Mathematische Annalen

CrossMark

General and refined Montgomery Lemmata

Dmitriy Bilyk[1] · Feng Dai[2] · Stefan Steinerberger[3] (ID)

Abstract

Montgomery's Lemma on the torus $\mathbb{T}^d$ states that a sum of N Dirac masses cannot be orthogonal to many low-frequency trigonometric functions in a quantified way. We provide an extension to general manifolds that also allows for positive weights: let (M, g) be a smooth compact d-dimensional manifold without boundary, let $(\phi_k)_{k=0}^{\infty}$ denote the Laplacian eigenfunctions, let $\{x_1, \ldots, x_N\} \subset M$ be a set of points and $\{a_1, \ldots, a_N\} \subset \mathbb{R}_{\geq 0}$ be a sequence of nonnegative weights. Then, for all $X \geq 0$,

$$\sum_{k=0}^{X} \left| \sum_{n=1}^{N} a_n \phi_k(x_n) \right|^2 \gtrsim_{(M,g)} \left(\sum_{i=1}^{N} a_i^2 \right) \frac{X}{(\log X)^{\frac{d}{2}}}.$$

This result is sharp up to the logarithmic factor. Furthermore, we prove a refined spherical version of Montgomery's Lemma, and provide applications to estimates of discrepancy and discrete energies of N points on the sphere $\mathbb{S}^d$.

Mathematics Subject Classification 11L99 · 11K38 · 35B05 · 35B40 · 42B05 · 52C35

✉ Stefan Steinerberger
    stefan.steinerberger@yale.edu

    Dmitriy Bilyk
    dbilyk@math.umn.edu

    Feng Dai
    fdai@ualberta.ca

[1] School of Mathematics, University of Minnesota,, Minneapolis MN 55408, USA

[2] Department of Mathematical and Statistical Sciences, University of Alberta, Alberta, Edmonton T6G 2G1, Canada

[3] Department of Mathematics, Yale University,, New Haven CT 06510, USA

1 Introduction

1.1 Montgomery's Lemma

The lemma, which constitutes the main subject of our investigation, has its origins in the theory of irregularities of distribution.

Let $\{x_1, \ldots, x_N\} \subset \mathbb{T}^2 \cong [0, 1)^2$ be a set of N points. Montgomery's theorem [16] (see also Beck [4,6]) guarantees the existence of a disk $D \subset \mathbb{T}^2$ with radius $1/4$ or $1/2$ such that the proportion of points in the disk is either much larger or much smaller than what is predicted by the area

$$\left| \frac{1}{N} \cdot \# \{1 \leq i \leq N : x_i \in D\} - |D| \right| \gtrsim N^{-3/4}. \tag{1.1}$$

A higher-dimensional version of this statement for sets in $\mathbb{T}^d$ holds with the right-hand side of the order $N^{-\frac{1}{2} - \frac{1}{2d}}$. The proof of Montgomery's argument proceeds as follows: we first bound the L^∞-norm of the 'discrepancy function' trivially from below by the L^2-norm and then use Parseval's identity to multiplicatively separate the Fourier transform of the characteristic function of the geometric shape (in the example above: a disk) and the Fourier coefficients of the Dirac measures located at $\{x_1, \ldots, x_N\} \subset \mathbb{T}^2$

$$\left(\widehat{\sum_{n=1}^{N} \delta_{x_n}} \right)(k) = \sum_{n=1}^{N} e^{-2\pi i \langle k, x_n \rangle} \quad \text{for } k \in \mathbb{Z}^2.$$

A fundamental ingredient of the method is the fact that the Fourier transform of finite set of Dirac measures cannot be too small on low frequencies.

Lemma (Montgomery [16]) *For any* $\{x_1, \ldots, x_N\} \subset \mathbb{T}^2$ *and* $X \geq 0$

$$\sum_{|k_1| \leq X} \sum_{|k_2| \leq X} \left| \sum_{n=1}^{N} e^{2\pi i \langle k, x_n \rangle} \right|^2 \geq N X^2. \tag{1.2}$$

This inequality is a two-dimensional analogue of an earlier result of Cassels [10] and related to a result of Siegel [19]. Montgomery's Lemma is essentially sharp. Generalizations of the statement to $\mathbb{T}^d$ are straightforward. This discussion suggests that expression akin to the left-hand side of (1.2) can be used as measures of uniformity of discrete sets of points, much like the discrepancy (1.1), see [14].

1.2 Related recent results

A slight sharpening of Montgomery's Lemma has recently been given by the third author in [20] (we only describe the result on $\mathbb{T}^2$, but higher-dimensional versions also hold): for all $\{x_1, \ldots, x_N\} \subset \mathbb{T}^2$ and $X \geq 0$

$$\sum_{\|k\| \leq X} \left| \sum_{n=1}^{N} e^{2\pi i \langle k, x_n \rangle} \right|^2 \gtrsim \sum_{i,j=1}^{N} \frac{X^2}{1 + X^4 \|x_i - x_j\|^4}. \tag{1.3}$$

This quantifies the natural notion that any type of clustering of the points is going to decrease the orthogonality to trigonometric functions. Montgomery's Lemma has usually been regarded as an inequality on the torus as opposed to a more general principle. However, in the study of irregularities of distribution on the sphere $\mathbb{S}^d$, the natural analogue of Fourier series is given by harmonic polynomials which are also well understood and allow for fairly explicit analysis. In [7] the first and second author proved a generalization of (1.1) on $\mathbb{S}^d$, which essentially boiled down to a spherical analogue of (1.2). Namely, denoting the eigenfunctions of the spherical Laplacian (i.e. spherical harmonics) by $\phi_0, \ldots, \phi_k, \ldots$, this inequality states

$$\sum_{k=0}^{X} \left| \sum_{n=1}^{N} \phi_k(x_n) \right|^2 \gtrsim_d NX. \tag{1.4}$$

We observe that ϕ_0 is constant and thus the first term is already of size $\sim N^2$. Exactly like on $\mathbb{T}^d$, for $k = 0$ the inner sum is of size N^2 and the inequality is only interesting when the number of eigenfunction X starts to outnumber the number of points $X \gtrsim N$. This is also necessary because there are point sets that are orthogonal to the first $\sim N$ eigenfunctions (this is classical on $\mathbb{T}^d$ and a substantial result on $\mathbb{S}^d$, see [2,3]; it is likely to hold at a much greater level of generality).

2 Main results

In the present paper we further extend Montgomery's Lemma in two different directions. First, we extend and generalize the statement of Montogomery's lemma (1.2) to general manifolds (with a logarithmic loss). Second, in the case of the sphere $\mathbb{S}^d$, we combine the ideas of (1.3)–(1.4) and prove a spherical analogue of (1.3), which refines (1.4). We also provide several applications of this result to irregularities of distribution and energy minimization on the sphere: a notable example is a refinement of Beck's lower bound on the L^2-spherical cap discrepancy.

2.1 Montgomery Lemma on general manifolds.

We now phrase a general version of Montgomery's Lemma on compact manifolds. It relates to various natural questions and we believe that a sharper form would be quite desirable.

Theorem 1 *Let (M, g) be a smooth compact d-dimensional manifold, let $(\phi_k)_{k=0}^{\infty}$ denote the L^2-normalized Laplacian eigenfunctions of $-\Delta_g$ with the corresponding eigenvalues arranged in increasing order. Let $\{x_1, \ldots, x_N\} \subset M$, and let $(a_i)_{i=1}^{N}$ be a set of nonnegative weights. Then, for all $X \geq 0$,*

$$\sum_{k=0}^{X}\left|\sum_{n=1}^{N}a_n\phi_k(x_n)\right|^2 \gtrsim_{(M,g)} \left(\sum_{i=1}^{N}a_i^2\right)\frac{X}{(\log X)^{\frac{d}{2}}}.$$

It seems likely that the logarithm is an artifact of the method; the result is more general (but logarithmically worse) than the classical Montgomery Lemma on $\mathbb{T}^d$ and the version on the sphere [7] since it allows for nonnegative weights: the classical proofs of Montgomery's Lemma, both on $\mathbb{T}^d$ and $\mathbb{S}^d$, fails in this more general setting. The last author has shown [21] that in the setting of compact manifolds, for N sufficiently large, there exists $k \le c_d N + o(N)$ such that

$$\left|\sum_{n=1}^{N}a_n\phi_k(x_n)\right| > 0,$$

where c_d is a constant that only depends on the dimension of the manifold but not the manifold itself (on the sphere $\mathbb{S}^d$ this has implications for the existence of spherical designs, see [21]).

Theorem 1 has various implications: one would naturally assume that as soon as $X \gtrsim N$, the eigenfunctions should be fairly decoupled from the set of points and each single summand should be roughly of order $\sim N$: the theorem shows this basic intuition to be true up to logarithmic factors. Another application concerns the limits of numerical integration: the Laplacian eigenfunctions ϕ_k have mean value 0 as soon as $k \ge 1$ and are oscillating rather slowly. One would, of course, expect it to be possible for N points to integrate $\sim N$ functions exactly but, simultaneously, one would not expect such a rule to be able to do well on a larger set of (mutually orthogonal) functions. If the logarithmic factor in Theorem 1 could be improved, this would amount to a precise quantitative form of this basic intuitive principle.

2.2 Spherical extensions of Montgomery's Lemma

We now restrict our attention to the case when $M = \mathbb{S}^d$ is the unit sphere in $\mathbb{R}^{d+1}$ equipped with the normalized Haar measure σ. Denote by $\mathcal{H}_n$ the space of all spherical harmonics of degree n on $\mathbb{S}^d$, and let $\{Y_{n,k} : k = 1, 2, \cdots, d_n\}$ be a real orthonormal basis of $\mathcal{H}_n$ (recall dim $\mathcal{H}_n \sim n^{d-1}$). We have the following spherical analogue of (1.3).

Theorem 2 *For $\{x_1, \cdots, x_N\} \subset \mathbb{S}^d$, we have for all $L \in \mathbb{N}$*

$$\sum_{n=0}^{L}\sum_{k=1}^{d_n}\left|\sum_{j=1}^{N}Y_{n,k}(x_j)\right|^2 \ge c_d L^d \sum_{i,j=1}^{N}\frac{\log\left(2+L\left\|x_i-x_j\right\|\right)}{\left(1+L\left\|x_i-x_j\right\|\right)^{d+1}}. \tag{2.1}$$

We observe that the left-hand side runs over $\sim L^d$ terms. Leaving just the diagonal terms ($i = j$) on the right-hand side one finds that the right-hand side is at least of the order $\sim NL^d$, i.e. (2.1) is stronger than (1.4). Similar to the case of the torus, this result

has immediate applications irregularities of distribution on the sphere. We provide refinements of both classical [5] and recent [7] discrepancy bounds. Moreover, with the help of the Stolarsky principle and its generalizations [7,22], see (5.4)–(5.5), we obtain estimates on the the difference between discrete energies and energy integrals. These corollaries are gathered and proved in Sect. 5.

2.3 L^2-spherical cap discrepancy

We wish to highlight a particular implication that refines of a famous result of Beck [5]. The L^2-spherical cap discrepancy is defined as the L^2-norm of the spherical cap discrepancy (i.e. the difference between the empirical distribution of N points and the uniform distribution) integrated over all radii (we refer to Sect. 5 for a more formal definition). The result of Beck states that for any set Z of N points on $\mathbb{S}^d$

$$D_{L^2,\mathrm{cap}}(Z) \gtrsim_d N^{-\frac{1}{2}-\frac{1}{2d}},$$

and this is sharp up to a logarithmic factor. Our approach yields a slight refinement.

Theorem 3 *For any set of N points $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$*

$$D_{L^2,\mathrm{cap}}(Z) \gtrsim_d N^{-\frac{1}{2}-\frac{1}{2d}} \left(\frac{1}{N} \sum_{i,j=1}^{N} \frac{\log\left(2 + N^{1/d}\|z_i - z_j\|\right)}{(1 + N^{1/d}\|z_i - z_j\|)^{d+1}} \right)^{1/2}.$$

We remark that summing over the diagonal $i = j$ shows that the additional factor is $\gtrsim 1$ implying Beck's original result. However, as soon as there is subtle clustering of points, the off-diagonal terms may actually contribute a nontrivial quantity.

3 Montgomery Lemma on general manifolds: proof of Theorem 1.

Proof We first observe that the eigenfunction $\phi_0 \equiv 1/\sqrt{|M|}$ is constant and thus

$$\sum_{k=0}^{X} \left| \sum_{i=1}^{N} a_i \phi_k(x_i) \right|^2 \gtrsim_{(M,g)} \left(\sum_{i=1}^{N} a_i \right)^2 = \frac{\left(\sum_{i=1}^{N} a_i\right)^2}{\sum_{i=1}^{N} a_i^2} \sum_{i=1}^{N} a_i^2$$

and it thus suffices to prove the statement for

$$X \gtrsim \frac{\left(\sum_{i=1}^{N} a_i\right)^2}{\sum_{i=1}^{N} a_i^2}.$$

The proof starts by bounding the desired quantity from below; here, we let $t > 0$ be an arbitrary number that will be fixed later.

$$
\sum_{k=0}^{X} \left| \sum_{i=1}^{N} a_i \phi_k(x_i) \right|^2 \geq \sum_{k=0}^{X} e^{-\lambda_k t} \left| \sum_{i=1}^{N} a_i \phi_k(x_i) \right|^2
$$

$$
= \sum_{k=0}^{X} e^{-\lambda_k t} \sum_{i,j=1}^{N} a_i \phi_k(x_i) a_j \phi_k(x_j)
$$

$$
= \sum_{i,j=1}^{N} a_i a_j \sum_{k=0}^{X} e^{-\lambda_k t} \phi_k(x_i) \phi_k(x_j).
$$

Here and throughout the proof, the λ_k denote the eigenvalues of $-\Delta_g$ such that $-\Delta_g \phi_k = \lambda_k \phi_k$ and $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$. The inner sum is now close to a classical expansion for the heat kernel

$$
p_t(x, y) = \sum_{k=0}^{\infty} e^{-\lambda_k t} \phi_k(x) \phi_k(y).
$$

This means that we can replace the inner sum by the heat kernel while incurring an error that only depends on the size of X. We will now make this precise: the main ingredients are Weyl's law $\lambda_k \sim c_M k^{2/d}$, where c_M only depends on the volume of the manifold M and Hörmander's estimate [12]

$$
\|\phi_k\|_{L^\infty} \lesssim_{(M,g)} \lambda_k^{\frac{d-1}{4}}.
$$

Combining these two inequalities, we can now estimate the tail:

$$
\left| \sum_{k=X+1}^{\infty} e^{-\lambda_k t} \phi_k(x_i) \phi_k(x_j) \right| \lesssim_{(M,g)} \sum_{k=X+1}^{\infty} \left| e^{-ck^{\frac{2}{d}} t} \phi_k(x_i) \phi_k(x_j) \right|
$$

$$
\leq \sum_{k=X+1}^{\infty} e^{-ck^{\frac{2}{d}} t} \|\phi_k\|_{L^\infty}^2
$$

$$
\lesssim_{(M,g)} \sum_{k=X+1}^{\infty} e^{-ck^{\frac{2}{d}} t} \lambda_k^{\frac{d-1}{2}}
$$

$$
\lesssim_{(M,g)} \sum_{k=X+1}^{\infty} e^{-ck^{\frac{2}{d}} t} k^{1-\frac{1}{d}}.
$$

This quantity can be bounded from above by an integral which, after substitution, reduces to the incomplete Gamma function:

$$\sum_{k=X+1}^{\infty} e^{-ck^{2/d}t}k^{1-\frac{1}{d}} \leq \int_X^{\infty} e^{-\left(\frac{y}{t^{-d/2}}\right)^{\frac{2}{d}}} y^{1-\frac{1}{d}}\,dy$$

$$= \frac{1}{t^{d-\frac{1}{2}}} \int_{cXt^{d/2}}^{\infty} e^{-z^{\frac{2}{d}}} z^{1-\frac{1}{d}}\,dz$$

$$= \frac{d}{2}\frac{1}{t^{d-\frac{1}{2}}}\Gamma\left(d-\frac{1}{2}, cX^{\frac{2}{d}}t\right).$$

We will end up working in the regime $X^{\frac{2}{d}}t \gg 1$. In this regime, there is a classical asymptotic (see e.g. Abramowitz and Stegun [[1], Sect. 6.5]), valid for $a \gg 1$,

$$\Gamma\left(d-\frac{1}{2}, a\right) \lesssim_d a^{d-\frac{3}{2}}e^{-a}.$$

Altogether, this implies, since we may assume that

$$X \gtrsim \frac{\left(\sum_{i=1}^N a_i\right)^2}{\sum_{i=1}^N a_i^2},$$

the bound

$$\sum_{k=0}^{X}\left|\sum_{i=1}^N a_i\phi_k(x_i)\right|^2 \gtrsim \sum_{i,j=1}^N a_i a_j p_t(x_i,x_j) - C\sum_{i,j=1}^N \frac{a_i a_j}{t^{d-\frac{1}{2}}}\left(X^{\frac{2}{d}}t\right)^{d-\frac{3}{2}}\exp\left(-cX^{\frac{2}{d}}t\right)$$

$$= \left(\sum_{i,j=1}^N a_i a_j p_t(x_i,x_j)\right) - C\frac{\left(\sum_{i=1}^N a_i\right)^2}{t^{d-\frac{1}{2}}}\left(X^{\frac{2}{d}}t\right)^{d-\frac{3}{2}}\exp\left(-cX^{\frac{2}{d}}t\right)$$

$$\gtrsim \sum_{i,j=1}^N a_i a_j p_t(x_i,x_j) - C\frac{X\sum_{i=1}^N a_i^2}{t^{d-\frac{1}{2}}}\left(X^{\frac{2}{d}}t\right)^{d-\frac{3}{2}}\exp\left(-cX^{\frac{2}{d}}t\right).$$

We will end up working at time $t \sim X^{-\frac{2}{d}}\log X \ll 1$ which, for X sufficiently large, enables us to make use of Varadhan's short-time asymptotics

$$p_t(x,y) \sim \frac{1}{(4\pi t)^{d/2}}\exp\left(-\frac{\|x-y\|^2}{4t}\right)$$

to argue that

$$\sum_{i,j=1}^N a_i a_j p_t(x_i,x_j) \geq \sum_{i=1}^N a_i^2 p_t(x_i,x_i) \gtrsim t^{-\frac{d}{2}}\sum_{i=1}^N a_i^2.$$

Summarizing, we have

$$\sum_{k=0}^{X} \left| \sum_{i=1}^{N} a_i \phi_k(x_i) \right|^2 \gtrsim_{(M,g)} \sum_{i=1}^{N} a_i^2 \left[t^{-\frac{d}{2}} - \frac{CX}{t^{d-\frac{1}{2}}} \left(X^{\frac{2}{d}} t \right)^{d-\frac{3}{2}} \exp\left(-c X^{\frac{2}{d}} t \right) \right].$$

Setting $t = A X^{-\frac{2}{d}} \log X$ with $A = \frac{1}{c}(1 - \frac{1}{d}) + 1$ now implies the result. $\qquad\square$

4 An improved Montgomery Lemma on the sphere:proof of Theorem 2

Let C_n^λ denote the Gegenbauer (ultraspherical) polynomials of degree n, which are orthogonal on $[-1, 1]$ with respect to the weight $w_\lambda(t) = (1 - t^2)^{\lambda-1/2}$ (see [11] for the backgound information). Since we are working on $\mathbb{S}^d$, we set $\lambda = \frac{d-1}{2}$. Denote also $E_n^\lambda(t) = \frac{n+\lambda}{\lambda} C_n^\lambda(t)$. For $\delta > 0$, we define the Cesàro-type kernel

$$K_L^\delta(t) := \sum_{k=0}^{L} \frac{A_{L-k}^\delta}{A_L^\delta} E_k^\lambda(t), \quad \text{with } A_j^\delta = \frac{\Gamma(j + \delta + 1)}{\Gamma(j + 1)\Gamma(\delta + 1)}.$$

It is a classical result of Kogbetliantz [13] (see also [18]) that $K_L^\delta(t) \geq 0$ on $[-1, 1]$, whenever $\delta \geq d$.

Lemma 1 *For $\{x_1, \cdots, x_N\} \subset \mathbb{S}^d$ and any $\delta > 0$, we have*

$$\sum_{k=1}^{d_n} \left| \sum_{j=1}^{N} Y_{n,k}(x_j) \right|^2 = \sum_{i,j=1}^{N} E_n^\lambda(x_i \cdot x_j) \geq 0, \quad n = 0, 1, \cdots,$$

and

$$\sum_{n=0}^{L} \sum_{k=1}^{d_n} \left| \sum_{j=1}^{N} Y_{n,k}(x_j) \right|^2 \geq \sum_{i,j=1}^{N} K_L^\delta(x_i \cdot x_j). \tag{4.1}$$

This lemma follows directly from the addition formula for spherical harmonics. We include the proof here for the sake of completeness.

Proof By the addition formula for spherical harmonics, we have

$$\sum_{k=1}^{d_n} \left| \sum_{j=1}^{N} Y_{n,k}(x_j) \right|^2 = \sum_{k=1}^{d_n} \sum_{i=1}^{N} \sum_{j=1}^{N} Y_{n,k}(x_i) Y_{n,k}(x_j) = \sum_{i,j=1}^{N} \sum_{k=1}^{d_n} Y_{n,k}(x_i) Y_{n,k}(x_j)$$

$$= \sum_{i,j=1}^{N} E_n^\lambda(x_i \cdot x_j).$$

This also implies that, using $A_{L-n}^\delta \le A_L^\delta$ for all $0 \le n \le L$,

$$\sum_{n=0}^{L} \sum_{k=1}^{d_n} \left| \sum_{j=1}^{N} Y_{n,k}(x_j) \right|^2 = \sum_{n=0}^{L} \sum_{i,j=1}^{N} E_n^\lambda(x_i \cdot x_j) \ge \sum_{n=0}^{L} \frac{A_{L-n}^\delta}{A_L^\delta} \sum_{i,j=1}^{N} E_n^\lambda(x_i \cdot x_j)$$

$$= \sum_{i,j=1}^{N} \sum_{n=0}^{L} \frac{A_{L-n}^\delta}{A_L^\delta} E_n^\lambda(x_i \cdot x_j) = \sum_{i,j=1}^{N} K_L^\delta(x_i \cdot x_j).$$

□

Numerical experiments suggest that K_n^d is not just non-negative, but is actually strictly positive and should satisfy favorable lower bounds. However, we could not prove it, hence, as in [20], we shall make use of additional rounds of averaging. Define

$$G_n^{d+1}(t) = \frac{1}{n+1} \sum_{j=0}^{n} K_j^d(t) \quad \text{and} \quad G_n^{d+2}(t) = \frac{1}{n+1} \sum_{j=0}^{n} G_j^{d+1}(t).$$

Lemma 2 *For $n \in \mathbb{N}$ and $\theta \in (0, \pi)$,*

$$G_n^{d+2}(\cos\theta) \ge Cn^d (1 + n\theta)^{-d-1} \log(2 + n\theta). \tag{4.2}$$

Remark It seems that (4.2) with G_n^{d+1} in place of G_n^{d+2} remains true, but the proof would be more involved [we prove a slightly weaker bound (4.4)].

Proof First, we recall that $K_n^d(\cos\theta) \ge 0$ for $\theta \in [0, \pi]$, and $\|K_n^d\|_\infty = K_n^d(1) \sim (n+1)^d$. It follows that for $\delta = d+1$ or $d+2$,

$$\|G_n^\delta\|_\infty = G_n^\delta(1) \sim (n+1)^d.$$

By Bernstein's inequality for trigonometric polynomials, this also implies that for $F_n(t) := K_n^d(t)$ or $G_n^{d+1}(t)$ or $G_n^{d+2}(t)$, we have

$$F_n(\cos\theta) \ge \frac{1}{2} \|F_n\|_\infty \sim (n+1)^d, \quad 0 \le \theta \le \frac{1}{2n}. \tag{4.3}$$

Next, we show that

$$G_n^{d+1}(\cos\theta) \ge cn^d (1 + n\theta)^{-d-1}, \quad n \ge 1, \quad \theta \in [0, \pi]. \tag{4.4}$$

If $0 \le \theta \le \frac{1}{2n}$, then (4.4) follows directly from (4.3). For $\frac{1}{2n} \le \theta \le \pi$, we have

$$G_n^{d+1}(\cos\theta) = \frac{1}{n+1} \sum_{j=0}^{n} K_j^d(\cos\theta) \ge \frac{1}{n+1} \sum_{0 \le j \le \frac{1}{2\theta}} K_j^d(\cos\theta)$$

$$\ge c\frac{1}{n+1} \sum_{0 \le j \le \frac{1}{2\theta}} (j+1)^d \sim n^{-1}\theta^{-d-1} \sim n^d(1+n\theta)^{-d-1}.$$

Finally, we prove estimate (4.2). Note that (4.4) with G_n^{d+2} in place of G_n^{d+1} remains true. Thus, without loss of generality, we may assume that $\frac{2}{n} \le \theta \le \pi$ and $n \ge 10$. We then have

$$G_n^{d+2}(\cos\theta) = \frac{1}{n+1} \sum_{j=0}^{n} G_j^{d+1}(\cos\theta) \ge cn^{-1} \sum_{j=0}^{n} j^d(1+j\theta)^{-d-1}$$

$$\ge cn^{-1} \sum_{\theta^{-1} \le j \le n} j^{-1}\theta^{-d-1} \ge cn^{-1}\theta^{-d-1} \int_{\theta^{-1}+1}^{n} \frac{dt}{t}$$

$$= cn^{-1}\theta^{-d-1} \int_{1+\theta}^{n\theta} \frac{dt}{t} \sim n^d(1+n\theta)^{-d-1}\log(n\theta+2).$$

$\square$

Proof of Theorem 2 Using Lemma 1, we have

$$\sum_{n=0}^{L} \sum_{k=1}^{d_n} \left| \sum_{j=1}^{N} Y_{n,k}(x_j) \right|^2 \ge \frac{1}{L} \sum_{m=0}^{L} \sum_{n=0}^{m} \sum_{k=1}^{d_n} \left| \sum_{j=1}^{N} Y_{n,k}(x_j) \right|^2$$

$$\ge \frac{1}{L} \sum_{m=0}^{L} \sum_{i,j=1}^{N} K_m^d(x_i \cdot x_j) = \sum_{i,j=1}^{N} G_L^{d+1}(x_i \cdot x_j). \quad (4.5)$$

Using (4.5) and averaging once again, we have

$$\sum_{n=0}^{L} \sum_{k=1}^{d_n} \left| \sum_{j=1}^{N} Y_{n,k}(x_j) \right|^2 \ge \frac{1}{L} \sum_{m=0}^{L} \sum_{n=0}^{m} \sum_{k=1}^{d_n} \left| \sum_{j=1}^{N} Y_{n,k}(x_j) \right|^2$$

$$\ge \frac{1}{L} \sum_{m=0}^{L} \sum_{i,j=1}^{N} G_m^{d+1}(x_i \cdot x_j) = \sum_{i,j=1}^{N} G_L^{d+2}(x_i \cdot x_j),$$

which, using (4.2), implies the desired estimate (2.1). $\square$

5 Some corollaries for discrepancy and discrete energy of point distributions on the sphere

For a finite set of points $Z = \{z_1, \cdots, z_N\} \subset \mathbb{S}^d$, its L^2-discrepancy with respect to a function $f : [-1, 1] \to \mathbb{R}$ is defined as

$$
D_{L^2, f}(Z) = \left(\int_{\mathbb{S}^d} \left| \frac{1}{N} \sum_{j=1}^{N} f(x \cdot z_j) - \int_{\mathbb{S}^d} f(x \cdot y) \, d\sigma(y) \right|^2 d\sigma(x) \right)^{\frac{1}{2}}. \tag{5.1}
$$

In particular, when $f(t) = f_\tau(t) = \mathbf{1}_{[\tau, 1]}(t)$, one obtains the discrepancy with respect to spherical caps $C(x, \tau) = \{y \in \mathbb{S}^d : x \cdot y \geq \tau\}$ of aperture $\arccos \tau$, i.e.

$$
D_{L^2, f_\tau}^2(Z) = \int_{\mathbb{S}^d} \left| \frac{1}{N} \sum_{j=1}^{N} \mathbf{1}_{C(x,\tau)}(z_j) - \sigma(C(x, \tau)) \right|^2 d\sigma(x), \tag{5.2}
$$

Its L^2-average over the parameter τ yields the classical L^2-spherical cap discrepancy

$$
D_{L^2, \text{cap}}^2(Z) = \int_{-1}^{1} D_{L^2, f_\tau}^2(Z) \, d\tau, \tag{5.3}
$$

which has been extensively studied [4,5]. In particular, this quantity satisfies the following identity known as the *Stolarsky principle* [22], which relates it to a certain discrete energy.

$$
c_d \, D_{L^2, \text{cap}}^2(Z) = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \|x - y\| \, d\sigma(x) \, d\sigma(y) \; - \; \frac{1}{N^2} \sum_{i,j=1}^{N} \|z_i - z_j\|, \tag{5.4}
$$

where c_d is a dimensional constant. It has been established in [7,8] that Stolarsky principle can be generalized in the following way: for $f \in L^2\big([-1, 1], w_\lambda\big)$

$$
D_{L^2, f}^2(Z) = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} F(z_i \cdot z_j) - \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} F(x \cdot y) \, d\sigma(x) d\sigma(y), \tag{5.5}
$$

where the function $F : [-1, 1] \to \mathbb{R}$ is defined through the identity

$$
\widehat{F}(n, \lambda) = \big(\widehat{f}(n, \lambda)\big)^2. \tag{5.6}
$$

Here and throughout the proof,

$$\widehat{f}(n, \lambda) := \frac{(n + \lambda)\Gamma(\lambda)}{\sqrt{\pi}\Gamma(\lambda + \frac{1}{2})} \int_{-1}^{1} f(t)C_n^{\lambda}(t)(1 - t^2)^{\lambda - \frac{1}{2}} \, dt,$$

where the notation used is reminiscent of that of Fourier transform because spherical harmonics diagonalize the Laplacian on the sphere. It is now easy to see that the refined spherical Montgomery Lemma, Theorem 2, provides new estimates both for the discrepancy and discrete energies. Setting

$$G(x) = \frac{1}{N} \sum_{j=1}^{N} f(x \cdot z_j), \quad \text{we see that} \quad D_{L^2, f}(Z) = \|G - \widehat{G}(0, \lambda)\|_{L^2(\mathbb{S}^d, d\sigma)},$$

and, according to the Funk–Hecke formula, for any spherical harmonic $Y_n \in \mathcal{H}_n$

$$\langle G, Y_n \rangle = \frac{1}{N} \sum_{j=1}^{N} \int_{\mathbb{S}^d} f(x \cdot z_j) Y_n(x) d\sigma(x) = \frac{1}{N} \widehat{f}(n, \lambda) \sum_{j=1}^{N} Y_n(z_j). \qquad (5.7)$$

Thus we find that

$$\begin{aligned}
D_{L^2, f}^2(Z) &= \|G - \widehat{G}(0, \lambda)\|_2^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{d_n} |\langle G, Y_{n,k} \rangle|^2 \\
&= \frac{1}{N^2} \sum_{n=1}^{\infty} |\widehat{f}(n, \lambda)|^2 \sum_{k=1}^{d_n} \left| \sum_{j=1}^{N} Y_{n,k}(z_j) \right|^2 \\
&\geq \frac{1}{N^2} \cdot \min_{1 \leq n \leq L} |\widehat{f}(n, \lambda)|^2 \cdot \sum_{n=1}^{L} \sum_{k=1}^{d_n} \left| \sum_{j=1}^{N} Y_{n,k}(z_j) \right|^2 \\
&= \frac{1}{N^2} \cdot \min_{1 \leq n \leq L} |\widehat{f}(n, \lambda)|^2 \cdot \left(\sum_{n=0}^{L} \sum_{k=1}^{d_n} \left| \sum_{j=1}^{N} Y_{n,k}(z_j) \right|^2 - N^2 \right), \qquad (5.8)
\end{aligned}$$

where we used the fact that the term, corresponding to $n = 0$, is N^2. If we set $L = C'N^{\frac{1}{d}}$ with C' being a large dimensional constant, and leave just the diagonal terms in (2.1), we see that

$$\sum_{n=1}^{L} \sum_{k=1}^{d_n} \left| \sum_{j=1}^{N} Y_{n,k}(z_j) \right|^2 \geq c''N^2.$$

Therefore, again applying (2.1) of Theorem 2, we arrive at the following corollary:

Corollary 1 *Let* $f \in L^2\big([-1, 1], (1 - t^2)^{\lambda - \frac{1}{2}}\big)$. *For* $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$ *we have*

$$D^2_{L^2, f}(Z) \gtrsim \frac{1}{N} \cdot \min_{1 \leq n \leq C' N^{\frac{1}{d}}} |\widehat{f}(n, \lambda)|^2 \cdot \sum_{i, j=1}^{N} \frac{\log(2 + N^{1/d} \|z_i - z_j\|)}{(1 + N^{1/d} \|z_i - z_j\|)^{d+1}}, \quad (5.9)$$

where C' *is a large constant depending only on the dimension.*

Such lower bounds, which show that finite point sets cannot be distributed too uniformly, are a common theme in the subject of *irregularities of distribution*. Using the generalized Stolarsky principle (5.5) and relation (5.6) we can also obtain a similar corollary for the discrete energy:

Corollary 2 *Assume that* $F \in C[-1, 1]$ *and* $\widehat{F}(n, \lambda) \geq 0$ *for all* $n \geq 1$ *(i.e., up to the constant term, F is a positive definite function on the sphere $\mathbb{S}^d$). Then for any point distribution* $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$

$$\frac{1}{N^2} \sum_{i, j=1}^{N} F(z_i \cdot z_j) - I_F(\sigma) \gtrsim \frac{1}{N} \cdot \min_{1 \leq n \leq C' N^{1/d}} \widehat{F}(n, \lambda) \cdot \sum_{i, j=1}^{N} \frac{\log(2 + N^{1/d} \|z_i - z_j\|)}{(1 + N^{1/d} \|z_i - z_j\|)^{d+1}},$$
$$(5.10)$$

where C' *is a large constant depending only on the dimension, and* $I_F(\sigma) = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} F(x \cdot y) \, d\sigma(x) d\sigma(y)$ *denotes the energy integral with potential given by* F.

Remark The fact that every continuous positive definite function on the sphere can be represented by (5.6), i.e. has appropriate decay of $\widehat{F}(n, \lambda)$, has been discussed in ([7], Lemma 2.3).

It is known (see e.g. [7,8]) that for positive definite functions F, the uniform surface measure σ minimizes the energy with potential F over all Borel probability measures on $\mathbb{S}^d$. Thus Corollary 2 states, in a quantitative way, that the energy of finite atomic measures with equal weights cannot be too close to the minimum.

We observe that leaving just the N diagonal terms ($i = j$) in the right-hand sides of (5.9) and (5.10) we recover the bounds obtained in ([7],Theorem 4.2):

$$D_{L^2, f}(Z) \gtrsim \min_{1 \leq n \leq C' N^{\frac{1}{d}}} |\widehat{f}(n, \lambda)|,$$

$$\frac{1}{N^2} \sum_{i, j=1}^{N} F(z_i \cdot z_j) - I_F(\sigma) \gtrsim \min_{1 \leq n \leq C' N^{\frac{1}{d}}} \widehat{F}(n, \lambda). \quad (5.11)$$

Corollaries 1 and 2 add more subtle information to these lower bounds.

Returning to the classical case of the spherical cap discrepancy (5.3), recall that Beck's famous result [6], which states that

$$D_{L^2, \text{cap}}(Z) \gtrsim N^{-\frac{1}{2} - \frac{1}{2d}}, \quad (5.12)$$

for any N-point set in the sphere $\mathbb{S}^d$ (and this is optimal up to a logarithmic factor). Using the fact that (see e.g. [23] or [7])

$$\int_{-1}^{1} \left| \widehat{f_\tau}(n, \lambda) \right|^2 d\tau \approx n^{-d-1}, \tag{5.13}$$

and repeating the arguments above almost verbatim, but with an additional averaging in τ, one obtains a refinement of Beck's original estimate (this refinement has been stated in §2 as Theorem 3).

Corollary 3 *For any point distribution* $Z = \{z_1, \dots, z_N\} \subset \mathbb{S}^d$

$$D_{L^2,\mathrm{cap}}^2(Z) \gtrsim_d N^{-2-\frac{1}{d}} \cdot \sum_{i,j=1}^{N} \frac{\log\left(2 + N^{1/d} \|z_i - z_j\|\right)}{(1 + N^{1/d} \|z_i - z_j\|)^{d+1}}. \tag{5.14}$$

As before, by considering only the diagonal terms one recovers Beck's result (5.12), and the bound (5.15) provides more information: in particular, if the order of magnitude of the energy on the right-hand side is significantly greater than N, then the spherical cap discrepancy of Z is necessarily too big. The original Stolarsky principle (5.4) then leads to the following corollary concerning the sum of Euclidean distances between N points on the sphere:

Corollary 4 *For any point distribution* $Z = \{z_1, \dots, z_N\} \subset \mathbb{S}^d$

$$\mathcal{J}_d - \frac{1}{N^2} \sum_{i,j=1}^{N} \|z_i - z_j\| \gtrsim_d N^{-2-\frac{1}{d}} \cdot \sum_{i,j=1}^{N} \frac{\log\left(2 + N^{\frac{1}{d}} \|z_i - z_j\|\right)}{(1 + N^{\frac{1}{d}} \|z_i - z_j\|)^{d+1}}, \tag{5.15}$$

where

$$\mathcal{J}_d = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \|x - y\| \, d\sigma(x) \, d\sigma(y) = \frac{2^d \left[\Gamma\left(\frac{d+1}{2}\right)\right]^2}{\sqrt{\pi} \, \Gamma\left(d + \frac{1}{2}\right)}.$$

References

1. Abramowitz, M., Stegun, I.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. National Bureau of Standards Applied Mathematics Series 55. US Government Printing Office, Washington (1964)
2. Ahrens, C., Beylkin, G.: Rotationally invariant quadratures for the sphere. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **465**(2110), 3103–3125 (2009)
3. Bondarenko, A., Radchenko, D., Viazovska, M.: Optimal asymptotic bounds for spherical designs. Ann. Math. **178**(2), 443–452 (2013)

4. Beck, J.: Some upper bounds in the theory of irregularities of distribution. Acta Arith. **43**, 115–130 (1983)
5. Beck, J.: Sums of distances between points on a sphere-an application of the theory of irregularities of distribution to discrete geometry. Mathematika **31**(1), 33–41 (1984)
6. Beck, J.: Irregularities of distribution I. Acta Math. **159**, 1–49 (1987)
7. Bilyk, D., Dai, F.: Geodesic distance Riesz energy on the sphere (2016). arXiv:1612.08442
8. Bilyk, D., Dai, F., Matzke, R.: Stolarsky princuple and energy optimization on the sphere. Constr. Approx. **48**, 31–60 (2018). https://doi.org/10.1007/s00365-017-9412-4
9. Bilyk, D., Lacey, M.: One bit sensing, discrepancy and Stolarsky principle. Mat. Sbornik. **208**, 744 (2017)
10. Cassels, J.W.S.: On the sum of complex numbers. Acta Math. Hungar. **7**, 283–289 (1957)
11. Dai, F., Xu, Y.: Approximation Theory and Harmonic Analysis on Spheres and Balls. Springer Monographs in Mathematics. Springer, New York (2013)
12. Hörmander, L.: The spectral function of an elliptic operator. Acta Math. **88**, 341–370 (1968)
13. Kogbetliantz, E.: Recherches sur la sommabilité des sériesultra-sphériques par la méthode des moyennes arithmetiques. J. Math. Pures Appl. **3**, 107–188 (1924)
14. Lu J., Steinerberger, S.: Riesz Energy on the Torus: Regularity of Minimizers (2017). arXiv:1710.08010
15. Montgomery H.L.: The pair correlation of zeros of the zeta function. In: Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), pp. 181–193. Amer. Math. Soc., Providence, R.I. (1973)
16. Montgomery, H.: Irregularities of Distribution by Means of Power Sums, Congress of Number Theory (Zarautz 1984), vol. 1989, pp. 11–27. Universidad del Paıs Vasco Bilbao, Lejona (1984)
17. Montgomery, H.: Ten Lectures at the Interface of Harmonic Analysis and Number Theory. American Mathematical Society, Providence (1994)
18. Reimer, M.: A short proof of a result of Kogbetliantz on the positivity of certain Cesáro means. Math. Z. **221**(2), 189–192 (1996)
19. Siegel, C.L.: Uber Gitterpunkte in convexen Korpern und ein damit zusammenhangendes Extremalproblem. Acta Math. **65**, 307–323 (1935)
20. Steinerberger, S.: Exponential Sums and Riesz energies. J. Number Theory **182**, 37–56 (2018)
21. Steinerberger, S.: Spectral Limitations of Quadrature Rules and Generalized Spherical Designs (2017). arXiv:1708.08736
22. Stolarsky, K.B.: Sums of distances between points on a sphere. II. Proc. Am. Math. Soc. **41**, 575–582 (1973)
23. Szegö, G.: Orthogonal Polynomials, vol. 23, 4th edn. Am. Math. Soc. Colloq. Publ., Providence (1975)