SPECTRAL TRANSITIONS FOR THE SQUARE FIBONACCI HAMILTONIAN

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Abstract. We study the spectrum and the density of states measure of the square Fibonacci Hamiltonian. We describe where the transitions from positive-measure to zero-measure spectrum and from absolutely continuous to singular density of states measure occur. This shows in particular that for almost every parameter from some open set, a positive-measure spectrum and a singular density of states measure coexist. This provides the first physically relevant example exhibiting this phenomenon.

1. Introduction

The square Fibonacci Hamiltonian is the bounded self-adjoint operator

$$\begin{aligned}[H^{(2)}_{\lambda_1,\lambda_2,\omega_1,\omega_2}\psi](m,n) &= \psi(m+1,n)+\psi(m-1,n)+\psi(m,n+1)+\psi(m,n-1)\\ &\quad+\Big(\lambda_1\chi_{[1-\alpha,1)}(m\alpha+\omega_1 \mathrm{mod}\ 1)+\lambda_2\chi_{[1-\alpha,1)}(n\alpha+\omega_2 \mathrm{mod}\ 1)\Big)\psi(m,n)\end{aligned}$$

in $\ell^2(\mathbb{Z}^2)$, with $\alpha=\frac{\sqrt{5}-1}{2}$, coupling constants $\lambda_1,\lambda_2>0$ and phases $\omega_1,\omega_2\in\mathbb{T}=\mathbb{R}/\mathbb{Z}$.

The square Fibonacci Hamiltonian is the natural two-dimensional analog of the standard Fibonacci Hamiltonian, which is the bounded self-adjoint operator

$$[H^{(1)}_{\lambda,\omega}\psi](n)=\psi(n+1)+\psi(n-1)+\lambda\chi_{[1-\alpha,1)}(n\alpha+\omega \mod 1)\psi(n)$$

in $\ell^2(\mathbb{Z})$, again with the coupling constant $\lambda>0$ and the phase $\omega\in\mathbb{T}$.

The origin of these operators is twofold. On the one hand, the Fibonacci Hamiltonian was proposed in 1983 as a model whose self-similarity leads to an exact renormalization group approach [26, 41], connecting its spectral properties to dynamical properties of the associated renormalization map. On the other hand, since the discovery of quasicrystals in 1982 (published in 1984; see [48]), the Fibonacci Hamiltonian has served as the most prominent model for the study of electronic transport properties in one-dimensional quasi-crystalline environments. There is obvious interest in removing the restriction to one dimension, and a natural analogous model in two dimensions is given by the square Fibonacci Hamiltonian. It is then straightforward to generalize this construction to higher dimensions. For the sake of simplicity we will limit our discussion to the case of two dimensions in this paper. For a recent survey of the spectral theory of the Fibonacci Hamiltonian and the square Fibonacci Hamiltonian, see [8].

Date: May 2, 2018.

Key words and phrases. Square Fibonacci Hamiltonian, Spectral Transitions, Quasicrystals.

D. D. was supported in part by NSF grant DMS–1361625.

A. G. was supported in part by NSF grant DMS–1301515.

Using the minimality of an irrational rotation and strong operator convergence, one can readily see that the spectra of these operators are phase-independent. That is, there are compact subsets Σ_λ and $\Sigma_{\lambda_1,\lambda_2}$ of $\mathbb{R}$ such that

$$\sigma(H^{(1)}_{\lambda,\omega}) = \Sigma_\lambda \qquad \text{for every } \omega \in \mathbb{T},$$
$$\sigma(H^{(2)}_{\lambda_1,\lambda_2,\omega_1,\omega_2}) = \Sigma_{\lambda_1,\lambda_2} \qquad \text{for every } \omega_1, \omega_2 \in \mathbb{T}.$$

The density of states measures associated with these operator families are defined as follows,

$$\int_{\mathbb{R}} g(E)\, d\nu_{\lambda_1,\lambda_2}(E) = \int_{\mathbb{T}} \int_{\mathbb{T}} \langle \delta_0, g(H^{(2)}_{\lambda_1,\lambda_2,\omega_1,\omega_2}) \delta_0 \rangle_{\ell^2(\mathbb{Z}^2)}\, d\omega_1\, d\omega_2.$$

and

$$\int_{\mathbb{R}} g(E)\, d\nu_\lambda(E) = \int_{\mathbb{T}} \langle \delta_0, g(H^{(1)}_{\lambda,\omega}) \delta_0 \rangle_{\ell^2(\mathbb{Z})}\, d\omega.$$

It is a standard result from the theory of ergodic Schrödinger operators that $\Sigma_\lambda = \operatorname{supp} \nu_\lambda$ and $\Sigma_{\lambda_1,\lambda_2} = \operatorname{supp} \nu_{\lambda_1,\lambda_2}$, where $\operatorname{supp} \nu$ denotes the topological support of the measure ν.

The theory of separable operators (see, e.g., [13, Appendix] and [46, Sections II.4 and VIII.10]) quickly implies that

$$\Sigma_{\lambda_1,\lambda_2} = \Sigma_{\lambda_1} + \Sigma_{\lambda_2} \tag{1}$$

and

$$\nu_{\lambda_1,\lambda_2} = \nu_{\lambda_1} * \nu_{\lambda_2}, \tag{2}$$

where the convolution of measures is defined by

$$\int_{\mathbb{R}} g(E)\, d(\mu * \nu)(E) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(E_1 + E_2)\, d\mu(E_1)\, d\nu(E_2).$$

The rigorous spectral analysis of the Fibonacci Hamiltonian was begun in the 1986 paper [6] by Casdagli and the 1987 paper [53] by Sütő and, in a certain sense, it was recently completed in [14]. The latter paper proved many of the (then) remaining conjectures about this operator and in particular gave rise to a global picture in that it proved results for all values of the coupling constant, while most of the previous results were restricted to either sufficiently small or sufficiently large values of the coupling constant. The fact that these results are now known globally is crucial to what we do in this paper. For example, it was shown in [14] that for every $\lambda > 0$, the spectrum Σ_λ is a dynamically defined Cantor set. In particular, its box counting dimension exists, coincides with its Hausdorff dimension, and the common value belongs to $(0,1)$. Moreover, it was also shown in [14] that for every $\lambda > 0$, the density of states measure ν_λ is exact-dimensional, and the respective dimensions obey

$$0 < \dim_{\mathrm{H}} \nu_\lambda < \dim_{\mathrm{H}} \Sigma_\lambda < 1. \tag{3}$$

The spectral analysis of the square Fibonacci Hamiltonian, on the other hand, is still in its early stages. The first rigorous result was obtained in 2011 in [11], where it was shown that for λ sufficiently small, $\Sigma_{\lambda,\lambda}$ has no gaps at all; compare Figure 1. While not stated in [11] explicitly, the results there (in particular, Theorem 1.2 and Lemma 6.2) also imply that for λ_1, λ_2 sufficiently small, the set $\Sigma_{\lambda_1,\lambda_2}$ is an interval. Moreover, it is not hard to show that for any given λ_1 and then

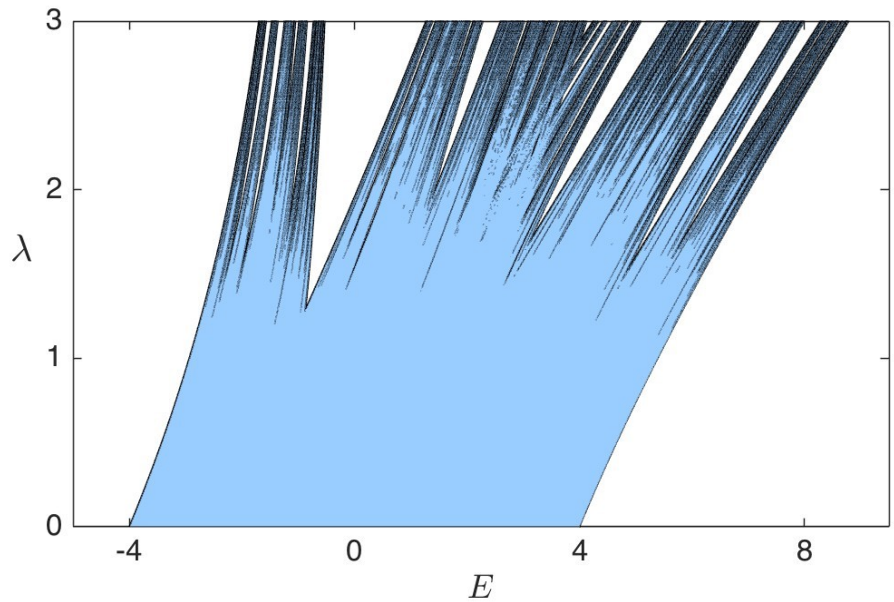

FIGURE 1. The spectrum $\Sigma_{\lambda,\lambda}$ of the square Fibonacci Hamiltonian; image courtesy of Mark Embree.

λ_2 sufficiently small, the set $\Sigma_{\lambda_1,\lambda_2}$ has at most one gap. That is, while Cantor spectrum is persistent in one dimension, the spectrum fails to be a Cantor set in two dimensions if the coupling constants are sufficiently small. On the other hand, even in two dimensions, the spectrum is a Cantor set of zero Lebesgue measure if both λ_1, λ_2 are sufficiently large. This follows quickly from the fact, shown in [9], that the dimension of Σ_λ goes to zero as $\lambda \to \infty$. In particular, $\Sigma_{\lambda_1,\lambda_2}$ undergoes two interesting transitions as the coupling constants are increased: from non-Cantor to Cantor, and from positive measure to zero measure. This shows that the two-dimensional case is richer and more interesting than the one-dimensional case, where no such transitions occur. In this paper we will study the transition from positive-measure spectrum to zero-measure spectrum and describe precisely where it occurs.

Let us now turn to the density of states measure $\nu_{\lambda_1,\lambda_2}$. Given the fact that the spectrum $\Sigma_{\lambda_1,\lambda_2}$ is the topological support of $\nu_{\lambda_1,\lambda_2}$ and in the regime of small λ_1, λ_2 this set is a non-degenerate interval as discussed above, folk wisdom should lead one to expect that $\nu_{\lambda_1,\lambda_2}$ is absolutely continuous. On the other hand, since both measures ν_{λ_1}, ν_{λ_2} are singular and it is notoriously difficult to establish the absolute continuity of a convolution of two singular measures, it is initially far from obvious how to establish this property. Nevertheless, by developing new tools in the study of convolutions of singular measures, it was shown in [13] that for Lebesgue almost all pairs (λ_1, λ_2) in the small coupling regime, the density of states measure $\nu_{\lambda_1,\lambda_2}$ is absolutely continuous.

However, and this fact will be a central theme of this paper, a nice structure of the spectrum, while often indicative of the absolute continuity of the density of states measure, is in fact in general not sufficient to ensure the absolute continuity of this measure.

In the regime of large λ_1, λ_2, the density of states measure $\nu_{\lambda_1,\lambda_2}$ is clearly singular since its topological support $\Sigma_{\lambda_1,\lambda_2}$ has zero Lebesgue measure as discussed above. This gives again rise to an interesting transition as the coupling constants are increased: from an absolutely continuous density of states measure to a singular one, and this transition is again a phenomenon that was not present in the one-dimensional case.

We are now ready to formulate our main result. Note that it follows from (3) that the sets

$$\{(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 : \dim_{\mathrm{H}} \nu_{\lambda_1} + \dim_{\mathrm{H}} \nu_{\lambda_2} = 1\}$$

and

$$\{(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 : \dim_{\mathrm{H}} \Sigma_{\lambda_1} + \dim_{\mathrm{H}} \Sigma_{\lambda_2} = 1\}$$

are disjoint. The complement of the union of these sets consists of three regions, in which we have three different kinds of spectral behavior:

Theorem 1.1. *Consider the following three regions in $\mathbb{R}_+^2$:*

$$U_{\mathrm{acds}} = \{(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 : \dim_{\mathrm{H}} \nu_{\lambda_1} + \dim_{\mathrm{H}} \nu_{\lambda_2} > 1\},$$
$$U_{\mathrm{pmsd}} = \{(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 : \dim_{\mathrm{H}} \Sigma_{\lambda_1} + \dim_{\mathrm{H}} \Sigma_{\lambda_2} > 1 \textit{ and } \dim_{\mathrm{H}} \nu_{\lambda_1} + \dim_{\mathrm{H}} \nu_{\lambda_2} < 1\},$$
$$U_{\mathrm{zmsp}} = \{(\lambda_1, \lambda_2) \in \mathbb{R}_+^2 : \dim_{\mathrm{H}} \Sigma_{\lambda_1} + \dim_{\mathrm{H}} \Sigma_{\lambda_2} < 1\}.$$

Then, the following statements hold:

(a) *The regions U_{acds}, U_{pmsd}, U_{zmsp} are disjoint and the union of their closures covers the parameter space $\mathbb{R}_+^2$.*
(b) *Each of the regions U_{acds}, U_{pmsd}, U_{zmsp} is open and non-empty.*
(c) *For Lebesgue almost every $(\lambda_1, \lambda_2) \in U_{\mathrm{acds}}$, $\nu_{\lambda_1,\lambda_2}$ is absolutely continuous, and hence $\Sigma_{\lambda_1,\lambda_2}$ has positive Lebesgue measure.*
(d) *For every $(\lambda_1, \lambda_2) \in U_{\mathrm{pmsd}}$, $\nu_{\lambda_1,\lambda_2}$ is singular, but for Lebesgue almost every $(\lambda_1, \lambda_2) \in U_{\mathrm{pmsd}}$, $\Sigma_{\lambda_1,\lambda_2}$ has positive Lebesgue measure.*
(e) *For every $(\lambda_1, \lambda_2) \in U_{\mathrm{zmsp}}$, $\Sigma_{\lambda_1,\lambda_2}$ has zero Lebesgue measure, and hence $\nu_{\lambda_1,\lambda_2}$ is singular.*

Remark 1.2. (a) The coexistence of positive measure spectrum and singular density of states measure is a rather unusual phenomenon. Until very recently it was an open problem whether this can even occur in the context of Schrödinger operators. The existence of Schrödinger operators with quasi-periodic potentials exhibiting this phenomenon was shown in [2]. However, the examples given in that paper are somewhat artificial, and "typical" quasi-periodic Schrödinger operators are not expected to have these two properties. The examples provided by the square Fibonacci Hamiltonian with parameters in U_{pmsd}, on the other hand, are not artificial at all, but rather correspond to operators that are arguably physically relevant. Moreover the phenomenon is made possible by and is closely connected to the strict inequality between $\dim_{\mathrm{H}} \nu_\lambda$ and $\dim_{\mathrm{H}} \Sigma_\lambda$, as stated in (3), which was originally conjectured by Barry Simon and finally proved in [14] (see [12] for an earlier partial result for sufficiently small values of λ).

(b) The potential of the Fibonacci Hamiltonian may be generated by the Fibonacci substitution $a \mapsto ab$, $b \mapsto a$. This substitution is the most prominent example of an invertible two-letter substitution. We believe that, using [22, 32], the results above may be generalized to the case where the Fibonacci substitution is replaced by a general primitive invertible two-letter substitution.

(c) We expect that similar phenomena can appear also in other models, such as for example the labyrinth model, or the square off-diagonal (or tridiagonal) Fibonacci Hamiltonian, see [54, 55] for the description of the models and some partial results.

(d) For other work on the square Fibonacci Hamiltonian and related models, see [16, 17, 18, 21, 50, 51, 56, 57, 59].

(e) Results analogous to Theorem 1.1 also hold for the cubic Fibonacci Hamiltonian (and even higher dimensional versions of operators with separable Fibonacci potential). Indeed, due to [24], the sum of two dynamically defined Cantor sets C_1 and C_2 with $\dim_{\mathrm{H}} C_1 + \dim_{\mathrm{H}} C_2 < 1$ generically must be a dynamically defined Cantor set with $\dim_{\mathrm{H}}(C_1 + C_2) = \dim_{\mathrm{H}} C_1 + \dim_{\mathrm{H}} C_2$. Similarly, the Hausdorff dimension of the convolution of two singular measures of maximal entropy (that correspond to the density of states measures) is typically equal to the sum of dimensions of the initial measures. This reduces the consideration of the cubic Fibonacci Hamiltonian to the results of the current paper.

(f) It would be interesting to understand the topological structure of the spectrum of the Square Fibonacci Hamiltonian in the "intermediate coupling" regime. We conjecture that there exists an open set in the space of parameters $(\lambda_1, \lambda_2) \in \mathbb{R}^2_+$ for which the spectrum of the corresponding Square Fibonacci Hamiltonian is a Cantorval[1]. The conjecture is supported by the results from [35]. They claim that there is an open set $\mathcal{U}$ in the space of dynamically defined Cantor sets such that for generic $C_1, C_2 \in \mathcal{U}$, the sum $C_1 + C_2$ is a Cantorval. Unfortunately, this result does not provide any specific and verifiable genericity conditions that would allow one to check that the sum of two given specific Cantor sets is indeed a Cantorval.

The structure of the paper is as follows. In Section 2 we discuss sums of dynamically defined Cantor sets, and in particular the question of when such a sum has positive Lebesgue measure. The main result, Theorem 2.1, provides sufficient conditions and may be of independent interest since this question arises in a variety of settings, not only in the study of the spectrum of the square Fibonacci Hamiltonian. Then, in Section 3, we return to our discussion of the square Fibonacci Hamiltonian and show that Theorem 2.3, which is a generalization of Theorem 2.1, is applicable in this context and yields the key input in our study of the transition from positive-measure spectrum to zero-measure spectrum. We also discuss the transition of the type of the density of states measure and then conclude the section with a proof of Theorem 1.1.

2. Sums of Dynamically Defined Cantor sets

In this section we work in a general setting and prove a result that provides criteria for certain sum sets to have positive Lebesgue measure. We will eventually apply this to the spectrum of the square Fibonacci Hamiltonian which, as pointed out in the previous section, is given by the sum of two spectra of one-dimensional operators, but as this result may be of independent interest, we present it in the

[1] A compact set $C \subset \mathbb{R}^1$ is a Cantorval if it has a dense interior, i.e., $\overline{\mathrm{int}(C)} = C$, has a continuum of connected components, and none of them are isolated.

appropriate general setting, where it becomes clear what precisely is needed in the proof.

The study of the structure and the properties of sums of Cantor sets is motivated by applications in dynamical systems [38, 39, 40, 42], number theory [7, 27, 33], harmonic analysis [3, 4], and spectral theory [16, 17, 18, 21, 59]. In many cases *dynamically defined* Cantor sets are of special interest.

Definition 1. *A dynamically defined* (*or regular*) *Cantor set of class C^r is a Cantor subset $C \subset \mathbb{R}$ of the real line such that there are disjoint compact intervals $I_1, \dots, I_l \subset \mathbb{R}$ and an expanding C^k function $\Phi : I_1 \cup \dots \cup I_l \to I$ from the disjoint union $I_1 \cup \dots \cup I_l$ to its convex hull I with*

$$C = \bigcap_{n \in \mathbb{N}} \Phi^{-n}(I).$$

In the case when the restriction of the map Φ to each of the intervals $I_j, j = 1, \dots, l$, is affine, the corresponding Cantor set is also called *affine.* If all these affine maps have the same expansion rate (i.e., $|\Phi'(x)| = const$ for all $x \in I_1 \cup \dots \cup I_l$), the Cantor set is called *homogeneous.* A specific example of a homogeneous Cantor set, a *middle-α* Cantor set[2] C_a, is defined by $\Phi : [0, a] \cup [1 - a, 1] \to [0, 1]$, where $\Phi(x) = \frac{x}{a}$ for $x \in [0, a]$, and $\Phi(x) = \frac{x}{a} - \frac{1}{a} + 1$ for $x \in [1 - a, 1]$. For example, $C_{1/3}$ is the standard middle-third Cantor set.

Considering the sum $C + C'$ of two Cantor sets C, C', defined by

$$C + C' = \{c + c' : c \in C, \ c' \in C'\},$$

it is not hard to show (see, e.g., Chapter 4 in [42]) that if the Cantor sets C and C' are dynamically defined, one has $\dim_{\mathrm{H}}(C+C') \le \min(\dim_{\mathrm{H}} C + \dim_{\mathrm{H}} C', 1)$. Hence in the case $\dim_{\mathrm{H}} C + \dim_{\mathrm{H}} C' < 1$, the sum $C + C'$ must be a Cantor set, and an interesting question here is whether the identity $\dim_{\mathrm{H}}(C+C') = \dim_{\mathrm{H}} C + \dim_{\mathrm{H}} C'$ holds. This question was addressed for homogeneous Cantor sets in [44] (see also [37]), and some explicit criteria were provided in [24].

In the case when $\dim_{\mathrm{H}} C + \dim_{\mathrm{H}} C' > 1$, a major result was obtained by Moreira and Yoccoz in [36]. They showed that for a *generic* pair of Cantor sets (C, C') in this regime, the sum $C + C'$ contains an interval. The genericity assumptions there are quite non-explicit, and cannot be verified in a specific case. This does not allow one to apply this result when a specific pair or a specific family of Cantor sets is given (which is often the case in applications), which therefore motivates further investigations in this direction. For example, while [36] solves one part of the Palis conjecture on sums of Cantor sets ("generically the sum of two dynamically defined Cantor sets either has zero measure or contains an interval"), the second part of the conjecture ("generically the sum of two affine Cantor sets either has zero measure or contains an interval") is still open.

An important characteristic of a Cantor set related to questions about intersections and sum sets is the *thickness*, usually denoted by $\tau(C)$. This notion was introduced by Newhouse in [38]; for a detailed discussion, see [42]. The famous *Newhouse Gap Lemma* asserts that if $\tau(C) \cdot \tau(C') > 1$, then $C + C'$ contains an interval. This allowed for essential progress in dynamics [39, 40, 15], and found an application in number theory [1]. Nevertheless, in some cases $\tau(C) \cdot \tau(C') < 1$, while $\dim_{\mathrm{H}} C + \dim_{\mathrm{H}} C' > 1$, and other arguments are needed.

[2] It is standard to denote the *middle-α* Cantor set by C_a, where $a = \frac{1}{2}(1 - \alpha)$.

In [52] Solomyak studied the sums $C_a + C_b$ of middle-α type Cantor sets. He showed that in the regime when $\dim_{\mathrm{H}} C_a + \dim_{\mathrm{H}} C_b > 1$, for almost every pair of parameters (a, b), one has $\mathrm{Leb}(C_a + C_b) > 0$. Similar results for sums of homogeneous Cantor sets (parameterized by the expansion rate) with a fixed compact set were obtained in [44].

In this paper we are able to work in far greater generality and prove the following:

Theorem 2.1. *Let $\{C_\lambda\}$ be a family of dynamically defined Cantor sets of class C^2 (i.e., $C_\lambda = C(\Phi_\lambda)$, where Φ_λ is an expansion of class C^2 both in $x \in \mathbb{R}$ and in $\lambda \in J = (\lambda_0, \lambda_1)$) such that $\frac{d}{d\lambda} \dim_{\mathrm{H}} C_\lambda \neq 0$ for $\lambda \in J$. Let $K \subset \mathbb{R}$ be a compact set such that*

$$\dim_{\mathrm{H}} C_\lambda + \dim_{\mathrm{H}} K > 1 \quad \textit{for all} \quad \lambda \in J.$$

Then $\mathrm{Leb}(C_\lambda + K) > 0$ for a.e. $\lambda \in J$.

Remark 2.2. It would be interesting to relax the assumptions in Theorem 2.1 and to show that the same statement holds for $C^{1+\alpha}$ Cantor sets. We conjecture that this is indeed the case (possibly under some extra conditions on the dependence of Φ and $\frac{\partial}{\partial x}\Phi$ on λ).

In the case when the dynamically defined Cantor sets $\{C_\lambda\}$ are affine (or nonlinear, but C^2-close to affine), a statement analogous to Theorem 2.1 was obtained in [23]. The case of a sum of homogeneous (affine with the same contraction rate for each of the generators) Cantor sets with a dynamically defined Cantor set was considered in Theorem 1.4 in [49]; in this case the set of exceptional parameters has zero Hausdorff dimension.

In many applications a dynamically defined Cantor sets appears as the intersection of the stable lamination of some hyperbolic horseshoe with a transversal. More specifically, suppose that $f : M^2 \to M^2$ is a C^r-diffeomorphism, $r \geq 2$, and $\Lambda \subset M^2$ is a hyperbolic horseshoe (i.e., a totally disconnected locally maximal invariant compact set such that there exists an invariant splitting $T_\Lambda = E^s \oplus E^u$ so that along the stable subbundle $\{E^s\}$, the differential Df contracts uniformly, and along $\{E^u\}$, the differential of the inverse Df^{-1} contracts uniformly). Then

$$W^s(\Lambda) = \{x \in M^2 : \mathrm{dist}\,(f^n(x), \Lambda) \to 0 \text{ as } n \to +\infty\}$$

consists of stable manifolds $W^s(\Lambda) = \bigcup_{x\in\Lambda} W^s(x)$ and locally looks like a product of a Cantor set with an interval. If $f = f_{\lambda^*} \in \{f_\lambda\}_{\lambda \in J = (\lambda_0, \lambda_1)}$ is an element of a smooth family of diffeomorphisms, then there exists a family of horseshoes $\{\Lambda_\lambda\}, f_\lambda(\Lambda_\lambda) = \Lambda_\lambda$, for parameters λ sufficiently close to the initial $\lambda^* \in J$. Suppose that $L \subset M^2$ is a line transversal to every leaf in $W^s(\Lambda_\lambda)$, $\lambda \in J$, with compact intersection $L \cap W^s(\Lambda_\lambda)$. The intersection $C_\lambda = L \cap W^s(\Lambda_\lambda)$ is a λ-dependent dynamically defined Cantor set. The lamination $\{W^s(x)\}$ consists of C^r leaves, but in general one cannot include it in a foliation of smoothness better than $C^{1+\alpha}$ (even for C^∞ or real analytic f). That justifies the traditional assumption on $C^{1+\alpha}$ smoothness of generators of a dynamically defined Cantor set[3]. This prevents us from using Theorem 2.1 in the context above. Nevertheless, the analog of Theorem 2.1 holds for families of Cantor sets $\{C_\lambda\}$ obtained via the described construction:

[3]Notice that C^1-smoothness is usually too weak since it does not allow one to use distortion property arguments; see [34, 58] for some results on sums of C^1 Cantor sets.

Theorem 2.3. *Suppose that* $\{f_\lambda\}_{\lambda\in J=(\lambda_0,\lambda_1)}$, $f_\lambda : M^2 \to M^2$, *is a* C^2*-family of* C^2*-diffeomorphisms with uniformly (in* λ*) bounded* C^2 *norms. Let* $\{\Lambda_\lambda\}_{\lambda\in J}$ *be a family of hyperbolic horseshoes, and* $\{L_\lambda\}_{\lambda\in J}$ *be a smooth family of curves parameterized by* $\gamma_\lambda : \mathbb{R} \to M^2$, *transversal to* $W^s(\Lambda_\lambda)$, *with compact* $C_\lambda = \gamma_\lambda^{-1}(L_\lambda \cap W^s(\Lambda_\lambda))$. *Assume that*

$$\frac{d}{d\lambda}\dim_{\mathrm{H}} C_\lambda \neq 0 \text{ for all } \lambda \in J. \tag{4}$$

If $K \subset \mathbb{R}$ *is a compact set such that*

$$\dim_{\mathrm{H}} C_\lambda + \dim_{\mathrm{H}} K > 1 \quad \text{for all} \quad \lambda \in J, \tag{5}$$

then $\mathrm{Leb}(C_\lambda + K) > 0$ *for Lebesgue almost every* $\lambda \in J$.

Notice that Theorem 2.3 implies Theorem 2.1. Indeed, under the assumptions of Theorem 2.1, one can construct a family of horseshoes and curves as in Theorem 2.3 that produce the same family of Cantor sets $\{C_\lambda\}$. Namely, if C is a dynamically defined Cantor set as in Definition 1, one can consider l disjoint closed intervals $J_1, J_2, \dots, J_l \subset \mathbb{R}$, with convex hull J, and l contacting mappings $f_i : J \to J_i$, $i = 1, \dots, l$. The map $\Psi : \cup_{s=1,\dots,l,\ k=1,\dots,l} I_s \times J_k \to I \times J$, defined by $\Psi(x, y) = (\Phi(x), f_s(y))$ if $x \in I_s$, has an invariant hyperbolic set Λ such that its unstable set intersects the line $\mathbb{R} \times \{0\}$ by the set $C \times \{0\}$. For a more detailed discussion of the relation between dynamically defined Cantor sets and hyperbolic invariant sets of diffeomorphisms, see, for example, [42, Chapter 4].

The proof of Theorem 2.3 is based on Theorem 3.7 from [13]. The setting there is the following.

Suppose $J \subset \mathbb{R}$ is a compact interval, and $f_\lambda : M^2 \to M^2$, $\lambda \in J$, is a smooth family of smooth surface diffeomorphisms. Specifically, we require $f_\lambda(p)$ to be C^2-smooth with respect to both λ and p, with a finite C^2-norm. Also, we assume that $f_\lambda : M^2 \to M^2$, $\lambda \in J$, has a locally maximal transitive totally disconnected hyperbolic set Λ_λ that depends continuously on the parameter.

Let $\gamma_\lambda : \mathbb{R} \to M^2$ be a family of smooth curves, smoothly depending on the parameter, and $L_\lambda = \gamma_\lambda(\mathbb{R})$. Suppose that the stable manifolds of Λ_λ are transversal to L_λ.

Lemma 2.4 (Lemma 3.1 from [13])**.** *There is a Markov partition of* Λ_λ *and a continuous family of projections* $\pi_\lambda : \Lambda_\lambda \to L_\lambda$ *along stable manifolds of* Λ_λ *such that for any two distinct elements of the Markov partition, their images under* π_λ *are disjoint.*

Suppose $\sigma_A : \Sigma_A^\ell \to \Sigma_A^\ell$ is a topological Markov chain, which for every $\lambda \in J$ is conjugated to $f_\lambda : \Lambda_\lambda \to \Lambda_\lambda$ via the conjugacy $H_\lambda : \Sigma_A^\ell \to \Lambda_\lambda$. Let μ be an ergodic probability measure for $\sigma_A : \Sigma_A^\ell \to \Sigma_A^\ell$ such that $h_\mu(\sigma_A) > 0$. Set $\mu_\lambda = H_\lambda(\mu)$, then μ_λ is an ergodic invariant measure for $f_\lambda : \Lambda_\lambda \to \Lambda_\lambda$.

Let $\pi_\lambda : \Lambda_\lambda \to L_\lambda$ be the continuous family of continuous projections along the stable manifolds of Λ_λ provided by Lemma 2.4. Set $\nu_\lambda = \gamma_\lambda^{-1} \circ \pi_\lambda(\mu_\lambda) = \gamma_\lambda^{-1} \circ \pi_\lambda \circ H_\lambda(\mu)$.

In this setting the following theorem holds.

Theorem 2.5 (Theorem 3.7 from [13])**.** *Suppose that* J *is a compact interval so that* $\left|\frac{d}{d\lambda}\mathrm{Lyap}^u(\mu_\lambda)\right| \geq \delta > 0$ *for some* $\delta > 0$ *and all* $\lambda \in J$. *Then for any compactly supported exact-dimensional measure* η *on* $\mathbb{R}$ *with*

$$\dim_{\mathrm{H}} \eta + \dim_{\mathrm{H}} \nu_\lambda > 1$$

for all $\lambda \in J$, the convolution $\eta * \nu_\lambda$ is absolutely continuous with respect to Lebesgue measure for Lebesgue almost every $\lambda \in J$.

Remark 2.6. In fact, in Theorem 2.5 the assumptions on the measure η can be replaced by the following weaker ones:

- There are $C > 0$ and $d > 0$ such that for every $x \in \mathbb{R}$ and $r > 0$, we have $\eta(B_r(x)) \le Cr^d$ (this is the only consequence of exact dimensionality of η that was used in the proof of Theorem 2.5 in [13]),
- $d + \dim_{\mathrm{H}} \nu_\lambda > 1$.

Proof of Theorem 2.3. The condition $\dim_{\mathrm{H}} C_\lambda + \dim_{\mathrm{H}} K > 1$ trivially implies that $\dim_{\mathrm{H}} K > 0$. By Frostman's Lemma (see, e.g., [30, Theorem 8.8]), for every $d < \dim_{\mathrm{H}} K$, there exist a Borel measure η on $\mathbb{R}$ with $\eta(K) = 1$ and a constant C such that

$$\eta(B_r(x)) \le Cr^d \quad \text{for every } x \in \mathbb{R} \text{ and } r > 0. \tag{6}$$

We will show that for every $\lambda_0 \in J$, there exists $\varepsilon = \varepsilon(\lambda_0) > 0$ such that $\mathrm{Leb}(C_\lambda + K) > 0$ for Lebesgue almost every $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \cap J$. This will imply Theorem 2.3.

Fix $\lambda_0 \in J$. Let μ_{λ_0} be the equilibrium measure on Λ_{λ_0} that corresponds to the potential $-\dim_{\mathrm{H}} C_{\lambda_0} \log |Df_{\lambda_0}|_{E^u}|$. Then (see [31]), the measure μ_{λ_0} is a measure of maximal (unstable) dimension, that is, $\dim_{\mathrm{H}} \pi_{\lambda_0}(\mu_{\lambda_0}) = \dim_{\mathrm{H}} C_{\lambda_0}$. Denote by ν_{λ_0} the projection $\pi_{\lambda_0}(\mu_{\lambda_0})$. In order to mimic the setting of Theorem 2.5, set $\mu = H_{\lambda_0}^{-1}(\mu_{\lambda_0})$. Then μ is an invariant probability measure for the shift $\sigma_A : \Sigma_A^l \to \Sigma_A^l$. Let us denote $\mu_\lambda = H_\lambda(\mu)$ and $\nu_\lambda = \pi_\lambda(\mu_\lambda)$. There exists a canonical family of conjugacies $\mathcal{H}_{\lambda_1,\lambda_2} : \Lambda_{\lambda_1} \to \Lambda_{\lambda_2}$, $(\lambda_1, \lambda_2) \in J \times J$, so that $\mathcal{H}_{\lambda_1,\lambda_2} \circ f_{\lambda_1} = f_{\lambda_2} \circ \mathcal{H}_{\lambda_1,\lambda_2}$. It is well known (see, for example, Theorem 19.1.2 from [25]) that each of the maps $\mathcal{H}_{\lambda_1,\lambda_2}$ is Hölder continuous. Moreover, the Hölder exponent tends to one as $|\lambda_1 - \lambda_2| \to 0$; see [43]. As a result, we conclude that for any λ sufficiently close to λ_0, we have

$$\dim_{\mathrm{H}} \nu_\lambda + d > 1$$

for a suitable d that is chosen sufficiently close to $\dim_{\mathrm{H}} K$ and for which we have (6) with suitable η and C.

In order to apply Theorem 2.5 we need to show that $\left|\frac{d}{d\lambda}\mathrm{Lyap}^u(\mu_\lambda)\right| \ge \delta > 0$. But due to [29] we know that

$$\mathrm{Lyap}^u \mu_\lambda = \frac{h_{\mu_\lambda}}{\dim_{\mathrm{H}} \nu_\lambda},$$

where $h_{\mu_{\lambda_0}} = h_\mu$ is the entropy of the invariant measure μ_λ (which is by construction independent of λ). Notice also that $\mathrm{Lyap}^u \mu_\lambda$ is a C^1 smooth function of λ. Indeed, the center-stable and center-unstable manifolds of the partially hyperbolic invariant set of the map $(\lambda, p) \mapsto (\lambda, f_\lambda(p))$ are C^2-smooth, hence

$$\mathrm{Lyap}^u \mu_\lambda = \int_{\Lambda_\lambda} \log |Df_\lambda|_{E^u}|\, d\mu_\lambda = \int_{\Sigma_A^l} \log |Df_\lambda(H_\lambda(\omega))|_{E^u}|\, d\mu(\omega)$$

is a C^1-smooth function of $\lambda \in J$.

Finally, consider $\dim_{\mathrm{H}} C_\lambda$ and $\dim_{\mathrm{H}} \nu_\lambda$ as functions of λ; see Fig. 2. Due to [28] we know that $\dim_{\mathrm{H}} C_\lambda$ is a C^1-function of λ. Without loss of generality we can assume that $\frac{d}{d\lambda} \dim_{\mathrm{H}} C_\lambda \ge \delta > 0$ for some $\delta > 0$. Since $\mathrm{supp}\, \nu_\lambda \subseteq C_\lambda$, we have $\dim_{\mathrm{H}} \nu_\lambda \le \dim_{\mathrm{H}} C_\lambda$. By construction we have $\dim_{\mathrm{H}} \nu_{\lambda_0} = \dim_{\mathrm{H}} C_{\lambda_0}$. This implies

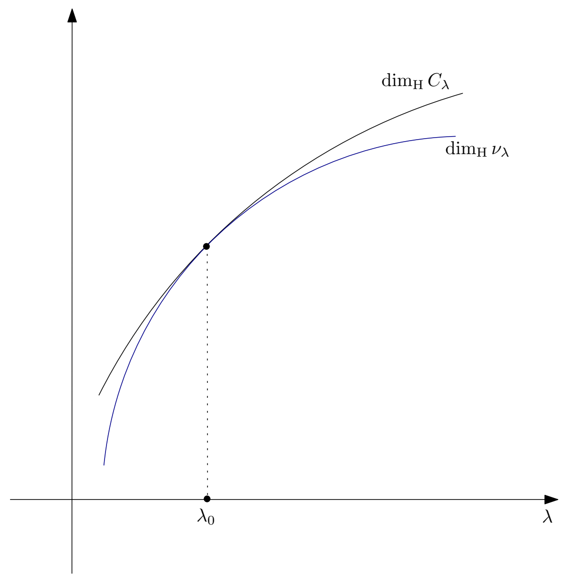

FIGURE 2. Graphs of $\dim_{\mathrm{H}} C_\lambda$ and $\dim_H \nu_\lambda$ as functions of λ

that $\frac{d}{d\lambda}|_{\lambda=\lambda_0} \dim_{\mathrm{H}} \nu_\lambda = \frac{d}{d\lambda}|_{\lambda=\lambda_0} \dim_{\mathrm{H}} C_\lambda \geq \delta > 0$, and hence for some $\varepsilon > 0$, $\frac{d}{d\lambda} \dim_{\mathrm{H}} \nu_\lambda \geq \frac{\delta}{2} > 0$ for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. Now we can apply Theorem 2.5 to the measures η and ν_λ, and get that for Lebesgue almost every $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, the convolution $\eta * \nu_\lambda$ is absolutely continuous with respect to Lebesgue measure, and hence $\mathrm{Leb}(C_\lambda + K) > 0$. □

3. The Square Fibonacci Hamiltonian

The ultimate goal of this section is to prove Theorem 1.1. We will first recall the dynamical description of the spectrum of the Fibonacci Hamiltonian via the trace map.

3.1. **The Dynamical Description of the Spectrum.** There is a fundamental connection between the spectral properties of the Fibonacci Hamiltonian and the dynamics of the *trace map*

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \ T(x, y, z) = (2xy - z, x, y). \tag{7}$$

The function $G(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1$ is invariant[4] under the action of T, and hence T preserves the family of cubic surfaces[5]

$$S_\lambda = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - 2xyz = 1 + \frac{\lambda^2}{4} \right\}. \tag{8}$$

It is therefore natural to consider the restriction T_λ of the trace map T to the invariant surface S_λ. That is, $T_\lambda : S_\lambda \to S_\lambda$, $T_\lambda = T|_{S_\lambda}$. We denote by Λ_λ the set of points in S_λ whose full orbits under T_λ are bounded (it follows from [5, 47] that Λ_λ is equal to the non-wandering set of T_λ; compare the discussion in [12]).

[4] G is usually called the Fricke-Vogt invariant.
[5] The surface S_0 is known as Cayley cubic.

Denote by ℓ_λ the line

$$\ell_\lambda = \left\{ \left(\frac{E-\lambda}{2}, \frac{E}{2}, 1 \right) : E \in \mathbb{R} \right\}. \tag{9}$$

It is easy to check that $\ell_\lambda \subset S_\lambda$. The key to the fundamental connection between the spectral properties of the Fibonacci Hamiltonian and the dynamics of the trace map is the following result of Sütő [53]. An energy $E \in \mathbb{R}$ belongs to the spectrum Σ_λ of the Fibonacci Hamiltonian if and only if the positive semiorbit of the point $(\frac{E-\lambda}{2}, \frac{E}{2}, 1)$ under iterates of the trace map T is bounded.

It turns out that for every $\lambda > 0$, Λ_λ is a locally maximal compact transitive hyperbolic set of $T_\lambda : S_\lambda \to S_\lambda$; see [5, 6, 10]. Moreover, it was shown in [14] that for every $\lambda > 0$, the line of initial conditions ℓ_λ intersects $W^s(\Lambda_\lambda)$ transversally. Thus, we are essentially in the setting in which Theorem 2.3 applies. The only minor difference is that in the present setting, the surface S_λ depends formally on λ, while it is λ-independent in the setting of Theorem 2.3. After partitioning the parameter space into smaller intervals if necessary, we can then consider a small λ-interval, choose a λ_0 in it, and then conjugate with smooth projections of S_λ to S_{λ_0}.

3.2. **The Measure of the Spectrum.** As was pointed out above, it was shown in [14] that the box counting dimension of Σ_λ exists and is equal to the Hausdorff dimension of Σ_λ. A particular consequence of this is the following:

Proposition 3.1. *If $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ is such that $\dim_{\mathrm{H}} \Sigma_{\lambda_1} + \dim_{\mathrm{H}} \Sigma_{\lambda_2} < 1$, then $\Sigma_{\lambda_1,\lambda_2}$ has zero Lebesgue measure.*

Here we are able to prove the following companion result:

Proposition 3.2. *Suppose that for all pairs (λ_1, λ_2) in some open set $U \subset \mathbb{R}_+^2$, we have $\dim_{\mathrm{H}} \Sigma_{\lambda_1} + \dim_{\mathrm{H}} \Sigma_{\lambda_2} > 1$. Then, for Lebesgue almost all pairs $(\lambda_1, \lambda_2) \in U$, $\Sigma_{\lambda_1,\lambda_2}$ has positive Lebesgue measure.*

Proof. It clearly suffices to work locally in U (compare with the first steps in the proof of Theorem 2.3). That is, we consider a rectangular box $B = \{(\lambda_1, \lambda_2) : a < \lambda_1 < b, c < \lambda_2 < d\}$ inside U and prove that for Lebesgue almost every $(\lambda_1, \lambda_2) \in B$, $\Sigma_{\lambda_1,\lambda_2}$ has positive Lebesgue measure. To accomplish this, it suffices to show that for every fixed $\lambda_2 \in (c, d)$, $\Sigma_{\lambda_1,\lambda_2}$ has positive Lebesgue measure for Lebesgue almost every $\lambda_1 \in (a, b)$.

The set Σ_{λ_2} will play the role of the set K in Theorem 2.3. By the analyticity of $\lambda_1 \mapsto \dim_{\mathrm{H}} \Sigma_{\lambda_1}$, we can subdivide (a, b) into intervals, on the interiors of which we have the condition

$$\frac{d}{d\lambda_1} \dim_{\mathrm{H}} \Sigma_{\lambda_1} \neq 0.$$

This ensures that condition (4) in Theorem 2.3 holds. Condition (5) in Theorem 2.3 holds since we work inside U. All the other assumptions in Theorem 2.3 hold by the discussion in the previous subsection. Thus we may apply Theorem 2.3 and obtain the desired statement. $\square$

3.3. **The Density of States Measure.** Combining results from [13] and [14], we obtain the following statement:

Proposition 3.3. *For Lebesgue almost all pairs $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ in the region where $\dim_{\mathrm{H}} \nu_{\lambda_1} + \dim_{\mathrm{H}} \nu_{\lambda_2} > 1$, the measure $\nu_{\lambda_1,\lambda_2}$ is absolutely continuous.*

Let us prove the following companion result:

Proposition 3.4. *Suppose that* $\dim_{\mathrm{H}} \nu_{\lambda_1} + \dim_{\mathrm{H}} \nu_{\lambda_2} < 1$. *Then,* $\nu_{\lambda_1,\lambda_2}$ *is singular, that is, it is supported by a set of zero Lebesgue measure.*

We begin by recalling some basic concepts from measure theory and fractal geometry; the standard texts [19, 30] can be consulted for background information. Suppose μ is a finite Borel measure on $\mathbb{R}^d$. The lower Hausdorff dimension, resp. the upper Hausdorff dimension, of μ are given by

$$\dim_{\mathrm{H}}^{-}(\mu) = \inf\{\dim_{\mathrm{H}}(S) : \mu(S) > 0\}, \tag{10}$$

$$\dim_{\mathrm{H}}^{+}(\mu) = \inf\{\dim_{\mathrm{H}}(S) : \mu(\mathbb{R}^d \setminus S) = 0\}. \tag{11}$$

Thus, the measure μ gives zero weight to every set S with $\dim_{\mathrm{H}}(S) < \dim_{\mathrm{H}}^{-}(\mu)$ and, for every $\varepsilon > 0$, there is a set S with $\dim_{\mathrm{H}}(S) < \dim_{\mathrm{H}}^{+}(\mu) + \varepsilon$ that supports μ (i.e., $\mu(\mathbb{R} \setminus S) = 0$).

For $x \in \mathbb{R}^d$ and $\varepsilon > 0$, we denote the open ball with radius ε and center x by $B(x,\varepsilon)$. The lower scaling exponent of μ at x is given by

$$\alpha_\mu^-(x) = \liminf_{\varepsilon \to 0} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon}.$$

For μ-almost every x, $\alpha^-(x) \in [0,d]$. Moreover, we have

$$\dim_{\mathrm{H}}^{-}(\mu) = \mu - \operatorname{essinf} \alpha_\mu^- \equiv \sup\{\alpha : \alpha_\mu^-(x) \ge \alpha \text{ for } \mu\text{-almost every } x\}, \tag{12}$$

$$\dim_{\mathrm{H}}^{+}(\mu) = \mu - \operatorname{esssup} \alpha_\mu^- \equiv \inf\{\alpha : \alpha_\mu^-(x) \le \alpha \text{ for } \mu\text{-almost every } x\}, \tag{13}$$

compare [20, Propositions 10.2 and 10.3].

One can also consider the upper scaling exponent of μ at x,

$$\alpha_\mu^+(x) = \limsup_{\varepsilon \to 0} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon},$$

which also belongs to $[0,d]$ for μ-almost every x. The measure μ is called exact-dimensional if there is a number $\dim \mu \in [0,d]$ such that $\alpha_\mu^+(x) = \alpha_\mu^-(x) = \dim \mu$ for μ-almost every $x \in \mathbb{R}^d$. In this case, it of course follows that $\dim_{\mathrm{H}}^{+}(\mu) = \dim_{\mathrm{H}}^{-}(\mu) = \dim \mu$, and tangentially we note that the common value also coincides with the upper and lower packing dimension of μ, which are defined analogously by replacing the Hausdorff dimension of a set in the above definitions by the packing dimension; see [19, 20, 30] for further details.

We are now ready to prove Proposition 3.4. In fact, the statement will follow quickly from known results once we have established the following simple lemma.

Lemma 3.5. *Suppose* ν_1 *and* ν_2 *are compactly supported exact-dimensional measures on* $\mathbb{R}$ *of dimension* d_1 *and* d_2*, respectively. If* $d_1 + d_2 < 1$*, then the convolution* $\nu_1 * \nu_2$ *is singular.*

Proof. Note first that the product measure $\nu_1 \times \nu_2$ is exact-dimensional with dimension $d_1 + d_2$. Moreover, the convolution $\nu_1 * \nu_2$ can be obtained from $\nu_1 \times \nu_2$ by projection, that is,

$$\nu_1 * \nu_2(B) = \nu_1 \times \nu_2\{(x,y) \in \mathbb{R}^2 : x + y \in B\}.$$

It follows that for $\nu_1 * \nu_2$-almost every $x \in \mathbb{R}$, the lower scaling exponent

$$\alpha^-_{\nu_1 * \nu_2}(x) = \liminf_{\varepsilon \downarrow 0} \frac{\log(\nu_1 * \nu_2((x-\varepsilon, x+\varepsilon)))}{\log \varepsilon}$$

is bounded from above by $d_1 + d_2$. This implies that the upper Hausdorff dimension of $\nu_1 * \nu_2$,

$$\begin{aligned} \dim^+_{\mathrm{H}}(\nu_1 * \nu_2) &= \inf\{\dim_{\mathrm{H}}(S) : \nu_1 * \nu_2(\mathbb{R} \setminus S) = 0\} \\ &= \nu_1 * \nu_2 - \operatorname{esssup} \alpha^-_{\nu_1 * \nu_2} \\ &\equiv \inf\{d : \alpha^-_{\nu_1 * \nu_2}(x) \le d \text{ for } \nu_1 * \nu_2\text{-almost every } x\}, \end{aligned}$$

is bounded from above by $d_1 + d_2$ (here we used (11) and (13)). Since $d_1 + d_2 < 1$ by assumption, $\nu_1 * \nu_2$ has a support of Hausdorff dimension strictly less than one and hence of Lebesgue measure zero. This shows that $\nu_1 * \nu_2$ is singular. $\square$

Proof of Proposition 3.4. It was shown in [14] that for every $\lambda > 0$, the density of states measure ν_λ is exact-dimensional. Thus, Proposition 3.4 is an immediate consequence of Lemma 3.5. $\square$

3.4. **Putting It All Together.** We are now in a position to prove our main result, Theorem 1.1.

Proof of Theorem 1.1. (a) The regions U_{acds}, U_{pmsd}, U_{zmsp} are clearly disjoint due to their definition. Moreover, the union of their closures covers the parameter space $\mathbb{R}^2_+$ due to the analyticity of both $\dim_{\mathrm{H}} \nu_\lambda$ and $\dim_{\mathrm{H}} \Sigma_\lambda$; compare [45].

(b) It was shown in [14] that $\dim_{\mathrm{H}} \nu_\lambda$ and $\dim_{\mathrm{H}} \Sigma_\lambda$ obey the inequalities (3). This, together with the continuity of these functions, implies that each of the regions U_{acds}, U_{pmsd}, U_{zmsp} is open and non-empty.

(c) Proposition 3.3 shows that for Lebesgue almost every $(\lambda_1, \lambda_2) \in U_{\mathrm{acds}}$, $\nu_{\lambda_1,\lambda_2}$ is absolutely continuous, and hence $\Sigma_{\lambda_1,\lambda_2}$ has positive Lebesgue measure.

(d) On the other hand, for every $(\lambda_1, \lambda_2) \in U_{\mathrm{pmsd}}$, $\nu_{\lambda_1,\lambda_2}$ is singular by Proposition 3.4, while for Lebesgue almost every $(\lambda_1, \lambda_2) \in U_{\mathrm{pmsd}}$, $\Sigma_{\lambda_1,\lambda_2}$ has positive Lebesgue measure due to Proposition 3.2.

(e) Finally, it follows from Proposition 3.1 that for every $(\lambda_1, \lambda_2) \in U_{\mathrm{zmsp}}$, $\Sigma_{\lambda_1,\lambda_2}$ has zero Lebesgue measure, and hence $\nu_{\lambda_1,\lambda_2}$ is singular. $\square$

Acknowledgment

We are grateful to Mark Embree for generating the image shown in Figure 1.

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