

# ANDERSON LOCALIZATION FOR RADIAL TREE GRAPHS WITH RANDOM BRANCHING NUMBERS

DAVID DAMANIK AND SELIM SUKHTAIEV

ABSTRACT. We prove Anderson localization for the discrete Laplace operator on radial tree graphs with random branching numbers. Our method relies on the representation of the Laplace operator as the direct sum of half-line Jacobi matrices whose entries are non-degenerate, independent, identically distributed random variables with singular distributions.

## 1. INTRODUCTION

The main goal of this paper is to prove Anderson localization for the discrete Laplace operator on rooted radial tree graphs with random branching numbers. These are tree graphs with a fixed vertex  $o$ , the root, such that every vertex  $v$  at a distance  $n$  from the root  $o$  is connected with  $b_n \geq 2$  vertices at a distance  $n+1$  from  $o$ , cf. Figure 1. Assuming that  $\{b_n\}_{n=0}^\infty$  is a sequence of non-degenerate i.i.d. random variables we show that, almost surely, the Laplace operator  $\Delta$ , cf. (2.1), has pure point spectrum and admits a basis of exponentially decaying eigenfunctions. More concretely, our main result is the following Theorem.

**Theorem 1.1.** *Suppose that  $\{\Delta(\omega)\}_{\omega \in \Omega}$  is a family of Laplace operators on radial tree graphs with random branching numbers. Suppose that the branching numbers are given by non-degenerate independent identically distributed random variables*

$$\{b_n(\omega)\}_{n=0}^\infty \subset \{2, \dots, d\} \text{ for some fixed } d > 2.$$

*Then  $\Delta(\omega)$  exhibits Anderson localization at all energies. That is, almost surely,  $\Delta(\omega)$  has pure point spectrum and possesses a basis of exponentially decaying eigenfunctions.*

The spectral theory of Schrödinger operators on tree graphs has attracted a lot of attention; see, for example, [1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 17, 18, 19, 20, 21] and references therein. When studying the effects of randomness, one can for example consider random geometry (and then one typically studies the Laplacian), or one can consider a random potential (and then one typically fixed regular geometry). The effects of disorder in the geometry of trees have been studied in [1, 13]. In [1] the authors consider trees with edge lengths given by  $\ell_e(\omega) = \ell e^{\lambda \omega_e}$ , where  $\ell > 0$  is fixed, and  $\lambda \in [0, 1]$  determines the strength of the disorder and  $\{\omega_e\}_{e \in \mathcal{E}}$  are i.i.d. random variables. It is proved in [1] that the absolutely continuous spectrum of the Laplace operator is continuous (in the sense of [1, Theorem 1.1]) at  $\lambda = 0$  almost surely. In the same work it is conjectured that such a continuity property fails in

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*Date:* October 30, 2018.

*Key words and phrases.* Anderson localization, Laplace operator, tree graphs.

D.D. was supported in part by NSF grant DMS-1700131.

S.S. was supported in part by an AMS-Simons travel grant, 2017-2019.

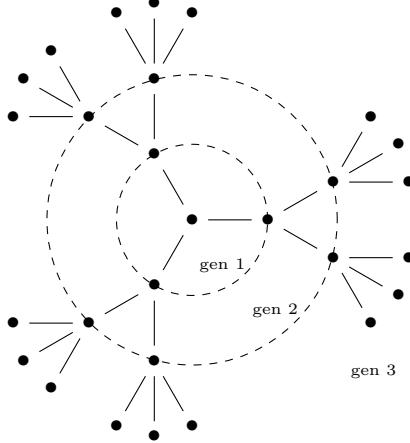


FIGURE 1. The first three generations of a radial rooted tree graph  $\Gamma$  with branching numbers  $b_0 = 3$ ,  $b_1 = 2$ ,  $b_2 = 3$ .

the case of radial disorder. This conjecture was settled in [13], where the authors showed that in the radial case the spectrum is almost surely pure point. In fact, they proved Anderson localization for the random length and random Kirchhoff models by essentially the same method. Our work is motivated by that of P. Hislop and O. Post. The random branching model considered in this paper naturally complements the two models considered in [13]. However, it is worth noting that the methods of [13] are not applicable in the present setting since they are based on spectral averaging and hence rely heavily on the assumption that the random variables are absolutely continuous. Of course, in the case of random branching numbers such a hypothesis cannot be made. We therefore turn to a recent work [7] (which was inspired by [5]) that offers a new proof of Anderson localization for random Schrödinger operators on  $\mathbb{Z}$ . Their methods can be adapted to show localization for the random half-line Jacobi matrices (2.2) that naturally arise in the context of Laplace operators on radial tree graphs. Namely, we show that almost surely for every generalized eigenvalue  $E$ , cf. (3.21), one has

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{M}_n^E(\omega)\| = L(E),$$

where  $\mathcal{M}_n^E(\omega)$  is the  $n$ -step transfer matrix, cf. (3.3), (3.4), and  $L(E)$  is the Lyapunov exponent which is shown to be positive for  $E \neq 0$ . Once we have established localization for these Jacobi matrices, we then show that the Laplace operator  $\Delta$  almost surely has a basis of exponentially decaying eigenfunctions. Due to the exponential growth of the surface area of spheres, exponential decay of eigenfunctions is not automatically sufficient to ensure their square summability. However, the decay rate that we establish, cf. (3.62)–(3.64), is sufficient to draw this conclusion. Interestingly, a similar issue does not arise in the setting of  $\mathbb{Z}^d$  since in this case the cardinality of spheres grows polynomially and exponential decay of any rate is sufficient for square summability.<sup>1</sup>

<sup>1</sup>We thank the referee for bringing this to our attention.

## 2. THE LAPLACIAN ON RADIAL TREE GRAPHS

In this section we consider the Laplace operator on discrete rooted tree graphs  $\Gamma = (\mathcal{V}, \mathcal{E})$ , with the set of vertices  $\mathcal{V}$  and the set of edges  $\mathcal{E}$ . A tree graph  $\Gamma$  is a graph without nontrivial closed paths. A tree graph  $\Gamma$  together with a fixed vertex  $o \in \mathcal{V}$  is called a rooted tree graph. The distance  $d(u, v)$  between two vertices  $u, v \in \mathcal{V}$  is defined as the number of edges of the unique (non-repetitive) path connecting  $u$  and  $v$ . The generation of a vertex  $v$  is defined by  $\text{gen}(v) := d(o, v)$  and its branching number is given by  $b(v) := \deg v - 1$  if  $v \neq o$ , and  $b(o) = \deg o$ . A rooted tree graph is called radial if the branching number is a function of the generation only, that is  $b(v) = b_n \geq 0$  whenever  $\text{gen}(v) = n$ , for some sequence  $\{b_n\}_{n=0}^{\infty} \subset \mathbb{N}$ . In other words, cf. [6, Definition 2.1], a rooted graph is radial if any vertex  $v$  in generation  $n$  is connected with  $b_n$  vertices in generation  $n + 1$ .

Assuming that the sequence  $\{b_n\}_{n=1}^{\infty}$  is bounded, we define the Laplace operator via the adjacency matrix of the tree. Concretely, we let

$$(\Delta f)(v) := \sum_{u \in \mathcal{V} : d(u, v) = 1} f(u), \quad f \in \ell^2(\mathcal{V}). \quad (2.1)$$

The boundedness of the sequence of branching numbers yields  $\Delta \in \mathcal{B}(\ell^2(\mathcal{V}))$ ; moreover, since  $\Delta$  is clearly symmetric, one has  $\Delta = \Delta^*$  in  $\ell^2(\mathcal{V})$ .

An important step in the proof of Anderson localization is representing  $\Delta$  as the direct sum of half-line Jacobi matrices given by

$$J_n := \begin{pmatrix} 0 & \sqrt{b_n} & 0 & & \\ \sqrt{b_n} & 0 & \sqrt{b_{n+1}} & \ddots & \\ 0 & \sqrt{b_{n+1}} & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \end{pmatrix}, \quad n \in \mathbb{Z}^+ \quad (2.2)$$

and acting in  $\ell^2(\mathbb{Z}^+)$ .

We assume that non-trivial branching occurs in every generation.

**Hypothesis 2.1.** *Suppose that for some  $d \in \mathbb{N}$ ,  $b_n \in \{2, \dots, d\}$ ,  $n \in \mathbb{Z}^+$ .*

**Theorem 2.2.** [6, Theorem 2.4] *Assume Hypothesis 2.1. Let us define*

$$\beta_k := \begin{cases} (b_{k-1} - 1) \prod_{j=0}^{k-2} b_j, & k \geq 2, \\ b_0 - 1, & k = 1, \\ 1, & k = 0. \end{cases} \quad (2.3)$$

*Then one has*

$$\ell^2(\mathcal{V}) = \bigoplus_{N=0}^{\infty} \bigoplus_{k=1}^{\beta_N} H_{N,k}, \quad \Delta \restriction_{H_{N,k}} \cong J_N. \quad (2.4)$$

*where every subspace  $H_{N,k}$  reduces  $\Delta$ , for all admissible  $N, k$ .*

Concretely, there exists an orthonormal basis

$$\{\varphi_{N,k,j}\}_{N \in \mathbb{Z}^+, 1 \leq k \leq \beta_N, j \in \mathbb{Z}^+} \text{ in } \ell^2(\mathcal{V}),$$

so that

- 1)  $\text{supp}(\varphi_{N,k,j}) \subset \{v \in \mathcal{V} : \text{gen}(v) = N + j\}$ , for all admissible  $N, k, j$ ,
- 2)  $\{\varphi_{N,k,j}\}_{j \in \mathbb{Z}^+}$  is an orthonormal basis for  $H_{N,k}$ , for all admissible  $N, k$ .
- 3)  $\|\varphi_{N,k,j+1}\|_{\ell^{\infty}(\mathcal{V})} = b_j^{-1/2} \|\varphi_{N,k,j}\|_{\ell^{\infty}(\mathcal{V})}$  for all admissible  $N, k, j$ .

Associated with this basis is a unitary operator  $\Phi$  given by

$$\Phi : \ell^2(\mathcal{V}) \rightarrow \bigoplus_{N=0}^{\infty} \bigoplus_{k=1}^{\beta_N} \ell^2(\mathbb{Z}^+); \quad \varphi_{N,k,j} \mapsto \delta_j(N, k), \quad j \in \mathbb{Z}^+, \quad (2.5)$$

where  $\delta_j(N, k) = \delta_j$  should be viewed as a vector in the  $N, k$ -th copy of  $\ell^2(\mathbb{Z}^+)$ . Then one has

$$\Phi \Delta \Phi^{-1} = \bigoplus_{N=0}^{\infty} \bigoplus_{k=1}^{\beta_N} J_N(k), \quad (2.6)$$

where  $J_N(k) = J_N$  is the Jacobi matrix acting in the  $N, k$ -th copy of  $\ell^2(\mathbb{Z}^+)$  and given by (2.2).

### 3. ANDERSON LOCALIZATION FOR HALF-LINE JACOBI MATRICES

In this section we turn to radial trees with branching numbers given by i.i.d. random variables which take values in the set  $\mathcal{A} = \{2, \dots, d\}$  for some fixed  $d > 2$ . Suppose that  $(\mathcal{A}, \tilde{\mu})$  is a probability space and that  $\text{supp } \tilde{\mu}$  contains at least two elements,  $\#\text{supp } \tilde{\mu} \geq 2$ . Furthermore, let us denote  $\Omega := (\text{supp } \tilde{\mu})^{\mathbb{Z}^+}$ ,  $\mu := \tilde{\mu}^{\mathbb{Z}^+}$  and define the left shift

$$(T\omega)_n := \omega_{n+1}, \quad \omega \in \Omega, \quad n \in \mathbb{Z}^+.$$

For a given  $\omega \in \Omega$ ,  $\Delta(\omega)$  denotes the Laplace operator on the radial tree graph  $\Gamma_\omega$  with branching numbers defined by  $b_n(\omega) := \omega_n$ .

As discussed in Theorem 2.2, the operator  $\Delta(\omega)$  gives rise to a sequence of Jacobi matrices. The  $n$ -th matrix of this sequence is given by  $J_0(T^n\omega)$ , where the self-adjoint operator  $J_0(\omega)$  is acting in  $\ell^2(\mathbb{Z}^+)$  and given by

$$[J_0(\omega)u](n) := \begin{cases} \sqrt{\omega_{n-1}}u(n-1) + \sqrt{\omega_n}u(n+1), & n \in \mathbb{N}, \\ \sqrt{\omega_0}u(1), & n = 0. \end{cases} \quad (3.1)$$

A sequence  $u = \{u_n\}_{n=0}^{\infty}$  satisfies

$$\begin{cases} \sqrt{\omega_{n-1}}u(n-1) + \sqrt{\omega_n}u(n+1) = Eu(n), & n \in \mathbb{N}, \\ \sqrt{\omega_0}u(1) = Eu(0), \end{cases} \quad (3.2)$$

$E \in \mathbb{R}$  if and only if

$$\begin{bmatrix} u_{n+1} \\ \sqrt{\omega_n}u_n \end{bmatrix} = \mathcal{M}^E(T^{n-1}\omega) \begin{bmatrix} u_n \\ \sqrt{\omega_{n-1}}u_{n-1} \end{bmatrix}, \quad \mathcal{M}^E(\omega) := \begin{bmatrix} \frac{E}{\sqrt{\omega_1}} & -\frac{1}{\sqrt{\omega_1}} \\ \sqrt{\omega_1} & 0 \end{bmatrix}, \quad (3.3)$$

$n \geq 1$ . The map  $\mathcal{M}^E : \Omega \rightarrow \text{SL}(2, \mathbb{R})$  determines an  $\text{SL}(2, \mathbb{R})$ -cocycle in a canonical way

$$(T, \mathcal{M}^E) : \Omega \times \mathbb{R}^2 \rightarrow \Omega \times \mathbb{R}^2, \quad (T, \mathcal{M}^E)(\omega, v) = (T\omega, \mathcal{M}^E(\omega)v).$$

The iterates over the skew product are given by  $(T, \mathcal{M}^E)^n := (T^n, \mathcal{M}_n^E)$ , where  $\mathcal{M}_0^E := I_2$  and

$$\mathcal{M}_n^E(\omega) = \prod_{i=n-1}^0 \mathcal{M}^E(T^i\omega), \quad n \in \mathbb{N}. \quad (3.4)$$

The Lyapunov exponent of this cocycle is defined by

$$L(E) := \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} \log \|\mathcal{M}_n^E(\omega)\| d\mu(\omega). \quad (3.5)$$

By Kingman's Subadditive Ergodic Theorem,

$$L(E) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{M}_n^E(\omega)\| \quad (3.6)$$

for  $\mu$ -almost every  $\omega$ .

Our proof of positivity of the Lyapunov exponent is based on the following facts.

**Theorem 3.1.** *Let  $\nu$  be a probability measure on  $\mathrm{SL}(2, \mathbb{R})$  that satisfies*

$$\int \log \|M\| d\nu(M) < \infty.$$

*Let  $G_\nu$  be the smallest closed subgroup of  $\mathrm{SL}(2, \mathbb{R})$  that contains  $\mathrm{supp} \nu$ .*

*(i) [10, Theorem 8.6] Assume that  $G_\nu$  is not compact and that it is strongly irreducible (cf. the definition preceding Theorem 2.1 in [7]). Then the Lyapunov exponent  $L(\nu)$  associated with  $\nu$  is positive.*

*(ii) [11, Theorem B] Assume that the set*

$$\mathrm{Fix}(G_\nu) := \{V \in \mathbb{RP}^1 : MV = V \text{ for every } M \in G_\nu\}$$

*contains at most one element. If  $\nu_k \rightarrow \nu$  weakly and boundedly (cf. the definitions preceding Theorem 2.5 in [7]), then  $L(\nu_k) \rightarrow L(\nu)$  as  $k \rightarrow \infty$ .*

A subgroup of  $\mathrm{SL}(2, \mathbb{R})$  is called contracting if there exists a sequence  $\{g_n\}_{n=1}^\infty$  of its elements such that  $\|g_n\|g_n^{-1}$  converges to a rank-one matrix.

**Theorem 3.2.** *Let  $\nu^E$  denote the push-forward measure of  $\tilde{\mu}$  under the map  $\mathcal{M}^E$ , cf. (3.3). Suppose that  $G_{\nu^E}$  is the smallest subgroup of  $\mathrm{SL}(2, \mathbb{R})$  that contains  $\mathrm{supp} \nu^E$ . Then  $G_{\nu^E}$  is a non-compact, strongly irreducible, contracting subgroup for every  $E \in \mathbb{R} \setminus \{0\}$ . Moreover, one has*

$$\mathrm{Fix}(G_{\nu^E}) := \{V \in \mathbb{RP}^1 : MV = V \text{ for every } M \in G_{\nu^E}\} = \emptyset, \quad (3.7)$$

for all  $E \in \mathbb{R}$ . Furthermore, the Lyapunov exponent is a continuous function of the energy  $E$ . In addition,  $L(E) > 0$  if  $E \neq 0$  and  $L(0) = 0$ .

*Proof.* For all  $E \in \mathbb{R}$ ,  $G_{\nu^E}$  contains at least 2 distinct elements of the form

$$M_\alpha := \begin{bmatrix} \frac{E}{\sqrt{\alpha}} & -\frac{1}{\sqrt{\alpha}} \\ \sqrt{\alpha} & 0 \end{bmatrix}. \quad (3.8)$$

Let us pick arbitrary  $\alpha \neq \beta$  and observe that the following sequence of matrices is unbounded and belongs to  $G_{\nu^E}$

$$A_n := (M_\alpha M_\beta^{-1})^n = \begin{bmatrix} \alpha^{-\frac{n}{2}} \beta^{\frac{n}{2}} & 0 \\ 0 & \alpha^{\frac{n}{2}} \beta^{-\frac{n}{2}} \end{bmatrix}, n \in \mathbb{Z}^+. \quad (3.9)$$

Hence,  $G_{\nu^E}$  is not compact.

Let us fix  $E \neq 0$ . In order to prove that  $G_{\nu^E}$  is strongly irreducible, we observe that for any  $V \in \mathbb{RP}^1$ , one has

$$\lim_{n \rightarrow \pm\infty} A_n V \in \{\mathrm{span}\{e_1\}, \mathrm{span}\{e_2\}\}, \quad (3.10)$$

where  $e_1 = (1, 0)^\top$ ,  $e_2 = (0, 1)^\top$ . Therefore, every finite subset  $\mathcal{F} \subset \mathbb{RP}^1$  which is invariant under  $G_{\nu^E}$  must be equal to one of the following sets:

$$V_1 := \mathrm{span}\{e_1\}, \quad V_2 := \mathrm{span}\{e_2\}, \quad \{V_1, V_2\}. \quad (3.11)$$

However, none of these sets is invariant when  $E \neq 0$ . Indeed, in this case one has

$$M_\alpha V_1 \notin \{V_1, V_2\}, \quad M_\alpha^{-1} V_2 \notin \{V_1, V_2\}. \quad (3.12)$$

To show (3.7), let us assume that there exists  $V \in \text{Fix}(G_{\nu_E})$ . Then  $A_n V = V$  for each  $n \in \mathbb{N}$ , hence, using (3.10) one infers that either  $V = V_1$  or  $V = V_2$ . On the other hand, neither of  $V_1, V_2$  belongs to  $\text{Fix}(G_{\nu_E})$ , since  $M_\alpha V_k \neq V_k$  for  $k \in 1, 2$ .

By Theorem 3.1, the Lyapunov exponent is continuous everywhere and positive at nonzero energies.

Next, we prove that  $L(0) = 0$ . To this end we explicitly compute the limit in (3.6). One has

$$\begin{aligned} \mathcal{M}_{2n}^0(\omega) &= \prod_{i=2n-1}^0 \mathcal{M}^0(T^i \omega) \\ &= \prod_{i=n}^1 \mathcal{M}^0(T^{2i-1} \omega) \mathcal{M}^0(T^{2i-2} \omega) \\ &= (-1)^n \begin{bmatrix} \left(\prod_{i=n}^1 \frac{\omega_{2i-1}}{\omega_{2i}}\right)^{1/2} & 0 \\ 0 & \left(\prod_{i=n}^1 \frac{\omega_{2i}}{\omega_{2i-1}}\right)^{1/2} \end{bmatrix}. \end{aligned} \quad (3.13)$$

Denoting

$$\xi_i := \frac{1}{2} \log(\omega_{2i-1} \omega_{2i}^{-1}), \quad (3.14)$$

we notice that  $\{\xi_i\}_{i=1}^\infty$  is a sequence of i.i.d. random variables satisfying  $\int_{\mathcal{A}} \xi_i d\tilde{\mu}^2 = 0$  for all  $i \in \mathbb{N}$ . By the law of large numbers, one has

$$\frac{1}{n} \sum_{i=1}^n \xi_i \rightarrow 0, \text{ as } n \rightarrow \infty, \text{ almost surely.} \quad (3.15)$$

Combining (3.13), (3.14), and (3.15), one infers

$$L(0) = \lim_{n \rightarrow \infty} \frac{1}{2n} \log \|\mathcal{M}_{2n}^0(\omega)\| = 0,$$

for  $\mu$ -almost every  $\omega \in \Omega$ .

Finally,  $G_{\nu_E}$  is contracting since  $A_n/\|A_n\|$  converges to a rank-one matrix as  $n \rightarrow \infty$ .  $\square$

**Theorem 3.3.** *Assume Hypothesis 2.1, then there exists a full-measure set  $\Omega_0$  such that*

$$\sigma(J_0(\omega)) = [-2\sqrt{d_\mu}, 2\sqrt{d_\mu}], \text{ for all } \omega \in \Omega_0, \quad (3.16)$$

where  $d_\mu := \max(\text{supp } \tilde{\mu})$ .

*Proof.* Clearly, for every  $\omega \in \Omega$  one has

$$\sigma(J_0(\omega)) \subset [-2\sqrt{d_\mu}, 2\sqrt{d_\mu}]. \quad (3.17)$$

Next, we prove that the opposite inclusion holds for  $\mu$ -a.e.  $\omega \in \Omega$ . To this end, let us consider the full-measure set (cf., e.g., [15, Proposition 3.8])

$$\Omega_0 := \bigcap_{R \in \mathbb{Z}^+} \bigcup_{k \in \mathbb{Z}^+} \{\omega \in \Omega : \omega_j = d_\mu \text{ for all } k \leq j < k + R\}. \quad (3.18)$$

Pick an arbitrary  $E \in [-2\sqrt{d_\mu}, 2\sqrt{d_\mu}]$  and define  $\theta = \theta(E) \in [0, \pi]$  by

$$E = \sqrt{d_\mu} (e^{i\theta} + e^{-i\theta}).$$

For every  $\omega \in \Omega_0$  we can find  $k_l \in \mathbb{Z}^+$  and  $R_l \rightarrow \infty$  as  $l \rightarrow \infty$  such that

$$\sqrt{\omega_j} = \sqrt{d_\mu} \text{ for all } k_l \leq j < k_l + R_l, \quad l \in \mathbb{N}.$$

The inclusion  $E \in \sigma(J_0(\omega))$  follows from the Weyl Criterion provided

$$\|(J_0(\omega) - E)\psi_l\| \rightarrow 0, \quad \ell \rightarrow \infty. \quad (3.19)$$

where, for each  $l \in \mathbb{N}$ ,

$$\psi_l(j) := \begin{cases} R_l^{-1/2} e^{ij\theta}, & \text{for all } k_l \leq j < k_l + R_l, \\ 0, & \text{otherwise.} \end{cases} \quad (3.20)$$

To prove (3.19) let us notice that

$$((J_0(\omega) - E)\psi_l)_j = \begin{cases} R_l^{-1/2} \omega_j^{1/2} e^{i(j+1)\theta}, & \text{if } j = k_l - 1, \\ R_l^{-1/2} (\omega_j^{1/2} e^{i(j+1)\theta} - E e^{ij\theta}), & \text{if } j = k_l, \\ 0, & \text{if } k_l < j < k_l + R_l - 1, \\ R_l^{-1/2} (\omega_{j-1}^{1/2} e^{i(j-1)\theta} - E e^{ij\theta}), & \text{if } j = k_l + R_l - 1, \\ R_l^{-1/2} \omega_{j-1}^{1/2} e^{i(j-1)\theta}, & \text{if } j = k_l + R_l, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\|(J_0(\omega) - E)\psi_l\|^2 \lesssim_{d_\mu, E} R_l^{-1} \underset{l \rightarrow \infty}{=} o(1),$$

concluding the proof.  $\square$

Next we focus on proving Anderson localization for  $J_0(\omega)$ . Our proof closely follows the main line of arguments from [7]. Let  $\mathcal{G}(J_0(\omega))$  denote the set of energies for which (3.2) admits non-trivial solutions satisfying

$$|u(n)| \leq C_u(1+n), \quad C_u > 0, n \in \mathbb{Z}^+. \quad (3.21)$$

A key ingredient of the proof of Anderson localization is the following theorem.

**Theorem 3.4.** *There exists a full-measure set  $\Omega_1 \subset \Omega$ , such that for all  $\omega \in \Omega_1$  and every  $E \in \mathcal{G}(J_0(\omega))$ , one has*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{M}_n^E(\omega)\| = L(E). \quad (3.22)$$

In particular,  $J_0(\omega)$  is spectrally localized for all  $\omega \in \Omega_1$ .

Let

$$I_l := [-2\sqrt{d}, 2\sqrt{d}] \setminus (-1/l, 1/l), \quad l > 1. \quad (3.23)$$

Then Theorem 3.4 follows from a slightly weaker result, Theorem 3.7, upon taking the intersection of all  $l$ -dependent full measure sets from Theorem 3.7. Hence, we proceed by discussing the latter theorem. The first item in the program is positivity and continuity of the Lyapunov exponent. By Theorem 3.2, the subgroup  $G_{\nu^E}$  is noncompact, strongly irreducible, contracting, and  $L(E) > 0$  for all  $E \in I_l$ ,  $l > 1$ .

Thus, the results of [7] concerning the products of i.i.d. random  $\text{SL}(2, \mathbb{R})$  matrices are applicable in the present setting. To record these results let us introduce some notation. For a given  $n \in \mathbb{N}$ , let

$$F_n(\omega, E) := \frac{1}{n} \log \|\mathcal{M}_n^E(\omega)\|, \quad (3.24)$$

$$J_0^n(\omega) := J_0(\omega) \upharpoonright_{[1, n]}. \quad (3.25)$$

For any  $E \notin \sigma(J_0^n(\omega))$ , we set

$$G_{\omega,n}^E := (J_0^n(\omega) - E)^{-1} \text{ and } G_{\omega,n}^E(j,k) := \langle \delta_j, G_{\omega,n}^E \delta_k \rangle, \quad j, k \in [1, n]. \quad (3.26)$$

Similarly, given  $a, b \in \mathbb{Z}^+$ ,  $a < b$ , we denote  $G_{\omega,[a,b]}^E := (J_0(\omega)|_{[a,b]} - E)^{-1}$  for  $E \notin \sigma(J_0(\omega)|_{[a,b]})$ .

**Theorem 3.5.** *Fix arbitrary  $\varepsilon \in (0, 1)$ ,  $\zeta \in \mathbb{Z}^+$ , and  $l > 1$ . Then there exists a subset  $\Omega_+(\varepsilon) \subset \Omega$  of full  $\mu$ -measure such that for all  $\omega \in \Omega_+(\varepsilon)$ , there exist  $n_0 = n_0(\omega, \varepsilon, l)$ ,  $n_1 = n_1(\omega, \varepsilon, l)$  so that the following statements hold.*

(i) [7, Proposition 5.2] *For all  $n \geq \max(n_0, (\log(|\zeta| + 1))^{2/3})$ , and  $E \in I_l$  one has*

$$\left| L(E) - \frac{1}{n^2} \sum_{s=0}^{n^2-1} F_n(T^{\zeta+sn}\omega, E) \right| < \varepsilon, \quad (3.27)$$

(ii) [7, Corollary 5.3] *For all  $n \geq \max(n_1, \log^2(|\zeta| + 1))$ , and  $E \in I_l$  one has*

$$\frac{1}{n} \|\log \mathcal{M}_n^E(T^\zeta \omega)\| \leq L(E) + 2\varepsilon. \quad (3.28)$$

(iii) *For all  $n \geq \varepsilon^{-1} \max(n_1, 2\log^2(|\zeta| + 1))$ ,  $E \in I_l \setminus \sigma(J_0(T^\zeta \omega))$ , and  $1 \leq j, k \leq n$ , one has*

$$\left| G_{T^\zeta \omega, n}^E(j, k) \right| \lesssim_d \frac{\exp[(n - |j - k|)L(E) + C_0 \varepsilon n]}{|\det(J_0^n(T^\zeta \omega) - E)|} \prod_{i=0}^n \sqrt{\omega_{i+\zeta}}, \quad (3.29)$$

where  $C_0 = C_0(\tilde{\mu}) > 0$ .

*Proof.* These results were obtained in [7] for Schrödinger operators on the whole line. Let us briefly point out a minor change in (3.29). The new term is the last factor in the right-hand side of (3.29). It appears due to the following relations between the transfer matrices and the Green's functions of truncated Jacobi matrices, cf. [14],

$$\mathcal{M}_n^E(\omega) = \frac{1}{\prod_{i=0}^n \sqrt{\omega_i}} \mathcal{A}(n) \begin{bmatrix} \mathcal{P}_n^E(\omega) & \sqrt{\omega_0} \mathcal{P}_{n-1}^E(T\omega) \\ -\sqrt{\omega_n} \mathcal{P}_{n-1}^E(\omega) & -\sqrt{\omega_n \omega_0} \mathcal{P}_{n-1}^E(T\omega) \end{bmatrix} \mathcal{A}^{-1}(0), \quad (3.30)$$

where

$$\begin{aligned} \mathcal{P}_n^E(\omega) &:= \det([E - J_0(\omega)]|_{[1,n]}), \quad n \geq 1, \\ \mathcal{P}_0^E(\omega) &:= 0, \quad \mathcal{P}_{-1}^E(\omega) := 1, \\ \mathcal{A}(n) &:= \text{diag}\{1, \sqrt{\omega_n}\}, \quad n \in \mathbb{Z}^+. \end{aligned} \quad (3.31)$$

Moreover, one has

$$G_{\omega,n}^E(j, k) = \frac{\mathcal{P}_{j-1}^E(\omega) \mathcal{P}_{n-k-1}^E(T^{k+1}\omega)}{\mathcal{P}_n^E(\omega)} \prod_{k \leq i < j} \sqrt{\omega_i}, \quad (3.32)$$

where  $1 \leq j \leq k \leq n$  and the vacuous product that occurs for  $j = k$  is defined to be equal to one. Using (3.30), (3.32), and [7, (5.13)] and following the proof of [7, Corollary 5.3], one infers (3.29).  $\square$

A crucial element of the proof of Anderson localization is the elimination of double resonances. We recall it in the following theorem.

**Theorem 3.6.** [7, Proposition 6.1] *Let us fix arbitrary  $\varepsilon > 0$ ,  $N \in \mathbb{N}$ , and define*

$$\mathcal{D}_N(\varepsilon, l) := \left\{ \omega \in \Omega \left| \begin{array}{l} \|G_{\omega, [0, n]}^E\| \geq e^{K^2} \text{ and } |F_n(T^r \omega, E)| \leq L(E) - \varepsilon, \\ \text{for some } K \geq N, 0 \leq n \leq K^9, E \in I_l, \\ K^{10} \leq r \leq \bar{K} := \lfloor K^{\log K} \rfloor \end{array} \right. \right\} \quad (3.33)$$

*Then there exist  $C > 0, \eta > 0$  that do not depend on  $N$  such that*

$$\mu(\mathcal{D}_N(\varepsilon, l)) \leq Ce^{-\eta N}. \quad (3.34)$$

*In particular,*

$$\Omega_-(\varepsilon) := \Omega \setminus \limsup_{N \rightarrow \infty} \mathcal{D}_N(\varepsilon, l), \quad (3.35)$$

*is a full-measure set.*

**Theorem 3.7.** *For every  $l > 1$ ,  $\mu$ -almost every  $\omega \in \Omega$  and every  $E \in \mathcal{G}(J_0(\omega)) \cap I_l$ , one has*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{M}_n^E(\omega)\| = L(E). \quad (3.36)$$

*Proof.* Let us define the full-measure set

$$\Omega_* := \Omega_0 \bigcap_{m \in \mathbb{N}} \Omega_+(m^{-1}) \cap \Omega_-(m^{-1}), \quad (3.37)$$

where  $\Omega_0, \Omega_-(\cdot), \Omega_+(\cdot)$  are defined in Theorem 3.3, Theorem 3.5, Theorem 3.6 respectively.

Next, let us fix arbitrary  $l > 1$ ,  $\omega \in \Omega_*$ , and  $E \in \mathcal{G}(J_0(\omega)) \cap I_l$ . Our objective is to show

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{M}_n^E(\omega)\| \leq L(E), \quad (3.38)$$

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{M}_n^E(\omega)\| \geq L(E). \quad (3.39)$$

The first inequality follows from (3.28). The second one requires a more subtle analysis. We follow the proof of [7, Theorem 1.2], modifying some model-specific arguments.

Our goal is to show that for a given  $\varepsilon > 0$  and  $\omega \in \Omega_*$  one has

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \|\mathcal{M}_n^E(\omega)\| \geq L(E) - \varepsilon. \quad (3.40)$$

To this end, let  $u$  be the generalized eigenfunction satisfying (3.2), (3.21) and normalized by  $u(0) = 1$ . Let

$$K := \lceil \varepsilon^{-1} \max\{n_0(\omega, \varepsilon, l), n_2(\omega, \varepsilon, l), N_0\} \rceil,$$

where  $n_0(\omega, \varepsilon, l)$  is defined in Theorem 3.5,  $n_2(\omega, \varepsilon, l)$  is the smallest natural number for which

$$\omega \in \bigcap_{i \geq n_2(\omega, \varepsilon, l)} (\Omega_-(\varepsilon) \setminus D_i(\varepsilon, l)), \quad (3.41)$$

and  $N_0 \geq 2$  is a sufficiently large number that will be defined later.

**Step 1.** There exist  $\{a, b\} \subset \mathbb{Z}^+$  such that  $a \leq K^9$ ,  $a + K^3 - 2 \leq b \leq a + K^3$  such that

$$\left| G_{\omega, [a, b]}^E(j, k) \right| \leq d^2 \exp(-|j - k|L(E) + C_0 \varepsilon K^3), \quad (3.42)$$

for all  $j, k \in [a, b]$ , and some  $C_0 > 0$  dependent only on the measure  $\tilde{\mu}$ .

Using (3.27) with  $n = K^3$ , one has

$$\sum_{s=0}^{K^6-1} \frac{1}{K^6} \left( L(E) - \frac{1}{K^3} \log \|\mathcal{M}_{K^3}^E(T^{sK^3}\omega)\| \right) < \varepsilon. \quad (3.43)$$

Hence, for some  $0 \leq t \leq K^3(K^6 - 1)$ , one has

$$L(E) - \frac{1}{K^3} \log \|\mathcal{M}_{K^3}^E(T^t\omega)\| < \varepsilon. \quad (3.44)$$

Combining (3.30)–(3.32), (3.44), and the fact that the norm of  $\mathcal{M}_{K^3}^E(T^t\omega)$  is at most four times its largest entry and assuming that  $N_0 \geq (\varepsilon^{-1} \log(4d^2))^{1/3}$ , one infers

$$L(E) - \frac{1}{K^3} \log \frac{\mathcal{P}_{b-a}^E(T^a\omega)}{\prod_{i=0}^{K^3} \sqrt{\omega_{i+t}}} < 2\varepsilon, \quad (3.45)$$

for some  $a \in \{t, t+1\}$  and  $b \in \{K^3 + a, K^3 + a - 1, K^3 + a - 2\}$ . Next, we pick  $N_0$  sufficiently large so that  $K^3 \geq 2 \log^2(1+a)$  to ensure applicability of (3.29) with  $\zeta$  replaced by  $a$  and  $n$  replaced by  $b-a$ . Using this modification of (3.29) and (3.45) we obtain

$$|G_{T^a\omega, b-a}^E(j, k)| \leq \frac{\exp[(b-a-|j-k|)L(E) + C_0\varepsilon(b-a)]}{\mathcal{P}_{b-a}^E(T^a\omega)} \prod_{i=0}^{b-a} \sqrt{\omega_{i+a}} \quad (3.46)$$

$$\leq d^2 \frac{\exp[(K^3 - |j-k|)L(E) + C_0\varepsilon K^3]}{\exp(K^3(L(E) - 2\varepsilon))} \quad (3.47)$$

$$= d^2 \exp[-|j-k|L(E) + (C_0 + 2)\varepsilon K^3], \quad (3.48)$$

for all  $a \leq j, k \leq b$ .

**Step 2.** Let  $\ell := \lfloor \frac{a+b}{2} \rfloor$ , then

$$|u(\ell)| \leq d^{-\frac{1}{2}} e^{-2K^2}. \quad (3.49)$$

Indeed, utilizing (3.21), (3.42), and the standard representation of  $u$  in terms of Green's function on  $[a, b]$  and the boundary values  $u(a-1), u(b+1)$ , one infers

$$\begin{aligned} |u(\ell)| &\leq \sqrt{\omega_{a-1}} |u(a-1)| \left| G_{\omega, [a, b]}^E(a, \ell) \right| + \sqrt{\omega_b} |u(b+1)| \left| G_{\omega, [a, b]}^E(\ell, b) \right| \\ &\leq d^{5/2} C_u (K^9 + K^3 + 1) e^{C_0\varepsilon K^3} (e^{-|\ell-a|L(E)} + e^{-|b-\ell|L(E)}) \\ &\leq 2d^{5/2} C_u (K^9 + K^3 + 1) e^{-K^3 L(E)/3 + C_0\varepsilon K^3} < e^{-2K^2}, \end{aligned} \quad (3.50)$$

where we assumed that  $N_0$  is sufficiently large so that (3.50) holds whenever  $K \geq N_0$ .

**Step 3.** Let us recall the normalization  $u(0) = 1$ . One therefore has

$$1 = u(0) \leq \sqrt{\omega_b} |u(\ell)| \left| G_{\omega, [0, \ell-1]}^E(0, \ell-1) \right|. \quad (3.51)$$

Combining (3.49), (3.51) and denoting  $p := \ell - 1$  we get

$$\left\| G_{\omega, [0, p]}^E \right\| \geq e^{K^2}. \quad (3.52)$$

Using the inclusion (3.41),  $0 \leq p \leq K^9$  and (3.33), we infer

$$\frac{1}{j} \log \|\mathcal{M}_j^E(T^r\omega)\| > L(E) - \varepsilon. \quad (3.53)$$

for all  $j \in \{K, 2K\}$ ,  $r \in [K^{10}, \bar{K}]$ . This fact together with the Avalanche principle yield

$$\frac{1}{m} \log \|\mathcal{M}_m^E(\omega)\| \geq L(E) - \varepsilon, \quad (3.54)$$

for all  $m \in [K^{11} + K^{10}, \bar{K}]$  (cf. [7, (6.17)–(6.19)]). Since every sufficiently large integer belongs to one of these intervals we obtain (3.40).  $\square$

A function  $f \in \ell^2(\mathcal{V})$  is said to have almost exponential decay rate  $L > 0$  if for all  $\varepsilon > 0$  there exists  $C_\varepsilon > 0$  such that

$$|f(v)| \leq C_\varepsilon e^{-(L-\varepsilon)d(o,v)}, \quad v \in \Gamma.$$

*Proof of Theorem 1.1.* First, we define the full measure set

$$\tilde{\Omega} := \bigcap_{k=0}^{\infty} T^{-k}(\Omega_0 \cap \Omega_1), \quad (3.55)$$

where  $\Omega_0, \Omega_1$  are defined in Theorem 3.3 and Theorem 3.4 respectively. Then for all  $\omega \in \tilde{\Omega}$  and  $k \in \mathbb{Z}^+$ , one has

$$\sigma(J_0(T^k \omega)) = \sigma(\Delta(\omega)) = [-2\sqrt{d_\mu}, 2\sqrt{d_\mu}], \quad d_\mu := \max(\text{supp } \tilde{\mu}) \quad (3.56)$$

$$\sigma_c(J_0(T^k \omega)) = \emptyset. \quad (3.57)$$

It follows from (3.57) (combined with the general direct sum decomposition given by Theorem 2.2) that

$$\sigma_c(\Delta(\omega)) = \emptyset, \quad (3.58)$$

and in particular  $\Delta(\omega)$  possesses a basis of eigenfunctions.

Next, we prove that this basis can be chosen so that each of its elements decays exponentially. To this end, let

$$\{u^{N,k,r}\}_{r=1}^{\infty} \subset \ell^2(\mathbb{Z}^+), \quad (3.59)$$

be a basis of exponentially decaying eigenfunctions of  $J_N(k)$  (where  $J_N(k)$  is the  $\omega$ -dependent Jacobi matrix associated with  $J_0(\omega)$  as in Section 2).

It is sufficient to show that for arbitrary admissible  $N, k, r$ , the function  $\Phi^{-1}u^{N,k,r}$  has almost exponential decay rate  $L(E) + \frac{\log 2}{2}$  on the tree graph  $\Gamma$ . To that end, let us fix arbitrary vertex  $v$ ,  $\text{gen}(v) = m$  and notice that

$$[\Phi^{-1}u^{N,k,r}](v) = \begin{cases} \sum_{j=0}^{\infty} u_j^{N,k,r} \varphi_{N,k,j}(v) = u_{m-N}^{N,k,r} \varphi_{N,k,m-N}(v) & m \geq N, \\ 0, & m < N, \end{cases}$$

since  $\varphi_{N,k,j}(v) = 0$  whenever  $\text{gen}(v) \neq N + j$ . Furthermore, by the Osceledec Theorem and Theorem 3.4, for all  $\varepsilon > 0$  there exists  $C(\varepsilon) > 0$  such that

$$|u_j^{N,k,r}| \leq C(\varepsilon) e^{-(L(E)-\varepsilon)j}, \quad j \in \mathbb{Z}^+. \quad (3.60)$$

Hence, using

$$\|\varphi_{N,k,j+1}\|_{\ell^\infty(\mathcal{V})} = \frac{\|\varphi_{N,k,j}\|_{\ell^\infty(\mathcal{V})}}{\sqrt{\omega_j}} \leq \frac{\|\varphi_{N,k,j}\|_{\ell^\infty(\mathcal{V})}}{\sqrt{2}}, \quad (3.61)$$

we obtain

$$|[\Phi^{-1}u^{N,k,r}](v)| = |u_{m-N}^{N,k,r} \varphi_{N,k,m-N}(v)| \quad (3.62)$$

$$\leq \frac{C(\varepsilon) e^{(L(E)-\varepsilon)N} e^{-(L(E)-\varepsilon)m}}{\sqrt{\omega_N \omega_{N+1} \dots \omega_{m-1}}} \|\varphi_{N,k,0}\|_{\ell^\infty(\mathcal{V})} \quad (3.63)$$

$$\lesssim_N C(\varepsilon) e^{-\left(L(E) + \frac{\log 2}{2} - \varepsilon\right)m}. \quad (3.64)$$

□

## ACKNOWLEDGMENT

We are grateful to Peter Hislop for raising our interest in the problem studied in this paper and Jake Fillman for numerous helpful comments on an earlier draft of this paper.

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DEPARTMENT OF MATHEMATICS, RICE UNIVERSITY, HOUSTON, TX 77005, USA  
*E-mail address:* `damanik@rice.edu`

DEPARTMENT OF MATHEMATICS, RICE UNIVERSITY, HOUSTON, TX 77005, USA  
*E-mail address:* `sukhtaiev@rice.edu`