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MIXED RAY TRANSFORM ON SIMPLE 2-DIMENSIONAL RIEMANNIAN MANIFOLDS

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ABSTRACT. We characterize the kernel of the mixed ray transform on simple 2-dimensional Riemannian manifolds, that is, on simple surfaces for tensors of any order.

1. Introduction

We provide a characterization of the kernel of the mixed ray transform on simple 2-dimensional Riemannian manifolds for tensors of any order. The key application pertains to elastic qS-wave tomography [3] in weakly anisotropic media.

We let (M,g) be a smooth, compact, connected 2-dimensional Riemannian manifold with smooth boundary ∂M. We assume that (M,g) is simple; that is, ∂M is strictly convex with respect to g and $\exp_p : \exp_p^{-1}(M) \to M$ is a diffeomorphism for every $p \in M$. We let $SM = \{(x,v) \in TM; \|v\|_g = 1\}$ be the unit sphere bundle. We use the notation ν for the outer unit normal vector field to ∂M. We write $\partial_{in}(SM) = \{(x,v) \in SM; x \in \partial M, \langle v, \nu\rangle_g \leq 0\}$ for the vector bundle of inward pointing unit vectors on ∂M. For $(x,v) \in SM$, $\gamma_{x,v}(t)$ is the geodesic starting from x in direction v, and $\tau(x,v)$ is the time when $\gamma_{x,v}$ exits M. Since (M,g) is simple $\tau(x,v) < \infty$ for all $(x,v) \in \partial_{in}(SM)$, and the *exit time function* τ is smooth in $\partial_{in}(SM)$ [15, Section 4.1].

We use the notation S^kM, $k \in \mathbf{N}$, for the space of smooth symmetric tensor fields on M. We also use the notation $S^kM \times S^\ell M$, $k, \ell \geq 1$, for the space of smooth tensor fields that are symmetric with respect to first k and last ℓ variables. *The mixed ray transform* $L_{k,\ell}$ of a tensor field $f \in S^kM \times S^\ell M$ is given by the formula

$$L_{k,\ell} f(x,v) = \int_0^{\tau(x,v)} f_{i_1,\dots,i_k j_1,\dots,j_\ell}(\gamma(t))\dot\gamma(t)^{i_1}\cdots\dot\gamma(t)^{i_k}\eta(t)^{j_1}\cdots\eta(t)^{j_\ell}\mathrm{d}t, \quad (x,v) \in \partial_{in}(SM), \quad \gamma = \gamma_{x,v}, \tag{1.1}$$

where we used the summation convention, while $\eta(t)$ is some unit length vector field on γ that is parallel and perpendicular to $\dot\gamma(t)$ and depends smoothly on

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$(x,v) \in \partial_{in}(SM)$. We note that the definition of the mixed ray transform is different in higher dimensions, due to the freedom in the choice of η (see [15, Section 7.2]). We consider the choice of $\eta(t)$ and the mapping properties of $L_{k,\ell}$ in dimension 2.

We define two linear operators, the images of which are contained in the kernel of $L_{k,\ell}$. For a $(k\times\ell)$-tensor, $f_{i_1,\dots,i_k j_1,\dots,j_\ell}$, we introduce the symmetrization operator as

$$(\mathrm{Sym}(i_1,\dots,i_k)f)_{i_1,\dots,i_k j_1,\dots,j_\ell} := \frac{1}{k!}\sum_\sigma f_{i_{\sigma(1)},\dots,i_{\sigma(k)} j_1,\dots,j_\ell}, \tag{1.2}$$

where σ runs over all permutations of $(1,2,\dots,k)$. This operator symmetrizes f with respect to the first k indices. We define the symmetrization operator $\mathrm{Sym}(j_1,\dots,j_\ell)$ for the last ℓ indices analogously.

We introduce a *first* operator λ, the image of which is contained in the kernel of $L_{k,\ell}$. The operator $\lambda : S^{k-1}M\times S^{\ell-1}M \to S^kM\times S^\ell M$ is defined by

$$(\lambda w)_{i_1,\dots,i_k j_1,\dots,j_\ell} := \mathrm{Sym}(i_1,\dots,i_k)\,\mathrm{Sym}(j_1,\dots,j_\ell)(g_{i_1 j_1} w_{i_2,\dots,i_k j_2,\dots,j_\ell}). \tag{1.3}$$

Using (1.2) and (1.3) it is straightforward to verify that

$$(\lambda w)_{i_1,\dots,i_k j_1,\dots,j_\ell} v^{i_1}\cdots v^{i_k}(v^\perp)^{j_1}\cdots(v^\perp)^{j_\ell} = 0, \quad v\in TM, \tag{1.4}$$

where $v^\perp$ is any vector orthogonal to v. Therefore (1.4) implies that

$$\mathrm{Im}(\lambda)\subset \ker(L_{k,\ell}).$$

We use the notation $u_{i_1,\dots,i_k;h}$ for the (h) component functions of the covariant derivative ∇u of the tensor field u. We define the *second* operator, d' say, by the formula

$$\begin{gathered} d' : S^{k-1}M\times S^\ell M \to S^kM\times S^\ell M, \\ (d'u)_{i_1,\dots,i_k j_1,\dots,j_\ell} := \mathrm{Sym}(i_1,\dots,i_k) u_{i_2,\dots,i_k j_1,\dots,j_\ell; i_1}. \end{gathered} \tag{1.5}$$

Then the following holds for any $u\in S^{k-1}M\times S^\ell M$:

$$\begin{aligned} &\frac{d}{dt}\left(u_{i_1,\dots,i_{k-1} j_1,\dots,j_\ell}(\gamma(t))\dot\gamma(t)^{i_1}\cdots\dot\gamma(t)^{i_{k-1}}\eta(t)^{j_1}\cdots\eta(t)^{j_\ell}\right) \\ &\qquad = (d'u)_{i_1,\dots,i_k j_1,\dots,j_\ell}\dot\gamma(t)^{i_1}\cdots\dot\gamma(t)^{i_k}\eta(t)^{j_1}\cdots\eta(t)^{j_\ell}. \end{aligned}$$

If $u|_{\partial M}=0$ (in the sense that all component functions of u vanish at ∂M), then $L_{k,\ell}(d'u)=0$ by the fundamental theorem of calculus. Thus

$$\{d'u :\ u\in S^{k-1}M\times S^\ell M,\ u|_{\partial M}=0\}\subset \ker(L_{k,\ell}).$$

Our main result shows that the kernel of $L_{k,\ell}$ is spanned by the images of these two linear operators.

Theorem 1.1. *Let (M,g) be a simple 2-dimensional Riemannian manifold. Let $f\in S^kM\times S^\ell M$, $k,\ell\geq 1$. Then*

$$L_{k,\ell}f(x,v)=0, \qquad (x,v)\in\partial_{in}(SM),$$

if and only if

$$f = d'u+\lambda w, \quad u\in S^{k-1}M\times S^\ell M,\ u|_{\partial M}=0, \quad w\in S^{k-1}M\times S^{\ell-1}M.$$

The key observation needed to prove this theorem is that the mixed ray transform and the geodesic ray transform can be transformed to one another, for arbitrary $k, \ell \geq 1$, if (M, g) is a 2-dimensional simple Riemannian manifold. A similar observation has already been obtained for the transverse ray transform by Sharafutdinov [15, Chapter 5]. The work by Paternain, Salo, and Uhlmann [10] proved the s-injectivity of the geodesic ray transform on simple manifolds in dimension 2. In Theorem 1.1, we characterize the kernel of $L_{k,\ell}$ using their results.

2. Relation with elastic qS-wave tomography

We describe a mixed ray transform arising from elastic wave tomography. We follow the presentation in [15, Chapter 7], wherein one can find more details. Let (x^1, x^2) be any curvilinear coordinate system in $\mathbb{R}^2$, where the Euclidean metric is

$$ds^2 = g_{jk} dx^j dx^k.$$

The elastic wave equations

$$\rho \frac{\partial^2 u_j}{\partial t^2} = \sigma_{jk;}{}^{k} := \sigma_{jk;l} g^{kl} \tag{2.1}$$

describe the waves traveling in a 2-dimensional elastic body $M \subset \mathbf{R}^2$. Here $u(x,t) = (u^1, u^2)$ is the displacement vector. The strain tensor is given by

$$\varepsilon_{jk} = \frac{1}{2}(u_{j;k} + u_{k;j}),$$

while the stress tensor is

$$\sigma_{jk} = C_{jklm} \varepsilon^{lm},$$

where $\mathbf{C}(x) = (C_{jklm})$ is the elastic tensor and $\rho(x)$ is the density of mass. Here ε^{lm} is obtained by raising indices with respect to the metric g_{jk}. The elastic tensor has the following symmetry properties:

$$C_{jklm} = C_{kjlm} = C_{lmjk}. \tag{2.2}$$

We assume that the elastic tensor is weakly anisotropic; that is, it can be represented as

$$C_{jklm} = \lambda g_{jk} g_{lm} + \mu(g_{jl} g_{km} + g_{jm} g_{kl}) + \delta c_{jklm},$$

where λ and μ are positive functions called the Lamé parameters and $\mathbf{c} = (c_{jklm})$ is an anisotropic perturbation. Here, δ is a small positive real number. We note here that $\delta = 0$ corresponds to an isotropic medium.

We construct geometric optics solutions to system (2.1) using the parameter $\omega = \omega_0/\delta$, where ω_0 is a constant,

$$u_j = e^{i\omega\iota} \sum_{m=0}^{\infty} \frac{u_j^{(m)}}{(i\omega)^m}, \quad \varepsilon_{jk} = e^{i\omega\iota} \sum_{m=-1}^{\infty} \frac{\varepsilon_{jk}^{(m)}}{(i\omega)^m}, \quad \sigma_{jk} = e^{i\omega\iota} \sum_{m=-1}^{\infty} \frac{\sigma_{jk}^{(m)}}{(i\omega)^m},$$

and $\iota(x)$ is a real function.

We substitute the above solutions into equation (2.1), assume $u^{(-1)} = \varepsilon^{(-2)} = \sigma^{(-2)} = 0$, and equate the terms of the order -2 and -1, respectively, in ω to obtain

$$(\lambda + \mu)\langle u^{(0)}, \nabla\iota\rangle_g \nabla\iota + (\mu\|\nabla\iota\|_g^2 - \rho) u^{(0)} = 0.$$

If we take

$$\|\nabla\iota\|_g^2 = \frac{\rho}{\mu}, \tag{2.3}$$

then

$$\langle u^{(0)}, \nabla \iota \rangle_g = 0.$$

The solutions $u_j^{(0)}$ represent shear waves (S-waves), and the displacement vector $u^{(0)}$ is orthogonal to $\nabla\iota$. We denote $n_s = \rho/\mu$ and $v_s = 1/n_s$. The characteristics of the eikonal equation (2.3) are geodesics of the Riemannian metric $n_s^2 ds^2 = n_s^2 g_{jk} dx^j dx^k$.

We choose a geodesic γ of metric $n_s^2 ds^2$ and apply the change of variables,

$$u_j^{(0)} = A_s n_s^{-1} \zeta_j,$$

where

$$A_s = \frac{C}{\sqrt{J\rho v_s}}, \quad J^2 = n_s^2 \det(g_{jk}), \quad C \text{ is a constant.}$$

Then it is shown in [15, Section 7.1.5] that ζ satisfies *Rytov's law*,

$$\left(\frac{D\zeta}{d\tau}\right)_j = -i\frac{1}{\rho v_s^6}(\delta_j^q - \dot\gamma_j\dot\gamma^q)\omega_0 c_{qklm}\dot\gamma^k\dot\gamma^m\zeta^l, \tag{2.4}$$

where $\frac{D}{d\tau}$ is the covariant derivative along γ. We note that $c_{qklm}\dot\gamma^k\dot\gamma^m$ is quadratic in $\dot\gamma$ and symmetric in k, m, so the solution ζ of (2.4) depends only on the symmetrization

$$f_{jklm} = -i\frac{1}{4\rho v_s^6}(c_{jlkm} + c_{jmkl}).$$

We assume that for every unit speed geodesic $\gamma : [a, b] \to M$ (in Riemannian manifold $(M, n_s^2 ds^2)$) with endpoints in ∂M, the value $\zeta(b)$ of a solution to equation (2.4) is known as $\zeta(b) = U(\gamma)\zeta(a)$, where $U(\gamma)$ is the solution operator of (2.4) and $\zeta(a)$ is the initial value. We formulate an inverse problem.

Inverse Problem 2.1. Determine tensor field f from $U(\gamma)$.

We linearize this problem as in [15, Chapter 5]. Take a unit vector field $\xi(t) \perp \dot\gamma(t)$ (with respect to metric $n_s^2 ds^2$), which is also parallel along γ. Then $e_1(t) = \xi(t)$ and $e_2(t) = \dot\gamma(t)$ form an orthonormal frame along γ. In this basis, equation (2.4) is

$$\dot\zeta_1 = -i\frac{1}{\rho v_s^6}\omega_0 c_{1l1m}\dot\gamma^l\dot\gamma^m\zeta^1, \quad \dot\zeta_2 = 0. \tag{2.5}$$

We denote $F(t) = -i\frac{1}{\rho v_s^6}\omega_0 c_{1l1m}(\gamma(t))\dot\gamma^l(t)\dot\gamma^m(t)$. Since (2.5) is a separable first-order ordinary differential equation, its solution is

$$\zeta_1(b) = e^{\int_a^b F(t)dt}\zeta_1(a).$$

We take the first-order Taylor expansion of the right-hand side of the equation above to obtain

$$\zeta_1(b) - \zeta_1(a) \sim \int_a^b F(t)\zeta^1(a)dt.$$

Multiplying this equation by $\zeta^1(a)$, we get

$$\begin{aligned}(\zeta_1(b) - \zeta_1(a))\zeta^1(a) &\sim \int_a^b F(t)\zeta^1(a)\zeta^1(a)dt \\ &= \int_a^b \omega_0 f_{11lm}(\gamma(t))\zeta^1(a)\zeta^1(a)\dot\gamma^l(t)\dot\gamma^m(t)dt.\end{aligned} \tag{2.6}$$

We denote the vector field $\eta(t) = \zeta^i(a)e_i(t)$, $\zeta^2(a) = 0$, and observe that it is parallel along γ and perpendicular to $\dot\gamma(t)$. We emphasize that $\eta(t)$ does not need to solve (2.5). The right-hand side of (2.6) then takes the form

$$\int_a^b \omega_0 f_{11lm}(\gamma(t))\eta^1(t)\eta^1(t)\dot\gamma^l(t)\dot\gamma^m(t)dt.$$

We arrive at the inverse problem.

Inverse Problem 2.2. Determine the tensor field f from

$$L_{2,2}(f) = \int_a^b f_{jklm}(\gamma(t))\eta^j(t)\eta^k(t)\dot\gamma^l(t)\dot\gamma^m(t)dt$$

for all γ and $\eta \perp \gamma$, where η is parallel along γ.

Remark 2.3. The tensor field f possesses the same symmetry properties (2.2) as $\mathbf{C}$. Therefore $f \in S^2M \times S^2M$. Since

$$L_{2,2}(f + d'u + \lambda w) = L_{2,2}(f) \quad \text{for any } u \in S^1M \times S^2M, \quad w \in S^1M \times S^1M,$$

we can only recover the tensor f up to the kernel of $L_{2,2}$. Thus Inverse Problem 2.2 is a special case of Theorem 1.1.

3. Context and previous work

We note that if $\ell = 0$ in (1.1), the operator $L_{k,0}$ is the geodesic ray transform I_k for a symmetric k-tensor f. It is well known that $\mathrm{Sym}(i_1, \dots, i_k)\nabla u$ is in the kernel of I_k, where u is a symmetric $(k-1)$-tensor with $u|_{\partial\Omega} = 0$. If $I_k f = 0$ implies $f = \mathrm{Sym}(i_1, \dots, i_k)\nabla u$, we say that I_k is s-injective.

When (M, g) is a 2-dimensional simple manifold, Paternain, Salo, and Uhlmann [10] proved the s-injectivity of I_k for arbitrary k. The standard way to prove s-injectivity of I_0 and I_1 is to use an energy identity known as the Pestov identity. If $k \geq 2$ this identity alone is not sufficient to prove the s-injectivity. The special case $k = 2$ was proved earlier [16] using the proof for boundary rigidity [14].

In dimension 3 or higher, it was proved that I_0 is injective [7,8] and that I_1 is s-injective [2]. The s-injectivity of I_k for $k \geq 2$ is still open for simple Riemannian manifolds. Under certain curvature conditions, the s-injectivity of I_k, $k \geq 2$, was proved in [4,12,13,15]. Without any curvature condition, I_2 has a finite-dimensional kernel [18]. If g is in a certain open and dense subset of simple metrics in C^r, $r \gg 1$, containing analytic metrics, the s-injectivity was obtained by analytic microlocal analysis for $k = 2$ [17]. Under a different assumption, namely, that M can be foliated by strictly convex hypersurfaces, the s-injectivity was established for $k = 0$ [21] and $k = 1, 2$ [19].

The mixed ray transform ($\ell \neq 0$, $k \neq 0$) has not been studied as extensively as the geodesic ray transform. In dimension 2 or higher, a result similar to Theorem 1.1 was obtained under a restrictive curvature condition [15].

When $k = 0$, $L_{0,\ell}$ is called the transverse ray transform, also denoted by J_ℓ. For J_ℓ, the situations are quite different for dimension 2 and higher dimensions. In dimension 3 or higher, J_ℓ is injective for $\ell < \dim M$ under certain curvature conditions [15]. However, J_ℓ has a nontrivial kernel in dimension 2. This problem is related to *polarization* tomography, for which some results are given under different conditions [5, 9, 11].

In a recent paper [6] the authors studied the s-injectivity of attenuated geodesic ray transforms. In [6, Theorem 5] the authors proved that for a given $k \in \mathbf{N}$ the family $\{L_{k-\ell,\ell} : \ell \in \{0,\dots,k\}\}$ of mixed ray transforms determines a symmetric tensor $f \in S^kM$ uniquely if (M,g) is a simple surface or if M is a unit disk of $\mathbf{R}^2$ and g is a radial metric $c(r)e$ satisfying Herglotz' nontrapping condition

$$\frac{d}{dr}\left(\frac{r}{c(r)}\right) > 0.$$

4. Proof of Theorem 1.1

Since (M,g) is a 2-dimensional simple Riemannian manifold, there exists a diffeomorphism ϕ from M onto a closed unit disc $\overline{\mathbb{D}}$ of $\mathbf{R}^2$. If g' is the pullback of metric g under ϕ^{-1} on $\overline{\mathbb{D}}$, then g' is conformally Euclidean, meaning that there exists a change of coordinates after which $g' = he$, where h is some positive function and e is the Euclidean metric; this was shown in [1, Theorem 4] and [20, Proposition 1.3]. Therefore, there exist global isothermal coordinates (x_1,x_2) on M so that the metric g can be written as $e^{2\alpha(x)}(\mathrm{d}x_1^2+\mathrm{d}x_2^2)$, where $\alpha(x)$ is a smooth real-valued function of x.

The global isothermal coordinate structure makes it possible to define a smooth rotation,

$$\sigma : TM \to TM, \quad \sigma(v) := (v_2, -v_1),$$

where $v = (v_1,v_2)$ in these coordinates. This map satisfies

$$v \perp \sigma(v) \quad \text{and} \quad \|v\|_g = \|\sigma(v)\|_g. \tag{4.1}$$

Moreover, there exists a linear map

$$\begin{aligned} &\Phi : S^kM \times S^\ell M \to C^\infty(SM), \\ &(\Phi f)(x,v) := f_{i_1,\dots,i_k j_1,\dots,j_\ell}(x) v^{i_1}\cdots v^{i_k}\sigma(v)^{j_1}\cdots\sigma(v)^{j_\ell}. \end{aligned} \tag{4.2}$$

Thus each tensor field $f \in S^kM \times S^\ell M$ is related to a smooth function on SM via (4.2). We note that Φ is not one-to-one since $\Phi(\lambda w) = 0$ for any $w \in S^{k-1}M \times S^{\ell-1}M$, where λ is as in (1.3). We have the following:

Lemma 4.1. *For any $f \in S^kM \times S^\ell M$ it holds that*

$$L_{k,\ell}f(x,v) = \int_0^{\tau(x,v)} (\Phi f)(\gamma_{x,v}(t), \dot\gamma_{x,v}(t))\mathrm{d}t, \qquad (x,v) \in \partial_{in}(SM), \tag{4.3}$$

and

$$L_{k,\ell} : S^kM \times S^\ell M \to C^\infty(\partial_{in}SM)$$

if we assume that

$$\eta(0) = \sigma(v), \quad (x,v) \in \partial_{in}(SM).$$

Proof. Let $(x,v) \in \partial_{in}SM$. We define $\eta = \sigma(v)$. Let $P_t(\eta)$ be the parallel transport of η from T_xM to $T_{\gamma_{x,v}(t)}M$, $t \in [0,\tau(x,v)]$. By the property of parallel translation, $P_t : T_xM \to T_{\gamma_{x,v}(t)}M$ is an isometry, whence $\|P_t\eta\|_g = 1$ and $\langle P_t\eta, \dot\gamma(t)\rangle_g = 0$. Since M is 2-dimensional, the continuity of $P_t\eta$ in t with (4.1) implies that

$$P_t\eta = \sigma(\dot\gamma_{x,v}(t)).$$

Because the functions Φf and τ are smooth in $\partial_{in}(SM)$, the function $L_{k,\ell}(f)$ is smooth in $\partial_{in}(SM)$ due to (4.3). $\square$

Let $f \in S^k M \times S^\ell M$. Simplifying the notation, from here on we do not distinguish tensor f from function $\Phi(f)$. We notice first that

$$f(x,v) = (-1)^{\ell - N(j_1,\dots,j_\ell)} f_{i_1,\dots,i_k j_1,\dots,j_\ell}(x) v^{i_1}\cdots v^{i_k} v_1^{\ell - N(j_1,\dots,j_\ell)} v_2^{N(j_1,\dots,j_\ell)}, \quad (x,v)\in SM, \tag{4.4}$$

where $N(j_1,\dots,j_\ell)$ is the number of 1's in $(j_1,\dots,j_\ell)$. We let δ be the map that maps 1's in $(j_1,\dots,j_\ell)$ to 2's and vice versa. We denote by $\delta(j_1,\dots,j_\ell)$ the ℓ-tuple obtained from applying δ to $(j_1,\dots,j_\ell)$. Then we define a linear operator

$$\begin{aligned} &A : S^k M \times S^\ell M \to S^k M \times S^\ell M, \\ &(Af)_{i_1,\dots,i_k j_1,\dots,j_\ell} = (-1)^{\ell - N(j_1,\dots,j_\ell)} f_{i_1,\dots,i_k \delta(j_1,\dots,j_\ell)}. \end{aligned} \tag{4.5}$$

We note that for any $k,\ell \geq 1$ we have

$$\begin{aligned} &A\Big(r(x)\big((\otimes^h dx^1)\otimes_s(\otimes^{k-h}dx^2)\big)\otimes\big((\otimes^a dx^1)\otimes_s(\otimes^{\ell-a}dx^2)\big)\Big) \\ &= r(x)\big((\otimes^h dx^1)\otimes_s(\otimes^{k-h}dx^2)\big)\otimes\big((\otimes^a(\star dx^1))\otimes_s(\otimes^{\ell-a}(\star dx^2))\big) \\ &= (-1)^{\ell-a} r(x)\big((\otimes^h dx^1)\otimes_s(\otimes^{k-h}dx^2)\big)\otimes\big((\otimes^{\ell-a}dx^1)\otimes_s(\otimes^{a}dx^2)\big), \quad r\in C^\infty(M), \end{aligned}$$

where $\star$ is the Hodge star operator. Formula (4.5) implies that A is invertible with the inverse

$$A^{-1} = (-1)^\ell A. \tag{4.6}$$

We then point out that

$$\begin{aligned} &(Af)_{i_1,\dots,i_k j_1,\dots,j_\ell}(x) v^{i_1}\cdots v^{i_k} v^{j_1}\cdots v^{j_\ell} \\ &= (\operatorname{Sym} Af)_{i_1,\dots i_k j_1,\dots,j_\ell}(x) v^{i_1}\cdots v^{i_k} v^{j_1}\cdots v^{j_\ell}. \end{aligned} \tag{4.7}$$

The notation $\operatorname{Sym} h$ stands for the full symmetrization of the tensor field h.

Using equations (4.4), (4.5), and (4.7), we find that

$$L_{k,\ell}(f) = I_{k+\ell}(\operatorname{Sym}(Af)), \tag{4.8}$$

where $I_{k+\ell}$ is the geodesic ray transform on symmetric tensor field $h \in S^{k+\ell}M$, defined by the formula

$$\begin{aligned} &I_{k+\ell}(h)(x,v) \\ &\quad = \int_0^{\tau(x,v)} h_{i_1,\dots,i_{k+\ell}}(\gamma_{x,v}(t))\dot\gamma_{x,v}(t)^{i_1}\cdots\dot\gamma_{x,v}(t)^{i_{k+\ell}}\mathrm{d}t, \quad (x,v)\in\partial_{in}(SM). \end{aligned}$$

By (4.8) and [10, Theorem 1.1] it holds that for any $h \in S^k M\times S^\ell M$,

$$L_{k,\ell}(h) = 0 \quad \text{if and only if } \operatorname{Sym} Ah = d^s v, \quad v\in S^{k+\ell-1}M, \quad v|_{\partial M} = 0. \tag{4.9}$$

In the above, d^s stands for the inner derivative, that is, the symmetrization of the covariant derivative

$$d^s u = \operatorname{Sym}(\nabla u), \quad u \in S^{k+\ell-1}M. \tag{4.10}$$

If $L_{k,\ell}(f) = 0$, then, with (4.6) and (4.9), we can write

$$f = (-1)^\ell A(\operatorname{Sym}(Af) + (Af - \operatorname{Sym}(Af))) = (-1)^\ell A(d^s u) + f + (-1)^{\ell+1}A(\operatorname{Sym}(Af)).$$

We conclude that the claim of Theorem 1.1 holds if

$$f + (-1)^{\ell+1}A(\operatorname{Sym}(Af)) = \lambda w, \quad A(d^s u - d'u) = \lambda w', \quad d'A = Ad'$$

for some $w, w' \in S^{k-1}M \times S^{\ell-1}M$ and $u \in S^{k+\ell-1}M$. These equations will be proved in the following subsections.

4.1. **Analysis of operator** $A\,\mathrm{Sym}\,A$**.** In this subsection, we prove the following identity for any $f \in S^kM \times S^\ell M$:

$$f + (-1)^{\ell+1} A(\mathrm{Sym}(Af)) = \lambda w \quad \text{for some } w \in S^{k-1}M \times S^{\ell-1}M. \tag{4.11}$$

We start with a lemma that characterizes the kernel of $A\,\mathrm{Sym}\,A$.

Lemma 4.2. *For the linear maps*

$$A\mathrm{Sym}A : S^kM \times S^\ell M \to S^kM \times S^\ell M$$

and

$$\lambda : S^{k-1}M \times S^{\ell-1}M \to S^kM \times S^\ell M,$$

the following holds:

$$\ker(A\mathrm{Sym}A) = \mathrm{Im}(\lambda).$$

Proof. We use the notation $\otimes_s$ for the symmetric product of tensors. We note that the choice of isothermal coordinates implies that

$$\lambda(a \otimes b) = e^{2\alpha(x)}\big((dx^1 \otimes_s a) \otimes (dx^1 \otimes_s b) + (dx^2 \otimes_s a) \otimes (dx^2 \otimes_s b)\big), \qquad a \otimes b \in S^{k-1}M \times S^{\ell-1}M. \tag{4.12}$$

Since A is a bijection, it suffices to prove that

$$\mathrm{Im}(\lambda) = \ker(\mathrm{Sym}\,A). \tag{4.13}$$

We prove first that $\mathrm{Im}(\lambda) \subset \ker(\mathrm{Sym}\,A)$. In the view of the $C^\infty(M)$-linearity of λ and $\mathrm{Sym}\,A$, it suffices to prove that $\lambda w \in \ker(\mathrm{Sym}\,A)$ when w is a $C^\infty(M)$-basis tensor field of the form

$$w = \big((\bigotimes^{h-1} dx^1) \otimes_s (\bigotimes^{k-h} dx^2)\big) \otimes \big((\bigotimes^{a-1} dx^1) \otimes_s (\bigotimes^{\ell-a} dx^2)\big), \qquad h \in \{1,\dots,k\}, \quad a \in \{1,\dots,\ell\}.$$

Then

$$\begin{aligned} e^{-2\alpha(x)} A\lambda w = (-1)^{\ell-a}\Bigg(&\big((\bigotimes^{h} dx^1) \otimes_s (\bigotimes^{k-h} dx^2)\big) \otimes \big((\bigotimes^{\ell-a} dx^1) \otimes_s (\bigotimes^{a} dx^2)\big) \\ &- \big((\bigotimes^{h-1} dx^1) \otimes_s (\bigotimes^{k-h+1} dx^2)\big) \otimes \big((\bigotimes^{\ell-a+1} dx^1) \otimes_s (\bigotimes^{a-1} dx^2)\big)\Bigg). \end{aligned} \tag{4.14}$$

Due to linearity of Sym and formula (4.16), which is given later in this proof, we have $\mathrm{Sym}\,A(\lambda w) = 0$. Therefore, $\mathrm{Im}(\lambda) \subset \ker(\mathrm{Sym}\,A)$.

Now we prove that $\ker(\mathrm{Sym}\,A) \subset \mathrm{Im}(\lambda)$. We note that any $f \in S^kM \times S^\ell M$ can be written as $f = \sum_{m=1}^M u_m$, where

$$\begin{aligned} &u_m = r_m b_m, \quad r_m \in C^\infty(M), \\ &b_m = \big((\bigotimes^{h_m} dx^1) \otimes_s (\bigotimes^{k-h_m} dx^2)\big) \otimes \big((\bigotimes^{\ell-a_m} dx^1) \otimes_s (\bigotimes^{a_m} dx^2)\big), \\ &h_m \in \{0,\dots,k\}, \quad a_m \in \{0,\dots,\ell\}. \end{aligned} \tag{4.15}$$

It is straightforward to show that for any $m, m' \in \{1, \dots, M\}$ it holds that

(4.16) $$(\operatorname{Sym} A) b_m(x) = (\operatorname{Sym} A) b_{m'}(x) \quad \text{if and only if } h_m + a_m = h_{m'} + a_{m'}.$$

Therefore, for a given $m \in \{1, \dots, M\}$, the sum $H := h_m + a_m$ is an important quantity associated with the tensor u_m from the point of view of the map $\operatorname{Sym} A$. However, for a given $H \in \{0, \dots, k+\ell\}$ there are usually several $(h, a) \in \{0, \dots, k\} \times \{0, \dots, \ell\}$ whose sum is H. Thus we define $h_H \in \{0, \dots, k\}$ to be the smallest integer such that there exists $a_H \in \{0, \dots, \ell\}$ that satisfies $h_H + a_H = H$. We note that for every H the pair (h_H, a_H) is unique.

Then we can write

$$f = \sum_{H=0}^{k+\ell} f_H, f_H = \sum_{r=0}^{R(H)} b_{H,r} f_{H,r}, \quad b_{H,r} \in C^\infty(M),$$

$$f_{H,r} := \big((\bigotimes^{h_H + r} dx^1) \otimes_s (\bigotimes^{k-(h_H+r)} dx^2) \big) \otimes \big((\bigotimes^{\ell-(a_H - r)} dx^1) \otimes_s (\bigotimes^{a_H - r} dx^2) \big),$$

and the summing limit $R(H)$ depends on k, ℓ, and H. Moreover the $C^\infty(M)$-linearity of $\operatorname{Sym} A$ and (4.16) imply that $f \in \ker(\operatorname{Sym} A)$ if and only if $f_H \in \ker(\operatorname{Sym} A)$ for every $H \in \{0, \dots, k+\ell\}$.

In the following, we study the tensor f_H for a given $H \in \{0, \dots, k+\ell\}$. For $r \in \{1, \dots, R(H)\}$ we define $w_r \in S^{k-1}M \times S^{\ell-1}M$ by the formula

$$w_r = \big((\bigotimes^{h_H + r - 1} dx^1) \otimes_s (\bigotimes^{k-(h_H+r)} dx^2) \big) \otimes \big((\bigotimes^{\ell-(a_H - r)-1} dx^1) \otimes_s (\bigotimes^{a_H - r} dx^2) \big).$$

Then (4.12) yields

$$\lambda w_r = e^{2\alpha(x)} (f_{H,r} + f_{H,r-1}).$$

This implies the recursive formula

$$f_{H,r} = \lambda(e^{-2\alpha(x)} w_r) - f_{H,r-1}.$$

Thus for every $r \in \{0, \dots, R(H)\}$ there exists $w_r' \in S^{k-1}M \times S^{\ell-1}M$ such that

(4.17) $$f_{H,r} = \lambda w_r' + (-1)^r f_{H,0}.$$

Therefore, there exists $w_H \in S^{k-1}M \times S^{\ell-1}M$ such that

$$f_H = \sum_{r=0}^{R(H)} b_{H,r} f_{H,r} = \lambda w_H + \Big(\sum_{r=0}^{R(H)} (-1)^r b_{H,r} \Big) f_{H,0}.$$

If $f_H \in \ker(\operatorname{Sym} A)$ it holds by the first part of this proof that

$$\operatorname{Sym} A f_H = \Big(\sum_{r=0}^{R(H)} (-1)^r b_{H,r} \Big) (\operatorname{Sym} A f_{H,0}) = 0.$$

Since $\operatorname{Sym} A f_{H,0} \neq 0$, it follows that $\sum_{r=0}^{R(H)} (-1)^r b_{H,r} = 0$, whence $f_H = \lambda w_H$. This implies that $f = \lambda w$ for some $w \in S^{k-1}M \times S^{\ell-1}M$.

This completes the proof of Lemma 4.2. □

By the proof of the previous lemma we can write any $f \in S^k M \times S^\ell M$ in the form

(4.18) $$f = \lambda w + \sum_{H=0}^{k+\ell} r_H f_{H,0}, \quad r_H \in C^\infty(M),$$

for some $w \in S^{k-1}M \times S^{\ell-1}M$. Next, we prove that

$$A\,\mathrm{Sym}\,Af_{H,0} = (-1)^{\ell} f_{H,0} + \lambda w, \quad H \in \{0, \dots, k+\ell\}. \tag{4.19}$$

We note that

$$\begin{aligned}
\mathrm{Sym}\,Af_{H,0} &= (-1)^{a_H} \big(\bigotimes^{H} dx^1 \otimes_s \big(\bigotimes^{k+\ell-H} dx^2\big)\big) \\
&= \frac{(-1)^{a_H}}{(k+\ell)!} \sum_{r=0}^{R(H)} A_r \big(\big(\bigotimes^{h_H+r} dx^1\big) \otimes_s \big(\bigotimes^{k-(h_H+r)} dx^2\big)\big) \\
&\quad \otimes \big(\big(\bigotimes^{a_H-r} dx^1\big) \otimes_s \big(\bigotimes^{\ell-(a_H-r)} dx^2\big)\big),
\end{aligned}$$

where $\sum_{r=0}^{R(H)} A_r = (k+\ell)!$. Using (4.17), we obtain

$$\begin{aligned}
A\,\mathrm{Sym}\,Af_{H,0} &= (-1)^{a_H} \frac{1}{(k+\ell)!} \sum_{r=0}^{R(H)} (-1)^{\ell-(a_H-r)} A_r f_{H,r} \\
&= (-1)^{\ell} \frac{1}{(k+\ell)!} \Big(\sum_{r=0}^{R(H)} A_r \Big) f_{H,0} + \lambda w \\
&= (-1)^{\ell} f_{H,0} + \lambda w.
\end{aligned}$$

Therefore, we have proved (4.19).

Equation (4.11) follows from Lemma 4.2 and (4.18)–(4.19).

4.2. **Analysis of operator Ad^s.** We note that $S^{k+\ell}M \subset S^kM \times S^{\ell}M$. Therefore, we can extend the inner derivative d^s to an operator $d^s : S^{k-1}M \times S^{\ell}M \to S^kM \times S^{\ell}M$ and evaluate $d^s - d'$. In this subsection, we show that for any $u \in S^{k-1}M \times S^{\ell}M$ the following equations hold:

$$A(d^s u - d' u) = \lambda w \quad \text{for some } w \in S^{k-1}M \times S^{\ell-1}M, \tag{4.20}$$

$$d'A = Ad'. \tag{4.21}$$

Since Ad^s and Ad' are linear it suffices to prove the claims for

$$u = r(x)\big(\big(\bigotimes^{h-1} dx^1\big) \otimes_s \big(\bigotimes^{k-h} dx^2\big)\big) \otimes \big(\big(\bigotimes^{a} dx^1\big) \otimes_s \big(\bigotimes^{\ell-a} dx^2\big)\big), \quad r \in C^{\infty}(M).$$

By (1.5) and (4.5) we have

$$\begin{aligned}
Ad'u = (-1)^{\ell-a} \bigg(&\Big(\frac{\partial}{\partial x^1} r(x) - R_1\Big) \big(\big(\bigotimes^{h} dx^1\big) \otimes_s \big(\bigotimes^{k-h} dx^2\big)\big) \otimes \big(\big(\bigotimes^{\ell-a} dx^1\big) \otimes_s \big(\bigotimes^{a} dx^2\big)\big) \\
&+ \Big(\frac{\partial}{\partial x^2} r(x) - R_2\Big) \big(\big(\bigotimes^{h-1} dx^1\big) \otimes_s \big(\bigotimes^{k-h+1} dx^2\big)\big) \otimes \big(\big(\bigotimes^{\ell-a} dx^1\big) \otimes_s \big(\bigotimes^{a} dx^2\big)\big) \bigg),
\end{aligned} \tag{4.22}$$

where $R_m = \sum_{s=1}^{k+\ell-1} r_{i_1,\dots,i_{s-1}p,i_{s+1},\dots,i_{k+\ell}} \Gamma^p_{mi_s}$, $m \in \{1,2\}$, $r_{i_1,\dots,i_{s-1}p,i_{s+1},\dots,i_{k+\ell}} \in \{0, r\}$ depending on $(i_1, \dots, i_{k+\ell})$ and $\Gamma^p_{mi_s}$ are the Christoffel symbols of metric g.

We write $H = h + a$ and denote $\widetilde{R}_m = \frac{\partial}{\partial x^m} r(x) - R_m$. Then we obtain from (4.5) and (4.10),

$$d^s u = \frac{1}{(k+\ell)!}\Big(\widetilde{R}_1 \sum_{r=0}^{R(H)} A_r\big((\bigotimes^{h_H+r} dx^1) \otimes_s (\bigotimes^{k-(h_H+r)} dx^2)\big) \otimes \big((\bigotimes^{a_H-r} dx^1) \otimes_s (\bigotimes^{\ell-(a_H-r)} dx^2)\big) + \widetilde{R}_2 \sum_{r=0}^{R(H-1)} B_r\big((\bigotimes^{h_{H-1}+r} dx^1) \otimes_s (\bigotimes^{k-(h_{H-1}+r)} dx^2)\big) \otimes \big((\bigotimes^{a_{H-1}-r} dx^1) \otimes_s (\bigotimes^{\ell-(a_{H-1}-r)} dx^2)\big)\Big),$$

where $\sum_{r=0}^{R(H)} A_r = \sum B_{r=0}^{R(H-1)} = (k+\ell)!$. This yields

$$Ad^s u = \widetilde{R}_1 \frac{(-1)^{\ell - a_H}}{(k+\ell)!} \sum_{r=0}^{R(H)} \big((-1)^r A_r\, g_{H,r}\big) + \widetilde{R}_2 \frac{(-1)^{\ell - a_{H-1}}}{(k+\ell)!} \sum_{r=0}^{R(H-1)} \big((-1)^r B_r\, g_{H-1,r}\big),$$

where the shorthand notation $g_{H,r}$, $g_{H-1,r}$ stand for

$$g_{H,r} := \big((\bigotimes^{h_H+r} dx^1) \otimes_s (\bigotimes^{k-(h_H+r)} dx^2)\big) \otimes \big((\bigotimes^{\ell-(a_H-r)} dx^1) \otimes_s (\bigotimes^{a_H-r} dx^2)\big),$$

$$g_{H-1,r} := \big((\bigotimes^{h_{H-1}+r} dx^1) \otimes_s (\bigotimes^{k-(h_{H-1}+r)} dx^2)\big) \otimes \big((\bigotimes^{\ell-(a_{H-1}-r)} dx^1) \otimes_s (\bigotimes^{a_{H-1}-r} dx^2)\big).$$

By an analogous argument as in the proof of Lemma 4.2 we obtain a recursive formula

$$g_{H,r} = \lambda w_{H,r} + (-1)^{R(H)-r} g_{H,R(H)} \quad \text{for some } w_{H,r} \in S^{k-1}M \times S^{\ell-1}M.$$

Thus for some $w', w'' \in S^{k-1}M \times S^{\ell-1}M$ it holds that

$$Ad' u = (-1)^{\ell - a}\Big(\widetilde{R}_1 g_{H,h-h_H} + \widetilde{R}_2 g_{H-1,h-1-h_{H-1}}\Big)$$
$$= (-1)^{\ell}\Big((-1)^{R(H)-h+h_H-a} \widetilde{R}_1 g_{H,R(H)} + (-1)^{R(H-1)-h+1+h_{H-1}-a} \widetilde{R}_2 g_{H-1,R(H-1)}\Big) + \lambda w',$$

and

$$Ad^s u = \widetilde{R}_1 \frac{(-1)^{\ell - a_H}}{(k+\ell)!} \sum_{r=0}^{R(H)} (-1)^r A_r (\lambda w_{H,r} + (-1)^{R(H)-r} g_{H,R(H)}) + \widetilde{R}_2 \frac{(-1)^{\ell - a_{H-1}}}{(k+\ell)!} \sum_{r=0}^{R(H-1)} (-1)^r B_r (\lambda w_{H-1,r} + (-1)^{R(H-1)-r} g_{H,R(H-1)})$$
$$= (-1)^{\ell}\Big((-1)^{R(H)-a_H} \widetilde{R}_1 g_{H,R(H)} + (-1)^{R(H-1)-a_{H-1}} \widetilde{R}_2 g_{H-1,R(H-1)}\Big) + \lambda w''.$$

Since we defined

$$a + h = H = a_H + h_H \quad \text{and} \quad H - 1 = a_{H-1} + h_{H-1},$$

the identities above imply that

$$A(d^s u - d' u) = \lambda w, \quad w \in S^{k-1}M \times S^{\ell-1}M.$$

Therefore we have proved (4.20).

Finally, we prove equation (4.21). We note that

$$\begin{aligned} d' A u = \ & (-1)^{\ell-a} \Big(\widetilde{R}_1 \big((\bigotimes^{h} dx^1) \otimes_s (\bigotimes^{k-h} dx^2) \big) \otimes \big((\bigotimes^{\ell-a} dx^1) \otimes_s (\bigotimes^{a} dx^2) \big) \\ & + \widetilde{R}_2 \big((\bigotimes^{h-1} dx^1) \otimes_s (\bigotimes^{k-h+1} dx^2) \big) \otimes \big((\bigotimes^{\ell-a} dx^1) \otimes_s (\bigotimes^{a} dx^2) \big) \Big). \end{aligned}$$

Thus (4.21) holds since the previous equation coincides with (4.22).

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