A Relaying Graph and Special Strong Product for Zero-error Problems in Primitive Relay Channels

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***Abstract*—A primitive relay channel (PRC) has one source (S) communicating a message to one destination (D) with the help of a relay (R). The link between R and D is considered to be noiseless, of finite capacity, and parallel to the link between S and (R,D). Prior work has established, for any fixed number of channel uses, the minimal R-D link rate needed so that the overall S-D message rate equals the *zero-error* single-input multiple output outer bound (Problem 1). The zero-error relaying scheme was expressed as a coloring of a carefully defined "relaying compression graph". It is shown here that this relaying compression graph for n channel uses is not obtained as a strong product from its $n = 1$ instance. Here we define a new graph, the "primitive relaying graph" and a new "special strong product" such that the n-channel use primitive relaying graph corresponds to the n-fold special strong product of the $n = 1$ graph. We show how the solution to Problem 1 can be obtained from this new primitive relaying graph directly. Further study of this primitive relaying graph has the potential to highlight the structure of optimal codes for zero-error relaying.**

I. Introduction and contribution

A primitive relay channel $(\mathcal{X}, p(y, y_R|x), \mathcal{Y} \times \mathcal{Y}_R, k)$ (PRC), shown in Fig. 1, consists of a family of conditional probability mass functions $p(y, y_R|x)$ relating inputs $x \in \mathcal{X}$ to outputs $y \in \mathcal{Y}$ and $y_R \in \mathcal{Y}_R$ at the destination and relay respectively, and an out-of-band link between the relay and destination able to support up to $k \in \mathbb{R}^+$ bits/channel use.

Quantities of interest may then be the maximal number of codewords (the *message rate*) that can be reliably communicated for a given R-D link rate k (the *relay rate*), or the minimal relay rate needed to transmit at a desired message rate (provided the desired message rate is feasible at all). When the R-D link rate k is large enough, the relay can forward its entire observation to the destination terminal. Thus, the primitive relay channel effectively turns into a point-to-point channel with a single input and two outputs, say $(\mathcal{X}, p(y, y_R|x), \mathcal{Y} \times \mathcal{Y}_R)$, for which we have finite n (or n-shot, where n is the number of channel uses) and asymptotic expressions (though they cannot currently be calculated / evaluated) for the zero-error capacity. This capacity is an upper bound to the message rate achievable for any finite relay rate. The question (Problem 1) posed and solved in [1] is, for fixed number of channel uses n, the minimal relay rate needed to ensure that the overall message rate achieves the capacity of the single-input multiple output (SIMO) channel $(\mathcal{X}, p(y, y_R|x), \mathcal{Y} \times \mathcal{Y}_R)$. The *small-error* version of this question was considered in [2], [3], with much interesting recent progress in [4], [5].

Contributions. The zero-error relaying scheme solving Problem 1 was expressed in [1] as a coloring of a carefully defined "relaying compression graph". It is shown here that this relaying compression graph for n channel uses *cannot be obtained by a strong product from its $n = 1$ instance*. In seeking to understand the structure of zero-error relaying, one question is whether a graph characterizing this form of zero-error relaying admits any strong product form at all. We answer this in the positive by defining a new graph, the "primitive relaying graph" (PRG) and a new "special strong product" such that the n-channel use PRG corresponds to the n-fold special strong product of the $n = 1$ PRG. We show how the solution to Problem 1 expressed as in [1] can be obtained from this new primitive relaying graph. For binary $||\mathcal{X}|| = ||\mathcal{Y}|| = ||\mathcal{Y}_R||$ we solve Problem 1 exactly. We provide several examples of how to use the newly proposed PRG and the special strong product. A full version of this paper (with proofs in the Appendix) is available at [6].

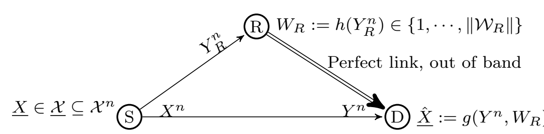

Fig. 1. An n-shot protocol $(n, \underline{\mathcal{X}}, h, g)$ for zero-error communication over a PRC: codebook $\underline{\mathcal{X}}$, relaying function h, decoding function g.

II. Problem definition

An *n-shot protocol* for $n \geq 1$ channel uses denoted by $(n, \underline{\mathcal{X}}, h, g)$ for communication over the PRC $(\mathcal{X}, p(y, y_R|x), \mathcal{Y} \times \mathcal{Y}_R, k)$, shown in Fig. 1 comprises a codebook $\underline{\mathcal{X}} \subseteq \mathcal{X}^n$, a relaying function $h : \mathcal{Y}_R^n \to \mathcal{W}_R$, and a decoding function $g : \mathcal{Y}^n \times \mathcal{W}_R \to \underline{\mathcal{X}}$.

When a conditional joint pmf $p(y, y_R|x)$ with support $\mathcal{X}$ and output $\mathcal{Y} \times \mathcal{Y}_R$ is restricted to input $\mathcal{K}$, we denote its *induced conditional pmf, support,* and *output* by $p_{\mathcal{K}}(y, y_R|x)$, $\mathcal{K}$ and $\mathcal{Y}|_{\mathcal{K}} \times \mathcal{Y}_R|_{\mathcal{K}}$ respectively.

Let $R_z^{(n)} := \frac{1}{n} \log \|\underline{\mathcal{X}}\|$ be the *message rate*, and $r_z^{(n)} := \frac{1}{n} \log ||\mathcal{W}_R||$ be the *relay rate*. Then, a rate pair $(R_z^{(n)}, r_z^{(n)})$ is said to be *achievable* if first, zero error communication is attainable, i.e. $\Pr[g(Y^n, W_R) \neq \underline{X} \mid \underline{X} \text{ sent}] = 0$ for all $\underline{X} \in \underline{\mathcal{X}}$, and second, there exists an n-shot protocol $(n, \underline{\mathcal{X}}, h, g)$ such that the relaying function uses less than k bits per channel use on average, i.e. $r_z^{(n)} := \frac{1}{n} \log ||\mathcal{W}_R|| \leq k$. For n channel uses, we set $C_z^{(n)}(k)$ to be the maximum $R_z^{(n)}$, such that there exists an achievable $(R_z^{(n)}, r_z^{(n)})$ pair (i.e. for which $r_z^{(n)} \le k$). The *zero-error capacity of the relay channel at R-D link rate k,* defined as $C_z(k)$, is the supremum over n of $C_z^{(n)}(k)$.

Some notation. We call one usage of the channel "one-shot", and this is governed by $p(y, y_R|x)$. If we use the channel twice, we use the notation $p(y_1y_2, y_{R_1}y_{R_2}|x_1x_2) = p(y_1, y_{R_1}|x_1) \cdot p(y_2, y_{R_2}|x_2)$ to denote two channel uses.

A graph $G(V, E)$ consists of a set V of vertices (sometimes denoted as $V(G)$) or nodes together with a set E of edges (sometimes denoted as $E(G)$), which are two-element subsets of V. Two nodes connected by an edge are called *adjacent*. We will usually drop the V, E indices in $G(V, E)$. Let $\cong$ denote the isomorphism relation between two graphs. The adjacency matrix A of a graph G is a matrix with rows and columns labeled by graph vertices, with a 1 or 0 in position (v_i, v_j) indicating whether v_i and v_j are adjacent or not. Our convention is to put 0's on the diagonal (nodes are not adjacent to themselves). The *strong product* $G \boxtimes H$ of two graphs G and H is defined as the graph with vertex set $V(G \boxtimes H) = V(G) \times V(H)$, in which two distinct vertices (g, h) and (g', h') are adjacent iff g is adjacent or equal to g' in G and h is adjacent or equal to h' in H. $G^{\boxtimes n}$ denotes the strong product of n copies of G.

A. Problem 1: Minimal R-D link rate $r_z^{*(n)}$ needed to achieve the SIMO bound for a fixed n:

For fixed number of channel uses n, by giving the destination the relay output y_R^n, one obtains the single-input multiple output outer bound SIMO(n) on $C_z^{(n)}(k)$:

$$\text{SIMO}(n) := \log \sqrt[n]{\alpha([G_{X|Y,Y_R}]^{\boxtimes n})},$$

where the confusability graph $G_{X|Y,Y_R}$ has vertices the input alphabet $\mathcal{X}$ and an edge between $x \ne x'$ if there exists a $(y, y_R) \in \mathcal{Y} \times \mathcal{Y}_R$ such that $p(y, y_R|x) \cdot p(y, y_R|x') > 0$. Clearly, if $k \ge \log ||\mathcal{Y}_R||$ this bound is achievable by simply having the relay forward the received Y_R^n sequence perfectly over the out-of-band link. An interesting question is how small this link rate k may be so as to hit this SIMO(n) upper bound, i.e. to find $r_z^{*(n)}$ defined as:

Definition 1 (The n-shot minimum R-D link rate $r_z^{*(n)}$ to achieve the n-shot SIMO bound)**.**

$$r_z^{*(n)} := \min\{r_z : C_z^{(n)}(r_z) = \text{SIMO}(n)\}. \tag{1}$$

This was solved for fixed n in [1], where $r_z^{*(n)}$ is exactly characterized as an n-letter extension of the one-shot "color-and-forward" scheme, outlined next.

Problem 1 can be re-stated and solved as follows. For the PRC $(\mathcal{X}, p(y, y_R|x), \mathcal{Y} \times \mathcal{Y}_R, \log ||\mathcal{Y}_R||)$ (where the R-D link rate is initially set to $\log ||\mathcal{Y}_R||$ for notational convenience), let $\mathcal{K}$ be a maximal independent set of the confusability graph $G_{X|Y,Y_R}$. Note that $\mathcal{K}$ need not be unique.

Question. Find the minimum cardinality relaying function $h : \mathcal{Y}_R|_\mathcal{K} \to \mathcal{W}_R$ (and its cardinality) such that $(Y, h(Y_R))$ can distinguish X without error, namely, for each $(y, h(y_R))$ with positive probability, there exists at most one $x \in \mathcal{K}$ for which $p(y, h(y_R)|x) > 0$. This is a reformulation of the problem of finding $r_z^{*(1)}$ for the PRC $(\mathcal{K}, p_\mathcal{K}(y, y_R|x), \mathcal{Y}|_\mathcal{K} \times \mathcal{Y}_R|_\mathcal{K}, \log ||\mathcal{Y}_R||)$. To answer this, we construct a new graph, the **relaying compression graph** $G_R^{(1)}|_\mathcal{K}$, for which a coloring provides the optimal relaying scheme. $G_R^{(1)}|_\mathcal{K}$ has:

- Vertices: $Y_R|_\mathcal{K}$
- Edges: vertices $y_R \ne y'_R$ both in $Y_R|_\mathcal{K}$ share an edge when $\exists x \in \mathcal{K}$ and $\exists x' \in \mathcal{K}$, $x \ne x'$ such that $p(y, y_R|x) > 0$ and $p(y, y'_R|x') > 0$ for some $y \in \mathcal{Y}|_\mathcal{K}$.

Answer: Coloring of the relaying compression graph $G^{(1)}_{R|\mathcal{K}}$. Color the graph at its chromatic number and let $h(y_R)$ be one of the corresponding minimal colorings. We term this processing and forwarding at the relay as "color-and-forward" relaying. It is easy to see how $G_R^{(1)}|_\mathcal{K}$ may be extended to n channel uses to obtain $G_R^{(n)}|_{\mathcal{K}^{(n)}}$. Then, [1] showed:

Theorem 1. *(Chen, Devroye [1]) For the PRC,*

$$r_z^{*(n)} = \min_{\mathcal{K}^{(n)}:\, \mathcal{K}^{(n)} \text{ is a max. ind. set of } G_{X|Y,Y_R}^{\boxtimes n}} \frac{1}{n} \log \chi(G_R^{(n)}|_{\mathcal{K}^{(n)}}), \tag{2}$$

where $\chi(G_R^{(n)}|_{\mathcal{K}^{(n)}})$ is the chromatic number of graph $G_R^{(n)}|_{\mathcal{K}^{(n)}}$, constructed via the algorithm described in the Answer above with restricted input / codebook $\mathcal{K}^{(n)}$.

III. Primitive Relaying Graph

The main contribution is the introduction of the primitive relaying graph-like structure (PRG), and a new "special strong product" (SSP) that allows the graph $G_R^{(n)}$ needed to describe the optimal relaying scheme (optimal in the sense of Problem 1) to be computed recursively. We note that what we define in the following is strictly speaking not a graph (hence the usage of graph-like structure) , since we will introduce different *types* of edges between vertices, but we have been unable to find a better terminology. We will use graph in the following rather than graph-like structure for brevity.

In [1], given an independent set $\mathcal{K}$ of confusability graph $G_{X|Y,Y_R}$ the relay compression graph $G_R^{(1)}|_\mathcal{K}$ places an edge between vertices $y_R \ne y'_R$ both in $Y_R|_\mathcal{K}$ when $\exists x \in \mathcal{K}$ and $\exists x' \in \mathcal{K}$, $x \ne x'$ such that $p(y, y_R|x) > 0$ and $p(y, y'_R|x') > 0$ for some $y \in \mathcal{Y}|_\mathcal{K}$. For $(n = 2)$ an edge between vertices $y_{R_1}y_{R_2} \ne y'_{R_1}y'_{R_2}$ both in $Y^{(2)}_{R|\mathcal{K}^{(2)}}$ in $G_R^{(2)}|_{\mathcal{K}^{(2)}}$ exists when $\exists x_1x_2 \in \mathcal{K}^{(2)}$ and $\exists x'_1x'_2 \in \mathcal{K}^{(2)}$, $x_1x_2 \ne x'_1x'_2$ such that both $p(y_1y_2, y_{R_1}y_{R_2}|x_1x_2) > 0$, and $p(y_1y_2, y'_{R_1}y'_{R_2}|x'_1x'_2) > 0$ for some $y_1y_2 \in \mathcal{Y}^2|_{\mathcal{K}^{(2)}}$. The question is whether $G_R^{(2)}$ can be obtained from a strong product of $G_R^{(1)}$ with itself. Example 3 shows that this is not possible even when the confusability graph is edge free, the main reason being that the regular strong product of the compression graph ignores the delicate interaction between the Y_R's and the channel inputs (it is the inputs we want to distinguish at the destination, not the Y_R's).

A. Graph Definition

From the PRC $(\mathcal{X}, p(y, y_R|x), \mathcal{Y} \times \mathcal{Y}_R, k)$, we introduce the *Primitive Relaying Graph* (PRG) - like structure denoted by $G_{X,Y_R|Y}$. The vertices of $G_{X,Y_R|Y}$ consist of a set $V \subseteq \mathcal{X} \times \mathcal{Y}_R$ of pairs $v = (x, y_R)$ corresponding to a given input $x \in \mathcal{X}$ and a relay channel output $y_R \in \mathcal{Y}_R$, so that $p(y, y_R|x) > 0$ for some $y \in \mathcal{Y}$. Let V_x and V_{y_R} denote the sets of all the x-coordinates, and y_R coordinates of V, respectively. The edge set E consists of pairs of vertices $e = (v, v') \triangleq (x, y_R)(x', y'_R)$ connected by an edge in the PRG, and is given by $E = E_s \cup E_d \cup E_c$, where E_s, E_d, E_c are mutually disjoint sets defined as:

1) Solid edge set $E_s \triangleq \{(x, y_R)(x', y'_R) : x \neq x', y_R \neq y'_R$ and $\exists y : p(y|x, y_R) \cdot p(y|x', y'_R) > 0\}$. A solid edge occurs whenever two different channel inputs x, x' produce different relay outputs y_R, y'_R for some y. These y_R's cannot be compressed.
2) Dotted edge set $E_d \triangleq \{(x, y_R)(x, y'_R) : y_R \neq y'_R$ and $\exists y : p(y|x, y_R) \cdot p(y|x, y'_R) > 0\}$. This type of edge arises when for a single channel input x and channel output y, there are two different y_R and y'_R relay outputs. These y_R's in theory be compressed / assigned the same color for a single channel use, but must be distinguished because when looking at multiple channel uses, when combined with a confusable or solid edge, may lead to a solid edge in which the y_R's must be distinguished.
3) Confusable edge set $E_c \triangleq \{(x, y_R)(x', y_R) : x \neq x'$ and $\exists y : p(y|x, y_R) \cdot p(y|x', y_R) > 0\}$. A confusable edge corresponds to the case of two different channel inputs x, x' that produce the same relay output y_R for the same channel output y. These x's cannot be distinguished even when y and y_R are both available. These edges will yield the confusability graph $G_{X|Y,Y_R}$.

Our notation for the graph $G_{X,Y_R|Y}$ is reminiscent of the confusability graph notation in point-to-point channels, $G_{X|Y}$, where an edge between two vertices $x \neq x'$ in $\mathcal{X}$ exists if there exists a $y : p(y|x)p(y|x') > 0$ (i.e. two inputs are confusable). In the above, we have similar notions for confusable (x, y_R) pairs, but since the destination, who has access to y, cares only about recovering x and not necessarily y_R (or only to the extent that it may help in recovering x), different types of edges are relevant. To the best of our knowledge, this type of graph has not been considered.

B. Special Strong Product

Due to presence of several types of edges in the PRG we introduce a new graph product, which we term the "special strong product" (SSP) and denoted by $\circledast$, for two PRG G and H ($G \circledast H$), as (following graph products definitions [7]):

- $V(G \circledast H)$ is the Cartesian product of $V(G)$ and $V(H)$.
- For any PRG G, there is an incidence function, $\delta_G : V(G) \times V(G) \to \{\Delta, s, d, c, 0\}$ defined as:

$$\delta_G((x_1, y_{R_1}),(x_2, y_{R_2}))=\begin{cases} \Delta & \text{if } (x_1, y_{R_1}) = (x_2, y_{R_2}) \\ s & \text{if } (x_1, y_{R_1})(x_2, y_{R_2}) \in E_s \\ d & \text{if } (x_1, y_{R_1})(x_2, y_{R_2}) \in E_d \\ c & \text{if } (x_1, y_{R_1})(x_2, y_{R_2}) \in E_c \\ 0 & \text{Otherwise} \end{cases}.$$

  The incidence function between two vertices indicates what type of edge exists between them (if any).
- The SSP multiplication table for the $\circledast$ operation is:

$\circledast$	Δ	s	d	c	0
Δ	Δ	s	d	c	0
s	s	s	s	s	0
d	d	s	d	s	0
c	c	s	s	c	0
0	0	0	0	0	0

  Notice that Δ behaves like a multiplicative identity.
- If $(x_1, y_{R_1}), (x_2, y_{R_2}) \in V(G)$, $(x'_1, y'_{R_1}), (x'_2, y'_{R_2}) \in V(H)$, then the incidence function for $G \circledast H$ is

$$\delta_{G\circledast H}((x_1x_2, y_{R_1}y_{R_2}), (x'_1x'_2, y'_{R_1}y'_{R_2})) \\ := \delta_G((x_1, y_{R_1}), (x_2, y_{R_2})) \circledast \delta_H((x'_1, y'_{R_1}), (x'_2, y'_{R_2}))$$

  and indicates the type of edge of $G \circledast H$: 0 means no edge, s solid, d dotted and c confusable edge.

All finite graphs can be represented by their adjacency matrix. Usually, this matrix consists of binary elements. However, given the distinct types of edges defined for the PRG, the adjacency matrix of a PRG must also carry this information. We do this by allowing elements of the form $0, s, d,$ or c, to indicate no, solid, dotted, or confusable edges, respectively. Moreover, the elements on the diagonal are all zeros by convention, and given that this is an undirected graph, the adjacency matrix is symmetric. An example of an adjacency matrix of a PRG can be seen in Equation (5). We denote the adjacency matrix of $G_{X^n,Y_R^n|Y^n}$ by $A_{G_{X^n,Y_R^n|Y^n}}$.

Properties of SSP. The SSP is commutative, associative, distributive over a disjoint union, and that K is a unit for the SSP if $|V(K)| = 1$. In addition, the SSP reduces to the regular strong product when all edges are of the "s" type (in E_s), i.e. if $E_d(G) = E_c(G) = E_d(H) = E_c(H) = \emptyset$, then $G \circledast H \cong G \boxtimes H$.

C. PRC related subgraphs and operations

We show how $G_{X^n|Y^n,Y_R^n}$ and $G_R^{(n)}$ from [1] may be obtained from the PRG and its SSP. Given PRG $G_{X,Y_R|Y}(V, E)$, consider subgraph H_c defined as $V(H_c) = V(G_{X,Y_R|Y})$, and $E(H_c) = E_c(G_{X,Y_R|Y}) \subseteq E(G_{X,Y_R|Y})$. Then one can show the following, which can be extended to any n.

Lemma 2. *Graph H_c is p_x-homomorphic to $G_{X|Y,Y_R}$, $p_x : V(G_{X,Y_R|Y}) \to V_x$ is the projection $p_x(x, y_R) = x$.*

For an independent set $\mathcal{K} \subseteq V_x(G_{X|Y,Y_R})$, let $H_s|_\mathcal{K}$ denote the subgraph with $V(H_s|_\mathcal{K}) = \{(x, y_R) \in V : x \in \mathcal{K}\}$, and $E(H_s|_\mathcal{K}) = \{(x, y_R)(x', y'_R) \in E_s : x \in \mathcal{K}\}$.

Lemma 3. *Given independent set $\mathcal{K}$ for confusability graph $G_{X|Y,Y_R}$, graph $H_s|_\mathcal{K}$ is p_{y_R}-homomorphic to the relaying compression graph $G_{R|\mathcal{K}}$, where $p_{y_R} : V \to V_{y_R}$ is the projection $p_{y_R}(x, y_R) = y_R$.*

Lemma 4. *Given PRG $G_{X,Y_R|Y}$, for any $n \geq 2$*

$$G_{X^n,Y_R^n|Y^n} = G_{X,Y_R|Y}^{\circledast n}, \tag{3}$$

where $G^{\boxtimes n}_{X,Y_R|Y}$ denotes the n-fold special strong product of PRG $G_{X,Y_R|Y}$ and mimics the notation $G^{\boxtimes n}$ for the conventional n-fold strong product of G. For adjacency matrices:

$$A^{(n)}_{G_{X,Y_R|Y}} = \left(A^{(1)}_{G_{X,Y_R|Y}} + \Delta\mathbb{I}\right) \otimes \left(A^{(n-1)}_{G_{X,Y_R|Y}} + \Delta\mathbb{I}\right) - \Delta\mathbb{I}, \tag{4}$$

where $A^{(n)}_G$ denotes the adjacency matrix of the n-fold SSP of G. The above shows that $A_{G_{X^n,Y^n_R|Y^n}} = A^{(n)}_{G_{X,Y_R|Y}}$.

In the Kronecker product in (4), multiplication follows the SSP multiplication table. We also note that if A_Γ and A_Φ are the adjacency matrices of graphs Γ and Φ then the adjacency matrix of the (regular) strong product may be obtained from these as $(A_\Gamma + \mathbb{I}) \otimes (A_\Phi + \mathbb{I}) - \mathbb{I}$. Our special strong product is of a similar form, with the identity matrix part changed to $\Delta\mathbb{I}$ (for Δ defined in the incidence function) and multiplication replaced by the SSP multiplication table.

Example 1. *Given the conditional probability mass function $p(y|x, y_R)$ shown in Fig. 2 we obtain the PRG $G_{X,Y_R|Y}$ using the definition in Section III-A. This example then shows how this PRG can be used to obtain $G_{X|Y,Y_R}$ (Lemma 2) and $G_{Y_R|\mathcal{K}_i}$ (Lemma 3), needed to solve Problem 1 as in [1] and Theorem 1. Here, there are two independent sets of $G_{X|Y,Y_R}$: $\mathcal{K}_1 = \{1, 3\}$ and $\mathcal{K}_2 = \{2, 3\}$. From these, Lemma 3 and the PRG allow one to obtain the two compression graphs $G_{Y_R|\mathcal{K}_i}$, for $i = 1, 2$, as in Fig. 2.*

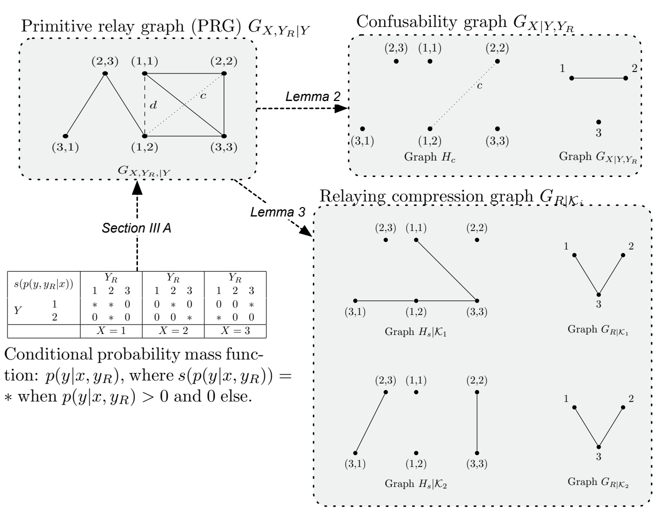

Fig. 2. Construction of the PRG $G_{X,Y_R|Y}$ from a probability mass function $p(y, y_R|x)$ for a primitive relay channel, and how to obtain graphs H_c and $G_{X|Y,Y_R}$ (Lemma 4) and graphs $H_s|_{\mathcal{K}_i}$ and associated $G_{R|_{\mathcal{K}_i}}$.

Example 2. *This example shows how to use Lemma 4 to obtain the PRG iteratively for any n. Consider the channel transition probability matrix presented in Table I. The adjacency matrix for the PRG $G_{X,Y_R|Y}$ is (5).*

TABLE I
CONDITIONAL PROBABILITY MASS FUNCTION: $p(y|x, y_R)$, WHERE $s(p(y|x, y_R)) = *$ WHEN $p(y|x, y_R) > 0$ AND 0 ELSE.

$s(p(y, y_R\|x))$		Y_R 1	2	3	4	Y_R 1	2	3	4
Y	1	$*$	0	$*$	0	0	$*$	0	$*$
	2	0	0	0	0	0	$*$	0	$*$
	3	$*$	0	$*$	0	0	0	0	0
		$X = 1$				$X = 2$			

$$A^{(1)}_{G_{X,Y_R|Y}} = \begin{array}{c} \begin{array}{cccc} (1,1) & (1,3) & (2,2) & (2,4) \end{array} \\ \begin{bmatrix} 0 & d & s & s \\ d & 0 & s & s \\ s & s & 0 & d \\ s & s & d & 0 \end{bmatrix} \end{array} \begin{array}{c} (1,1) \\ (1,3) \\ (2,2) \\ (2,4) \end{array} \tag{5}$$

Then, $A^{(n)}_{G_{X,Y_R|Y}}$ can be computed for any n, according to Lemma 4, and in particular Equation (4). We use the short notation $\mathbb{A}'$ to refer to $\left(A^{(n-1)}_{G_{X,Y_R|Y}} + \Delta\mathbb{I}\right)$. For this example, given the adjacency matrix (5), $A^{(n)}_{G_{X,Y_R|Y}}$ is given by

$$A^{(n)}_{G_{X,Y_R|Y}} = \begin{bmatrix} \mathbb{A}' & d\mathbb{A}' & s\mathbb{A}' & s\mathbb{A}' \\ d\mathbb{A}' & \mathbb{A}' & s\mathbb{A}' & s\mathbb{A}' \\ s\mathbb{A}' & s\mathbb{A}' & \mathbb{A}' & d\mathbb{A}' \\ s\mathbb{A}' & s\mathbb{A}' & d\mathbb{A}' & \mathbb{A}' \end{bmatrix} - \Delta\mathbb{I} \tag{6}$$

We notice in the above that only diagonal elements are subtracted (with the convention that $s - 0 = s, c - 0 = c, d - 0 = d$ and $\Delta - \Delta = 0$), and that the diagonal of is indeed zero.

Once the PRG graph is obtained for a particular n, it is possible to generate the confusability graph $G^{(n)}_{X|Y,Y_R}$ and all the corresponding compression graphs for each independent set $\mathcal{K}_i$ found in this graph. The benefit comes from having a recursive, compact way to build up the n-fold graphs needed. As a future potential benefit still under exploration, we believe this graph and SSP may point out the structure of optimal relaying / compression schemes.

IV. USING THE PRG TO SOLVE FOR $r_z^{*(n)}$

In this section we show how the SSP can be used to obtain the n-shot relaying compression graph $G^{(n)}_{R|\mathcal{K}}$ for any independent set $\mathcal{K}^{(n)}$ when $n \geq 1$, hence solving Problem 1 more easily than in [1] due to the smaller, and recursively defined PRG needed to compute the quantities in Corollary 1.

As an aside before tackling general alphabets, for binary primitive relay channels where $||\mathcal{X}|| = ||\mathcal{Y}_R|| = ||\mathcal{Y}|| = 2$, we are able to show the following. A binary PRC is called trivial if $\alpha(G(X^n|Y^n, Y^n_R)) = 1$ (the capacity of PRC is zero) or $\alpha(G(X^n|Y^n)) = 2^n$ (relaying is useless). The following proposition show and the definition of non-trivial PRCs imply that for binary alphabets and any n, the relay is either not used at all, or simply forwards its received signal, in order to be optimal in the sense of Problem 1.

Proposition 5. *For any non-trivial binary PRC channel, no compression is possible, i.e. $r_z^{*(n)} = 1$ for any n.*

Algorithm 1 SSP Algorithm for obtaining $r_z^{*(n)}$

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Input: p(y|y_R, x)
Output: r_z^{*(i)} , and R_z^{(i)} for 1 ≤ i ≤ n.
Derive PRG G_{X,Y_R|Y} corresponding to p(y|y_R, x)
for (i = 1 to i ≤ n) do
    Initialize r_z^{(i)} = i log |𝒴_R| and R_z^{(i)} = log |𝒳|.
    Compute the i−fold SSP (G_{X,Y_R|Y})^{⊠i}
    Obtain G_{X^i|Y^i,Y_R^i} = p_x (H_c(G_{X,Y_R|Y})^{⊠i})
    for each max ind. set 𝒦^i of G_{X^i|Y^i,Y_R^i} do
        Compute G_R^{(i)}|_{𝒦^{(i)}} = p_{y_R} (H_s(G_{𝒦^{(i)},Y_R|Y})^{⊠i}))
        if (1/i) log χ(G_R^{(i)}|_{𝒦^{(i)}}) < r_z^{(i)} then
            r_z^{(i)} = (1/i) log χ(G_R^{(i)}|_{𝒦^{(i)}}), R_z^{(i)} = (1/i) log |𝒦^{(i)}|.
        end if
    end for
end for
```

Algorithm 1 shows a greedy approach to obtain $r_z^{*(n)}$ for any finite n. It mimics that in [1], but instead of computing $G_{X^i|Y^i,Y_R^i}$ and $G_R^{(i)}|_{\mathcal{K}^{(i)}}$ directly, they are obtained more easily from projections of the PRG (via Lemmas 2 and 3). The benefit is that $G_{X,Y_R|Y}^{\boxtimes i}$ is obtained as a special strong product, whereas $G_R^{(i)}|_{\mathcal{K}^{(i)}}$ must be obtained directly from the channel for each i and for each $\mathcal{K}^{(i)}$ independent set of $G_{X^i|Y^i,Y_R^i}$. The computation of $G_{X,Y_R|Y}^{\boxtimes i}$ requires about the same computational complexity as a conventional strong product and may be iteratively computed. For any i, $G_{X^i|Y^i,Y_R^i} = p_x\left(H_c(G_{X,Y_R|Y})^{\boxtimes i}\right)$ and $G_R^{(i)}|_{\mathcal{K}^{(i)}} = p_{y_R}\left(H_s(G_{\mathcal{K}^{(i)},Y_R|Y})^{\boxtimes i}\right)$ may be obtained from $G_{X,Y_R|Y}^{\boxtimes i}$ by simple projections. The complexity is thus reduced, and the structure of the problem is simplified.

We next obtain a lower bound on $r_z^{*(n)}$ directly from the PRG, avoiding the need to repeatedly project.

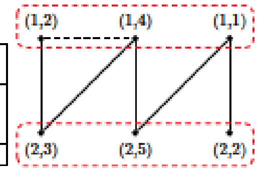
Proposition 6. *Given an independent set $\mathcal{K}^{(n)}$ of $G_{X^n|Y^n,Y_R^n}$, Let $H_s(G_{\mathcal{K}^{(n)},Y_R|Y})^{\boxtimes n}$ be the PRG restricted to input set $\mathcal{K}^{(n)}$. Then, $r_z^{*(n)} \geq \chi\left(H_2(G_{\mathcal{K}^{(n)},Y_R|Y})^{\boxtimes n}\right)$.*

Example 3. *We demonstrate a channel for which $r_z^{*(2)} < r_z^{*(1)}$, and for which relaying compression graph $G_R|_{\mathcal{K}} \boxtimes G_R|_{\mathcal{K}}$ is a strict subset of $G_R^{(2)}|_{\mathcal{K}\times\mathcal{K}}$. This illustrates that the relaying compression graph does not obey the (regular) strong product. Instead, we can use the method presented here to compute the SSP of the PRG, and then generate the compression graph for any n using homomorphic projections, as in Section III-C.*

Consider the channel transition probability matrix and its PRG shown in Fig. 3. Note that dashed boundaries have been drawn to identify node-pairs with the same inputs. Since there are no confusable edges, the maximal independent set of $G_{X|Y,Y_R}$ is the input set itself $\mathcal{K} = \{1,2\}$. We can construct $G_{\mathcal{R}|\mathcal{K}}$ by performing the homomorphic projection as in Section III-C and obtain the pentagon-shaped compression graph, which is minimally colored using 3 colors, and consequently, $r_z^{*(1)} = \log_2(3) = 1.5850$ bits.

$s(p(y, y_R\|x))$		Y_R 1	2	3	4	5	Y_R 1	2	3	4	5
Y	1	•	0	0	0	0	0	•	0	0	•
	2	0	•	0	•	0	0	0	•	0	0
	3	0	0	0	•	0	0	0	0	0	•
		$X=1$					$X=2$				

Conditional probability mass function: $p(y|x, x_R)$, where $s(p(y|x, x_R)) = \bullet$ when $p(y|x, x_R) > 0$, and 0 else.



Primitive relay graph (PRG) $G_{X,Y_R|Y}$

Fig. 3. Conditional probability mass function and its associated PRG.

Moreover, to illustrate how this method can be used for $n \geq 1$, consider $n = 2$. In particular, $\mathcal{K}^{(2)} = \mathcal{K} \times \mathcal{K}$ is the maximal independent set of of $G_{X^2|Y^2,Y_R^2}$. The compression graph $G_R^{(2)}|_{\mathcal{K}\times\mathcal{K}}$ next to its corresponding adjacency matrix (black squares denote the presence of an edge, white the absence of an edge) are shown in Fig. 4.

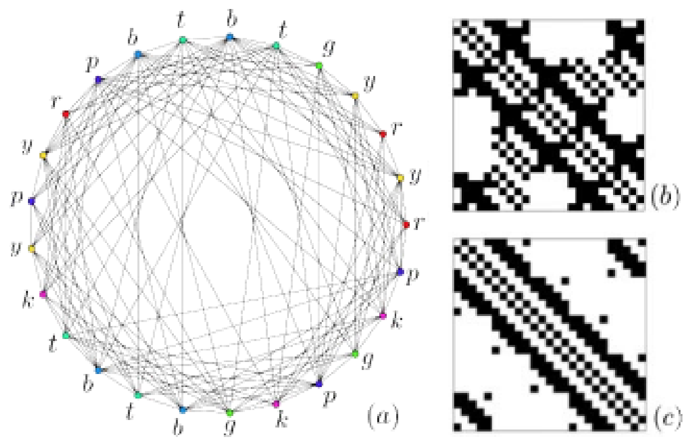


Fig. 4. (a) Compression graph $G_R^{(2)}|_{\mathcal{K}\times\mathcal{K}}$ for $n = 2$, and its adjacency matrix (b). The adjacency matrix of the strong product is shown in (c).

In particular, coloring $G_R^{(2)}|_{\mathcal{K}\times\mathcal{K}}$ requires 7 colors, and so $r_z^{*(2)} = \frac{1}{2}\log_2(7) = 1.4037$ bits. Next, we can visually observe that $G_R|_{\mathcal{K}} \boxtimes G_R|_{\mathcal{K}} \subset G_R^{(2)}|_{\mathcal{K}\times\mathcal{K}}$ It is clear that there are several edges that are not captured by simple performing the strong product, motivating the SSP developed.

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