Proceedings of Proceedings of the International Design Engineering Technical Conferences & Computers & Information in Engineering Conference
IDETC/CIE 2018
August 26-29, 2018, Quebec City, Canada

DETC2018-85655

IDETC: TOPOLOGY OPTIMIZATION OF CONFORMAL STRUCTURES USING EXTENDED LEVEL SET METHODS AND CONFORMAL GEOMETRY THEORY

Qian Ye[1], **Yang Guo**[2], **Shikui Chen**[1,*], **Xianfeng David Gu**[2,3]
Department of Mechanical Engineering[1]
Department of Computer Science[2]
Department of Applied Mathematics & Statistics[3]
State University of New York,Stony Brook,
New York, USA, 11794
Email:Qian.Ye, Shikui.Chen@stonybrook.edu
Yang.Guo, Gu@cs.stonybrook.edu

Na Lei
DUT-RU International School
of Information Science & Engineering
Dalian University of Technology
Dalian, 116620 China
Email: nalei@dlut.edu.cn

ABSTRACT

In this paper, we propose a new method to approach the problem of structural shape and topology optimization on manifold (or free-form surfaces). A manifold is conformally mapped onto a 2D rectangle domain, where the level set functions are defined. With conformal mapping, the corresponding covariant derivatives on a manifold can be represented by the Euclidean differential operators multiplied by a scalar. Therefore, the topology optimization problem on a free-form surface can be formulated as a 2D problem in the Euclidean space. To evolve the boundaries on a free-form surface, we propose a modified Hamilton-Jacobi equation and solve it on a 2D plane following the conformal geometry theory. In this way, we can fully utilize the conventional level-set-based computational framework. Compared with other established approaches which need to project the Euclidean differential operators to the manifold, the computational difficulty of our method is highly reduced while all the advantages of conventional level set methods are well preserved. We hope the proposed computational framework can provide a timely solution to increasing applications involving innovative structural designs on free-form surfaces in different engineering fields.

1 INTRODUCTION

Previous Work of Topology Optimization on Surface

Topology optimization aims to find the best geometry of a design in order to obtain an optimal performance under certain constraints. Topology optimization on shell structure has been studied extensively because of its broadly applications in engineering including architectural design, automotive and aviation industry and so on. Recently, due to the maturation of additive manufacturing technologies which provide the extra freedom on design space, people's desires to obtain shell structure designs on general surfaces are inflated. The density-based approach,such as the homogenization method [1,2,3] and the Solid Isotropic Material with Penalization (SIMP) method [4, 5], is the most popular way in doing topology optimization on surface. The key idea is to find the ideal material distribution of a predefined design domain. By using this method, Fauche et al. [6] obtained the optimal thickness distribution on a thin shell bridge. Moreover, Ansola et al. [7] proposed an integrated system to solve the shape and topology optimization problem on a surface shell structure. Afterwards, he extended the work to optimize the shape and reinforcement layout on a surface simultaneously [8]. In order to optimize on more general surfaces, Hassani et al. [9] introduced the NURBS (Non Uniform Rational B-Spline) technology into their SIMP model for surface generation. However, the design achieved by the density based optimization method may contain

[*] Address all correspondence to this author.

the checkerboard patterns or gray elements. Thus, a post processing approach like the noise cleaning technique [9] has to be considered. Alternatively, by adding a minimum length scale as a geometric constraints, Guest [10] and Zhou et al. [11] were able to filter the designs to obtain a relatively binary final design.

In contrast to the density-based method, the level set method can provide a clear boundary design. Moreover, since the level set functions are defined in the space with one higher dimension, the higher-order geometric information, such as curvatures and normal vectors, is embedded naturally in the geometric model. It allows the level set method for a exclusive capability of dealing with topological changes [12]. The level-set-based topology optimization (TO) approach has been considered as a powerful tool in generating innovative designs ever since the shape sensitivity analysis been casted into the framework [13, 14, 15, 16]. However, the conventional level set functions are defined in the Euclidean space $\mathbb{R}^2$ or $\mathbb{R}^3$ on a fixed Cartesian coordinate system, which cannot satisfy the demand of TO on free-form surfaces.

Solving Partial Differential Equations (PDEs) on Manifolds

Essentially, the level set based topology optimization is a PDE-driven approach [17]. Thus, the problem of level-set-based topology optimization on manifolds is equivalent to solving variation problems and PDEs on surfaces, which has been broadly studied in the fields of mathematics and computer graphics. One popular method is based on the numerical approximation. A manifold is discretized to a triangle mesh [18], point sets [19], NURBS or B-splines [20, 21], and the solution of the global PDEs are approximated by solving the local PDEs on each segments. Recently, the NURBS-based Isogeometric Analysis method has been used for solving higher order PDEs on manifolds [20,21]. The numerical approximation approach is straightforward and can be combined with the FEA or CAD solvers. However, the accuracy of this method is restricted to the quality of the geometrical representations.

An alternative way of solving PDEs on surface is the embedding method. The key idea is to construct a space surrounding the manifold on $\mathbb{R}^3$ explicitly (closest point method) [22] or implicitly (level set method) [23], and then replace the PDEs on surface by the standard representation defined in $\mathbb{R}^3$. Ruuth et al. [22] presented the closet point method to solve PDEs on surface as close as possible to the PDEs in $\mathbb{R}^3$. This approach is efficient for the reason that the computation is only carried out on a grid near the surface [22, 24]. However, since the embedding PDE is only valid initially, an extension step is needed to ensure the computational accuracy. Macdonald and Ruuth [25] combined the closest point method with level set functions to solve the PDEs as well as evolving the interfaces on the general surface. Similar to the closest point method, the implicit method solves embedding PDEs which defined in the embedded space. While the embedded space is defined implicitly on one higher dimension by using level set functions and the PDEs are solved in the Cartesian coordinate system [26, 23, 27]. This approach is both robust and accurate in dealing with deforming surfaces. Chen et al. [28] utilized the method [23,29] to find the point-wise correspondence of the manifolds during evolution. Nonetheless, as stressed in the work of King et al [24], the implicit method can not handle the complex surfaces as much as the closest point method.

In this work, the method we use to solve PDEs on surfaces is conformal mapping [30, 31], which is an explicit approach [32, 33]. With conformal parameterization, a manifold is mapped to a 2D domain. Meanwhile, the corresponding covariant derivatives on a surface can be represented by the Euclidean differential operators multiplied by a scalar factor [34]. In other words, the variation problems on surfaces is transformed to the 2D problems. This method offers us the significant advantage which the level set base topology optimization problem on a free-form surface can be reformulated as a 2D problem in the Euclidean space. Consequently, we propose a new framework to approach the problem of structural shape and topology optimization on manifold (or free-form surfaces) by using level set method and conformal mapping theory. The major contribution is that we extend the conventional level-set based topology optimization method from Euclidean space to surfaces with arbitrary topologies.

The paper is organized as follows: Section. 2 introduces the background regarding conventional level set method and conformal mapping theory. In Section. 3, we formulate the problem of compliance minimization problem on surface and provide the sensitivity analysis. The numerical implementation including the algorithm is presented in Section. 4, followed by the demonstration numerical experiments in Section. 5. Finally, in Section. 6, the conclusions are drawn and the future work has been briefly discussed.

2 METHOD OVERVIEW

In this paper, a computational framework is proposed for topology optimization on free-form surfaces (manifold), which hinges on the level-set-based topology optimization method and the conformal mapping theory. The geometry information is transported by using the conformal mapping between the manifold and a 2D domain on the Euclidean space. The level set is then defined on the 2D domain instead of on the surface. The design is evolved based on computing the modified Hamilton-Jacobi equation on 2D. The sensitivity analysis is done by using the strain energy field obtained in the 3D simulation model. The flow chart is shown in Fig. 1.

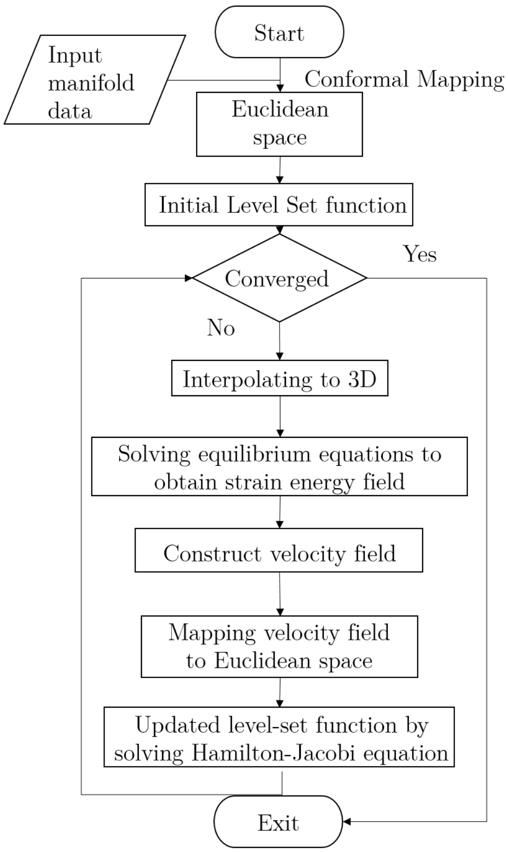


FIGURE 1: THE FLOW CHAT

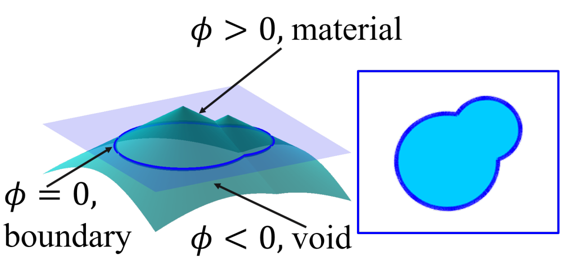


FIGURE 2: A SCHEMATIC OF LEVEL SET FUNCTIONS

2.1 The Conventional Level Set Based Topology Optimization Methodology

Conventionally, the level set function ϕ is defined in $\mathbb{R}^2$ or $\mathbb{R}^3$ as a implicit function on one higher dimension [13]. As demonstrated in Fig. 2, the level set function can implicitly represents the boundary. By slicing the level set function on the zero level, we can get the clear boundaries as shown in figure on the right hand side of Fig. 2. As expressed in Eq. 1, according to the value of the level set function, the domain is defined as three parts which are the material, the interface and the void, respectively .

$$\begin{cases} \phi(\mathbf{x},\mathbf{t}) > 0, & x \in \Omega, & \text{material} \\ \phi(\mathbf{x},\mathbf{t}) = 0, & x \in \bar{\Omega}, & \text{boundary} \\ \phi(\mathbf{x},\mathbf{t}) < 0, & x \in D/\Omega, & \text{void} \end{cases} \tag{1}$$

where D is a bounded area represents the design domain and $D \subset \mathbb{R}$. $\mathbf{x}$ is an point inside the design domain. As discussed in the Section. 1, the level set representation can spontaneously handle topological deformation. The evolution of level set function is governed by solving the Hamilton- Jacobi equation which is defined by differentiating the level set function with respect to time t [13].

$$\frac{\partial \phi}{\partial t} - \dot{\mathbf{x}} \cdot \nabla \phi = 0 \tag{2}$$

where $\dot{\mathbf{x}}$ is velocity field.

2.2 Conformal Mapping

Suppose given two Riemannian surfaces $(S_1, \mathbf{g_1})$ and $(S_2, \mathbf{g_2})$where $\mathbf{g_1}$ and $\mathbf{g_2}$ are Riemannian metric tensors, a C^1 smooth mapping $\varphi : S_1 \to S_2$ is called *conformal*, if the pull-back metric induced by φ and the original metric on the source differ by a scalar function. Specifically, there exists a real funciton $\lambda : S \to \mathbb{R}$, such that

$$\varphi^* \mathbf{g_2} = e^{2\lambda} \mathbf{g_1}.$$

Intuitively, the derivative map $d\varphi : TS_1(p) \to TS_2(\varphi(p))$ is a scaling transformation, which maps infinitesimal circles to infinitesimal circles. As shown in Fig. 3, a surface is conformally mapped onto a 2D disk, and the infinitesimal circles on surface as Fig. 3a are preserved on 2D as Fig. 3b. Therefore, φ preserves angles. In conclusion, the conformal mapping can be regarded as a local scaling process governed by the scalar function λ. It is proven [32] that by using conformal mapping the covariant derivatives on surface are equivalent to the differential operators on Euclidean apart from the scalar function. Thus, with conformal mapping, PDEs on surface can be formulated to 2D with a modified variational operators. For example, at each point $p \in (S, \mathbf{g})$, there is a neighbor $U(p)$, which can be conformally mapped onto the unit disk $\mathbb{D}^2$ on the plane. Suppose the planar coordinates are (u, v), then the Riemanian metric can be written as

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2),$$

(u, v) is called the *isothermal parameters* of the surface.

(a) Infinitesimal Circles on Surface

(b) Infinitesimal Circles on 2D Disk

FIGURE 3: CONFORMAL MAPPING FROM (a) TO (b) PRESERVES INFINITESIMAL CIRCLES

3 TOPOLOGY OPTIMIZATION

3.1 Problem Formulation

In this paper, a mean compliance minimization problem of free-form surface with a volume constraint is studied. The free-form surface is considered to be a linear elastic shell structure. The optimization problem is formulated as follows:

$$\begin{aligned} \text{Minimize} : J &= \int_\Omega \check{\varepsilon}_{ij}(u)\mathbb{C}_{ijkl}\check{\varepsilon}_{kl}(u)d\Omega \\ \text{Subject to} : a(u,v,\phi) &= l(v,\phi) \\ V(\Omega) &= V^* \end{aligned}$$

where V denotes the volume of the manifold shell, V* refers to the target volumn. $\mathbb{C}_{ijkl}$ is the fourth order constitutive tensor. Ω is the region occupied with linear elastic material.

$$\begin{aligned} V(\Omega) &= \int_D H(\phi)d\Omega \\ a(u,v,\phi) &= \int_\Omega \check{\varepsilon}_{ij}(u)\mathbb{C}_{ijkl}\check{\varepsilon}_{kl}(v)d\Omega \\ l(v) &= \int_\Omega f\cdot v d\Omega + \int_{\bar{\Omega}} g\cdot v d\bar{\Omega} \end{aligned}$$

where D is the design domain; $a(u,v)$ is a symmetric bilinear function, which means $a(u,v)$ is linear both in u and v. Thus, $a(u,v) = a(v,u)$. l is a linear function depending on the body force f and the traction force g as shown in Fig.4.

3.2 Shape Sensitivity Analysis

The Lagrangian of the optimization problem can be written as:

$$L(u,v) = J + \lambda(a(u,v) - l(v)) \tag{3}$$

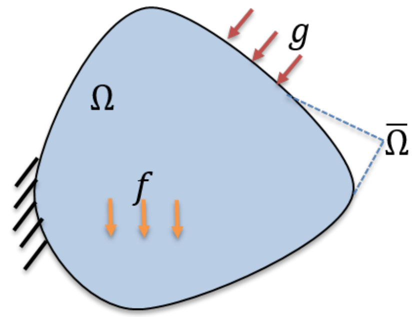


FIGURE 4: A SCHEMATIC OF GENERAL BOUNDARY CONDITION

where the λ is a Lagrange multiplier, and v is the test function. Since $a(u,v)$ and $l(v)$ are linear functions in terms of v,

$$L(u,v) = J + a(u,\lambda v) - l(\lambda v) \tag{4}$$

In equation (3), λv is in the same space of v. For simplicity we can denote λv as v from now on, and equation (3) can be reformulated as:

$$L(u,v) = a(u,u) + a(u,v) - l(v) \tag{5}$$

The material derivative [35, 36] of the equation (5) with respect to a pseudo time t is formulated as

$$\frac{dL(u,v)}{dt} = \frac{\partial L(u,v)}{\partial t} + \frac{\partial L(u,v)}{\partial \Omega} \tag{6}$$

where the partial derivative with respect to time results in the so called adjoint equation:

$$\frac{\partial L(u,v)}{\partial t} = L^{'} = a^{'}(u,u) + a^{'}(u,v), \tag{7}$$

and

$$\begin{aligned} a^{'}(u,u) &= 2\int_\Omega \check{\varepsilon}_{ij}(u')\mathbb{C}_{ijkl}\check{\varepsilon}_{kl}(u)d\Omega \\ a^{'}(u,v) &= \int_\Omega \check{\varepsilon}_{ij}(u')\mathbb{C}_{ijkl}\check{\varepsilon}_{kl}(v)d\Omega. \end{aligned} \tag{8}$$

The convection term of the material derivative forms the shape derivative which is formulated as follows:

$$\frac{\partial L(u,v)}{\partial \Omega} = \int_{\Gamma} \check{\varepsilon}_{ij}(u)\mathbb{C}_{ijkl}\check{\varepsilon}_{kl}(u)v_n ds + \int_{\Gamma} \check{\varepsilon}_{ij}(u)\mathbb{C}_{ijkl}\check{\varepsilon}_{kl}(v)v_n ds - \int_{\Gamma} f \cdot v v_n ds - \int_{\Gamma} [\frac{\partial(g\cdot v)}{\partial n} + \kappa g \cdot v] v_n ds. \tag{9}$$

Solving equation (7), we can get the adjoint variable $v = -2u$. Substitute $v = -2u$ to equation (9) and ignore the body force, we can get

$$\frac{\partial L(u,v)}{\partial \Omega} = \int_{\Gamma} \check{\varepsilon}_{ij}(u)\mathbb{C}_{ijkl}\check{\varepsilon}_{kl}(u)v_n ds \tag{10}$$

By using the steepest-descent method, we can construct the normal velocity filed as

$$v_n = -\check{\varepsilon}_{ij}(u)\mathbb{C}_{ijkl}\check{\varepsilon}_{kl}(u) \tag{11}$$

which is the strain energy density of the linear elastic structure. The volume constraint is considered by using the penalty Lagrangian method.

$$v_n = -\check{\varepsilon}_{ij}(u)\mathbb{C}_{ijkl}\check{\varepsilon}_{kl}(u) + \lambda_1(V - V^*) \tag{12}$$

4 NUMERICAL IMPLEMENTATION

4.1 Modified Hamilton-Jacobi Equation

By using our method, we can parameterize the manifold conformally onto the rectangular domain and evolve the level set function on 2D to optimize the design. Assume we have a level set function $\phi(\mathbf{x},t)$ where the boundary is defined as

$$\phi(\mathbf{x},t) = 0, \tag{13}$$

Conventionally, by differentiating Eq. 13 with respect to t, we get the Hamilton-Jacobi (H-J) equation. In our case, as stressed in Section. 2.2 , we can modify the H-J equation with the manifold version of gradient in order to solve the PDE on surface, which is shown as follows:

$$\frac{\partial \phi}{\partial t} - \dot{\mathbf{x}} \cdot \nabla_g \phi = 0 \tag{14}$$

where $\dot{\mathbf{x}}$ is the continuous velocity field,$\nabla_g \phi$ is the gradient of ϕ on manifold. According to [37], Only the normal component of velocity field plays a part in deforming the boundary.Thus, Eq. 14 can be rewritten as

$$\frac{\partial \phi}{\partial t} - v_n |\nabla_g \phi|_g = 0 \tag{15}$$

Let f be the conformal mapping between manifold M and 2-D domain: $f: M \to \mathbb{R}^2$. Considering [33,34]:

$$|\nabla_g \phi|_g = e^{-\lambda}|\nabla \phi| \tag{16}$$

where the λ is a conformal factor. Thus the Eq. 14 changes to

$$\frac{\partial \phi}{\partial t} - e^{-\lambda} v_n |\nabla \phi| = 0 \tag{17}$$

We define the Eq. 17 as modified Hamilton-Jacobi equation (M-H-J). The level set revolution on manifold can be successfully solved on the 2D domain by M-H-J equation.

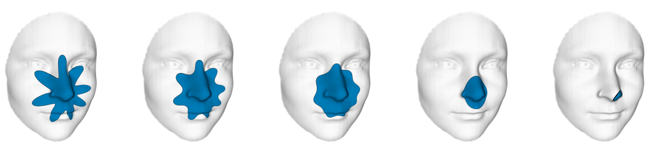
4.2 Algorithm

In our proposed method, the geometry information is transported by using conformal mapping from the manifold to the Euclidean space, as shown in Section. 2.2. Instead of solving the variation problem directly on the surface, a modified Hamilton-Jacobi equation is computed on 2D to involve the interface of level set function in order to optimize the design. The velocity is inherited from the sensitivity analysis results, which is equivalent to the strain energy density obtained in the simulation model on the surface. Intuitively, the method can guarantee to optimize the 3D design with a lower computational cost.

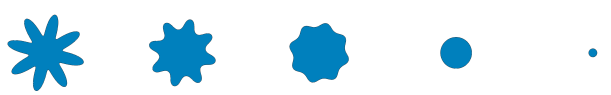
The framework can be decomposed into seven steps as shown in Alg. 1. The input is a triangle mesh surface S_1. In the first step, a 2D triangular mesh rectangle Q_1 is achieved utilizing the conformal mapping parametrization. Then, we construct a 2D quad mesh on Q where we define the level set function subsequently. The third step is to transport the level set values from 2D onto the surface. Firstly, the level set values on the 2D triangular mesh can be interpolated from the values on 2D quad mesh. By conformal mapping , the vertices's relationship on S_1 and Q_1 is given, which means a vertex on Q_1 is corresponding to one specific vertex on the surface S_1. Since the level set value on each vertices's can be regarded as a constant, the transportation from 2D to surface can be naturally made by the calculated conformal mapping parametrization. Step 4 and 5 are about to solve the equilibrium physics equations and do shape sensitivity analysis to construct the design velocity filed on the surface. Next, by solving the Modified Hamilton-Jacobi equation, the level set function is updated. The step 2 through 6 is repeated until the convergence criterion is fulfilled.

Algorithm 1: A Framework for Level-Set-Based Shape and Topology Optimization on Manifold

Input: A triangle meshed surface
Output: The 3D minimum compliance design
Given S_1, compute the global conformal parametrization from S_1 onto the 2D rectangle Q_1.;
Initialize the level set function ϕ on the 2D rectangle domain Q;
Transport the value of ϕ onto S_1 by using Barycentric interpolation method;
Solving equilibrium equation on S_1 to obtain strain energy field;
Shape sensitivity analysis to construct the design velocity filed;
Update level-set function by solving the M-H-J Equation on Q until getting converged;
Get the topology optimization design S_2 from ϕ;



(a) Mean curvature flow on the surface



(b) Mean curvature flow on 2D

FIGURE 5: MEAN CURVATURE FLOW

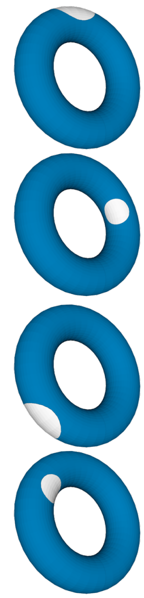
5 NUMERICAL EXAMPLES

5.1 Curvature Flow on Surface

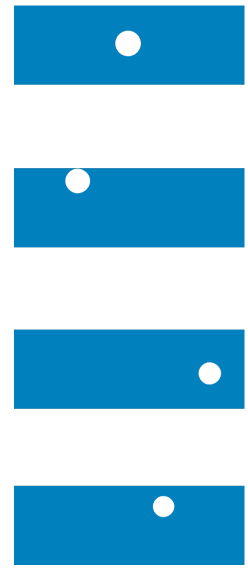
In this example, we test our level set algorithm on the motion with curvature-dependent acceleration. The interface moves in the normal direction with a velocity reciprocal to its curvature:

$$v_n = -\kappa = -\nabla \cdot n \tag{18}$$

where ∇ is the Laplace operator and n is the normal vector of the level set. Initially, the level set is designed in a star-shape [38] on the 2D rectangle domain as shown in Fig. 5b. By using conformal mapping, the star-shape interface is mapped to the surface as Fig. 5a. Eventually, on both 2D and the surface, the interface becomes to a circle and shrinks to a point until disappeared.



(a) Motion on the torus



(b) Motion on 2D

FIGURE 6: CONSTANT CONVECTION ON TORUS

5.2 Constant Convection Motion on the Torus (Genus one Surface)

In this example, a numerical experiment of interface moving on a torus is applied as to show the advantage of our proposed algorithm on handling boundary changes on manifolds with complex topologies. As shown in Fig. 6b a circle is defined by level set function on the zero level is moving under a constant velocity. The corresponding motion on torus surface is shown in Fig. 6a. The motion is driven by the H-J equation defined on 2D as

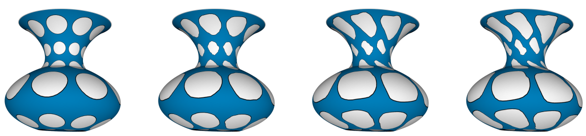
$$\frac{\partial \phi}{\partial t} - \dot{\mathbf{x}} \cdot \phi = 0 \tag{19}$$

Here, the velocity field $\dot{\mathbf{x}}$ is a constant along the specified direction on 2D. By applying the periodic boundary conditions [39], the circle can move continuously on the torus.

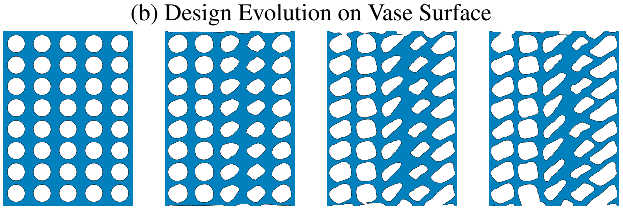
5.3 Vase Shape Surface Optimization

In the following example, the extended Level Set methods with conformal geometry theory is applied to the minimum mean compliance problem discussed in Section. 3 on a vase shape shell model. The volume target of the design is 40%. The linear elastic material with properties of Piosson's ratio $\nu = 0.3$ and the Young's modulus $E = 1$GPa is applied. In order to avoid singularity, a weak material with $E = 10^{-6}$GPa is set for the void. Fig. 7a shows the boundary conditions and the design domain. A vertical distributed load and a moment of force along -z direction

(a) Design Boundary Conditions



(b) Design Evolution on Vase Surface



(c) Design Evolution on 2D

FIGURE 7: TOPOLOGY OPTIMIZATION ON VASE SURFACE

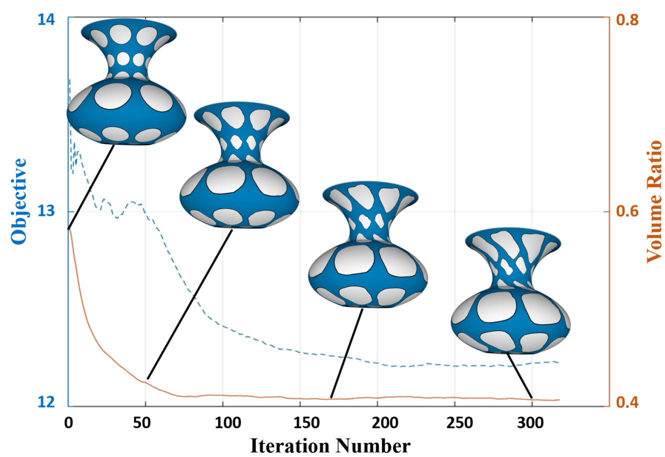
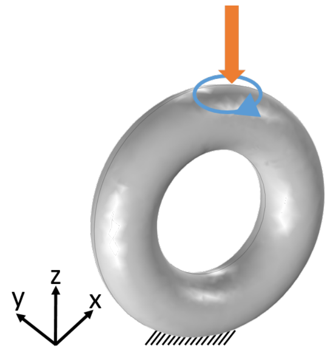
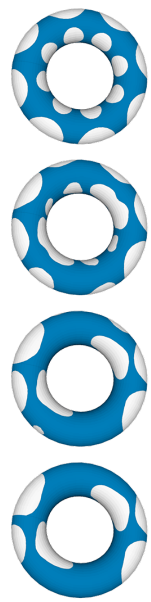


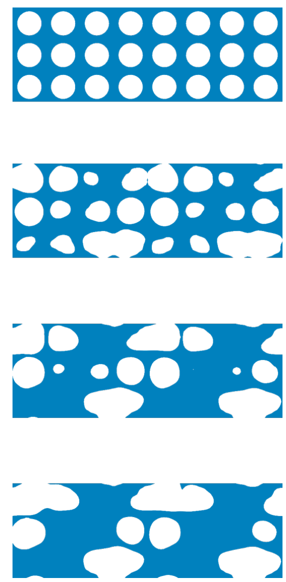
FIGURE 8: THE OPTIMIZATION HISTORY OF A VASE SURFACE DESIGN

are applied on the top of the vase with bottom boundary fixed. Following the Alg. 1, we map the 3D triangular meshed vase surface with 39764 elements onto a 2D rectangle domain Q_1. Then the level set function is constructed on Q_1 with a 101×75 quad mesh in order to evolve the design. By using this approach, the problem is notably simplified. The results on 2D and on the surface are shown in Fig. 7c as well as Fig. 7b. The initial design domain is set to be a surface with 40 circular holes. As presented in Fig. 8, the mean compliance of the vase is minimized and the volume constraint is satisfied after 300 iterations. In addition, it is noticeable that on the vase surface the local shape of the design is preserved from the 2D design which is in accord with the conformal mapping theory.



(a) Design Boundary Conditions





(b) Design Evolution on Torus (c) Design Evolution on 2D

FIGURE 9: TOPOLOGY OPTIMIZATION ON TORUS

5.4 Topology Optimization on a Torus Surface

A minimum compliance problem on a torus shape shell is considered by using the same method and material properties as Section. 5.3. The target volume ratio is 50%. The design domain along with the boundary conditions are shown in Fig. 9a. Similarly to the vase model, a concentrated load and a moment of force along -z direction are applied to the top of the model. A small area on the torus bottom is fixed. The mesh element on surface is 3200 and the quad mesh size on 2D is 101×36. The design results are shown in Fig. 9.

6 CONCLUSIONS

In this paper, a framework of structural shape and topology optimization on manifold (or free-form surfaces) is proposed. By using conformal mapping theory, we extend the level set based topology optimization approach from the Euclidean space $\mathbb{R}^2$ or $\mathbb{R}^3$ to surfaces with arbitrary topologies. A manifold is conformally mapped onto a 2D rectangle domain, where the level set functions are defined. With conformal mapping, the corresponding covariant derivatives on a manifold can be represented by the Euclidean differential operators multiplied by a scalar. Therefore, the TO problem on a free-form surface can be formulated as a 2D problem in the Euclidean space. To evolve the boundaries on a free-form surface, we propose a modified Hamilton-Jacobi equation and solve it on a 2D plane following the conformal geometry theory. In this way, we can fully utilize the conventional level-set-based computational framework. Compared with other established approaches, the computational difficulty of our method is highly reduced while all the advantages of conventional level set methods are well preserved. The results of numerical experiments indicate the robustness and effectiveness of our method in solving topology optimization problems on manifolds. Further work will be related to the innovative structural designs on free-form surfaces with different type of objection functions and constraints. For instance, the optimum design on free-form surfaces with multiple material and under multiple loading cases.

ACKNOWLEDGMENT

The authors acknowledge the support from the National Science Foundation (CMMI1462270), Ford University Research Program (URP), and the start-up from the State University of New York at Stony Brook.

REFERENCES

[1] Suzuki, K., and Kikuchi, N. "A homogenization method for shape and topology optimization". *Computer methods in applied mechanics and engineering,* ***93***(3), pp. 291–318.

[2] Allaire, G., and Kohn, R. "Topology optimization and optimal shape design using homogenization". In *Topology design of structures*. Springer, pp. 207–218.

[3] Nishiwaki, S., Frecker, M. I., Min, S., and Kikuchi, N. "Topology optimization of compliant mechanisms using the homogenization method".

[4] Rozvany, G. I., Zhou, M., and Birker, T. "Generalized shape optimization without homogenization". *Structural optimization,* ***4***(3-4), pp. 250–252.

[5] Bendsoe, M. P., and Sigmund, O., 2013. *Topology optimization: theory, methods, and applications*. Springer Science & Business Media.

[6] Fauche, E., Adriaenssens, S., and Prevost, J. H. "Structural optimization of a thin-shell bridge structure". *Journal of the International Association for Shell and Spatial Structures,* ***51***(2), p. 153.

[7] Ansola, R., Canales, J., Tarrago, J. A., and Rasmussen, J. "An integrated approach for shape and topology optimization of shell structures". *Computers & structures,* ***80***(5-6), pp. 449–458.

[8] Ansola, R., Canales, J., Tarrago, J., and Rasmussen, J. "Combined shape and reinforcement layout optimization of shell structures". *Structural and Multidisciplinary Optimization,* ***27***(4), pp. 219–227.

[9] Hassani, B., Tavakkoli, S. M., and Ghasemnejad, H. "Simultaneous shape and topology optimization of shell structures". *Structural and Multidisciplinary Optimization,* ***48***(1), pp. 221–233.

[10] Guest, J. K., Prévost, J. H., and Belytschko, T. "Achieving minimum length scale in topology optimization using nodal design variables and projection functions". *International journal for numerical methods in engineering,* ***61***(2), pp. 238–254.

[11] Zhou, M., Lazarov, B. S., Wang, F., and Sigmund, O. "Minimum length scale in topology optimization by geometric constraints". *Computer Methods in Applied Mechanics and Engineering,* ***293***, pp. 266–282.

[12] van Dijk, N. P., Maute, K., Langelaar, M., and Van Keulen, F. "Level-set methods for structural topology optimization: a review". *Structural and Multidisciplinary Optimization,* ***48***(3), pp. 437–472.

[13] Allaire, G., Jouve, F., and Toader, A.-M. "Structural optimization using sensitivity analysis and a level-set method". *Journal of computational physics,* ***194***(1), pp. 363–393.

[14] Osher, S. J., and Santosa, F. "Level set methods for optimization problems involving geometry and constraints: I. frequencies of a two-density inhomogeneous drum". *Journal of Computational Physics,* ***171***(1), pp. 272–288.

[15] Allaire, G., Jouve, F., and Toader, A.-M. "A level-set method for shape optimization". *Comptes Rendus Mathematique,* ***334***(12), pp. 1125–1130.

[16] Wang, M. Y., Wang, X., and Guo, D. "A level set method for structural topology optimization". *Computer methods in applied mechanics and engineering,* ***192***(1-2), pp. 227–246.

[17] Wang, M. Y., and Wang, X. "PDE-driven level sets, shape sensitivity and curvature flow for structural topology optimization". *Computer Modeling in Engineering and Sciences,* ***6***, pp. 373–396.

[18] Lopez-Perez, L., Deriche, R., and Sochen, N. "The beltrami flow over triangulated manifolds". In *Computer Vision and Mathematical Methods in Medical and Biomedical Image Analysis*. Springer, pp. 135–144.
[19] Clarenz, U., Rumpf, M., and Telea, A. "Surface processing methods for point sets using finite elements". *Computers & Graphics,* ***28***(6), pp. 851–868.
[20] Dedè, L., and Quarteroni, A. "Isogeometric Analysis for second order Partial Differential Equations on surfaces". *Computer Methods in Applied Mechanics and Engineering,* ***284***, pp. 807–834.
[21] Bartezzaghi, A., Dedè, L., and Quarteroni, A. "Isogeometric analysis of high order partial differential equations on surfaces". *Computer Methods in Applied Mechanics and Engineering,* ***295***, pp. 446–469.
[22] Ruuth, S. J., and Merriman, B. "A simple embedding method for solving partial differential equations on surfaces". *Journal of Computational Physics,* ***227***(3), pp. 1943–1961.
[23] Bertalmıo, M., Cheng, L.-T., Osher, S., and Sapiro, G. "Variational problems and partial differential equations on implicit surfaces". *Journal of Computational Physics,* ***174***(2), pp. 759–780.
[24] King, N. D., and Ruuth, S. J. "Solving variational problems and partial differential equations that map between manifolds via the closest point method". *Journal of Computational Physics,* ***336***, pp. 330–346.
[25] Macdonald, C. B., and Ruuth, S. J. "Level set equations on surfaces via the Closest Point Method". *Journal of Scientific Computing,* ***35***(2-3), pp. 219–240.
[26] Bertalmio, M., Sapiro, G., and Randall, G. "Region tracking on level-sets methods". *IEEE Transactions on Medical Imaging,* ***18***(5), pp. 448–451.
[27] Mémoli, F., Sapiro, G., and Osher, S. "Solving variational problems and partial differential equations mapping into general target manifolds". *Journal of Computational Physics,* ***195***(1), pp. 263–292.
[28] Chen, S., and Chen, W. "A new level-set based approach to shape and topology optimization under geometric uncertainty". *Structural and Multidisciplinary Optimization,* ***44***(1), pp. 1–18.
[29] Pons, J.-P., Hermosillo, G., Keriven, R., and Faugeras, O. "Maintaining the point correspondence in the level set framework". *Journal of Computational Physics,* ***220***(1), pp. 339–354.
[30] Gu, X., Wang, S., Kim, J., Zeng, Y., Wang, Y., Qin, H., and Samaras, D. "Ricci flow for 3D shape analysis". In Computer Vision, 2007. ICCV 2007. IEEE 11th International Conference on, IEEE, pp. 1–8.
[31] Gu, X., Wang, Y., Chan, T. F., Thompson, P. M., and Yau, S.-T. "Genus zero surface conformal mapping and its application to brain surface mapping". *IEEE Transactions on Medical Imaging,* ***23***(8), pp. 949–958.
[32] Lui, L. M., Gu, X., Chan, T. F., Yau, S.-T., et al. "Variational method on riemann surfaces using conformal parameterization and its applications to image processing". *Methods and Applications of Analysis,* ***15***(4), pp. 513–538.
[33] Gu, X., and Yau, S.-T. "Computing conformal structure of surfaces". *arXiv preprint cs/0212043*.
[34] Lui, L. M., Wang, Y., and Chan, T. F. "Solving PDEs on manifolds with global conformal parametriazation". In *Variational, Geometric, and Level Set Methods in Computer Vision*. Springer, pp. 307–319.
[35] Choi, K. K., and Kim, N.-H., 2006. *Structural sensitivity analysis and optimization 1: linear systems*. Springer Science & Business Media.
[36] Xia, Q., and Shi, T. "Optimization of structures with thin-layer functional device on its surface through a level set based multiple-type boundary method". *Computer Methods in Applied Mechanics and Engineering,* ***311***, pp. 56–70.
[37] Peng, D., Merriman, B., Osher, S., Zhao, H., and Kang, M. "A PDE-based fast local level set method". *Journal of computational physics,* ***155***(2), pp. 410–438.
[38] Fedkiw, S. O. R., and Osher, S. "Level set methods and dynamic implicit surfaces". *Surfaces,* ***44***, p. 77.
[39] Mitchell, I. M. "A toolbox of level set methods". *Dept. Comput. Sci., Univ. British Columbia, Vancouver, BC, Canada, http://www. cs. ubc. ca/˜ mitchell/ToolboxLS/toolboxLS. pdf, Tech. Rep. TR-2004-09*.