define the *arrival time* $\mathcal{T}_{p,s}(z)$ to be the infimum of times when a spherical wave emitted from (p,s) is observed at z. Hence $\mathcal{T}_{p,s}(z) = d_g(p,z)+s$. Although the arrival time function depends on the emission time s, it holds that *travel time difference function*

$$D_p(z_1,z_2) := \mathcal{T}_{p,s}(z_1) - \mathcal{T}_{p,s}(z_2), \quad z_1, z_2 \in \partial M,$$

is independent of s as it is given as the difference of the arrival times. Thus, the function D_p can be determined without the knowledge of the emission time s or the location of the origin p. This paper is devoted to the study of the inverse problem of travel time difference functions. This problem can be formulated as follows. Does the collection

$$\{D_p : p \in M^{\text{int}}\},$$

determine the Riemannian manifold (M,g) up to isometry?

Let $n \geq 2$ and (M,g) be a compact, connected, n-dimensional smooth Riemannian manifold with smooth boundary ∂M. We omit condition (1). Since M is compact for any points $p,q \in M$ there exists a distance minimizing C^1-smooth curve c from p to q, see [1]. Moreover for any $t_0 \in [0, d_g(p,q)]$ such that point $c(t_0)$ is an interior point of M there exists $\epsilon > 0$ such that $c : (t_0 - \epsilon, t_0 + \epsilon)$ is a geodesic. We use the notation SM for the unit sphere bundle of (M,g). Therefore, each $(p,v) \in SM$ determines the unique maximal unit speed geodesic, $\gamma_{p,v}$ say, of (M,g).

We note that $d_g(x,y),\ x,y \in M$ is not generally obtained by minimizing the lengths of geodesics connecting x to y. Therefore, we will redefine the travel time difference function without using the arrival times. For any $p \in M$, the corresponding *travel time difference function* is

$$D_p : \partial M \times \partial M \to \mathbb{R}, \quad D_p(z_1,z_2) := d_g(p,z_1) - d_g(p,z_2). \tag{2}$$

The function D_p is continuous. We assume that the following *travel time difference data*

$$(\partial M,\ \{D_p :\ p \in M^{\text{int}}\}), \tag{3}$$

is given. That is, we assume, that the $(n-1)$-dimensional smooth manifold ∂M without boundary and the collection of functions $\{D_p : \partial M \times \partial M \to \mathbb{R} \,|\, p \in M^{\text{int}}\}$ are given. We re-emphasize that a priori the points p related to D_p are unknown.

The aim of this paper is to prove that travel time difference data determine (M, g) up to isometry. Before stating our main theorem, we describe an additional geometric property for ∂M under which we can prove the uniqueness of the inverse problem. Let (N, G) be any smooth closed Riemannian manifold that extends (M, g), such that $g = G|_M$. We use the notation

$$\ell(x, v) := \inf\{t > 0 : \gamma_{x,v}(t) \in N \setminus M\}, \quad (x, v) \in SM,$$

for the *exit time function*. Thus, the domain of definition for $\gamma_{x,v}$ is $[-\ell(x, -v), \ell(x, v)]$. Moreover by [18, Lemma 1], $\ell(x, v)$ is independent of the extension. We note that $\gamma_{x,v}$ may intersect the boundary tangentially in many points.

We say that a point $q \in M$ is *not a cut point* to $p \in M$ along a distance minimizing geodesic $\gamma_{p,v}$, $(p, v) \in SM$ from p to q contained in M if there exists a closed Riemannian manifold (N, G) that is an extension of (M, g) satisfying the property

$$d_g(p, q) < \tau_G(p, v) := \sup\{t > 0 : d_G(p, \gamma_{p,v}(t)) = t\}. \tag{4}$$

Above d_G is the distance function of (N, G). The function τ_G is the cut distance function of (N, G).

Definition 1.1 We say that a smooth, complete, and connected Riemannian manifold (M, g) satisfies *the visibility* condition if the following holds: For every $p \in \partial M$, there exists $(p, \eta) \in \partial SM$, such that η is transverse to ∂M and $\ell(p, \eta) < \infty$. Geodesic $\gamma_{p,\eta} : [0, \ell(p, \eta)] \to M$ is a distance minimizer, and $\gamma_{p,\eta}(\ell(p, \eta))$ is not a cut point to p, $\dot{\gamma}_{p,\eta}(\ell(p, \eta))$ is transverse to ∂M and $\gamma_{p,\eta}((0, \ell(p, \eta))) \subset M^{\text{int}}$.

Similar type of condition on the boundary has been considered before in [18].

Let (M_1, g_1) and (M_2, g_2) be two smooth compact and connected Riemannian manifolds with smooth boundaries ∂M_1 and ∂M_2. The next definition formalizes what does it mean if the manifolds (M_1, g_1) and (M_2, g_2) have the same travel time difference data.

Definition 1.2 We say that the travel time difference data of (M_1, g_1) and (M_2, g_2) coincide if there exists a diffeomorphism $\phi : \partial M_1 \to \partial M_2$ such that

$$\{D_p(\phi^{-1}(\cdot), \phi^{-1}(\cdot)) : p \in M_1^{\text{int}}\} = \{D_q : q \in M_2^{\text{int}}\}. \tag{5}$$

Our main result is the following

Theorem 1.3 *Let $n \geq 2$ and (M_i, g_i), $i = 1, 2$ be compact, connected n-dimensional Riemannian manifolds with smooth boundaries ∂M_i. Suppose that (M_1, g_1) satisfy the visibility condition 1.1. If the travel time difference data of (M_1, g_1) and (M_2, g_2) coincide, then there exists a Riemannian isometry $\Psi : (M_1, g_1) \to (M_2, g_2)$ such that the restriction of Ψ on ∂M_1 coincides with ϕ.*

The proof of this theorem is given in Sect. 2. It consists of three steps. First, we construct a map $\Psi : (M_1, g_1) \to (M_2, g_2)$, as in Theorem 1.3, and show that this map is a homeomorphism. Then, we show that Ψ is a diffeomorphism. Finally, we will show that $g_1 = \Psi^* g_2$, where $\Psi^* g_2$ stands for the pullback of metric g_2.

While preparing this paper for submission, the authors became aware that S. Ivanov very recently posted a preprint [8] on ArXiv with a result related to Theorem 1.3 presented here. He extends the result of [12] in the following way. Let N be any complete, connected Riemannian manifold without boundary. Let $F, U \subset N$ be open. If the topology and differential structure of the observation domain F and $D_p : F \times F \to \mathbb{R}$, $p \in U$ are given then these data determine the geometry of the domain $(U, g|_U)$ uniquely up to a Riemannian isometry. Furthermore, he proves that the determination of (N, g) from travel time difference functions D_p is stable if the underlying manifold has a priori bounds on its diameter, curvature, and injectivity radius. Reference [8, Proposition 7.3] provides a result closest to our Theorem 1.3, for complete manifolds with nowhere concave boundary. We point out, also, that the proof given in [8] is different from ours in essential ways. S. Ivanov's proof is based on distance comparison inequalities implied by Toponogov's theorem and minimizing geodesic extension property. The latter property provides a lower bound on the length of a minimizing extension of a geodesic beyond a non-cut point in terms of the length of a minimizing extension beyond the other endpoint.

We end this section by comparing the visibility condition to the nowhere-concave condition for the boundary. Recall that the boundary ∂M of Riemannian manifold (M, g) is nowhere concave if for every $z \in \partial M$ the second fundamental form of ∂M at z, with respect to the inward-pointing normal vector, has at least one positive eigenvalue. This condition on the boundary was considered in [23]. If ∂M is nowhere concave, then by the proof of [23, Proposition 3.4] and [17, Sect. 4.1], it holds that (M, g) satisfies the visibility condition. Notice that an annulus, contained in Euclidean plane, satisfies the visibility condition, but not the nowhere-concave condition on the boundary. Therefore, the visibility condition is more general characteristic of these two.

Finally, we give an example of such geometry that does not satisfy either of these conditions. Let $M \subset S^2$ be a spherical cap larger than the hemisphere. If g is the round metric on M, then (M, g) does not satisfy the visibility condition, since any g-distance minimizing curve between boundary points lies in ∂M and therefore, it is not a geodesic of S^2. In this case, ∂M is not nowhere concave either.

Background

Four Geometric Inverse Problems Related to the Riemannian Wave Equation In this section, we assume that N, M, F and g are as in Sect. 1. There are four different data sets that are all related to Riemannian wave equation with the Dirac point source $(p, s) \in M \times \mathbb{R}$ with zero Cauchy data.

(1) The inverse problem of travel time functions have been considered in [9,11]. The authors study the properties of the map $\mathcal{R} : M \to C(\partial M)$, in which a point $p \in M$ is mapped into the corresponding travel time function $r_p : \partial M \to \mathbb{R}$, given by the formula

$$r_p(z) = d_g(p, z), \quad z \in \partial M.$$

The authors show that the data $(\partial M, \{r_p : p \in M\})$ determine a manifold (M, g) up to isometry. They use the map $\mathcal{R}$ to construct an isometric copy of M in $C(\partial M)$. They do not impose any restrictions to the geometry.

(2) In [12], the authors prove a result related to Theorem 1.3. In this paper, it is assumed that the travel time difference function is given in the *observation* set F with non-empty interior

$$D_p : F \times F \to \mathbb{R}.$$

In addition, they assume that the Riemannian structure of (F, g) is known. The proof of the main theorem in [12] is somehow similar to the proof of Theorem 1.3 presented here, and we will often refer to it for the details that are not presented in this paper.

(3) The inverse problem related to the set of exit directions

$$\Sigma_p = \{(\gamma_{p,v}(\ell(p, v)), \dot{\gamma}_{p,v}(\ell(p, v))) \in \partial SM : v \in S_p M\}$$

of geodesics emitted from p has been studied in [13]. Let

$$I(g, w, z, l) := \text{ number of } g\text{-geodesics of lenght } l \text{ connecting } w \text{ to } z,$$
$$w, z \in N, \ l > 0$$

The authors show that if (N, g) is a closed manifold such that

$$\sup_{w,z,\ell} I(g, w, z, l) < \infty, \tag{6}$$

M is non-trapping and ∂M is strictly convex, then the collection of exiting directions

$$\{\Sigma_p \subset \partial TM : p \in M^{\text{int}}\}$$

determine the manifold (M, g) up to isometry. Assumption (6) is needed to show that each set Σ_p is produced by the unique $p \in M$. As far as we know, it is not known if (6) follows from the convexity of the boundary and non-trapping properties. On the other hand, in [10], it is shown that (6) is a generic property in the space of all Riemannian metrics of N.

(4) The final data set is related to a *generalized sphere* of radius $r > 0$, that is given by the formula

$$S(p, r) =: \{\exp_p(v) : v \in T_p M, \ \|v\|_g = r, \ \exp_p \text{ is not singular at } v\}.$$

In [3], the authors show that the spherical surface data

$$\{S(q, t) \cap F : q \in M, \ t > 0\}$$

determine the universal cover space of N. If a generalized sphere $S(p, r)$ is given the authors show that there exists a specific coordinate structure in a neighborhood of any maximal normal geodesic to $S(p, r)$ such that in these coordinates metric tensor g can can be determined. However, this does not determine g globally. The authors provide an example of two different metric tensors which produce the same spherical surface data.

Microseismicity In this paper, the results in [12] are adapted, in a fundamental way, to data available from actual seismic surveys. The point sources are microseismic events detected in dense arrays at Earth's surface. In our theorem, we show that the data determine the metric up to change of coordinates. This implies that one can locate the closest surface point and determine the corresponding travel time to each event.

In the seismological literature, the problem of simultaneously determining the hypocentral parameters (location and origin times of interior point sources or events) and the wave speed has been formulated as double-difference (travel time) tomography. This problem is precisely addressed in this paper, but its resolution requires a key modification of the data. For reference, we describe the notion of double-difference travel times [20]: One takes the difference between the travel times of a pair of events located closely to one another and a common boundary observation point, and then collects this difference for a given pair at many of such observation points. A pair of events is also referred to as a doublet. The data are formed by combining all available pairs of events. The difference is typically linearized in the hypocentral parameters associated with both events relative to a reference configuration. In seismology, the idea to decouple the doublets to locate the events was introduced by Poupinet et al. [15]. Zhang and Thurber [21,22] extended this double-difference location method with an attempt to simultaneously solve for both wave speed structure and event locations by incorporating wave speed perturbations. They added the absolute arrival times to the data.

The travel time difference function, given in (2), is closer related to applications in exploration seismology with the purpose of locating microseismic events Grechka et al. [5]. In this paper, the authors assume that the travel time to the receivers and location of the master event is known. Notice that our result do not recover the locations of the events in Cartesian coordinates.

It was proven in [2] that if the metric is conformally Euclidean (isotropic medium), then a given compact set of sources and the corresponding set of wave speeds are determined up to an conformal isometry by the travel times.

2 Proof of the Main Theorem

In this section, we prove Theorem 1.3. Whenever it is not necessary to distinguish manifolds M_1 and M_2 from one other, we drop the subindices. In these cases, we work with the data (3).

2.1 Outline of the Proof of the Main Theorem

The proof consists of three steps. First, we use the data (3) to construct a mapping $\mathcal{D}$ from points of M to continuous functions on $\partial M \times \partial M$. We show that this mapping is a topological embedding. Then, we use the diffeomorphism $\phi : \partial M_1 \to \partial M_2$ and (5) to construct a homeomorphism $\Psi : M_1 \to M_2$ as in Theorem 1.3 (see (13) for the definition). In second part, we show that this mapping is a diffeomorphism. We prove the existence of such local coordinate maps that are determined by (3). In the third part, we first prove that the data (3) determine the images of geodesic segments that come to the boundary ∂M. Finally, we use this information to prove the uniqueness of Riemannian structure.

The outline of the proof of the main theorem is similar to the proof of the main theorem of [12]. The proof presented in this paper contains two key differences to the earlier result. The first one is the construction of the boundary coordinate system, in the beginning of Sect. 2.3. The determination of the boundary defining function (see (16) and (19)), only from the data (3), has not been presented in the literature before. The second difference, that is considered in the beginning of Sect. 2.4, is related to the construction of metric tensor from the data (3). In order to use the similar techniques as in [12], to prove that the metrics g_1 and $\Psi^* g_2$ coincide, we need to prove that the data (3) determine the full Taylor expansion of the metric tensor on ∂M in boundary normal coordinates. This makes it possible to extend M_1 to a closed manifold N given with two smooth metric tensors G and $\widetilde{G}$ that coincide in $F := N \setminus M_1^{\text{int}}$, and also satisfy $G|_{M_1} = g_1$ and $\widetilde{G}|_{M_1} = \Psi^* g_2$. Since we don't assume ∂M to be strictly convex, we need also to show that the travel time difference functions, measured on F, of metrics G and $\widetilde{G}$ coincide.

2.2 Topology

We start first extending the data to the boundary. If $p, w \in \partial M$ then by the triangle inequality it holds that

$$d_g(p, w) = \sup_{q \in M^{\text{int}}} D_q(p, w). \tag{7}$$

Thus, data (3) determine $d_g : \partial M \times \partial M \to \mathbb{R}$ and the extended data

$$(\partial M, \{D_p : \ p \in M\}). \tag{8}$$

Our first Lemma is

Lemma 2.1 *Let (M_i, g_i), $i = 1, 2$ be compact, connected n-dimensional Riemannian manifolds with smooth boundaries ∂M_i. If the travel time difference data of (M_1, g_1) and (M_2, g_2) coincide, then*

$$\{D_p(\phi^{-1}(\cdot), \phi^{-1}(\cdot)) : p \in M_1\} = \{D_q : q \in M_2\}. \tag{9}$$

Proof From (5) and (7), it follows that

$$d_1(\phi^{-1}(p), \phi^{-1}(q)) = d_2(p, q), \quad p, q \in \partial M_2. \tag{10}$$

Here, d_i is the distance function of g_i for $i \in \{1, 2\}$. Therefore, (9) holds. □

We study the properties of the mapping

$$\mathcal{D} : M \to C(\partial M \times \partial M), \quad \mathcal{D}(p) = D_p,$$

where the target space is equipped with the L^∞-norm.

Lemma 2.2 *The mapping $\mathcal{D}$ is a topological embedding.*

Proof Using triangle inequality, it is straightforward to see that $\mathcal{D}$ is 2–Lipschitz.

Next we prove that $\mathcal{D}$ is one-to-one. To show this, assume that $x, y \in M$ are such that $D_x = D_y$. We first show that this implies that the set $\{z_x\}$ of closest boundary points of x coincides with the set $\{z_y\}$ of closest boundary points of y. Let $w \in \partial M$ and define

$$f_{x,w} : \partial M \to \mathbb{R}, \quad f_{x,w}(z) := D_x(z, w). \tag{11}$$

Then $\{z_x\}$ is the set of minimizers of function $f_{x,w}$. Since $f_{x,w} = f_{y,w}$, we have proven that $\{z_x\} = \{z_y\}$. We will also use the function $f_{x,w}$ later when we construct a boundary defining function.

Let $z_0 \in \{z_x\}$ and denote $s_x = d_g(x, z_0)$ and $s_y = d_g(y, z_0)$. Without loss of generality, we can assume that $s_x \leq s_y$. Let ν be the inward-pointing unit normal vector field to ∂M. Then $\gamma_{z_0,\nu}$ is the distance minimizing geodesic from ∂M to x and y. Moreover

$$x = \gamma_{z_0,\nu}(s_x), \quad y = \gamma_{z_0,\nu}(s_y) \quad \text{and} \quad d_g(x, y) = s_y - s_x. \tag{12}$$

If $z \in \partial M \setminus \{z_0\}$ is close to z_0, the distance minimizing geodesic γ_x from z to x is not the same geodesic as $\gamma_{z_0,\nu}$, that is, the angle β of the curves γ_x and $\gamma_{z_0,\nu}$ at the point x is strictly between 0 and π. Let γ_y be a distance minimizing geodesic from y to z. We note that $D_x(z, z_0) = D_y(z, z_0)$ and (12) yields

$$\mathcal{L}(\gamma_y) = d_g(y, z) = d_g(y, x) + d_g(x, z) = \mathcal{L}(\gamma_{z_x,\nu}|_{[s_x,s_y]}) + \mathcal{L}(\gamma_x).$$

Thus, the union μ of the curves $\gamma_{z_x,\nu}([s_x, s_y])$ and γ_x is a distance minimizing curve from z to y, and hence it is a geodesic. However, as the angle β, defined above, is strictly between 0 and π, the curve μ is not smooth at x, and hence it is not possible that μ is a geodesic unless $x = y$. Thus, x and y have to be equal.

Since M is compact and we just proved that $\mathcal{D}$ is continuous and one–to–one, we have that mapping $\mathcal{D}$ is closed. Thus, the claim is proven. □

Since the mapping ϕ, given by Definition 1.2, is a diffeomorphism the mapping

$$\Phi : C(\partial M_1 \times \partial M_1) \to C(\partial M_2 \times \partial M_2), \quad \Phi(F) = F(\phi^{-1}(\cdot), \phi^{-1}(\cdot))$$

is an isometry. Let $\mathcal{D}_i,\ i \in \{1, 2\}$ be as $\mathcal{D}$ on (M_i, g_i). Now we are ready to define the mapping

$$\Psi : M_1 \to M_2, \quad \Psi = \mathcal{D}_2^{-1} \circ \Phi \circ \mathcal{D}_1. \tag{13}$$

Proposition 2.3 *Let $(M_i, g_i),\ i = 1, 2$ be compact, connected n-dimensional Riemannian manifolds with smooth boundaries ∂M_i. If the travel time difference data of (M_1, g_1) and (M_2, g_2) coincide, then the mapping Ψ given by (13) is a homeomorphism such that the restriction of Ψ on ∂M_1 coincides with ϕ.*

Proof By (9) and Lemma 2.2 it holds that the map Ψ is a well-defined homeomorphism. If $p \in \partial M_1$, then by (10) for any $z, w \in \partial M_2$, we have

$$\begin{aligned}(\mathcal{D}_2(\phi(p))(z, w) &= d_2(\phi(p), z) - d_2(\phi(p), w) = d_1(p, \phi^{-1}(z)) - d_1(p, \phi^{-1}(w)) \\ &= ((\Phi \circ \mathcal{D}_1)(p))(z, w).\end{aligned}$$

Applying $\mathcal{D}_2^{-1}$ to the both sides of the equation above implies $\Psi(p) = \phi(p)$. □

2.3 Smooth Structure

In this part, we show that the mapping Ψ given in (13) is a diffeomorphism. We consider separately the boundary and the interior cases.

We start with the boundary case. Let $\sigma_{\partial M}$ be the collection of all boundary cut points,

$$\begin{aligned}\sigma_{\partial M} &:= \{\gamma_{z,\nu}(\tau_{\partial M}(z)) \in M :\ z \in \partial M\}, \\ \tau_{\partial M}(z) &:= \sup\{t > 0 : d_g(\partial M, \gamma_{z,\nu}(t)) = t\}.\end{aligned}$$

By [16, Sect. III.4.] it holds that

$$\sigma_{\partial M} = \overline{\{p \in M : \#\{z \in \partial M : d_g(p, z) = d_g(p, \partial M)\} \geq 2\}}. \tag{14}$$

Choose $w \in \partial M$. Then by (14) and the Lemma 2.2 the data (8) determine the set

$$M \setminus \sigma_{\partial M} = \{p \in M :\ \text{The map } f_{p,w} \text{ has precisely one minimizer.}\}^{\text{int}}, \tag{15}$$

where $f_{p,w}$ is as in (11).

Lemma 2.4 *Let $(M_i, g_i),\ i = 1, 2$ be compact, connected n-dimensional Riemannian manifolds with smooth boundaries ∂M_i. If the travel time difference data of (M_1, g_1) and (M_2, g_2) coincide, then*

$$M_2 \setminus \sigma_{\partial M_2} = \Psi(M_1 \setminus \sigma_{\partial M_1}).$$

Proof By the definition of the mapping Ψ, we have for any $p \in M_1$ and $w \in \partial M_1$ that

$$f^1_{p,w}(z) = f^2_{\Psi(p),\phi(w)}(\phi(z)), \quad z \in \partial M_1,$$

where $f^1_{p,w}$ and $f^2_{\Psi(p),\phi(w)}$ are defined as $f_{p,w}$ in (11). Therefore, the claim follows from (15). $\square$

Next we construct a boundary defining function on $M \setminus \sigma_{\partial M}$. Let $p \in M \setminus \sigma_{\partial M}$ and denote by $Z(p)$ the closest boundary point of p. Thus, the map $x \mapsto Z(x) \in \partial M$ is smooth on $M \setminus \sigma_{\partial M}$. Define a function

$$f_p(z) := d_g(z, Z(p)) - D_p(z, Z(p)), \quad z \in \partial M. \tag{16}$$

Notice that this function is determined by the data (8), and by triangular inequality the function f_p is non-negative. If $p \in \partial M$ then f_p is a zero function. If $p \in M^{\text{int}} \setminus \sigma_{\partial M}$ then

$$f_p(z) > 0, \quad z \in (\partial M \setminus Z(p)). \tag{17}$$

If this is not true then there exists $\partial M \ni z \neq Z(p)$ such that

$$d_g(p, z) = d_g(p, Z(p)) + d_g(Z(p), z).$$

Which implies that there exists a distance minimizing curve from p to z, that goes through $Z(p)$, but is not C^1-smooth at $Z(p)$ (Fig. 1). By [1] this is not possible. Thus, (17) holds. Therefore, we have proven the following

$$\partial M = \{p \in M \setminus \sigma_{\partial M} : f_p \equiv 0\}. \tag{18}$$

Lemma 2.5 *Let (M, g) be a smooth, complete, and connected Riemannian manifold with smooth boundary for which the visibility condition 1.1 holds. Let $p \in \partial M$ and $(p, \eta) \in S_p M$ be as in Definition 1.1. We denote*

$$(w, \xi) = (\gamma_{p,\eta}(\ell(p, \eta)), \dot{\gamma}_{p,\eta}(\ell(p, \eta))) \in \partial SM.$$

There exists a neighborhood $W \subset SM$ of (p, η) such that for any $(x, v) \in W$ the distance function $d_g(x, \cdot)$ is smooth at $\gamma_{x,v}(t)$ if $0 < t \leq \ell(x, v)$. A point $\gamma_{x,v}(t) \in M^{\text{int}}$ if $0 < t < \ell(x, v)$ and the geodesic $\gamma_{x,v}$ is transverse to ∂M at $\gamma_{x,v}(\ell(x, v))$ and at x, if $x \in \partial M$.

Moreover there exists neighborhoods $U, V \subset M$ of p and w respectfully such that the distance functions $d_g(x, \cdot)$ and $d_g(y, \cdot)$ are smooth for any $(x, y) \in (U \times V)$.

Proof Since (p, η) and (w, ξ) are transverse to ∂M and $\gamma_{p,\eta}(0, \ell(p, \eta)) \subset M^{\text{int}}$ it follows from the continuity of the exponential map of (M, g) and the implicit function theorem that there exists an open neighborhood $W_1 \subset (SM \setminus S\partial M)$ of (p, η) such

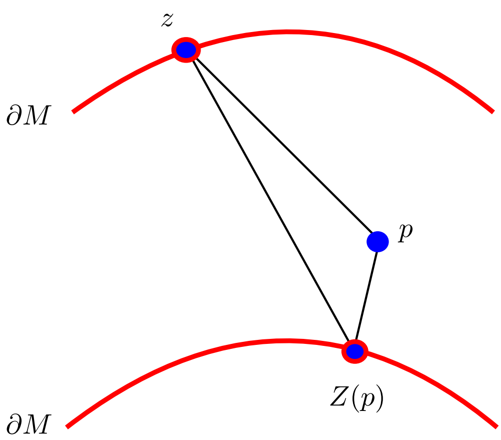

Fig. 1 The function f_p is non-negative, since the piecewice smooth curve from p to z that goes via $Z(p)$ is not C^1-smooth at $Z(p)$

that the exit time function ℓ restricted to this set is smooth and for any $(x, v) \in W_1$ and $t \in (0, \ell(x, v))$ it holds that $\gamma_{x,v}(t) \in M^{\text{int}}$. Moreover the geodesic $\gamma_{x,v}$ is transverse to ∂M at $\gamma_{x,v}(\ell(x, v))$ and at x, if $x \in \partial M$.

Let (N, G) be an extension of (M, g) for which (4) holds. Since w is not a cut point to p along $\gamma_{p,\eta}$ there exists a neighborhood $W_2 \subset SN$ of (p, η), such that for any $(x, v) \in W_2$ the distance function $d_G(x, \cdot)$ of (N, G) is smooth at $\gamma_{x,v}(t)$ if $0 < t < \tau_G(x, v)$. Due to Definition 1.1, we have that

$$\ell(p, \eta) = d_g(p, w) = d_G(p, w) < \tau_G(p, \eta).$$

Therefore, the continuity of functions τ_G and ℓ imply that there exists a neighborhood $W \subset W_1 \cap W_2$ of (p, η), such that for any $(x, v) \in W$, we have

$$\ell(x, v) < \tau_G(x, v).$$

Since any geodesic of (M, g) is a geodesic of (N, G), we have

$$d_g(x, \gamma_{x,v}(t)) = d_G(x, \gamma_{x,v}(t)), \quad \text{if } 0 < t \leq \ell(x, v).$$

Therefore, function $d_g(x, \cdot)$ is smooth at $\gamma_{x,v}(t)$ if $0 < t \leq \ell(x, v)$.

Since w is not a conjugate point to p along $\gamma_{p,\eta}$ the exponential map of (N, G) at p is an open map close to $\ell(p, \eta)\eta \subset T_pM$. This implies the existence of the set V as in the claim of this lemma. The proof for the existence of set U is similar. □

Suppose that (M, g) satisfies the visibility condition 1.1. Let $p \in \partial M$. By Lemma 2.5 there exists $w \in \partial M$ and $r > 0$ such that the distance functions $d_g(x, \cdot)$ and $d_g(y, \cdot)$ are smooth for any $(x, y) \in B(p, r) \times B(w, r)$ and $B(p, r) \cap B(w, r) = \emptyset$. Let $r_{\partial M} > 0$ be the minimum of r and the boundary injectivity radius. Choose

$$z_0 \in (\partial M \cap (B(w, r)) \text{ and } \delta \in (0, r_{\partial M}),$$

such that z_0 is not the closest boundary point for any $q \in B(p,\delta)$, $Z(q) \in B(p,r)$ and the distance minimizing geodesic from z_0 to p is not normal to ∂M at p. Then

$$E_{z_0} : B(p,\delta) \to [0,\infty), \quad E_{z_0}(q) := f_q(z_0) = d_g(z_0, Z(q)) - D_q(z_0, Z(q)) \tag{19}$$

is well-defined and smooth. Moreover, by (17), we have that $E_{z_0}(q) = 0$ if and only if $q \in B(p,\delta) \cap \partial M$. Thus, E_{z_0} is a boundary defining function. Denote (t, Z) for the boundary normal coordinates in $B(p,\delta)$, where $t(q) = d_g(\partial M, q)$ and $Z(q)$ is the closest boundary point to $q \in B(p,\delta)$. Then the map

$$W_{z_0} : B(p,\delta) \to [0,\infty) \times \partial M, \quad W_{z_0}(q) := (E_{z_0}(q), Z(q)), \tag{20}$$

is smooth.

We show that the Jacobian of this map with respect to boundary normal coordinates is invertible at p. By the inverse function theorem this yields the existence of a neighborhood $V \subset M$ of p such that the restriction of W_{z_0} to V is a coordinate map. The Jacobian of W_{z_0} at p is

$$\begin{pmatrix} \frac{\partial}{\partial t} E_{z_0} & \frac{\partial}{\partial t} Z \\ \frac{\partial}{\partial Z} E_{z_0} & \frac{\partial}{\partial Z} Z \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial t} E_{z_0} & \bar{0}^T \\ \frac{\partial}{\partial Z} E_{z_0} & Id_{n-1}. \end{pmatrix}$$

Notice

$$\frac{\partial}{\partial t} E_{z_0}(t, Z)\Big|_{(t,Z)=(0,p)} = 1 - g_p(\dot{\gamma}_{z_0,p}(d_g(p, z_0)), \nu) > 0.$$

The last inequality holds since the distance minimizing geodesic $\gamma_{z_0,p}$ from z_0 to p is not normal to the boundary at p. Thus, the Jacobian of W_{z_0} at p is invertible.

We use coordinates similar to W_{z_0} to show that $\Psi : M_1 \to M_2$ is a diffeomorphism near the boundary of M_1. In order to do so, we first prove the following

Lemma 2.6 *Let (M_i, g_i), $i = 1, 2$ be compact n-dimensional Riemannian manifolds with smooth boundaries ∂M_i. If the travel time difference data of (M_1, g_1) and (M_2, g_2) coincide, then*

$$g_1|_{\partial M_1} = \phi^*(g_2|_{\partial M_2}). \tag{21}$$

Proof Since (5) implies (10) the proof of this Lemma follows from the proof of [23, Proposition 3.3]. □

We find.

Lemma 2.7 *Let (M_i, g_i), $i = 1, 2$ be compact, connected n-dimensional Riemannian manifolds with smooth boundaries ∂M_i, whose travel time difference data coincide. Assume that (M_1, g_1) satisfy the visibility condition 1.1. Let $p \in \partial M_1$. There exists a neighborhood U of p in M_1 and $z_0 \in \partial M_1$ such that on U and $\Psi(U)$ the mappings $W^1_{z_0}(q_1) = (E^1_{z_0}(q_1), Z^1(q_1))$ and $W^2_{\phi(z_0)}(q_2) = (E^2_{\phi(z_0)}(q_2), Z^2(q_2))$, respectively,* *defined as in* (19) *and* (20)*, are smooth local boundary coordinate maps. Moreover, with respect to these coordinates, the local representation of* Ψ *is*

$$W^1_{z_0}(U) \ni (s, z) \mapsto (s, \phi(z)) \in W^2_{\phi(z_0)}(\Psi(U)). \tag{22}$$

Proof By Lemma 2.4, we have for any $q \in (M_1 \setminus \sigma_{\partial M_1})$ that the point $z \in \partial M_1$ is the closest boundary to q if and only if $\phi(z) \in \partial M_2$ is the closest boundary point to $\Psi(q) \in (M_2 \setminus \sigma_{\partial M_2})$. Thus,

$$\phi(Z^1(q)) = Z^2(\Psi(q)).$$

Therefore, using (10), we have that for all $q \in (M_1 \setminus \sigma_{\partial M_1}),\ z \in \partial M_1$

$$\begin{aligned} f^1_q(z) &:= d_1(z, Z^1(q)) - D_q(z, Z^1(q)) \\ &= d_2(\phi(z), Z^2(\Psi(q)) - D_{\Psi(q)}(\phi(z), Z^2(\Psi(q)) =: f^2_{\Psi(q)}(\phi(z)). \end{aligned} \tag{23}$$

We choose $w \in \partial M_1$ neighborhoods U and V for p and w respectively as in Lemma 2.5 for (M_1, g_1). Then function $(x, z) \mapsto d_1(x, z)$ is smooth in $(U \cap \partial M_1) \times (V \cap \partial M_1)$. Choose $(x, y) \in (U \cap \partial M_1) \times (V \cap \partial M_1)$. Let γ be the unique distance minimizing geodesic from x to y that is transversal to ∂M_1 at x and y and satifies $\gamma((0, d_1(p, w))) \subset M_1^{\text{int}}$. If c is any distance minimizing curve of M_2 from $\phi(x)$ to $\phi(y)$ it holds by (10) that $c((0, d_2(\phi(x), \phi(y)))) \subset M_2^{\text{int}}$. Therefore, c is a geodesic of g_2. Since $d_2(\phi(x), \cdot)|_{\partial M_2}$ is smooth at $\phi(y)$, c is the unique distance minimizing geodesic of (M_2, g_2) connecting $\phi(x)$ to $\phi(y)$. By (10) and (21) it follows that

$$\begin{aligned} D\phi\left(\operatorname{grad}'_1 d_1(\cdot, y)\Big|_x\right) &= \operatorname{grad}'_2 d_2(\cdot, \phi(y))\Big|_{\phi(x)} \quad \text{and} \\ D\phi\left(\operatorname{grad}'_1 d_1(\cdot, x)\Big|_y\right) &= \operatorname{grad}'_2 d_2(\cdot, \phi(x))\Big|_{\phi(y)}. \end{aligned} \tag{24}$$

Here $\operatorname{grad}'_i, i \in \{1, 2\}$ stands for the boundary gradient. Therefore, curve c is transversal to ∂M_2 at $\phi(x)$ and $\phi(y)$.

Due to (24) for any $(x, y) \in \phi(U) \times \phi(V)$ the initial direction of the distance minimizing geodesic $\gamma_{x,y}$, from x to y, depends smoothly on (x, y). Therefore, there exists $t_0 > 0$ such that the set

$$W := \{(\gamma_{x,y}(t), \dot{\gamma}_{x,y}(t) : t \in [0, t_0),\ (x, y) \in \phi(U) \times \phi(V)\} \subset SM_2$$

is a neighborhood of $(\phi(p), \dot{\gamma}_{\phi(p),\phi(w)}(0))$.

By the Implicit function theorem the exit time function ℓ_2 of (M_2, g_2) is smooth in a neighborhood $W_v \subset W$ for any $v \in W$. Since $v \in W$ is given by the unique $(t, x, y) \in [0, t_0) \times \phi(U) \times \phi(V)$ and $\gamma_{x,y} : [t, \ell_2(v)] \to M_2$ is the unique distance minimizing geodesic from $\pi(v)$ it holds that

$$\ell_2(v) = d_2(\pi(v), y). \tag{25}$$

Finally, we note that the function $(t, x, y) \to \pi(v(t, x, y))$ is smooth, and then (25) implies the smoothness of the distance function d_2 in $\pi(W) \times \phi(V) \subset M_2 \times \partial M_2$.

Therefore, we have proven that there exists $r_{\min} > 0$ smaller than the minimum of the boundary cut distances of g_1 and g_2, such that functions

$$(q, z) \mapsto d_1(q, Z^1(q)),\ d_1(q, z),\ d_1(z, Z^1(q)), \quad (q, z) \in B_1(p, r_{\min}) \times (B_1(w, r_{\min}) \cap \partial M_1)$$

and

$$(q', z') \mapsto d_2(q', Z^2(q')),\ d_2(q', z'),\ d_2(z', Z^2(q)), \quad (q', z') \in B_2(\phi(p), r_{\min}) \times (B_2(\phi(w), r_{\min}) \cap \partial M_2)$$

are smooth. Since Ψ is a homeomorphism the existence of set U and $z_0 \in \partial M_1$ as in the claim of this Lemma follow.

If $q \in U$, we obtain by (23) the following equation

$$E^1_{z_0}(q) = E^2_{\phi(z_0)}(\Psi(q)).$$

Therefore, we have proven that the map given in (22) and the mapping

$$W^2_{\phi(z_0)} \circ \Psi \circ (W^1_{z_0})^{-1} : W^1_{z_0}(U) \to W^2_{\phi(z_0)}(\Psi(U))$$

coincide. □

Next we consider the coordinates away from ∂M. Let $p \in M^{\text{int}}$ and choose any closest boundary point $z_p \in \partial M$ to p. By [9, Lemma 2.15] there exist neighborhoods $U \subset M^{\text{int}}$ of p and $W \subset \partial M$ of z_p such that the distance function $d_g : U \times W \to \mathbb{R}$ is smooth. Moreover for every $(q, w) \in U \times W$ the distance $d_g(q, w)$ is realized by the unique distance minimizing geodesic, contained in M^{int} if the end point w is excluded. We use a shorthand notation $v \in S_p M$ for the velocity $\dot{\gamma}_{z_p,\nu}(d_g(p, z_p))$. A similar argument as in [12, Lemma 2.6] yields to an existence of a neighborhood $V \subset W$ of z_p such that the set

$$\mathcal{V} = \{(z_i)_{i=1}^n \in V^n : \dim \operatorname{span}((F(z_i) - v)_{i=1}^n) = n\}$$

is open and dense in $V^n := V \times V \times \cdots \times V$. Here $F(q) := -\frac{(\exp_p)^{-1}(q)}{\|(\exp_p)^{-1}(q)\|_g}$, $q \in V$. Notice that this claim follows from [12, Lemma 2.6.] since $F(q) = -\frac{(\exp_p)^{-1}(q')}{\|(\exp_p)^{-1}(q')\|_g}$ for some $q' \in M$ if and only if there exists $0 < t < \tau_G(p, -F(q))$ such that $q' = \gamma_{p,-F(p)}(t)$. Here τ_G is the cut distance function of (N, G), where closed Riemannian manifold (N, G) is some extension of (M, g).

Moreover for every $(z_i)_{i=1}^n \in \mathcal{V}$ there exists an open neighborhood $U' \subset U$ of p such that

$$H : U' \to \mathbb{R}^n, \quad H(q) := (d_g(q, z_i) - d_g(q, z_p))_{i=1}^n$$

is a smooth coordinate mapping. This holds, since for any $(z_i)_{i=1}^n \in \mathcal{V}$ the Jacobian of H at p is invertible.

Lemma 2.8 *Let (M_i, g_i), $i = 1, 2$ be compact, connected n-dimensional Riemannian manifolds with smooth boundaries ∂M_i. Suppose that the travel time difference data of (M_1, g_1) and (M_2, g_2) coincide. Let $p \in M_1^{\text{int}}$. Let z_p be any closest boundary point to p. There exists a neighborhood U of p in M_1^{int} and a neighborhood $W \subset \partial M_1$ of z_p such that the distance functions $d_1 : U \times W$ of (M_1, g_1) and $d_2 : \Psi(U) \times \phi(W)$ of (M_2, g_2) are smooth.*

Moreover there exists points $z_1, \dots, z_n \in W$ and a neighborhood $V \subset U$ of p such that

$$H_1 : V \to \mathbb{R}^n, \quad H_1(x) = (d_1(x, z_i) - d_1(x, z_p))_{i=1}^n$$

and

$$H_2 : \Psi(V) \to \mathbb{R}^n, \quad H_2(q) = (d_2(q, \phi(z_i)) - d_2(q, \phi(z_p)))_{i=1}^n,$$

are smooth coordinate maps. We also have

$$H_1(V) = H_2(\Psi(V)) \text{ and } H_2 \circ \Psi \circ H_1^{-1} = Id_{\mathbb{R}^n}. \tag{26}$$

Proof Since Ψ is a homeomorphism, the first part of the claim follows from similar construction as done before this Lemma. The proof of the latter part is a modification of the proof of [12, Theorem 2.7]. □

Proposition 2.9 *Let (M_i, g_i), $i = 1, 2$ be compact, connected n-dimensional Riemannian manifolds with smooth boundaries ∂M_i whose travel time difference data coincide. If (M_1, g_1) satisfy the visibility condition 1.1, then mapping $\Psi : M_1 \to M_2$, given in (13), is a diffeomorphism.*

Proof The claim follows from Proposition 2.3 and Lemmas 2.7–2.8. □

2.4 Riemannian Structure

As we have proven that the map Ψ is diffeomorphism, we can define a pull back metric $\widetilde{g} := \Psi^* g_2$ on M_1. From now on, we only consider manifold $M := M_1$ with smooth boundary equipped with Riemannian metrics $g := g_1$ and $\widetilde{g}$. We need to show that $g = \widetilde{g}$. First, we notice that by the definitions of the diffeomorphism Ψ and metric $\widetilde{g}$ on M, we have by the data (8) that

$$D_p(z, w) = d_g(p, z) - d_g(p, w) = d_{\widetilde{g}}(p, z) - d_{\widetilde{g}}(p, w), \quad p \in M, \ z, w \in \partial M. \tag{27}$$

Lemma 2.10 *Let $p \in \partial M$ and $(x^1, \dots, x^n)$ be a boundary normal coordinate system of g near p and $\alpha \in \mathbb{N}^n$ any multi-index. Write $g = (g_{ij})_{i,j=1}^n$ and $\widetilde{g} = (\widetilde{g}_{ij})_{i,j=1}^n$. Then for all $i, j \in \{1, \dots, n\}$ it holds that*

$$\partial^\alpha g_{ij}|_{\partial M} = \partial^\alpha \widetilde{g}_{ij}|_{\partial M}, \quad \partial^\alpha := \prod_{k=1}^n \left(\frac{\partial}{\partial x^k}\right)^{\alpha_k}. \tag{28}$$

Proof We prove that the local lens relations (ℓ_g, σ_g) and $(\ell_{\widetilde{g}}, \sigma_{\widetilde{g}})$ of g and $\widetilde{g}$, respectively, coincide at some open set $\mathcal{D} \subset T\partial M$. After this the claim follows from the proof of [18, Theorem 1]. For the definitions of local lens relations, see [18].

Choose $q \in \partial M$ and neighborhoods $U, V \subset M$ of p and q be as in Lemma 2.5 for metric g. Let γ be the unique geodesic of g connecting p to q. Due to (10) and Lemma 2.5 it holds that $d_{\widetilde{g}}$ is smooth on $(U \cap \partial M) \times (V \cap \partial M)$. Therefore, (21) implies that for every $(x, y) \in (U \cap \partial M) \times (V \cap \partial M)$, we have that

$$\operatorname{grad}'_g d_g(\cdot, y)\Big|_x = \operatorname{grad}'_{\widetilde{g}} d_{\widetilde{g}}(\cdot, y)\Big|_x \quad \text{and} \quad \operatorname{grad}'_g d_g(\cdot, x)\Big|_y = \operatorname{grad}'_{\widetilde{g}} d_{\widetilde{g}}(\cdot, x)\Big|_y. \tag{29}$$

Denote $\dot{\gamma}(0) =: \eta$ and $\dot{\gamma}(d_g(p, q)) =: v$. Then (29) imply that $\dot{\widetilde{\gamma}}(0) = \eta$ and $\dot{\widetilde{\gamma}}(d_g(p, q)) = v$, where $\widetilde{\gamma}$ is the unique distance minimizing geodesic of $\widetilde{g}$ from p to q. By Lemma 2.5 it holds that η and v are transversal to ∂M.

Therefore, after possibly shrinking U and V, we have by [18, formula (10)] and formulas (21) and (29) that the local lens relations (ℓ_g, σ_g) and $(\ell_g, \sigma_{\widetilde{g}})$ coincide in the set

$$\mathcal{D} := \{\operatorname{grad}'_g d_g(\cdot, y)\Big|_x, \ \operatorname{grad}'_g d_g(\cdot, x)\Big|_y \in T\partial M : \ (x, y) \in (U \cap \partial M) \times (V \cap \partial M)\}.$$

The set $\mathcal{D}$ is open since it is an image of an open map, given by the composition of the diffeomorphism

$$W_\eta \ni (x, v) \mapsto \gamma_{x,v}(\ell(x, v)), \dot{\gamma}_{x,v}(\ell(x, v)) \in W_v$$

and the orthogonal projection from ∂SM to $T\partial M$. In the above $W_\eta \subset \partial SM$ is some open neighborhood of (p, η) and $W_v \subset \partial SM$ is some open neighborhood of (q, v). □

Let (N, G) be a smooth closed Riemannian manifold that is a smooth extension of (M, g). We write $F := N \setminus M^{\text{int}}$, as before. By Lemma 2.10 The Riemannian manifold $(N, \widetilde{G})$ is a smooth extension of $(M, \widetilde{g})$ if $\widetilde{G}$ is a Riemannian metric defined as

$$\widetilde{G}|_F = G|_F, \quad \widetilde{G}|_{M^{\text{int}}} = \widetilde{g}. \tag{30}$$

Lemma 2.11 *Let N, F, G and $\widetilde{G}$ be as above. Then*

$$d_G(p,z) - d_G(p,w) = d_{\widetilde{G}}(p,z) - d_{\widetilde{G}}(p,w) \quad p \in N,\ z, w \in F. \tag{31}$$

The functions d_G, $d_{\widetilde{G}}$ are the geodesic distances of G and $\widetilde{G}$, respectively.

Proof This proof is an adaptation of the proof of [8, Proposition 7.3]. If $p \in M$, we first give a proof for

$$d_G(p,z) - d_G(p,w) = d_{\widetilde{G}}(p,z) - d_{\widetilde{G}}(p,w), \quad z, w \in F. \tag{32}$$

If (32) holds for every $p \in M$ then (32) holds also for the case $p \in F$. The latter proof is given in [12, Proposition 1.2]. Therefore, Eq. (31) holds.

Let $p \in M$. Consider first the function $h_p(z) := d_g(p,z) - d_{\widetilde{g}}(p,z),\ z \in \partial M$. Let $w \in \partial M$. By (27) it holds that

$$h_p(z) = d_{\widetilde{g}}(p,w) - d_g(p,w).$$

Thus, h_p is a constant function.

We will prove that

$$d_G(p,z) = \inf\left\{ d_g(p,y_0) + \left(\sum_{j=1}^{N} d_F(y_{j-1},x_j) + d_g(x_j,y_j)\right) + d_F(x_N,z)\right\}, \tag{33}$$

where d_F is the distance function of the Riemannian manifold $(F, G|_F)$ and $\{y_0,\ldots,y_N,x_1,\ldots,x_N\} \subset \partial M$. We note that similar formula holds for $d_{\widetilde{G}}$, when d_g is replaced with $d_{\widetilde{g}}$. If (33) holds then, it follows from Eq. (10) that

$$d_G(p,z) - d_{\widetilde{G}}(p,z) = \text{constant with respect to } z.$$

This implies (32), in the case when $p \in M$.

Finally, we prove (33). Let $\epsilon > 0$. Since ∂M is a smooth co-dimension 1 submanifold of N, it follows from the definition of the Riemannian distance function d_G, that there exists a piecewise smooth curve c from p to q, that crosses the boundary finitely many times, and whose length is ϵ-close to $d_G(p,z)$. Then

$$d_g(p,y_0) + \left(\sum_{j=1}^{N} d_F(y_{j-1},x_j) + d_g(x_j,y_j)\right) + d_F(x_N,z) \le \mathcal{L}_G(c) \le d_G(p,z) + \epsilon,$$

where $\{y_0,\ldots,y_N,x_1,\ldots,x_N\} \subset \partial M$ are the points where c hits the boundary. Taking ϵ to 0 implies (33). □

In view of the previous lemma, it follows from [12, Sect. 2.4] that metric tensors g and $\widetilde{g}$ coincide. We will sketch here the main ideas for this proof.

First, we prove that the geodesics of metrics G and $\widetilde{G}$ agree up to reparametrization. Let $\tau_G : SN \to \mathbb{R}$ be the cut distance function of metric tensor G (see (4)). By [12, Lemma 2.9] the following equality holds for any $(z, v) \in SF^{\text{int}}$

$$\begin{aligned}&\gamma^G_{z,-v}((0, \tau_G(z, -v))\\&\quad = \{p \in N : D_p(\cdot, z) \text{ is smooth at } z \text{ and } \operatorname{grad}_G D_p(\cdot, z) \text{ at } z \text{ is } v\}\\&\quad =: \delta(z, v).\end{aligned} \tag{34}$$

Where $\gamma^G_{z,-v}$ is the geodesic of G with initial conditions $(z, -v)$. Since $G = \widetilde{G}$ on F^{int}, the formulas (31) and (34) imply

$$\gamma^G_{z,-v}((0, \tau_G(z, -v)) = \gamma^{\widetilde{G}}_{z,-v}((0, \tau_{\widetilde{G}}(z, -v)), \quad (z, v) \in SF^{\text{int}}, \tag{35}$$

where $\tau_{\widetilde{G}}$ is the cut distance function of $\widetilde{G}$. Therefore, for any $(z, v) \in SF^{\text{int}}$ there exists a diffeomorphism $\alpha_{z,v} : (0, \tau_G(z, -v)) \to (0, \tau_{\widetilde{G}}(z, -v))$ such that

$$\gamma^G_{z,-v}(t) = \gamma^{\widetilde{G}}_{z,-v}(\alpha_{z,v}(t)), \quad t \in (0, \tau_G(z, -v)),\ (z, v) \in SF^{\text{int}}. \tag{36}$$

Let $p \in M^{\text{int}}$. We denote the exponential map of G at p by $\exp_p$. Then the set,

$$\Omega_p := \{rv \in T_pN : r > 0,\ v = \exp_p^{-1}(z),\ p \in \delta(z, v),\ (z, v) \in SF^{\text{int}}\}^{int},$$

is not empty and, moreover—if we denote the exponential map of $\widetilde{G}$ at p by $\widetilde{\exp}_p$—in view of (36), we have

$$\Omega_p = \{rv \in T_pN : r > 0,\ v = \widetilde{\exp}_p^{-1}(z),\ p \in \delta(z, v),\ (z, v) \in SF^{\text{int}}\}^{int}. \tag{37}$$

Let (U, x) be a local coordindate chart of M^{int}. We denote the Christoffel symbols of G and $\widetilde{G}$ as Γ and $\widetilde{\Gamma}$, respectively. By (36), (37) and [12, Proposition 2.13] there exists a smooth 1-form β on U such that

$$\Gamma^k_{ij}(x) - \widetilde{\Gamma}^k_{ij}(x) = \delta^k_i \beta_j(x) + \delta^k_j \beta_i(x),$$

where δ^k_j is the Kronecker delta. This and [12, Lemma 2.14] imply that the geodesics of metric tensors G and $\widetilde{G}$ agree up to reparametrization. See also [14] for an earlier result. We arrive at

Lemma 2.12 *Suppose that N, F, G and $\widetilde{G}$ are as above. Then $G = \widetilde{G}$ in all of N.*

Proof Since geodesics of metric tensors G and $\widetilde{G}$ agree up to reparametrization the main result of [19] shows that the function

$$I_0((x, v)) = \left(\frac{\det(G(x))}{\det(\widetilde{G}(x))}\right)^{\frac{2}{n+1}} \widetilde{G}(x, v), \quad (x, v) \in TN, \tag{38}$$

where $\widetilde{G}(x, v) = \widetilde{G}_{jk}(x)v^j v^k$, is constant on the geodesic flow of G. Note that the function $F(x) := \frac{\det(G(x))}{\det(\widetilde{G}(x))}$ is coordinate invariant.

Let $\varphi_t : SN \to SN$, $t \in \mathbb{R}$ be the geodesic flow of G and $\pi : TN \to N$ the projection onto the base point. Since $G = \widetilde{G}$ on F^{int}, we have

$$G(\varphi_0(z, v)) = \|v\|_G^2 = I_0(\varphi_0(z, v)), \quad (z, v) \in TF^{\text{int}}.$$

Therefore, for any $t \in \mathbb{R}$ and for any $(z, v) \in TF^{\text{int}} \setminus \{0\}$ the following holds

$$G(\varphi_t(z, v)) = \|v\|_G^2 = I_0(\varphi_t(z, v)) = F(\pi(\varphi_t(z, v))\widetilde{G}(\varphi_t(z, v)).$$

This implies the claim. For more details, see [12, Lemma 2.15]. □

We conclude that the proof of Theorem 1.3 follows from Propositions 2.3, 2.9 and Lemma 2.12.

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