A Gradient-Free Three-Dimensional Source Seeking Strategy With Robustness Analysis

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***Abstract*—Distributed source seeking in a three-dimensional (3-D) environment without explicit estimation of the gradient of the field is challenging. Nevertheless, for a swarm of an arbitrary number of agents, we develop a strategy to perform a source seeking behavior by requiring agents to synchronize their direction of motion using only local interactions while modulating their speed based only on the field value measured by each agent. The agents collectively move toward the source without estimating gradient and without sharing measurements of the field. We formulate a cascaded input-to-state stability problem from which we obtain Lyapunov-based convergence and robustness results. We validate the convergence of the swarm to the minimum of the field through simulated source seeking behavior in a 3-D scalar field.**

***Index Terms*—Bioinspired swarm behavior, cascaded input-to-state stability, consensus-on-a sphere, Distributed source seeking.**

I. Introduction

Distributed source seeking is a collective behavior that has been studied in natural swarms such as flocks of bird and schools of fish, where the source value represents an environmental characteristic such as chemical concentration, light intensity or temperature, just to name a few [1]. Due to its distributed nature, this collective behavior has inspired researchers to develop strategies and algorithms for swarms of robots that are required to localize and identify a feature of interest. The majority of these source seeking strategies require either exact knowledge or explicit estimation of the gradient, which relies on the exchange of field measurements as is the case of [2]–[5]. Recently, a two-dimensional (2-D) bioinspired distributed source seeking strategy, called the speeding up and slowing down (SUSD) strategy has been developed in [6] where agents do not need to share their measurements, but still move towards the source collectively.

In this paper, we extend the 2-D SUSD strategy to the 3-D setting for an M-agent swarms. Motivated by certain biological swarming behaviors [7], [8], we consider three agents of the swarm to locally compute a 3-D time-varying moving frame based only on relative positions. On the other hand, the rest of the agents compute their body frame using a nonlinear consensus-on-a sphere. Biologically the three agents can be viewed as agents with older ages or more experience, which from a robotic point of view can be thought as robots with high sensing and computation capabilities. Each agent decomposes its velocity into forward motion in one direction of the frame (SUSD direction), and formation or connectivity-maintaining motion in the plane formed by the remaining components of the frame (formation plain). The forward motion speed (SUSD speed) depends only on the current field value measured by each agent, hence each agent speeds up or slows down as the field value changes.

The strategy results in a two-layer system: environmental and social layers. In the environmental layer, the agents interact with the environment by modulating their speed as a function of the field value. In the social layer, the agents interact with each other by means of implicit and explicit consensus and formation control laws. The implicit consensus law between the SUSD direction and the negative direction of the gradient remarkably emerges from the local interaction rules that the three agents apply to compute their body frame components. The explicit consensus is used by the remaining agents to align with the three agents. This results in a global synchronization behavior where the synchronized value is indirectly controlled by the field value in the environmental layer. This phenomenon can not be explained by known consensus algorithms because the consensus emerges not due to sharing the values or gradient of the field, but due to the SUSD strategy.

A difficulty this paper has overcome is to pursue source seeking in a 3-D space without assuming knowledge of the gradient as in [5], and without explicit gradient estimation that relies on the exchange of measurements as in [2]–[4]. The proposed SUSD strategy assumes that each agent is able to measure the field value only at its current position and does not require agents to exchange field measurements. Although extremum-based swarm source seeking approaches do not require explicit knowledge of the field gradient, they are designed to indirectly estimate the field gradient. Additionally, multiagent extremum seeking approaches require an exchange of the field measurements [9], [10]. Except for the three agents that are required to maintain an equilateral rigid body, the remaining agents are not required to form any specific formations or graph structure, as in [2] where a circular formation is required. This implies that the strategy is scalable to swarms of arbitrary numbers of agents and connected graphs.

Another challenge this paper has overcome is the convergence and robustness analysis of the nonlinear consensus-on-a sphere control laws. Although the general form of consensus-on-a-sphere is elegantly analyzed in [11] and [12], our analysis is different in that it deals with directed edges that complicate the analysis. Furthermore, since the swarm is continuously moving, the consensus considered in this paper is time varying with an input disturbance due to the change of the field gradient as the swarm navigates. Through a choice of collective states that represent the desired source seeking behavior, we are able to

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lump all the individual consensus states into an overall system of two cascaded subsystems, and then justify the robustness of the consensus law by formulating a cascaded input-to-state stability problem, using techniques as in [13]. The choice of the collective states allows us to analyze the cascaded input-to-state stability by only requiring connected graphs without any restriction on the graph structure. This is different than for example [14], where the analysis is initially conducted for specific graphs and then generalized to arbitrary graphs inductively.

The main contributions of this paper are as follows.

1) Extending the SUSD in [6] from 2-D to 3-D for three-agents formation.
2) Integrating a consensus-on-a sphere control law with the SUSD for a swarm of an arbitrary number of agents.
3) Proving the convergence and robustness of the proposed strategy through an input-to-state stability analysis.

The primary results of this paper are presented in [15] and [16]. Compared to [15], in this paper, we show that the dynamics of the SUSD direction is in the form of a consensus-on-a sphere. Compared to [16], where we used an ultimate boundedness convergence analysis, in this paper we prove the robustness of the consensus using a cascaded input-to-state stability framework.

The paper contributes to multiagent dynamics by providing a method to analyze the collective motion of agents that need to synchronize their direction of motion while modulating their speed. The selection of collective states is a key step to enable convergence and robustness results. Additionally, the resulted multilayer system can be a new framework to describe and analyze the behavior of biological swarms.

II. Problem Formulation

Consider a swarm of M agents in a 3-D space. Let $\mathbf{r}_i \in \mathbb{R}^3, i = 1, \dots, M$ be the position of the ith agent in the 3-D space. Let the interaction among the agents be described by a graph $\mathcal{G} \subseteq \mathcal{V} \times \mathcal{E}$, where $\mathcal{V}$ is the set of all agents and $\mathcal{E}$ is the set of all edges. Additionally, an edge $(i, j) \in \mathcal{E}$ is undirected if also $(j, i) \in \mathcal{E}$ where $i, j \in \mathcal{V}$. Consequently, a graph is undirected if all edges in $\mathcal{E}$ are undirected. A graph is connected if for all pairs of the agents in the graph, there exits a path connecting the two agents. A graph is complete, if each agent shares an edge with all other agents. The neighbor set of i is defined by $\mathcal{N}_i = \{j | (i, j) \in \mathcal{E}\}$. Additionally, if for each agent $\mathcal{N}_i$ is fixed, the graph is static, otherwise it is dynamic. In this paper, we assume the following.

Assumption 2.1: $\mathcal{G}$ is static and connected.

Assumption 2.2: Each agent i, is able to measure the relative displacement $(\mathbf{r}_j - \mathbf{r}_i)$ for all $j \in \mathcal{N}_i$.

In practice, robots can be equipped with sensors to measure the relative positions of their neighbors, which is less challenging than requiring the global positions [17].

Suppose each agent is able to measure a positive field value $f_i = f(\mathbf{r}_i) \in \mathbb{R}$ that represents an environmental characteristic such as temperature or light intensity, with the following assumption.

Assumption 2.3: 1) The field $f(\mathbf{r}_i)$ is smooth, time invariant and bounded, i.e., $0 \le f_{\min} \le f(\mathbf{r}_i) \le f_{\max}$.

2) The field has a unique minimum at the source location $\mathbf{r}_0$, i.e., $f(\mathbf{r}_0) = f_{\min}$.

Although not all of the real fields are smooth [18], this assumption does not limit the applicability of the proposed strategy. For nonsmooth fields, we can use stochastic models to transfer them into smooth fields. Indeed, in our preliminary work in [15], we use a Poisson counting process to transform a turbulent plume field into a smooth field.

Let the velocity of each agent be described by

$$\dot{\mathbf{r}}_i = \mathbf{v}_i. \tag{1}$$

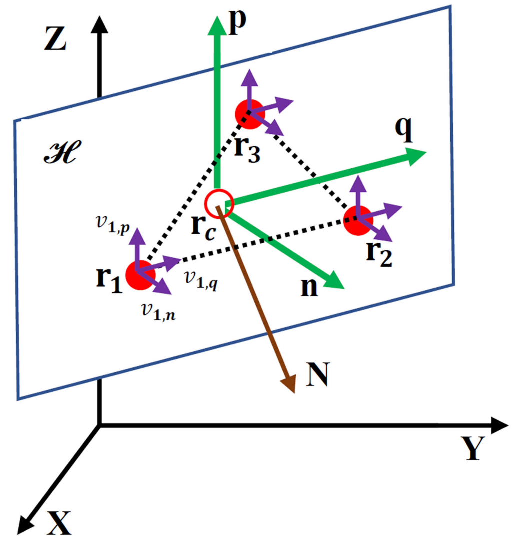

Fig. 1. Geometry of a three-agent group in inertial frame.

Then, the problem we want to solve is to design a velocity controller $\mathbf{v}_i$ such that the swarm converges to the source distributively. The problem is challenging since we want to solve it without explicitly estimating the field gradient and without sharing field measurements. This design might be used to model a biological swarm of a school of fish seeking dark areas [19], which presumably do not explicitly share field measurements. Additionally, this problem is important for swarm robotics with limited resources such as underwater robotics, and in environments where a field gradient is not possible to be estimated or not well-defined such as a turbulent field.

III. Velocity Control Law Design

In this section, we first design source seeking control laws for a three-agent swarm. We then design control laws that enable a swarm of arbitrary number of agents to get to the source distributively.

A. Swarm of Three Agents

Let the three agents form an equilateral formation, which means that $||\mathbf{r}_1 - \mathbf{r}_2|| = ||\mathbf{r}_2 - \mathbf{r}_3|| = ||\mathbf{r}_3 - \mathbf{r}_1||$. Define a right-handed orthonormal frame $(\mathbf{q}, \mathbf{p}, \mathbf{n})$, with an origin located at the center $\mathbf{r}_c = \frac{1}{3}\sum_{i=1}^{3} \mathbf{r}_i$, as

$$\mathbf{q} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{||\mathbf{r}_2 - \mathbf{r}_1||}, \quad \mathbf{p} = \frac{\mathbf{r}_3 - \mathbf{r}_c}{||\mathbf{r}_3 - \mathbf{r}_c||}, \quad \mathbf{n} = \mathbf{q} \times \mathbf{p}. \tag{2}$$

Fig. 1 illustrates the geometry of the three-agent group with the defined frame where $\mathbf{N} = \frac{\nabla f(\mathbf{r}_c)}{||\nabla f(\mathbf{r}_c)||}$ is a unit length vector pointing towards the field gradient at the center of the formation.

Assumption 3.1: The three agents are numbered from 1 to 3 such that each agent knows the numbers of the other agents.

Without explicit communication, this assumption can be satisfied by assigning each agent with a specific mark or a blinking LED that can be identified by each robot. Given the defined frame, let the velocity $\mathbf{v}_i$ in (1) be decomposed as

$$\mathbf{v}_i = \mathbf{v}_{i,\mathbf{n}} + \mathbf{v}_{i,\mathbf{q}} + \mathbf{v}_{i,\mathbf{p}} = v_{i,\mathbf{n}}\mathbf{n} + v_{i,\mathbf{q}}\mathbf{q} + v_{i,\mathbf{p}}\mathbf{p} \tag{3}$$

where $v_{i,\mathbf{n}}$, $v_{i,\mathbf{q}}$, and $v_{i,\mathbf{p}}$ represent the decoupled forward speed and formation speeds along the $\mathbf{n}$, $\mathbf{q}$, and $\mathbf{p}$ directions, respectively. Through this decomposition, we decouple the normal component of the velocity $\mathbf{v}_{i,\mathbf{n}}$ from the tangential components $\mathbf{v}_{i,\mathbf{q}}$ and $\mathbf{v}_{i,\mathbf{p}}$, which

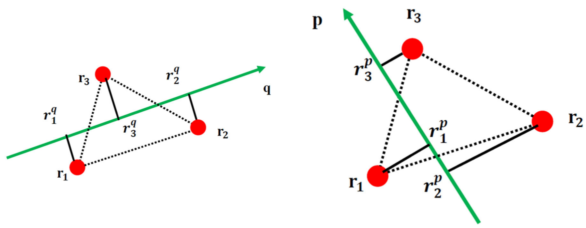

Fig. 2. Projections of $\mathbf{r}_i$ onto $\mathbf{q}$ (a) and $\mathbf{p}$ (b).

allows us to analyze the stability of the normal and tangential modes separately.

The objective for $v_{i,\mathbf{q}}$ and $v_{i,\mathbf{p}}$ is to maintain a rigid formation of an equilateral triangle in the plane $\mathcal{H}$, which is formed by the vectors $\mathbf{q}$ and $\mathbf{p}$. Let r_i^q be the projection of $\mathbf{r}_i$ onto vector $\mathbf{q}$, and r_i^p be the projection of $\mathbf{r}_i$ onto vector $\mathbf{p}$, as illustrated in Fig. 2(a) and (b), respectively. For agent i, we define sets $\mathcal{N}_i^q$ and $\mathcal{N}_i^p$ that contain the indices of the neighboring agents along directions $\mathbf{q}$ and $\mathbf{p}$, respectively. For example, for the three-agent group as shown in Fig. 2, $\mathcal{N}_1^q = \{3\}$, $\mathcal{N}_2^q = \{3\}$, $\mathcal{N}_3^q = \{1,2\}$, $\mathcal{N}_1^p = \{2,3\}$, $\mathcal{N}_2^p = \{1,3\}$, $\mathcal{N}_3^p = \{1,2\}$. The goal is to design $v_{i,\mathbf{q}}$ and $v_{i,\mathbf{p}}$ so that the relative distance from r_i^q to $r_j^q, i \neq j$, converges to a constant a_{ij}^0, and the relative distance from r_i^p to $r_j^p, i \neq j$, converges to a constant b_{ij}^0. Therefore, we design

$$v_{i,\mathbf{q}} = k_3 \sum_{j \in \mathcal{N}_i^q} [\langle \mathbf{r}_j - \mathbf{r}_i, \mathbf{q}\rangle - a_{j,i}^0] \tag{4}$$

$$v_{i,\mathbf{p}} = k_4 \sum_{j \in \mathcal{N}_i^p} [\langle \mathbf{r}_j - \mathbf{r}_i, \mathbf{p}\rangle - b_{j,i}^0] \tag{5}$$

where $k_3, k_4 > 0$ are formation gain constants, $a_{i,j}^0 = -a_{j,i}^0$ and $b_{i,j}^0 = -b_{j,i}^0$ are desired formation distances selected such that the three agents form an equilateral triangle.

Inspired by behaviors of fish schools [19], we design the forward speed $v_{i,\mathbf{n}}$ in the direction $\mathbf{n}$ to be proportional to the field value $f(\mathbf{r}_i)$ as following:

$$v_{i,\mathbf{n}} = k_1 f(\mathbf{r}_i) + k_2,\ i = 1, \ldots, M \tag{6}$$

where $k_1, k_2 \in \mathbf{R}$ are positive gain constants. Note that $v_{i,\mathbf{n}}$ depends only on the locally measured field value, $f(\mathbf{r}_i)$. Thus, the forward motion speed increases or decreases based on the field measurement, $f(\mathbf{r}_i)$, and hence it is called SUSD speed. From now and after, we call $v_{i,\mathbf{n}}$ the SUSD speed, and $\mathbf{n}$ the SUSD direction.

B. Swarm of More Than Three Agents

To enable a swarm of more than three agents to reach the source distributively, it is challenging to define the frame components $(\mathbf{q}, \mathbf{p}, \mathbf{n})$ based only on relative positions as in (2). Inspired by biological swarms where some agents are assumed to be more capable or more experienced [7], [20], we require three agents to define their SUSD direction (forward direction) as in (2), while the rest determine their SUSD direction using the following consensus law:

$$\dot{\mathbf{n}}_i = k_f \sum_{j \in \mathcal{N}_i} [\mathbf{n}_j - \langle \mathbf{n}_i, \mathbf{n}_j\rangle \mathbf{n}_i],\ i = 4, \ldots, M \tag{7}$$

where $k_f \in \mathbf{R}$ is a positive constant representing the consensus gain. This is a time-varying nonlinear consensus-on-a sphere, which preserves a unit length of its vectors $\mathbf{n}_i$.

Define $\mathcal{G}^l = (\mathcal{V}^l, \mathcal{E}^{ll})$ to be the complete undirected graph describing the interactions among agents $i = 1, 2, 3$. Similarly, define $\mathcal{G}^f = (\mathcal{V}^f, \mathcal{E}^{ff})$, to be the undirected graph describing interactions

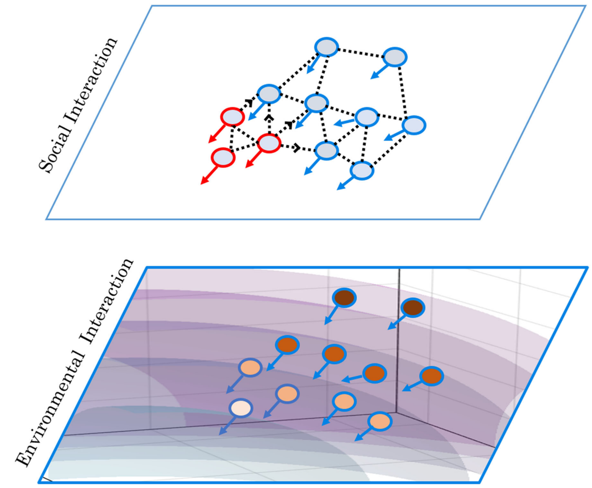

Fig. 3. Environmental and social interactions.

among agents $i = 4, \ldots, M$. Additionally, define $\mathcal{E}^{fl}$ to be the set of all edges that have $i \in \mathcal{V}^f$ as a tail and $j \in \mathcal{V}^l$ as a head. According to Assumption 2.1, we need $\mathcal{G}^f$ to be connected, and additionally we need to ensure that for $t \geq 0$ there exists at least one $i \in \mathcal{V}^f$ such that $(i, j) \in \mathcal{E}^{fl}$. These connectivity requirements are satisfied through formation control laws (9) and (10). Note that we require the undirected $\mathcal{G}^f$ to be only connected, but with any graph structure. For (7), we further assume the following.

Assumption 3.2: Each agent $i \in \mathcal{V}^f$ is able to obtain the SUSD directions $\mathbf{n}_j, \forall j \in \mathcal{N}_i$.

Relying on advanced vision techniques, agents can measure the headings of their neighbors [21]. With some treatment, agents can then satisfy Assumption 3.2 by obtaining $\mathbf{n}_j$ of their neighbors from their heading measurements.

Then, the velocity of each agent is decoupled as

$$\mathbf{v}_i = v_{i,\mathbf{n}} \mathbf{n}_i + \mathbf{v}_{i,\mathcal{H}}, \quad i = 1, \ldots, M \tag{8}$$

where the SUSD direction $\mathbf{n}_i$ and the SUSD speeds are as defined by (7) and (6), respectively. The formation term $\mathbf{v}_{i,\mathcal{H}}$ is defined to be

$$\mathbf{v}_{i,\mathcal{H}} = v_{i,\mathbf{q}} \mathbf{q} + v_{i,\mathbf{p}} \mathbf{p}, \qquad i = 1, \ldots, 3 \tag{9}$$

$$\mathbf{v}_{i,\mathcal{H}} = \sum_{j \in \mathcal{N}_i} w_{ij} \left(\mathbf{I} - \mathbf{n}_i \mathbf{n}_i^T\right) (\mathbf{r}_j - \mathbf{r}_i), \quad i = 4, \ldots, M \tag{10}$$

where $\mathbf{q}$ and $\mathbf{p}$ are as defined in (2), and $w_{ij} = ||\mathbf{r}_j - \mathbf{r}_i||^2 - \langle \mathbf{r}_j - \mathbf{r}_i, \mathbf{n}_i\rangle^2 - d_{ij}^2$ for an arbitrary desired interagent distances d_{ij}. Note that the formation term (10) acts only on the plane perpendicular to $\mathbf{n}_i$ so that the SUSD speed along the SUSD direction $\mathbf{n}_i$ is only affected by the field value. Additionally, (10) is required to only ensure connectivity maintenance without any specific rigid formations as in (9).

The resulted strategy may be viewed as a two-layer system composed of environmental and social interactions as shown in Fig. 3. In the environmental layer, all the agents modulate their SUSD speeds according to the environmental field value as indicated by (6). In the social layer, agents interact with each other to determine their SUSD directions as indicated by (2) and (7), and to maintain formations as indicated by (9) and (10).

Remark 1: Although three agents are enough to seek the source, the proposed strategy presents a new method that enables a swarm of more than three agents to navigate to the source with local information. This is particularly important in modeling large biological swarms. We discover in Section IV that using the three-agent local interactions (2), (6), (4), and (5) leads to a nonlinear consensus-on-a sphere (25) between the SUSD direction and the negative direction of the field gradient. This emergent consensus along with the explicit consensus law (7) of the rest of the agents in the social layer lead to a synchronization behavior where the synchronized value is indirectly controlled by the field value in the environmental layer of Fig. 3. This behavior is not achieved by the classical distance-based leader-follower approaches, where the headings of the followers are always pointing toward the leaders, not the field gradient. Hence, the proposed two-layer model might be more reasonable to describe the at least some synchronization behaviors of biological swarms.

IV. Convergence Analysis

In [15], given Assumptions 2.1, 2.2, and 2.3, the formation control laws (9) is proved to be exponentially stable. When the consensus law (7) converges, then all agents will have the same normal plane $\mathcal{H}$, and hence the formation control law (10) becomes a known formation problem that can be proved using methods in [15], or others in the related literature. However, during the transient time, each agent will have its own plane $\mathcal{H}_i$, and hence proving the convergence of (10) requires more treatment which is beyond the scope of this paper. Remark that all the following subsequent proofs do not require the convergence of (10).

A. Convergence of the SUSD Direction for $M = 3$

The goal of this section is to show that the SUSD direction $\mathbf{n}$ in (2) converges to the negative direction of the gradient, $\mathbf{N} = \frac{\nabla f(\mathbf{r}_c)}{\|\nabla f(\mathbf{r}_c)\|}$. In the inertial frame, once the formation converges, $v_{i,\mathbf{q}} = v_{i,\mathbf{p}} = 0$. Then, the velocity of the ith agent in the rigid body becomes $\mathbf{v}_i = v_{i,\mathbf{n}}\mathbf{n}$, and the velocity of the formation center is $\mathbf{v}_c = \frac{1}{3}\sum_{i=1}^{3} v_{i,\mathbf{n}}\mathbf{n} = v_{c,\mathbf{n}}\mathbf{n}$, which indicates that the moving direction of the rigid body coincides with the $\mathbf{n}$ axis of the body frame. Define shape variables $\langle \mathbf{N},\mathbf{n}\rangle$, $\langle \mathbf{N},\mathbf{q}\rangle$, and $\langle \mathbf{N},\mathbf{p}\rangle$ [22], [23]. The shape variables satisfy

$$\langle \mathbf{N},\mathbf{p}\rangle^2 = 1 - \langle \mathbf{N},\mathbf{q}\rangle^2 - \langle \mathbf{N},\mathbf{n}\rangle^2. \tag{11}$$

Since we have $\frac{d\langle \mathbf{N},\mathbf{n}\rangle}{dt} = \langle \mathbf{N},\dot{\mathbf{n}}\rangle + \langle \dot{\mathbf{N}},\mathbf{n}\rangle$, the first step is to derive $\dot{\mathbf{n}}$. In the frame $(\mathbf{q},\mathbf{p},\mathbf{n})$, we can write any vector $\mathbf{v}$ as

$$\mathbf{v} = \langle \mathbf{q},\mathbf{v}\rangle\mathbf{q} + \langle \mathbf{p},\mathbf{v}\rangle\mathbf{p} + \langle \mathbf{n},\mathbf{v}\rangle\mathbf{n}. \tag{12}$$

To find $\dot{\mathbf{n}}$, we apply (12) with $\mathbf{v} = \dot{\mathbf{n}}$, and calculate the coefficients $\langle \mathbf{q},\dot{\mathbf{n}}\rangle$, $\langle \mathbf{p},\dot{\mathbf{n}}\rangle$, and $\langle \mathbf{n},\dot{\mathbf{n}}\rangle$. In the inertial frame, define the rotation matrix of the rigid body as $\mathbf{g} = [\mathbf{q},\mathbf{p},\mathbf{n}] \in SO(3)$. Define a skew symmetric matrix $S(\omega)$, in which $\omega \in \mathbb{R}^3$ is the angular velocity of the rigid body. Then, we have $\dot{\mathbf{g}} = S(\omega)\mathbf{g}$, and from which we derive $\dot{\mathbf{n}} = \omega \times \mathbf{n}$. Since the speed of $\mathbf{r}_i$ along directions $\mathbf{q}$ and $\mathbf{p}$ are zero for the rigid body, we conclude that ω is confined in the plane $\mathcal{H}$. For the velocity of the agent in the inertial frame, $\mathbf{v}_i - \mathbf{v}_c$ satisfies

$$\mathbf{v}_i - \mathbf{v}_c = \omega \times (\mathbf{r}_i - \mathbf{r}_c). \tag{13}$$

Then, we have

$$(v_{i,\mathbf{n}} - v_{c,\mathbf{n}})\mathbf{n} = \omega \times (\mathbf{r}_i - \mathbf{r}_c). \tag{14}$$

Applying inner product with $\mathbf{n}$ on both sides of (14)

$$v_{i,\mathbf{n}} - v_{c,\mathbf{n}} = \langle \omega \times (\mathbf{r}_i - \mathbf{r}_c), \mathbf{n}\rangle. \tag{15}$$

Define $\omega_i = \langle \omega \times (\mathbf{r}_i - \mathbf{r}_c), \mathbf{n}\rangle$. We then have

$$\omega_i = -\langle \mathbf{r}_i - \mathbf{r}_c, \omega \times \mathbf{n}\rangle = -\langle \mathbf{r}_i - \mathbf{r}_c, \dot{\mathbf{n}}\rangle. \tag{16}$$

Using (2) and (16), we derive

$$\omega_3 = -\| \mathbf{r}_3 - \mathbf{r}_c \| \langle \mathbf{p},\dot{\mathbf{n}}\rangle \tag{17}$$

$$\omega_2 - \omega_1 = -\| \mathbf{r}_2 - \mathbf{r}_1 \| \langle \mathbf{q},\dot{\mathbf{n}}\rangle \tag{18}$$

which produces: $\langle \mathbf{p},\dot{\mathbf{n}}\rangle = -\frac{\omega_3}{\|\mathbf{r}_3 - \mathbf{r}_c\|}$ and $\langle \mathbf{q},\dot{\mathbf{n}}\rangle = -\frac{\omega_2 - \omega_1}{\|\mathbf{r}_2 - \mathbf{r}_1\|}$. Since $\mathbf{n}$ is a unit vector, we have $\langle \mathbf{n},\dot{\mathbf{n}}\rangle = 0$. Therefore,

$$\dot{\mathbf{n}} = -\frac{\omega_2 - \omega_1}{\| \mathbf{r}_2 - \mathbf{r}_1 \|}\mathbf{q} - \frac{\omega_3}{\| \mathbf{r}_3 - \mathbf{r}_c \|}\mathbf{p}. \tag{19}$$

From (15), we have $\omega_i = v_{i,\mathbf{n}} - v_{c,\mathbf{n}}$. Since the field $f(\mathbf{r})$ is at least class C^1, then, from the Taylor expansion, we have

$$v_{i,\mathbf{n}} = k_1(f(\mathbf{r}_c) + \langle \nabla f(\mathbf{r}_c), \mathbf{r}_i - \mathbf{r}_c\rangle) + k_2 + H.O.T \tag{20}$$

where $H.O.T$ represents higher order terms. In addition,

$$v_{c,\mathbf{n}} = \frac{1}{3}\sum_{i=1}^{3} v_{i,\mathbf{n}} = \frac{k_1}{3}\sum_{i=1}^{3} f(\mathbf{r}_i) + k_2$$

$$= k_1 f(\mathbf{r}_c) + \left\langle \frac{k_1}{3}\nabla f(\mathbf{r}_c), \sum_{i=1}^{3}\mathbf{r}_i - 3\mathbf{r}_c \right\rangle + k_2 = k_1 f(\mathbf{r}_c) + k_2.$$

Therefore, if the agents are close enough to each other such that the higher order terms are insignificant, we derive

$$\omega_i = v_{i,\mathbf{n}} - v_{c,\mathbf{n}} = k_1 \| \nabla f(\mathbf{r}_c) \| \langle \mathbf{N}, \mathbf{r}_i - \mathbf{r}_c\rangle \tag{21}$$

which leads to

$$\omega_2 - \omega_1 = k_1 \| \nabla f(\mathbf{r}_c) \| \| \mathbf{r}_2 - \mathbf{r}_1 \| (\mathbf{N}\cdot\mathbf{q}) \tag{22}$$

$$\omega_3 = k_1 \| \nabla f(\mathbf{r}_c) \| \| \mathbf{r}_3 - \mathbf{r}_c \| (\mathbf{N}\cdot\mathbf{p}). \tag{23}$$

Substituting (22) and (23) into (19), we obtain

$$\dot{\mathbf{n}} = -k_1 \| \nabla f(\mathbf{r}_c) \| (\langle \mathbf{N},\mathbf{q}\rangle\mathbf{q} + \langle \mathbf{N},\mathbf{p}\rangle\mathbf{p}). \tag{24}$$

Lemma 4.1: The dynamics (24) represents a consensus-on-a sphere control law between the SUSD direction and negative direction of the gradient. In particular, we can rewrite (24) as

$$\dot{\mathbf{n}} = -k_l||\nabla f(\mathbf{r}_c)||(\mathbf{I} - \mathbf{n}\mathbf{n}^T)\mathbf{N}. \tag{25}$$

Proof: From (11) we can write

$$\mathbf{N}^T(\mathbf{q}\mathbf{q}^T + \mathbf{p}\mathbf{p}^T)\mathbf{N} = \mathbf{N}^T\mathbf{N} - \mathbf{N}^T\mathbf{n}\mathbf{n}^T\mathbf{N} = \mathbf{N}^T(\mathbf{I} - \mathbf{n}\mathbf{n}^T)\mathbf{N}.$$

From which, we obtain

$$(\mathbf{q}\mathbf{q}^T + \mathbf{p}\mathbf{p}^T)\mathbf{N} = (\mathbf{I} - \mathbf{n}\mathbf{n}^T)\mathbf{N}. \tag{26}$$

Finally, plug (26) into (24) to get (25). ■

Note that the consensus (25) is a time-varying since $\nabla f(\mathbf{r}_c)$ is changing as the center $\mathbf{r}_c$ moves around. We want to show that the consensus law asymptotically converges to the agreement $\mathbf{n} = -\mathbf{N}$ as $t \to \infty$. Let $\theta = \langle \mathbf{N},\mathbf{n}\rangle + 1$ and $\delta = \langle \mathbf{n},\dot{\mathbf{N}}\rangle$. Then using (25), we derive

$$\begin{aligned}\dot{\theta} &= \frac{d\langle \mathbf{N},\mathbf{n}\rangle}{dt} = \langle \mathbf{N},\dot{\mathbf{n}}\rangle + \langle \mathbf{n},\dot{\mathbf{N}}\rangle \\ &= -k_l \| \nabla f(\mathbf{r}_c) \| (1 - \langle \mathbf{N},\mathbf{n}\rangle^2) + \langle \mathbf{n},\dot{\mathbf{N}}\rangle \\ &= -k_l||\nabla f(\mathbf{r}_c)||\theta(2-\theta) + \delta \triangleq h(t,\theta,\delta).\end{aligned} \tag{27}$$

The unforced system $h(t,\theta,0)$, has two equilibriums: $\theta = 0$ and $\theta = 2$, where $\theta = 0$ corresponds to the desired equilibrium $\langle \mathbf{N},\mathbf{n}\rangle = -1$, and $\theta = 2$ corresponds to the undesired equilibrium $\langle \mathbf{N},\mathbf{n}\rangle = 1$. Since we do not know $\dot{\mathbf{N}}$, we view δ as an input disturbance and analyze the system convergence using an input-to-state stability framework.

Note that since $\mathbf{N}$ is perpendicular to $\dot{\mathbf{N}}$, then $\delta = 0$ when $\theta = 0, 2$. Theorem 4.1 summarizes the stability results of (27).

Theorem 4.1: Consider (27). Assume that $||\nabla f(\mathbf{r}_c)||$ is bounded below along the trajectory of the formation center, i.e., $||\nabla f(\mathbf{r}_c)|| \geq \epsilon_c$ for a small constant $\epsilon_c > 0$ everywhere except at the source location where $f(\mathbf{r}_c) = 0$. If initially $\theta(0) \neq 2$, then the equilibrium $\theta = 0$ of the unforced system, $h(t, \theta, 0)$ is asymptotically stable. Moreover, whenever $\theta(0) \neq 2$ and assuming $|\delta| < 2k\epsilon\epsilon_c$ for a small $\epsilon < 1$, system (27) is input-to-state stable.

Proof: Let $\mathbf{D} = \{\theta \in \mathbb{R} | 0 \leq \theta < 2\}$. Let $V(\theta) : \mathbf{D} \to \mathbf{R}$ be a Lyapunov candidate function defined as follows:

$$V(\theta) = \frac{\theta}{2-\theta}. \tag{28}$$

Note that $V \geq 0$ and $V = 0$ if and only if $\theta = 0$. Furthermore, $V \to \infty$ as $\theta \to 2$. Then,

$$\dot{V} = \frac{\partial V}{\partial \theta}\dot{\theta} = \frac{2}{(2-\theta)^2}\dot{\theta}. \tag{29}$$

For the unforced system $h(t, \theta, 0)$, we have

$$\dot{V} = \frac{-2k||\nabla f(\mathbf{r}_c)||\theta(2-\theta)}{(2-\theta)^2} = -2k||\nabla f(\mathbf{r}_c)||V \leq 0. \tag{30}$$

Since $V \geq 0$ and $V = 0$ if and only if $\theta = 0$, then the equilibrium $\theta = 0$ of the unforced system is asymptotically stable. Moreover, since $\dot{V}$ is negative definite and $V \to \infty$ whenever $\theta \to 2$, then $\mathbf{D} = \{\theta \in \mathbb{R} | 0 \leq \theta < 2\}$ is a positively invariant set which implies that trajectories start inside $\mathbf{D}$ will stay there forever. For the forced system $h(t, \theta, \delta)$

$$\begin{aligned} \dot{V} &= \frac{-2k||\nabla f(\mathbf{r}_c)||\theta(2-\theta)}{(2-\theta)^2} + \frac{2\delta}{(2-\theta)^2} \\ &\leq -2k||\nabla f(\mathbf{r}_c)||(1-\epsilon)V, \ \forall\, |\theta| > \frac{|\delta|}{2k\epsilon\epsilon_c}. \end{aligned} \tag{31}$$

Let $\alpha_1(|\theta|) = \alpha_2(|\theta|) = \frac{|\theta|}{2-|\theta|}$, which are class $\mathcal{K}_\infty$ functions on $\mathbf{D}$ and satisfy: $\alpha_1(|\theta|) \leq V(\theta) \leq \alpha_2(|\theta|)$. Additionally, $\alpha_3(\theta) = 2k||\nabla f(\mathbf{r}_c)||(1-\epsilon)\frac{\theta}{2-\theta}$ and $\rho(|\delta|) = \frac{|\delta|}{2k\epsilon\epsilon_c}$ are class $\mathcal{K}$ functions. Therefore, according to [24, Th. 4.19], the system (27) is input-to-state stable with gain $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho = \frac{|\delta|}{2k\epsilon\epsilon_c}$. Additionally, since δ is vanishing at the equilibrium, then the system is asymptotically stable. ■

B. Convergence of the SUSD Directions for $M > 3$

The goal of this section is to show that the consensus law (7) converges to the solution $\mathbf{n}_i = \mathbf{n}_j = \mathbf{n}\ \forall\, i, j \in \mathcal{V}^f$.

Let $\mathbf{u}_i = \sum_{k\in\mathcal{N}_i} \mathbf{n}_k$. Then rewrite the consensus law (7) as

$$\dot{\mathbf{n}}_i = k_f \sum_{k\in\mathcal{N}_i} \left(\mathbf{n}_k - \mathbf{n}_k^T\mathbf{n}_i\mathbf{n}_i\right) = k_f \left(\mathbf{I} - \mathbf{n}_i\mathbf{n}_i^T\right)\mathbf{u}_i. \tag{32}$$

The consensus (32) has the following three equilibrium sets:

$$(\mathbf{n}_i, \mathbf{u}_i) \in \left\{\left(-\frac{\mathbf{u}_i}{||\mathbf{u}_i||}, \mathbf{u}_i\right), \left(\frac{\mathbf{u}_i}{||\mathbf{u}_i||}, \mathbf{u}_i\right), (\mathbf{n}_i, \mathbf{0})\right\}. \tag{33}$$

As proved in [12], the undesired equilibrium sets $(\mathbf{n}_i, \mathbf{u}_i) = (-\frac{\mathbf{u}_i}{||\mathbf{u}_i||}, \mathbf{u}_i)$ and $(\mathbf{n}_i, \mathbf{u}_i) = (\mathbf{n}_i, \mathbf{0})$ are unstable. The unique asymptotically stable equilibrium is the set $(\mathbf{n}_i, \mathbf{u}_i) = (\frac{\mathbf{u}_i}{||\mathbf{u}_i||}, \mathbf{u}_i)$, which is equivalent to $\{\mathbf{n}_i = \mathbf{n}_j = \mathbf{n}_l$ for all $i, j \in \mathcal{V}^f\}$ where by $\mathbf{n}_l$ we denote the SUSD direction of the three agents defined in (2).

Since three agents have their own SUSD dynamics given by (25), then the analysis of (32) is different from the one analyzed in [12]. In particular, in [12] all the edges are undirected whereas in this paper we need to consider the directed edges $(i, l) \in \mathcal{E}^{fl}$. Additionally, (25) is time varying, which produces a time-varying nonlinear consensus-on-a sphere problem compared to the time invariant form considered in [12]. To overcome these difficulties, we first construct collective states that represent the desired equilibrium and then derive their dynamics. We then formulate a cascaded input-to-state stability problem to analyze the stability of (32). Consider the following collective states:

$$\theta_f = \sum_{(i,j)\in\mathcal{E}^{ff}} (1 - \langle\mathbf{n}_i, \mathbf{n}_j\rangle) \tag{34}$$

$$\theta_l = \sum_{(i,l)\in\mathcal{E}^{fl}} (1 - \langle\mathbf{n}_i, \mathbf{n}_l\rangle) \tag{35}$$

$$\theta_N = 1 + \langle\mathbf{n}_l, \mathbf{N}\rangle \tag{36}$$

where $\theta_f = 0$ if and only if $\mathbf{n}_i = \mathbf{n}_j\ \forall\ (i, j) \in \mathcal{E}^{ff}$, $\theta_l = 0$ if and only if $\mathbf{n}_i = \mathbf{n}_l\ \forall\ (i, l) \in \mathcal{E}^{fl}$, and $\theta_N = 0$ if and only if $\mathbf{n}_l \to \mathbf{N}$ as $t \to \infty$. Hence, the convergence of these collective states, $(\theta_f, \theta_l, \theta_N)$ to the origin, $(0, 0, 0)$ represents the desired objective of $\mathbf{n}_i \to \mathbf{n}_l \to -\mathbf{N}$ for all agents. The dynamics of the collective state θ_N is given by (27). In the following, we derive the dynamics of the collective states θ_f and θ_l. Taking time derivative of (34), and using (32)

$$\begin{aligned} \dot{\theta}_f &= -\sum_{(i,j)\in\mathcal{E}^{ff}} [\langle\mathbf{n}_i, \dot{\mathbf{n}}_j\rangle + \langle\mathbf{n}_j, \dot{\mathbf{n}}_i\rangle] \\ &= -k_f \sum_{(i,j)\in\mathcal{E}^{ff}} [\langle\mathbf{n}_i, (\mathbf{I} - \mathbf{n}_j\mathbf{n}_j^T)\mathbf{u}_j\rangle + \langle\mathbf{n}_j, \left(\mathbf{I} - \mathbf{n}_i\mathbf{n}_i^T\right)\mathbf{u}_i\rangle]. \end{aligned}$$

To continue, we first prove the following *Lemma*.

Lemma 4.2: For a connected and undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $\sum_{(i,j)\in\mathcal{E}}[\langle\mathbf{n}_i, (\mathbf{I} - \mathbf{n}_j\mathbf{n}_j^T)\mathbf{u}_j\rangle + \langle\mathbf{n}_j, (\mathbf{I} - \mathbf{n}_i\mathbf{n}_i^T)\mathbf{u}_i\rangle] = \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} \langle\mathbf{n}_j, (\mathbf{I} - \mathbf{n}_i\mathbf{n}_i^T)\mathbf{u}_i\rangle$.

Proof: Since (i, j) is an undirected edge, then: $\sum_{(i,j)\in\mathcal{E}} [\langle\mathbf{n}_i, (\mathbf{I} - \mathbf{n}_j\mathbf{n}_j^T)\mathbf{u}_j\rangle + \langle\mathbf{n}_j, (\mathbf{I} - \mathbf{n}_i\mathbf{n}_i^T)\mathbf{u}_i\rangle] = 2\sum_{(i,j)\in\mathcal{E}}\langle\mathbf{n}_j, (\mathbf{I} - \mathbf{n}_i\mathbf{n}_i^T)\mathbf{u}_i\rangle = 2(\frac{1}{2})\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\langle\mathbf{n}_j, (\mathbf{I} - \mathbf{n}_i\mathbf{n}_i^T)\mathbf{u}_i\rangle = \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} \langle\mathbf{n}_j, (\mathbf{I} - \mathbf{n}_i\mathbf{n}_i^T)\mathbf{u}_i\rangle$. ■

Note that, the Lemma requires (i, j) to be an undirected edge. The fact that some directed edges appear under $\mathbf{u}_i = \sum_{k\in\mathcal{N}_i} \mathbf{n}_k$ or $\mathbf{u}_j = \sum_{k\in\mathcal{N}_j} \mathbf{n}_k$ for some i, j does not violate the lemma, since they are still captured under $\mathbf{u}_i$. Since $\mathcal{E}^{ff}$ is the set of all undirected edges, we use Lemma 4.2 to get

$$\begin{aligned} \dot{\theta}_f &= -k_f \sum_{i\in\mathcal{V}^f}\sum_{j\in\mathcal{N}_i} \left\langle\mathbf{n}_j, \left(\mathbf{I} - \mathbf{n}_i\mathbf{n}_i^T\right)\mathbf{u}_i\right\rangle \\ &= -k_f \sum_{i\in\mathcal{V}^f} \left\langle \sum_{j\in\mathcal{N}_i}\mathbf{n}_j, \left(\mathbf{I} - \mathbf{n}_i\mathbf{n}_i^T\right)\mathbf{u}_i \right\rangle. \end{aligned} \tag{37}$$

Note that $\sum_{j\in\mathcal{N}_i} \mathbf{n}_j \neq \mathbf{u}_i = \sum_{k\in\mathcal{N}_i} \mathbf{n}_k$ since $k \in \{1, 2, 3, \ldots, M\}$, while $j \in \{4, \ldots, M\}$. For this, we write the following: $\mathbf{u}_i = a_i\mathbf{n}_l + \sum_{j\in\mathcal{N}_i} \mathbf{n}_j = a_i\mathbf{n}_l + \hat{\mathbf{u}}_i$, where $a_i = 1$, if one of the agents in the set $\{1, 2, 3\}$ is a neighbor to the ith agent, and $a_i = 0$, otherwise. Then, we continue to obtain the following:

$$\begin{aligned}\dot{\theta}_f &= -k_f \sum_{i\in\mathcal{V}^f} \left\langle \hat{\mathbf{u}}_i, \left(\mathbf{I}-\mathbf{n}_i\mathbf{n}_i^T\right)(a_i\mathbf{n}_l+\hat{\mathbf{u}}_i)\right\rangle \\ &= k_f \left[\sum_{i\in\mathcal{V}^f}\left(\langle\hat{\mathbf{u}}_i,\mathbf{n}_i\rangle^2 - ||\hat{\mathbf{u}}_i||^2\right) - \sum_{i\in\mathcal{V}^f}\left\langle\hat{\mathbf{u}}_i, a_i\left(\mathbf{I}-\mathbf{n}_i\mathbf{n}_i^T\right)\mathbf{n}_l\right\rangle\right] \\ &= k_f \left[\sum_{i\in\mathcal{V}^f}\left(\langle\hat{\mathbf{u}}_i,\mathbf{n}_i\rangle^2 - ||\hat{\mathbf{u}}_i||^2\right) - \sum_{(i,l)\in\mathcal{E}^{fl}}\left\langle\hat{\mathbf{u}}_i, \left(\mathbf{I}-\mathbf{n}_i\mathbf{n}_i^T\right)\mathbf{n}_l\right\rangle\right].\end{aligned} \tag{38}$$

Similarly, taking time derivative of (35), and using (32) and (25)

$$\begin{aligned}\dot{\theta}_l &= -\sum_{(i,l)\in\mathcal{E}^{fl}}\left[\langle\mathbf{n}_l,\dot{\mathbf{n}}_i\rangle + \langle\mathbf{n}_i,\dot{\mathbf{n}}_l\rangle\right] \\ &= -k_f\sum_{(i,l)\in\mathcal{E}^{fl}}\left\langle\mathbf{n}_l,\left(\mathbf{I}-\mathbf{n}_i\mathbf{n}_i^T\right)\mathbf{u}_i\right\rangle \\ &\quad + k_l||\nabla f(\mathbf{r}_c)||\sum_{(i,l)\in\mathcal{E}^{fl}}\left\langle\mathbf{n}_i,\left(\mathbf{I}-\mathbf{n}_l\mathbf{n}_l^T\right)\mathbf{N}\right\rangle \\ &= -k_f\sum_{(i,l)\in\mathcal{E}^{fl}}\left\langle\mathbf{n}_l,\left(\mathbf{I}-\mathbf{n}_i\mathbf{n}_i^T\right)(\hat{\mathbf{u}}_i+\mathbf{n}_l)\right\rangle \\ &\quad + k_l||\nabla f(\mathbf{r}_c)||\sum_{(i,l)\in\mathcal{E}^{fl}}\left\langle\mathbf{n}_i,\left(\mathbf{I}-\mathbf{n}_l\mathbf{n}_l^T\right)\mathbf{N}\right\rangle \\ &= k_f\sum_{(i,l)\in\mathcal{E}^{fl}}\left(\langle\mathbf{n}_i,\mathbf{n}_l\rangle^2-1\right) - k_f\sum_{(i,L)\in\mathcal{E}^{fl}}\left\langle\hat{\mathbf{u}}_i,\left(\mathbf{I}-\mathbf{n}_i\mathbf{n}_i^T\right)\mathbf{n}_l\right\rangle \\ &\quad + k_l||\nabla f(\mathbf{r}_c)||\sum_{(i,l)\in\mathcal{E}^{fl}}\left\langle\mathbf{n}_i,\left(\mathbf{I}-\mathbf{n}_l\mathbf{n}_l^T\right)\mathbf{N}\right\rangle.\end{aligned} \tag{39}$$

Let $x_1 = \theta_f + \theta_l$ and $x_2 = \theta_N$. We then view the system as a cascade of two systems

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_2, \delta).\end{aligned} \tag{40}$$

Note that the f_1 system represents the consensus among all SUSD directions of the agents in the set $\{4,\ldots,M\}$ with x_2 represents the input disturbance due to the dynamics of the agents in the set $\{1,2,3\}$. On the other hand, the f_2 system represents the consensus of the SUSD direction of the agents in the set $\{1,2,3\}$ with the negative direction of the gradient in which δ represents the input disturbance due to the dynamics of the gradient. Using (38) and (39), we derive the dynamics of x_1

$$\begin{aligned}\dot{x}_1 &= \dot{\theta}_f + \dot{\theta}_l \\ &= k_f\sum_{i\in\mathcal{V}^f}\left(\langle\hat{\mathbf{u}}_i,\mathbf{n}_i\rangle^2 - ||\hat{\mathbf{u}}_i||^2\right) + k_f\sum_{(i,l)\in\mathcal{E}^{fl}}\left(\langle\mathbf{n}_i,\mathbf{n}_l\rangle^2 - 1\right) \\ &\quad - 2k_f\sum_{(i,l)\in\mathcal{E}^{fl}}\left\langle\hat{\mathbf{u}}_i,\left(\mathbf{I}-\mathbf{n}_i\mathbf{n}_i^T\right)\mathbf{n}_l\right\rangle \\ &\quad + k_l||\nabla f(\mathbf{r}_c)||\sum_{(i,l)\in\mathcal{E}^{fl}}\left\langle\mathbf{n}_i,\left(\mathbf{I}-\mathbf{n}_l\mathbf{n}_l^T\right)\mathbf{N}\right\rangle.\end{aligned} \tag{41}$$

To further simplify, we utilize the following lemma.

Lemma 4.3:

$$\begin{aligned}&-2\left\langle\hat{\mathbf{u}}_i,\left(\mathbf{I}-\mathbf{n}_i\mathbf{n}_i^T\right)\mathbf{n}_l\right\rangle = \left[\langle\mathbf{n}_i,\mathbf{u}_i\rangle^2 - ||\mathbf{u}_i||^2\right] \\ &+ \left[||\hat{\mathbf{u}}_i||^2 - \langle\hat{\mathbf{u}}_i,\mathbf{n}_i\rangle^2\right] + \left[1 - \langle\mathbf{n}_i,\mathbf{n}_l\rangle^2\right].\end{aligned} \tag{42}$$

Proof:

$$\begin{aligned}&-2\left\langle\hat{\mathbf{u}}_i,\left(\mathbf{I}-\mathbf{n}_i\mathbf{n}_i^T\right)\mathbf{n}_l\right\rangle = 2\langle\hat{\mathbf{u}}_i,\mathbf{n}_i\rangle\langle\mathbf{n}_i,\mathbf{n}_l\rangle - 2\langle\hat{\mathbf{u}}_i,\mathbf{n}_l\rangle \\ &= \left(\langle\hat{\mathbf{u}}_i,\mathbf{n}_i\rangle + \langle\mathbf{n}_i,\mathbf{n}_l\rangle\right)^2 - \langle\hat{\mathbf{u}}_i,\mathbf{n}_i\rangle^2 - \langle\mathbf{n}_i,\mathbf{n}_l\rangle^2 \\ &- \langle\hat{\mathbf{u}}_i+\mathbf{n}_l,\hat{\mathbf{u}}_i+\mathbf{n}_l\rangle + ||\hat{\mathbf{u}}_i||^2 + 1 = \langle\mathbf{n}_i,\hat{\mathbf{u}}_i+\mathbf{n}_l\rangle^2 \\ &- ||\hat{\mathbf{u}}_i+\mathbf{n}_l||^2 + ||\hat{\mathbf{u}}_i||^2 - \langle\hat{\mathbf{u}}_i,\mathbf{n}_i\rangle^2 + 1 - \langle\mathbf{n}_i,\mathbf{n}_l\rangle^2.\end{aligned} \tag{43}$$

Since $\mathbf{u}_i = \hat{\mathbf{u}}_i + \mathbf{n}_l \ \forall\ (i,l)\in\mathcal{E}^{fl}$, then the lemma follows directly from the last step. ■

Therefore, applying Lemma 4.3 in (41)

$$\begin{aligned}\dot{x}_1 &= k_f\left[\sum_{i\in\mathcal{V}^f}\left(\langle\hat{\mathbf{u}}_i,\mathbf{n}_i\rangle^2 - ||\hat{\mathbf{u}}_i||^2\right) + \sum_{(i,l)\in\mathcal{E}^{fl}}\left(||\hat{\mathbf{u}}_i||^2 - \langle\hat{\mathbf{u}}_i,\mathbf{n}_i\rangle^2\right)\right. \\ &\quad \left. + \sum_{(i,l)\in\mathcal{E}^{fl}}\left(\langle\mathbf{n}_i,\mathbf{u}_i\rangle^2 - ||\mathbf{u}_i||^2\right)\right] \\ &\quad + k_l||\nabla f(\mathbf{r}_c)||\sum_{(i,l)\in\mathcal{E}^{fl}}\left\langle\mathbf{n}_i,\left(\mathbf{I}-\mathbf{n}_l\mathbf{n}_l^T\right)\mathbf{N}\right\rangle \\ &= k_f\left[\sum_{i\in\tilde{\mathcal{V}}^f}\left(\langle\hat{\mathbf{u}}_i,\mathbf{n}_i\rangle^2 - ||\hat{\mathbf{u}}_i||^2\right) + \sum_{(i,l)\in\mathcal{E}^{fl}}\left(\langle\mathbf{n}_i,\mathbf{u}_i\rangle^2 - ||\mathbf{u}_i||^2\right)\right] \\ &\quad + k_l||\nabla f(\mathbf{r}_c)||\sum_{(i,l)\in\mathcal{E}^{fl}}\left\langle\mathbf{n}_i,\left(\mathbf{I}-\mathbf{n}_l\mathbf{n}_l^T\right)\mathbf{N}\right\rangle\end{aligned} \tag{44}$$

where $\tilde{\mathcal{V}}^f = \mathcal{V}^f - \{i|(i,l)\in\mathcal{E}^{fl}\}$. The following theorem summarizes the convergence results of the cascaded system (40).

Theorem 4.2: Consider (40) with $f_1(x_1,x_2)$ and $f_2(x_2,\delta)$ are as defined in (44) and (27), respectively. Assume that $||\nabla f(\mathbf{r}_c)||$ is bounded below along the trajectory of agents $\{1,2,3\}$, i.e., $||\nabla f(\mathbf{r}_c)|| \geq \epsilon_c$ for a small constant $\epsilon_c > 0$ everywhere except at the source location where $f(\mathbf{r}_c) = 0$. Then, the f_1 system is input-to-state stable with respect to the input disturbance x_2. Furthermore, the overall system (40) is input-to-state stable with respect to the field input disturbance δ.

Proof: We already proved in Theorem 4.1 that the f_2 system is input-to-state stable w.r.t. δ. What remains is to prove that the f_1 system is input-to-state stable w.r.t. x_2. Then, we use [13, Th. 3] to conclude that the overall interconnected system is input-to-state stable. For the unforced system, $f_1(x_1,0)$, $x_2 = \theta_N = 0$. This implies that $\mathbf{n}_l = -\mathbf{N}$, and hence $\left\langle\mathbf{n}_i,\left(\mathbf{I}-\mathbf{n}_l\mathbf{n}_l^T\right)\mathbf{N}\right\rangle = \langle\mathbf{n}_i,(\mathbf{I}-\mathbf{N}\mathbf{N}^T)\mathbf{N}\rangle = 0$. Therefore,

$$\begin{aligned}\dot{x}_1 &= f_1(x_1,0) \\ &= k_f\left[\sum_{i\in\tilde{\mathcal{V}}^f}\left(\langle\hat{\mathbf{u}}_i,\mathbf{n}_i\rangle^2 - ||\hat{\mathbf{u}}_i||^2\right) + \sum_{(i,l)\in\mathcal{E}^{fl}}\left(\langle\mathbf{n}_i,\mathbf{u}_i\rangle^2 - ||\mathbf{u}_i||^2\right)\right].\end{aligned}$$

Let $V(x_1) = \frac{1}{2}x_1^2$, be a Lyapunov candidate function. Note that V is positive definite and $V = 0$ if and only if $\theta_f = \theta_l = 0$, which implies that $\mathbf{n}_i = \mathbf{n}_l \; \forall \, i \in \mathcal{V}^f$. Therefore,

$$\begin{aligned}\dot{V} = &-k_f x_1 \sum_{i \in \tilde{\mathcal{V}}^f} \left(||\hat{\mathbf{u}}_i||^2 - \langle \hat{\mathbf{u}}_i, \mathbf{n}_i \rangle^2 \right) \\ &- k_f x_1 \sum_{(i,l) \in \mathcal{E}^{fl}} \left(||\mathbf{u}_i||^2 - \langle \mathbf{u}_i, \mathbf{n}_i \rangle^2 \right) \leq 0. \end{aligned} \quad (45)$$

Recall that $x_1 \geq 0$ and $x_1 = 0$ if and only if $\mathbf{n}_i = \mathbf{n}_j = \mathbf{n}_l$ for all $i, j \in \mathcal{V}^f$. Furthermore, since the undesired equilibrium sets in (33) are proved to be unstable in [12], then by LaSalle's invariance principal, the unforced system, $\dot{x}_1 = f_1(x_1, 0)$, is asymptotically stable. For the forced system, $\dot{x}_1 = f_1(x_1, x_2)$, we first prove the following lemma.

Lemma 4.4:

$$\left| \langle \mathbf{n}_i, \left(\mathbf{I} - \mathbf{n}_l \mathbf{n}_l^T \right) \mathbf{N} \rangle \right| \leq \left(\theta_N (2 - \theta_N) \right)^{\frac{1}{2}}. \quad (46)$$

Proof: Using the Cauchy–Schwartz inequality: $|\langle \mathbf{n}_i, (\mathbf{I} - \mathbf{n}_l \mathbf{n}_l^T) \mathbf{N} \rangle| \leq \|(\mathbf{I} - \mathbf{n}_l \mathbf{n}_l^T)\mathbf{N}\| = \langle (\mathbf{I} - \mathbf{n}_l \mathbf{n}_l^T)\mathbf{N}, (\mathbf{I} - \mathbf{n}_l \mathbf{n}_l^T)\mathbf{N} \rangle^{\frac{1}{2}} = (1 + \langle \mathbf{N}, \mathbf{n}_l \rangle^2 - 2\langle \mathbf{N}, \mathbf{n}_l \rangle^2)^{\frac{1}{2}} = (1 - \langle \mathbf{N}, \mathbf{n}_l \rangle^2)^{\frac{1}{2}} = (\theta_N(2 - \theta_N))^{\frac{1}{2}}$. ■

Hence, for the forced system, $\dot{x}_1 = f_1(x_1, x_2)$

$$\begin{aligned}\dot{V} \leq x_1 \Bigg[&- k_f \sum_{i \in \tilde{\mathcal{V}}^f} \left(||\hat{\mathbf{u}}_i||^2 - \langle \hat{\mathbf{u}}_i, \mathbf{n}_i \rangle^2 \right) \\ &- k_f \sum_{(i,l) \in \mathcal{E}^{fl}} \left(||\mathbf{u}_i||^2 - \langle \mathbf{u}_i, \mathbf{n}_i \rangle^2 \right) \\ &+ k_l ||\nabla f(\mathbf{r}_c)|| \sum_{(i,l) \in \mathcal{E}^{fl}} \left(\theta_N (2 - \theta_N) \right)^{\frac{1}{2}} \Bigg] \\ \leq &-k_f (1 - \epsilon) x_1 \Bigg[\sum_{i \in \tilde{\mathcal{V}}^f} \left(||\hat{\mathbf{u}}_i||^2 - \langle \hat{\mathbf{u}}_i, \mathbf{n}_i \rangle^2 \right) \\ &+ \sum_{(i,l) \in \mathcal{E}^{fl}} \left(||\mathbf{u}_i||^2 - \langle \mathbf{u}_i, \mathbf{n}_i \rangle^2 \right) \Bigg], \quad \forall \\ &- k_f \epsilon \Bigg[\sum_{i \in \tilde{\mathcal{V}}^f} \left(||\hat{\mathbf{u}}_i||^2 - \langle \hat{\mathbf{u}}_i, \mathbf{n}_i \rangle^2 \right) \\ &+ \sum_{(i,l) \in \mathcal{E}^{fl}} \left(||\mathbf{u}_i||^2 - \langle \mathbf{u}_i, \mathbf{n}_i \rangle^2 \right) \Bigg] \\ &+ k_l ||\nabla f(\mathbf{r}_c)|| \sum_{(i,l) \in \mathcal{E}^{fl}} \left(\theta_N (2 - \theta_N) \right)^{\frac{1}{2}} \leq 0. \end{aligned} \quad (47)$$

What remains is to find a sufficient function $\rho(|x_2|)$ such that the above-mentioned condition is satisfied whenever $|x_1| \geq \rho(|x_2|)$. Note that although $x_1, x_2 \geq 0$ by design, we use the absolute function to agree with the standard input-to-state stability (ISS) analysis. Let $\alpha_1(|x_1|) = \alpha_2(|x_1|) = \frac{1}{2} x_1^2$ be class $\mathcal{K}_\infty$ functions. After several steps of simplification and rearrangements that are omitted for space limitation, we can verify that

$$\begin{aligned} &\alpha_1(|x_1|) \leq V(x_1) \leq \alpha_2(|x_1|) \\ &\dot{V} \leq -W(x_1), \quad \forall \quad |x_1| \geq \rho(|x_2|) \end{aligned} \quad (48)$$

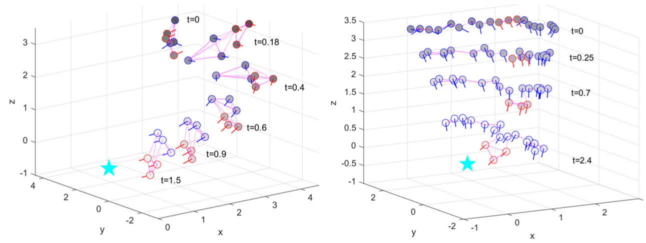

Fig. 4. Simulation of swarms of 6 and 20 agents.

where $W(x_1) = W(|x_1|)$ is a class $\mathcal{K}$ function defined by

$$\begin{aligned} W(|x_1|) = k_f (1 - \epsilon) |x_1| \Bigg[&\sum_{i \in \mathcal{V}^f, l \notin \mathcal{N}_i} \left(||\hat{\mathbf{u}}_i||^2 - \langle \hat{\mathbf{u}}_i, \mathbf{n}_i \rangle^2 \right) \\ &+ \sum_{(i,l) \in \mathcal{E}^{fl}} \left(||\mathbf{u}_i||^2 - \langle \mathbf{u}_i, \mathbf{n}_i \rangle^2 \right) \Bigg] \end{aligned} \quad (49)$$

and $\rho(|x_2|)$ is a class $\mathcal{K}$ function defined by

$$\rho(|x_2|) = \frac{k_l ||\nabla f(\mathbf{r}_c)||}{k_f \epsilon} |\mathcal{E}^{fl}| \sqrt{|x_2|(2 - |x_2|)}. \quad (50)$$

Therefore, according to [24, Th. 4.19], the forced system $h(t, \theta, \delta)$ is input-to-state stable. Note that, since $x_2 \to 0$ as $\mathbf{n}_l \to -\mathbf{N}$, $\rho(|x_2|) \to 0$. This implies that f_1 system is asymptotically stable. Additionally, since the f_2 system is proved in Theorem 4.1 to be input-to-state stable with respect to δ, then according to [13, Th. 3], the overall system is input-to-state stable. ■

This implies that the SUSD directions of agents $\{4, \ldots, M\}$ asymptotically converge to that of agents $\{1, 2, 3\}$, which in turn converge to the negative direction of the gradient. Since the source is located at the minimum of the field, then Theorem 4.2 implies that all agents converge to the source location.

V. Simulation Results

The SUSD source seeking strategy is simulated for two swarms of 6 and 20 agents as shown in Fig. 4. The field is represented by: $f(\mathbf{r}_i) = 0.5 * (x_i^2 + y_i^2 + z_i^2)$, which is minimum at the origin, as indicated by a star. The circular red discs represent agents in the set $\{1, 2, 3\}$, while the circular blue discs represent agents in the set $\{4, \ldots, M\}$. The colors of the discs change from dark to light mapping the intensity of the field. The arrows attached to each agent represent the SUSD direction of each agent, while the magenta dotted lines represent the edges of the graph. We use $k_1 = 1.1$ for $i \in \{1, 2, 3\}$ and $k_1 = 1$ for $i \in \{4, \ldots, M\}$, while $k_2 = 0$ for all agents. The separation distances in (4) and (5) are selected to be $a_{31}^0 = -a_{23}^0 = 0.25$ and $a_{31}^0 = -a_{32}^0 = \frac{\sqrt{2}}{4}$. The separation distance for $w_{i,j}$ in (10) is selected to be $\sqrt{0.7}$. The consensus gain of (7) is chosen to be $k_f = 8$, which we made it large to balance with the high SUSD speeds especially when the swarm is away from the source. As shown by the swarm trajectories in Fig. 4, given the initial random SUSD directions, the strategy successfully steers the swarm toward the source in a relatively short time.

VI. Conclusion

In this paper, we presented the SUSD strategy for source seeking in a 3-D space. We showed that through a mechanism of synchronizing direction of motion while varying the speed of each agent based only on the measured field value, the strategy successfully steers the swarm toward the minimum of the field. Future work may focus on the robustness of the proposed strategy against noisy field measurements. Additionally, considering a moving source will be an interesting problem where in this case the swarm not only has to locate the source but in addition to track it.

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