Performance of First and Second Order Linear Networked Systems Over Digraphs

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Abstract— **In this paper we investigate the performance of linear networked dynamical systems over digraphs with a globally reachable node. We consider first and second order systems subject to distributed disturbances and define an output that quantifies the performance through the input-output $\mathcal{H}_2$ norm of the system. We develop a generalized framework for computing the $\mathcal{H}_2$ norm for this class of systems, and apply this framework to evaluate two performance measures for systems whose underlying network graphs result in normal weighted graph Laplacian matrices. We find closed-form solutions for the measure that quantifies the total deviation of the states from the average, and bounds on the measure that quantifies the weighted squared difference between the states of neighboring nodes. Numerical examples indicate that a second order system connected over a cycle graph may have better performance when its underlying graph is directed due to complex eigenvalues of the Laplacian. The results also indicate that the $\mathcal{H}_2$ norm of a symmetric system is less than or equal to that of the corresponding perturbed non-symmetric system for either line or complete graphs when the network size is sufficiently large.**

I. INTRODUCTION

The most commonly studied aspect of linear networked dynamical systems is stability, i.e. determining conditions under which the agents in the network reach a synchronized state [1]–[3]. Once it is known that synchrony can be achieved, it is worthwhile to investigate the synchronization performance, e.g. the effort required to restore and/or maintain synchrony in the presence of persistent stochastic disturbances. This synchronization performance can, for example, be a measure of the system's efficiency and/or robustness and can be evaluated in terms of coherence or the degree of disorder in distributed consensus [4], [5], and linear oscillator networks [6], [7]. Performance has also been assessed in terms of transient real power losses in transmission [8] and renewable energy integrated power networks [9], as well as in microgrids [10]. It has also been evaluated in terms of the effective resistance of undirected graphs [7], which allows one to leverage efficient computational approaches [11]. Recent progress has been also made for computing the effective resistance of directed graphs [12], [13]. The effect of the graph topology on performance has also been investigated [14], [15], but directed graphs were not considered.

Networked dynamical systems in many contexts can be modeled as first or second order systems that interact over a graph through coupling functions, and are subjected to distributed disturbances. Performance measures of such systems are often obtained by defining a system output such that the desired performance is quantified through the input-output (IO) $\mathcal{H}_2$ norm of the system. Certain $\mathcal{H}_2$ norm based performance measures for systems with underlying graphs described by a symmetric weighted graph Laplacian can be obtained in closed form [7]–[10]. Closed form solutions have also been obtained for first order consensus systems with normal weighted graph Laplacian matrices [5]. The question of whether or not one can obtain a closed form expression for the IO $\mathcal{H}_2$ norm for related second order systems that interact over general digraphs remains open.

In this work, we take a step toward answering this question by providing a means to compute the performance of linear networked dynamical systems over digraphs with a globally reachable node. We consider first and second order systems subject to distributed disturbance inputs, and specify the system output so that the squared $\mathcal{H}_2$ norm quantifies the performance of the IO system. We define two performance measures; one that quantifies the weighted squared difference between the states of neighboring nodes, and one that measures the deviation of the states from the average. The related system outputs can be described in terms of a weighted symmetric Laplacian matrix that is associated with an undirected and connected output graph.

We derive a framework for computing the $\mathcal{H}_2$ norm for the class of systems discussed above by first decomposing the IO system into two subsystems that are associated with the zero and non-zero eigenvalues of the weighted graph Laplacian matrix. We show that the states of the subsystem associated with the zero eigenvalue are unobservable from the output, and use this fact to provide analytical expressions for the $\mathcal{H}_2$ norm as a function of the observability Gramian of the subsystem associated with the non-zero eigenvalues of this Laplacian matrix. Then we apply this framework to systems whose underlying graphs emit weighted graph Laplacians that are normal matrices. We obtain the closed-form $\mathcal{H}_2$ norm of the second order system for the case in which the performance measure quantifies the deviation from the state average. We obtain bounds on the $\mathcal{H}_2$ norm of the first order system with an arbitrary undirected and connected output graph. Finally, we discuss how our results generalize some of the existing literature concerning systems with normal and symmetric weighted Laplacian matrices, and demonstrate that directed communication can improve performance.

Numerical examples illustrate how the bounds for the first order system and the exact solutions for both the first and the second order systems scale with network size for

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the special cases of directed and undirected cycle graphs. Results from a second numerical study suggest that the $\mathcal{H}_2$ norm of a symmetric system might be a lower bound for the corresponding perturbed non-symmetric system for sufficiently large networks whose coupling is represented by either a line or a complete graph.

The remainder of this paper is organized as follows. Section 2 provides mathematical preleminaries and presents the system models to be studied. In section 3, we derive generalized analytical expressions for the $\mathcal{H}_2$ norm and apply these expressions to first and second order systems to evaluate their performance. Section 4 presents numerical examples that examine the effect of network size and compare the performance of symmetric and non-symmetric systems. Section 5 concludes the paper.

II. PROBLEM FORMULATION

A. Mathematical Preliminaries

Given an IO system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{w}, \qquad (1)$$
$$\mathbf{y} = C\mathbf{x},$$

where $\mathbf{x} \in \mathbb{C}^n$ is the state, $\mathbf{w} \in \mathbb{C}^q$ is the input, and $\mathbf{y} \in \mathbb{C}^p$ is the output, the $\mathcal{H}_2$ norm of the corresponding impulse response function $G(t) \in \mathbb{C}^{p\times q}$ is defined as $\|G(t)\|_2^2 := \operatorname{tr}\int_0^\infty G(\tau)^* G(\tau) d\tau = \operatorname{tr}\int_0^\infty G(\tau)G(\tau)^* d\tau$ [16]. The $\mathcal{H}_2$ norm of $G(t)$ can be computed as

$$\|G(t)\|_2^2 = \operatorname{tr}(B^* X B), \qquad (2)$$

where X is the observability Gramian of (1) given by $X := \int_0^\infty e^{A^*\tau} C^* C e^{A\tau} d\tau$ [16], if the integral converges. X can be computed from the Lyapunov equation [16]:

$$A^* X + XA = -C^* C. \qquad (3)$$

Uniqueness of the solution to (3) is determined by the well-known results given in the following propositions.

Proposition 1: The Lyapunov equation (3) has a unique solution if and only if $\sigma(A^*) \cap \sigma(-A) = \emptyset$.

This result follows directly from Theorem 2.4.4.1 in [17].

Proposition 2: X is the unique solution to the Lyapunov equation (3) if A is Hurwitz [18].

B. System Models

1) Dynamics: Consider n dynamical systems that communicate over a digraph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}\}$ with a globally reachable node. Here, $\mathcal{N} = \{1, ..., n\}$ is the set of nodes and $\mathcal{E} = \{(i,k) \mid i,k \in \mathcal{N}, i \neq k\}$ is the set of edges. The edge weight associated with $(i,k) \in \mathcal{E}$ is denoted by $r_{ik} > 0$, and $r_{ik} = 0$ if and only if $(i,k) \notin \mathcal{E}$.

We consider first order systems of this form with dynamics

$$\dot{x}_i = -\sum_{k=1}^{n} r_{ik}(x_i - x_k) + w_i,$$

at each node $i \in \mathcal{N}$, where w_i denotes the disturbance to the i^{th} system. In matrix form

$$\dot{\mathbf{x}} = -L\mathbf{x} + \mathbf{w}, \qquad (4)$$

where L denotes the weighted graph Laplacian matrix given by $[L]_{ii} = \sum_{k=1}^n r_{ik}$, and $[L]_{ik} = -r_{ik}$ if $i \neq k$, $\forall i,k \in \mathcal{N}$. Second order systems of this form have dynamics

$$\ddot{x}_i + k_d \dot{x}_i + k_p x_i = u_i + w_i,$$

at each node $i \in \mathcal{N}$, with $u_i = -\gamma_p \sum_{k=1}^n r_{ik}(x_i - x_k) - \gamma_d \sum_{k=1}^n r_{ik}(\dot{x}_i - \dot{x}_k)$ $\forall i \in \mathcal{N}$. Here, $k_p, k_d, \gamma_p, \gamma_d \geq 0$, and w_i denotes the disturbance to the i^{th} system. In matrix form

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -k_p I - \gamma_p L & -k_d I - \gamma_d L \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \mathbf{w}, \qquad (5)$$

where $\mathbf{v} := \dot{\mathbf{x}}$.

2) Performance Measures and IO Systems: We are interested in evaluating the performance of systems that can be represented by (4) or (5). We consider two performance measures described based on the interaction of the systems over an undirected and connected output graph $\mathcal{G}^{out} = \{\mathcal{N}, \mathcal{E}^{out}\}$. The first measure is a function of the states x_i, and the edge weights $g_{ik} > 0$ associated with $(i,k) \in \mathcal{E}^{out}$:

$$P := \lim_{t\to\infty} \mathbb{E}\left\{ \sum_{i,k=1}^{n} g_{ik}(x_i - x_k)^2 \right\}, \qquad (6)$$

where $g_{ik} = 0$ if and only if $(i,k) \notin \mathcal{E}^{out}$, and $\mathbb{E}\{\cdot\}$ denotes the expected value. Since $\mathcal{G}^{out}$ is undirected, $g_{ik} = g_{ki}$ $\forall (i,k) \in \mathcal{E}^{out}$, and since it is connected, there exists a path between any $i,k \in \mathcal{N}$. Therefore,

$$P = \lim_{t\to\infty} \mathbb{E}\{\mathbf{x}^\intercal L_g \mathbf{x}\}, \qquad (7)$$

where L_g denotes the symmetric weighted Laplacian matrix associated with $\mathcal{G}^{out}$, given by $[L_g]_{ii} = \sum_{k=1}^n g_{ik}$, and $[L_g]_{ik} = -g_{ik}$ if $i \neq k$, $\forall i,k \in \mathcal{N}$.

The second performance measure is the total steady-state variance of the difference between each state x_i and the average:

$$P_\Pi := \lim_{t\to\infty} \mathbb{E}\left\{ \sum_{i=1}^{n} [x_i - (\frac{1}{n}\sum_{i=1}^{n} x_i)]^2 \right\}, \qquad (8)$$

which can equivalently be written as

$$P_\Pi = \lim_{t\to\infty} \mathbb{E}\left\{\mathbf{x}^\intercal \Pi \mathbf{x}\right\}, \qquad (9)$$

where $\Pi := (I - \frac{1}{n}\mathbf{1}\mathbf{1}^\intercal)$ and $\mathbf{1} = [1 \dots 1]^\intercal$.

Remark 1: P_Π can be interpreted as a special case of P in (7) where $L_g = \Pi$ is the weighted Laplacian matrix of an undirected complete graph with uniform edge weights $\frac{1}{n}$.

The performance measure in (7) is the aggregate weighted local state error of neighboring nodes in $\mathcal{G}^{out}$ and it reduces to (9), which is the total error from the global mean, when $\mathcal{G}^{out}$ is such that any two nodes are neighbors with equal weight. When the systems (4) and (5) are subject to persistent stochastic disturbances, (7) and (9) are both measures of the degree of disorder and robustness in distributed consensus [4], [5], and linear oscillator networks [6], [7]. The metric in (7) also quantifies the total transient real power losses in transmission [8] and renewable energy integrated power networks [9], as well as in inverter based microgrids [10].

If we define the system output $\mathbf{y} := L_g^{1/2}\mathbf{x}$, we have

$$\begin{aligned}\dot{\mathbf{x}} &= -L\mathbf{x} + \mathbf{w} \\ \mathbf{y} &= L_g^{1/2}\mathbf{x},\end{aligned} \tag{10}$$

and

$$\begin{bmatrix}\dot{\mathbf{x}} \\ \dot{\mathbf{v}}\end{bmatrix} = \begin{bmatrix}0 & I \\ -k_p I - \gamma_p L & -k_d I - \gamma_d L\end{bmatrix}\begin{bmatrix}\mathbf{x} \\ \mathbf{v}\end{bmatrix} + \begin{bmatrix}0 \\ I\end{bmatrix}\mathbf{w} \tag{11}$$
$$\mathbf{y} = \begin{bmatrix}L_g^{1/2} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x} \\ \mathbf{v}\end{bmatrix}.$$

We will refer to the IO systems (10) and (11) by their impulse response functions. The functions for both systems will be denoted by $G_\Pi(t)$ if $L_g = \Pi$, and $G(t)$ if L_g is an arbitrary symmetric weighted Laplacian matrix associated with $\mathcal{G}^{out}$.

The squared $\mathcal{H}_2$ norm quantifies the steady-state variance of the output if the input is white noise with unit covariance $\mathbb{E}\{\mathbf{w}(\tau)\mathbf{w}(t)^\intercal\} = \delta(t-\tau)I$ [8]:

$$\|G(t)\|_2^2 = \lim_{t\to\infty} \mathbb{E}\{\mathbf{y}(t)^\intercal \mathbf{y}(t)\}. \tag{12}$$

Assuming an input of this type and using the interpretation in (12), the $\mathcal{H}_2$ norms of $G(t)$ and $G_\Pi(t)$ quantify the performance measures given by (7) and (9), respectively.

III. ANALYTICAL RESULTS

In this section, we block-diagonalize systems (10) and (11). Then, we decompose the dynamics into two subsystems; one associated with the zero eigenvalue of the weighted graph Laplacian matrix L, and one associated with the rest of the eigenvalues. The states of the subsystem associated with the zero eigenvalue are shown to be unobservable from the output. We then exploit this fact to derive analytical expressions for the $\mathcal{H}_2$ norm in terms of the parameters of the second subsystem. Finally, the framework is applied to systems whose underlying interconnection can be represented by normal weighted Laplacian matrices. We obtain a closed-form solution for P_Π in (9) for the second order system, and bounds on P in (7) for the first order system.

Prior to stating the main results, we investigate the spectral properties of L and L_g through the following lemmas.

Lemma 1: L can be decomposed as $L = RJR^{-1}$, where $R \in \mathbb{C}^{n\times n}$ is invertible, $J \in \mathbb{C}^{n\times n}$ is given by

$$J = \begin{bmatrix}0 & \mathbf{0}_{n-1}^\intercal \\ \mathbf{0}_{n-1} & J_2\end{bmatrix}, \tag{13}$$

and $J_2 \in \mathbb{C}^{n-1\times n-1}$ is in Jordan Canonical Form. If we define $\sigma(J) := \{\lambda_i | i = 1,\ldots,n\}$ and $\lambda_1 = 0$, then $\mathrm{Re}[\lambda_i] \in \mathbb{R}_{>0}$ for $i = 2,\ldots,n$. Furthermore, $R = \begin{bmatrix}\mathbf{1}/\sqrt{n} & R_2\end{bmatrix}$ and $R^{-1} = \begin{bmatrix}\mathbf{q} & Q_2^*\end{bmatrix}^*$, where $\mathbf{q}^* \in \mathbb{C}^{1\times n}$ is the normalized left eigenvector of $\lambda_1 = 0$, $R_2 \in \mathbb{C}^{n\times n-1}$, $Q_2 \in \mathbb{C}^{n-1\times n}$.

Proof: Since L is a Laplacian matrix, $L\mathbf{1} = 0$. Also, $\mathcal{G}$ having a globally reachable node implies that the zero eigenvalue has algebraic multiplicity 1 (Lemma 4 in [19]), hence $0 \notin \sigma(J_2)$. By the Gershgorin disk theorem [17], $\mathrm{Re}[\lambda_i] \in \mathbb{R}_{\geq 0}$, and λ_i is not purely imaginary $\forall i$. Therefore, $0 \notin \sigma(J_2)$ implies that $\mathrm{Re}[\lambda_i] \in \mathbb{R}_{>0}$, for $i = 2,\ldots,n$.

Since $L = RJR^{-1}$ is a similarity transform, the first column of R, which is $\mathbf{1}/\sqrt{n}$, and the first row of R^{-1} are the respective normalized right and left eigenvectors associated with $\lambda_1 = 0$. ■

Lemma 2: L_g can be decomposed as $L_g = UDU^\intercal$, where $U \in \mathbb{R}^{n\times n}$ is orthogonal, i.e., $UU^\intercal = I_n$ and

$$D = \begin{bmatrix}0 & \mathbf{0}_{n-1}^\intercal \\ \mathbf{0}_{n-1} & D_2\end{bmatrix}, \tag{14}$$

where $D_2 \in \mathbb{R}^{n-1\times n-1}$ is diagonal. If we define $\sigma(D) := \{\mu_i | i = 1,\ldots,n\}$ and $\mu_1 = 0$, then $0 \notin \{\mu_i | i = 2,\ldots,n\} = \sigma(D_2) \subseteq \mathbb{R}_{>0}$. Furthermore, $U = \begin{bmatrix}\mathbf{1}/\sqrt{n} & U_2\end{bmatrix}$, where $U_2 \in \mathbb{R}^{n\times n-1}$ has orthonormal columns.

Proof: The result follows from Lemma 1 and the fact that $\mathcal{G}^{out}$ is undirected and connected. ■

Remark 2: If $L_g = \Pi$, L_g is a projection matrix with eigenvalues 0, 1, …, 1, i.e. $D_2 = I_{n-1}$. Then, the columns of U_2 can be chosen as any orthonormal basis spanning the subspace of $\mathbb{R}^n$ that is orthogonal to $\mathbf{1}$.

Since $0 \in \sigma(L)$, by Proposition 1, the solution to the Lyapunov equation in (3) is not unique. In order to overcome this issue, we block-diagonalize the systems (10) and (11) by invoking lemmas 1 and 2 to show that the subsystem associated with the zero eigenvalue does not contribute to the $\mathcal{H}_2$ norm. The next two subsections use this approach to derive the main results of this work.

A. First Order Systems

The following theorem presents a result for the first order system in (10), which provides a means of obtaining the performance measures P and P_Π in (7) and (9).

Theorem 1: System (10) can be transformed into the block-diagonal system $\hat{G}(t)$ with subsystems $\hat{G}_1(t)$, $\hat{G}_2(t)$:

$$\dot{\hat{\mathbf{x}}} = \begin{bmatrix}\hat{A}_1 & \mathbf{0}_{n-1}^\intercal \\ \mathbf{0}_{n-1} & \hat{A}_2\end{bmatrix}\hat{\mathbf{x}} + \begin{bmatrix}\hat{B}_1 & \mathbf{0}_{n-1}^\intercal \\ \mathbf{0}_{n-1} & \hat{B}_2\end{bmatrix}\hat{\mathbf{w}} \tag{15}$$
$$\hat{\mathbf{y}} = \begin{bmatrix}\hat{C}_1 & \mathbf{0}_{n-1}^\intercal \\ \mathbf{0}_{n-1} & \hat{C}_2\end{bmatrix}\hat{\mathbf{x}},$$

through the relationship $G(t) = U\hat{G}(t)R^{-1}$, where $\hat{\mathbf{x}} := R^{-1}\mathbf{x}$ denotes the state, $\hat{\mathbf{w}} := R^{-1}\mathbf{w}$ denotes the disturbance input, and $\hat{\mathbf{y}} := U^\intercal\mathbf{y}$ denotes the output. $\hat{G}_1(t)$ is given by $\hat{A}_1 = 0$, $\hat{B}_1 = 1$, $\hat{C}_1 = 0$, and $\hat{G}_2(t)$ is given by $\hat{A}_2 = -J_2$, $\hat{B}_2 = I_{n-1}$, $\hat{C}_2 = D_2^{1/2}U_2^\intercal R_2$.

Furthermore, the squared $\mathcal{H}_2$ norm of $G(t)$ is given by:

$$\|G\|_2^2 = \mathrm{tr}\,(Q_2^*\hat{X}_2 Q_2), \tag{16}$$

where $\hat{X}_2$ is the observability Gramian of $\hat{G}_2(t)$, which is the unique solution to the Lyapunov equation:

$$J_2^*\hat{X}_2 + \hat{X}_2 J_2 = \hat{C}_2^*\hat{C}_2. \tag{17}$$

Proof: By lemmas 1 and 2, (10) can be rewritten as

$$\begin{aligned}\dot{\mathbf{x}} &= -RJR^{-1}\mathbf{x} + \mathbf{w} \\ \mathbf{y} &= UD^{1/2}U^\intercal\mathbf{x},\end{aligned}$$

which can be transformed into the system in (15) as

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= -J\hat{\mathbf{x}} + \hat{\mathbf{w}} \\ \hat{\mathbf{y}} &= D^{1/2}U^\intercal R\hat{\mathbf{x}},\end{aligned}$$

which is denoted by $\hat{G}(t)$. Using the definitions of R and U in lemmas 1 and 2, and the fact that $G(t) = U\hat{G}(t)R^{-1}$, we compute the $\mathcal{H}_2$ norm of $G(t)$. Since the state $\hat{x}_1$ associated with $\lambda_1 = 0$ is unobservable from the output $\hat{\mathbf{y}}$, $\hat{G}_1(t)$ does not contribute to the $\mathcal{H}_2$ norm and

$$\begin{aligned}\|G\|_2^2 &= \mathrm{tr}\int_0^\infty (R^{-1})^*\hat{G}(\tau)^*U^\intercal U\hat{G}(\tau)R^{-1}d\tau \\ &= \mathrm{tr}\int_0^\infty Q_2^* e^{-J_2^*\tau}\hat{C}_2^*\hat{C}_2 e^{-J_2\tau}Q_2 d\tau \\ &= \mathrm{tr}\left(Q_2^*\int_0^\infty e^{-J_2^*\tau}\hat{C}_2^*\hat{C}_2 e^{-J_2\tau}d\tau Q_2\right) \\ \|G\|_2^2 &= \mathrm{tr}\,(Q_2^*\hat{X}_2 Q_2).\end{aligned}$$

By Lemma 1, $\mathrm{Re}[\lambda_i] \in \mathbb{R}_{>0}$ for $i = 2, \ldots, n$, therefore $\hat{X}_2$ is the unique solution to (17) due to Proposition 2. ■

Now we use Theorem 1 to obtain bounds on P in (7) for the case of a normal weighted Laplacian matrix L.

Corollary 1: Consider the performance measure P in (7), and suppose that L in system (10) is normal. Then, the squared $\mathcal{H}_2$ norm of $G(t)$ is bounded from above by

$$\|G\|_2^2 \leq \frac{\sum_{i=2}^n \mu_i}{2\min_{i\geq 2}\{\mathrm{Re}\,[\lambda_i]\}}, \tag{18}$$

and bounded from below by

$$\|G\|_2^2 \geq \frac{\sum_{i=2}^n \mu_i}{2\max_{i\geq 2}\{\mathrm{Re}\,[\lambda_i]\}}. \tag{19}$$

Proof: Since L is normal, it is unitarily diagonalizable, i.e. $L = RJR^*$, and $R^{-1} = R^*$, which implies that $\mathbf{q} = \mathbf{1}$, i.e. L is weight balanced, and $Q_2 = R_2^*$ has orthonormal rows. Since we have $J_2^* = \bar{J}_2$, where $\bar{J}_2$ denotes the complex conjugate of J_2, the Lyapunov equation (17) becomes

$$\bar{J}_2\hat{X}_2 + \hat{X}_2 J_2 = \hat{C}_2^*\hat{C}_2.$$

Using the definition of $\hat{C}_2$, and taking the trace of both sides

$$\mathrm{tr}(2\,\mathrm{Re}\,[J_2]\hat{X}_2) = \mathrm{tr}(R_2^* U_2 D_2 U_2^\intercal R_2).$$

Since $U_2^\intercal R_2$ is unitary, $\mathrm{tr}(2\,\mathrm{Re}\,[J_2]\hat{X}_2) = \mathrm{tr}(D_2)$. Invoking lemmas 1 and 2, $-J_2$ is Hurwitz, and the Hermitian matrix $\hat{C}_2^*\hat{C}_2$ is positive definite. Combining these facts we have that $\hat{X}_2$ is Hermitian and positive definite. Using the trace inequality (4) in [20],

$$\frac{\mathrm{tr}(D_2)}{2\max_{i\geq 2}\{\mathrm{Re}\,[\lambda_i]\}} \leq \mathrm{tr}(\hat{X}_2) \leq \frac{\mathrm{tr}(D_2)}{2\min_{i\geq 2}\{\mathrm{Re}\,[\lambda_i]\}}. \tag{20}$$

Using the fact that $\|G\|_2^2 = \mathrm{tr}\,(\hat{X}_2 Q_2 Q_2^*) = \mathrm{tr}\,(\hat{X}_2 R_2^* R_2) = \mathrm{tr}\,(\hat{X}_2)$, we reach the desired result. ■

The rate of convergence to consensus, which is the rate at which all the states reach a consensus value, is determined by $\min_{i\geq 2}\{\mathrm{Re}\,[\lambda_i]\}$, therefore maximizing this value would result in a better rate of convergence. As the result of Corollary 1 indicates, maximizing this value would also decrease the upper bound on the $\mathcal{H}_2$ norm, resulting in a tighter performance threshold. Bounds also become tighter with a smaller value of $\max_{i\geq 2}\{\mathrm{Re}\,[\lambda_i]\} - \min_{i\geq 2}\{\mathrm{Re}\,[\lambda_i]\}$.

1) Relationship to Previous Results: We now present propositions showing that Theorem 1 reduces to previously known results under certain conditions. The result in Proposition 3 provides a first order version of Equation (13) in [8], which considers the total transient resistive losses in power networks.

Proposition 3: Consider the performance measure P in (7), and suppose that $L = L_g$ in system (10). Then, the squared $\mathcal{H}_2$ norm of $G(t)$ is given by

$$\|G\|_2^2 = \mathrm{tr}\,(\hat{X}_2) = \frac{1}{2}(n-1). \tag{21}$$

Proof: The result follows from applying Theorem 1, and using the fact that $R_2 = U_2$ and $Q_2 = U_2^\intercal$. ■

The result of Proposition 4, which was given in [5], provides a closed-form solution to P_Π for first order systems whose underlying interconnection can be represented by normal weighted Laplacian matrices.

Proposition 4: Consider the performance measure P_Π in (9), and suppose that L in system (10) is normal. Then, the squared $\mathcal{H}_2$ norm of $G_\Pi(t)$ is given by

$$\|G_\Pi\|_2^2 = \sum_{i=2}^n \frac{1}{2\,\mathrm{Re}[\lambda_i]}. \tag{22}$$

Proof: The result follows from invoking Remark 2, applying Theorem 1 and using the fact that J_2 is diagonal and $Q_2 = R_2^*$ has orthonormal rows. ■

Next, we apply a similar framework to obtain analogous results for second order systems of the form (11).

B. Second Order Systems

The second order system results are also based on the decomposition of $G(t)$ into subsystems associated with zero and non-zero eigenvalues of the weighted graph Laplacian L. Theorem 2 presents the associated result, which provides a means of obtaining the performance measures P and P_Π in (7) and (9) for second order systems of the form (11).

Theorem 2: System (11) can be transformed into the block-diagonal system $\hat{G}(t)$ with subsystems $\hat{G}_1(t)$, $\hat{G}_2(t)$:

$$\dot{\boldsymbol{\psi}} = \begin{bmatrix}\hat{A}_1 & 0_{2\times(2n-2)} \\ 0_{(2n-2)\times 2} & \hat{A}_2\end{bmatrix}\boldsymbol{\psi} + \begin{bmatrix}\hat{B}_1 & 0_{2\times(n-1)} \\ \mathbf{0}_{2n-2} & \hat{B}_2\end{bmatrix}\hat{\mathbf{w}} \tag{23}$$

$$\hat{\mathbf{y}} = \begin{bmatrix}\hat{C}_1 & \mathbf{0}_{2n-2}^\intercal \\ 0_{(n-1)\times 2} & \hat{C}_2\end{bmatrix}\boldsymbol{\psi}$$

through the relationship $G(t) = U\hat{G}(t)R^{-1}$, where $\boldsymbol{\psi} := T\begin{bmatrix}\hat{\mathbf{x}} \\ \hat{\mathbf{v}}\end{bmatrix}$ denotes the state with $\hat{\mathbf{x}} := R^{-1}\mathbf{x}$ and $\hat{\mathbf{v}} := R^{-1}\mathbf{v}$, $\hat{\mathbf{w}} := R^{-1}\mathbf{w}$ denotes the disturbance input, $\hat{\mathbf{y}} := U^\intercal\mathbf{y}$ denotes the output, and T denotes the permutation matrix

$$T := \begin{bmatrix}e_1 & e_{n+1} & e_3 & \ldots & e_n & e_2 & e_{n+2} & \ldots & e_{2n}\end{bmatrix}, \tag{24}$$

with the standard basis vectors e_i for $\mathbb{R}^{2n}$, for $i = 1, \ldots, 2n$.

The subsystem $\hat{G}_1(t)$ is given by $\hat{A}_1 = \begin{bmatrix}0 & 1 \\ -k_p & -k_d\end{bmatrix}$, $\hat{B}_1 = \begin{bmatrix}0 & 1\end{bmatrix}^\intercal$, and $\hat{C}_1 = \begin{bmatrix}0 & 0\end{bmatrix}$. The subsystem $\hat{G}_2(t)$

is given by $\hat{A}_2$[1], $\hat{B}_2 = \begin{bmatrix} 0_{(n-1)\times(n-1)} & I_{n-1} \end{bmatrix}^\intercal$, and $\hat{C}_2 = \begin{bmatrix} D_2^{1/2} U_2^\intercal R_2 T_2 & 0_{n-1\times n-1} \end{bmatrix}$, where T_2 denotes the permutation matrix $T_2 = \begin{bmatrix} \alpha_2 & \alpha_3 & \dots & \alpha_{n-1} & \alpha_1 \end{bmatrix}$, and α_i denotes the i^{th} standard basis vector for $\mathbb{R}^{n-1}$.

Furthermore, if the solutions to

$$\eta^2 + (k_d + \gamma_d \lambda_i)\eta + k_p + \gamma_p \lambda_i = 0, \quad i = 2, \dots, n.$$

are such that $\mathrm{Re}[\eta] < 0$, then $\hat{A}_2$ is Hurwitz, and the squared $\mathcal{H}_2$ norm of $G(t)$ is given by

$$\|G\|_2^2 = \mathrm{tr}\,(Q_2^* \hat{B}_2^\intercal \hat{X}_2 \hat{B}_2 Q_2). \tag{25}$$

Here $\hat{X}_2$ is the observability Gramian of $\hat{G}_2(t)$, which is the unique solution to the Lyapunov equation:

$$\hat{A}_2^* \hat{X}_2 + \hat{X}_2 \hat{A}_2 = -\hat{C}_2^* \hat{C}_2. \tag{26}$$

Proof: Invoking lemmas 1 and 2, and using the definitions of $\hat{\mathbf{x}}$, $\hat{\mathbf{v}}$, $\hat{\mathbf{w}}$ and $\hat{\mathbf{y}}$, we re-write (11) as

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}} \\ \dot{\hat{\mathbf{v}}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -k_p I - \gamma_p J & -k_d I - \gamma_d J \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{v}} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \hat{\mathbf{w}}$$
$$\hat{\mathbf{y}} = \begin{bmatrix} D^{1/2} U^\intercal R & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{v}} \end{bmatrix}.$$

Using (24) and the definition of $\boldsymbol{\psi}$, we obtain

$$\dot{\boldsymbol{\psi}} = T \begin{bmatrix} 0 & I \\ -k_p I - \gamma_p J & -k_d I - \gamma_d J \end{bmatrix} T^\intercal \boldsymbol{\psi} + T \begin{bmatrix} 0 \\ I \end{bmatrix} \hat{\mathbf{w}}$$
$$\hat{\mathbf{y}} = \begin{bmatrix} \begin{bmatrix} 0 & \mathbf{0}_{n-1}^\intercal \\ \mathbf{0}_{n-1} & D_2^{1/2} U_2^\intercal R_2 \end{bmatrix} & 0 \end{bmatrix} T^\intercal \boldsymbol{\psi}.$$

Evaluating this expression leads to the system (23), i.e. $\hat{G}(t)$.

Applying the definition of the impulse response function results in $G(t) = U\hat{G}(t)R^{-1}$. Since the states of $\hat{G}_1(t)$, i.e. the system associated with $\lambda_1 = 0$, are unobservable from the output $\hat{\mathbf{y}}$, $\hat{G}_1(t)$ does not contribute to the $\mathcal{H}_2$ norm:

$$\begin{aligned} \|G\|_2^2 &= \mathrm{tr} \int_0^\infty (R^{-1})^* \hat{G}(\tau)^* U^\intercal U \hat{G}(\tau) R^{-1} d\tau \\ &= \mathrm{tr} \int_0^\infty Q_2^* \hat{B}_2^\intercal e^{-\hat{A}_2^* \tau} \hat{C}_2^* \hat{C}_2 e^{-\hat{A}_2 \tau} \hat{B}_2 Q_2 d\tau \\ &= \mathrm{tr} \left(Q_2^* \hat{B}_2^\intercal \int_0^\infty e^{-\hat{A}_2^* \tau} \hat{C}_2^* \hat{C}_2 e^{-\hat{A}_2 \tau} d\tau \hat{B}_2 Q_2 \right) \\ \|G\|_2^2 &= \mathrm{tr}\,(Q_2^* \hat{B}_2^\intercal \hat{X}_2 \hat{B}_2 Q_2). \end{aligned}$$

It remains to show that $\hat{X}_2$, the observability Gramian of $\hat{G}_2(t)$, is the unique solution to the Lyapunov equation (26). The eigenvalues of $\hat{A}$ are the solutions to

$$\eta^2 + (k_d + \gamma_d \lambda_i)\eta + k_p + \gamma_p \lambda_i = 0, \quad i = 1, \dots, n. \tag{27}$$

where $\lambda_i \in \sigma(L) = \sigma(J)$. For $i = 1$, $\lambda_1 = 0$, which determines the characteristic equation of $\hat{A}_1$. Then, for $i = 2, \dots, n$, we have $n - 1$ equations characterizing $\sigma(\hat{A}_2)$, which by assumption satisfy $\mathrm{Re}[\eta] < 0$, so $\hat{A}_2$ is Hurwitz. Then by Proposition 2, $\hat{X}_2$ is unique. ■

[1] $\hat{A}_2$ is not given explicitly for the sake of brevity.

Theorem 2 shows that the $\mathcal{H}_2$ norm of system (11) is determined by the subsystem $\hat{G}_2(t)$ associated with the non-zero eigenvalues of L.

Remark 3: The change of basis applied in Theorem 2 is common in the literature and it is known that $\hat{A}_2$ characterizes convergence in consensus networks [1], [21]. Theorem 2 shows that the associated subsystem $\hat{G}_2$ also characterizes the performance of the full system.

Using Theorem 2, we now study the measures P in (7) and P_Π in (9) for the special case of normal L.

The following lemma, which will be useful for the analyses that follow, states a condition on k_p, k_d, γ_p and γ_d that is implied by $\hat{A}_2$ being Hurwitz.

Lemma 3: Let $\lambda \in \mathbb{C}$ such that $\mathrm{Re}[\lambda] > 0$. If the roots of $\eta^2 + (k_d + \gamma_d \lambda)\eta + k_p + \gamma_p \lambda = 0$ are such that $\mathrm{Re}[\eta] < 0$, then k_p, γ_p or k_d, γ_d are not both zero.

Proof: If $k_p = \gamma_p = 0$, then $\eta = 0$. If $k_d = \gamma_d = 0$, we have $\eta^2 + c = 0$, where $c = k_p + \gamma_p \lambda$. Since $k_p, \gamma_p \geq 0$, and $\mathrm{Re}[\lambda] > 0$ we can write c as $c = a + ib$ with $a \geq 0$, and $b \in \mathbb{R}$. $b = 0$ gives $\mathrm{Re}[\eta] = 0$, so we assume $b \neq 0$. We can also write $-c$ in phasor form as $-c = me^{j\theta}$, $m \geq 0$, and $\theta \in \mathbb{R}$, so that $\eta^2 = me^{j\theta}$, which gives

$$\eta = \pm\sqrt{m} e^{j\theta/2} = \pm\sqrt{m}(\cos(\theta/2) + j\sin(\theta/2)).$$

Since $b \neq 0$, the magnitude of c is non-zero, i.e. $m \neq 0$, then it cannot hold that $\mathrm{Re}[\eta] < 0$. ■

Corollary 2: Consider the performance measure P_Π in (9). Suppose L in system (11) is normal and the solutions to

$$\eta^2 + (k_d + \gamma_d \lambda_i)\eta + k_p + \gamma_p \lambda_i = 0, \quad i = 2, \dots, n$$

are such that $\mathrm{Re}[\eta] < 0$. Then, the squared $\mathcal{H}_2$ norm of $G_\Pi(t)$ is given by

$$\|G_\Pi\|_2^2 = \sum_{i=2}^n \frac{\phi_i}{2(\alpha_i \phi_i^2 + \beta_i \xi_i \phi_i - \beta_i^2)}, \tag{28}$$

where $\alpha_i = k_p + \gamma_p \mathrm{Re}[\lambda_i]$, $\phi_i = k_d + \gamma_d \mathrm{Re}[\lambda_i]$, $\beta_i = \gamma_p \mathrm{Im}[\lambda_i]$ and $\xi_i = \gamma_d \mathrm{Im}[\lambda_i]$.

Proof: Since L is normal, J in (13) is diagonal. By invoking Remark 2, $D_2 = I_{n-1}$, and we can choose $U = R$, so that $\hat{C}_2 = \begin{bmatrix} I_{n-1} & 0_{n-1\times n-1} \end{bmatrix}$.

Defining the states $\tilde{\boldsymbol{\psi}} := \tilde{T} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{v}} \end{bmatrix}$, where $\tilde{T}$ denotes

$$\tilde{T} := \begin{bmatrix} e_1 & e_{n+1} & e_2 & e_{n+2} & \dots & e_{n-1} & e_{2n-1} & e_n & e_{2n} \end{bmatrix}^\intercal,$$

and e_i are the standard basis vectors for $\mathbb{R}^{2n}$, for $i = 1, \dots, 2n$, we can block diagonalize the system into n subsystems, each of which is given by

$$\begin{bmatrix} \dot{\tilde{\psi}}_{2i-1} \\ \dot{\tilde{\psi}}_{2i} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_p - \gamma_p \lambda_i & -k_d - \gamma_d \lambda_i \end{bmatrix} \begin{bmatrix} \tilde{\psi}_{2i-1} \\ \tilde{\psi}_{2i} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{w}_i$$
$$\hat{y}_i = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\psi}_{2i-1} \\ \tilde{\psi}_{2i} \end{bmatrix}, \tag{29}$$

$\forall i \in \{1, \dots, n\}$. Note that, the IO system above corresponds to $\hat{G}_1(t)$ for $i = 1$, and $\hat{G}_2(t)$ for $i = 2, \dots, n$. $\hat{X}_2$ is also block diagonal per (26), and each of its $n-1$ blocks is given by the observability Gramian of (29), which we will denote by $\hat{X}_2^{(i)}$ $\forall i \in \{2, \ldots, n\}$.

Using the properties of the trace, (25) gives $\|G_\Pi\|_2^2 = \operatorname{tr}(\hat{B}_2^\intercal \hat{X}_2 \hat{B}_2 Q_2 Q_2^*) = \operatorname{tr}(\hat{B}_2^\intercal \hat{X}_2 \hat{B}_2 U_2^\intercal U_2) = \operatorname{tr}(\hat{B}_2^\intercal \hat{X}_2 \hat{B}_2) = \sum_{i=2}^n \begin{bmatrix} 0 & 1 \end{bmatrix} \hat{X}_2^{(i)} \begin{bmatrix} 0 & 1 \end{bmatrix}^\intercal$. Invoking Lemma 3 and solving the Lyapunov equation of (29):

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \hat{X}_2^{(i)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\phi_i}{2(\alpha_i \phi_i^2 + \beta_i \xi_i \phi_i - \beta_i^2)}.$$

Summing the above over $i = 2, \ldots, n$ gives the result. ∎

In contrast to the result for the first order system in Proposition 4, the $\mathcal{H}_2$ norm of (9) is a function of higher order terms that depend both on $\operatorname{Re}[\lambda_i]$ and $\operatorname{Im}[\lambda_i]$.

Remark 4: The $\mathcal{H}_2$ norm in (28) is independent of $\operatorname{Im}[\lambda_i]$ if and only if $\beta_i \xi_i \phi_i - \beta_i^2 = 0$. This holds if $\operatorname{Im}[\lambda_i] = 0$ or L is symmetric or $\gamma_p = 0$.

If $\beta_i \xi_i \phi_i - \beta_i^2 = 0$, Equation (28) reduces to

$$\|G_\Pi\|_2^2 = \sum_{i=2}^n \frac{1}{2(k_p + \gamma_p \operatorname{Re}[\lambda_i])(k_d + \gamma_d \operatorname{Re}[\lambda_i])}. \quad (30)$$

Depending on the values of k_p, k_d, γ_p and γ_d in (30), the denominator in (28) can be quadratic in $\operatorname{Re}[\lambda_i]$, which could indicate a smaller $\mathcal{H}_2$ norm for sufficiently large $\operatorname{Re}[\lambda_i]$, hence better performance compared to the first order system.

The following Corollary presents a special case that demonstrates the effect of the imaginary parts of the weighted Laplacian matrix eigenvalues on the computation of the $\mathcal{H}_2$ norm for second order systems.

Corollary 3: Consider the performance measure P_Π in (9). Let G_Π and G'_Π be two systems with realization (11), respective communication graphs $\mathcal{G}$ and $\mathcal{G}'$ and associated weighted Laplacian matrices L and L'. Suppose that L and L' are normal, $\sigma(L') = \{\operatorname{Re}[\lambda_i] | \lambda_i \in \sigma(L), i = 1, \ldots, n\}$ and the solutions to

$$\eta^2 + (k_d + \gamma_d \lambda_i)\eta + k_p + \gamma_p \lambda_i = 0, \quad i = 2, \ldots, n$$

are such that $\operatorname{Re}[\eta] < 0$. If

$$\gamma_d (k_d + \gamma_d \operatorname{Re}[\lambda_i]) - \gamma_p \geq 0, \quad i = 2, \ldots, n, \quad (31)$$

then $\|G_\Pi\|_2^2 \leq \|G'_\Pi\|_2^2$.

Proof: Invoking Remark 4, $\|G'_\Pi\|_2^2$ is given by (30). Condition (31) implies that $\beta_i \xi_i \phi_i - \beta_i^2 \geq 0$, therefore

$$\frac{\phi_i}{2(\alpha_i \phi_i^2 + \beta_i \xi_i \phi_i - \beta_i^2)} \leq \frac{1}{2\alpha_i \phi_i}, \quad i = 2, \ldots, n. \quad (32)$$

Summation of the inequalities in (32) gives the result. ∎

Corollary 3 shows one can achieve a more coherent system through an appropriate choice of control gains if the weighted Laplacian eigenvalues have non-zero imaginary parts. This idea will be studied further in Section 4.

1) Relationship to Previous Results: We now present propositions which show that Theorem 2 can also be used to obtain some previous results from the literature.

Propositions 5 and 6 show that under certain conditions, Theorem 2 leads to the results associated with linear oscillator networks over undirected graphs [7], [8].

Proposition 5: Consider the performance measure P in (7). Suppose $L = L_g$ in system (11) and the solutions to

$$\eta^2 + (k_d + \gamma_d \lambda_i)\eta + k_p + \gamma_p \lambda_i = 0, \quad i = 2, \ldots, n.$$

are such that $\operatorname{Re}[\eta] < 0$. Then, the squared $\mathcal{H}_2$ norm of $G(t)$ is given by

$$\|G\|_2^2 = \sum_{i=2}^n \frac{\lambda_i}{2(k_p + \gamma_p \lambda_i)(k_d + \gamma_d \lambda_i)}. \quad (33)$$

Proof: The result follows from the block-diagonalization framework used in the proof of Corollary 2 and the fact that $J = D$ and $R = U$. ∎

Equation (34) in Proposition 6 quantifies the transient resistive losses in power transmission networks [8].

Proposition 6: Consider the performance measure P in (7). Suppose that $L = L_g$ in system (11). If $k_p = \gamma_d = 0$, and $k_d, \gamma_p > 0$, the squared $\mathcal{H}_2$ norm of $G(t)$ is given by

$$\|G\|_2^2 = \frac{1}{2 k_d \gamma_p}(n-1). \quad (34)$$

If $k_p = k_d = 0$, and $\gamma_p, \gamma_d > 0$, the squared $\mathcal{H}_2$ norm of $G(t)$ is given by

$$\|G\|_2^2 = \frac{1}{2\gamma_p \gamma_d} \sum_{i=2}^n \frac{1}{\lambda_i}. \quad (35)$$

Proof: The result follows directly from (33). ∎

Next, we present numerical examples that provide further insight into the analytical results.

IV. NUMERICAL EXAMPLES

First, we simulate the systems given by (10) and (11) connected over a directed cycle graph with unit edge weights and an undirected cycle graph with edge weights $\frac{1}{2}$. The eigenvalues of L associated with the latter are the real parts of the eigenvalues of the former, which are given by $1 + e^{j\pi(1 - \frac{2k}{n})}, k = 0, \ldots, n-1$ [5]. We study the performance measure P_Π as a special case of P, since exact solutions to this $\mathcal{H}_2$ norm can be computed using Proposition 4 and Corollary 2. In Fig. 1(a), we examine how the bounds on the $\mathcal{H}_2$ norm for the first order system given in Corollary 1, scale compare to the exact solutions. The bounds become looser as the network size increases. Since the $\mathcal{H}_2$ norm only depends on the real parts of the weighted Laplacian matrix eigenvalues, the exact solutions and the bounds are equal for directed and undirected cycle graphs. In Fig. 1(b), we compare the $\mathcal{H}_2$ norm values for second order systems with directed and undirected cycle graphs, which are computed using Corollary 2. For $k_p = k_d = \gamma_p = \gamma_d = 1$, Corollary 3 indicates that the undirected cycle graph is an upper bound for the directed cycle graph. This result is confirmed in Fig. 1(b). Also note that the second order system has a smaller $\mathcal{H}_2$ norm compared to the first order system since the denominator in (30) is quadratic in $\operatorname{Re}[\lambda_i]$.

In Fig. 2, we study the performance measure P for the first and the second order systems over line and complete graphs. For the second order system, we focus on the cases

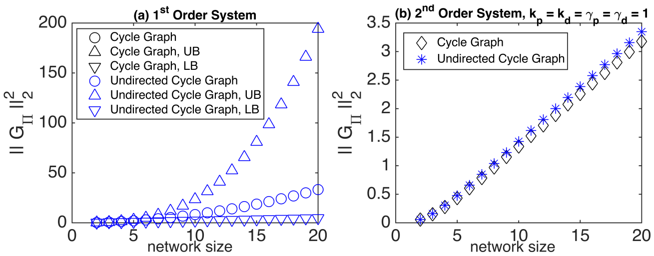

Fig. 1. Performance measure P_Π in (9) for directed and undirected cycle graphs for (a) first, and (b) second order systems (10) and (11). Upper and lower bounds are also provided for the first order system.

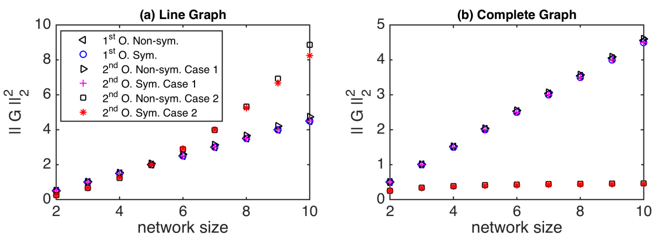

Fig. 2. Performance measure P in (7) for systems with non-symmetric (perturbed), and symmetric Laplacian matrices for (a) line, and (b) complete graphs with unit edge weights. For cases 1 and 2, $k_d = 1, \gamma_p = 1$ and $\gamma_p = \gamma_d = 1$, respectively, and the other parameters are zero.

given in Proposition 6. In each example, we compare the $\mathcal{H}_2$ norms of the systems with symmetric and non-symmetric Laplacian matrices. For the symmetric case, we set $L = L_g$, and $g_{ik} = 1, \forall (i,k) \in \mathcal{E}$. For the non-symmetric case, we define a perturbation to the edge weights $\Delta g_{ik} = 0.2$, so that the edge weights become $g_{ik} + \Delta g_{ik}$, if $i > k$, and $g_{ik} - \Delta g_{ik}$, if $i < k$ $\forall (i,k) \in \mathcal{E}$. Then, the resulting weighted Laplacian matrix is associated with a digraph for which $(i,k) \in \mathcal{E} \iff (k,i) \in \mathcal{E}$, $\forall i,k \in \mathcal{N}$. In this setting, every node is globally reachable, so the graph is strongly connected. Fig. 2 shows that the $\mathcal{H}_2$ norms of the systems with non-symmetric graph Laplacians are greater than or equal to those with symmetric graph Laplacians for systems connected over line graphs with a sufficiently large number of nodes. In all cases given for the complete graph in Fig. 2, the $\mathcal{H}_2$ norms of the non-symmetric systems are greater than or equal to that of their symmetric counterparts. Further analytical investigation of the conditions under which the $\mathcal{H}_2$ norm of a symmetric system can serve as a lower bound for that of the corresponding perturbed non-symmetric system is a direction for future work.

V. CONCLUSIONS AND FUTURE WORK

We have studied the performance of first and second order linear networked dynamical systems over digraphs with a globally reachable node. Our main results focus on systems whose underlying interconnection can be represented by normal weighted Laplacian matrices. For the second order system, we obtained a closed-form solution to the $\mathcal{H}_2$ norm that quantifies the deviation of the states from the average. We also obtained bounds on the $\mathcal{H}_2$ norm for the first order system, when the weighted graph Laplacian matrix associated with the output graph is symmetric. Our numerical results show that in certain cases the $\mathcal{H}_2$ norm of a symmetric system serves as a lower bound for that of the corresponding perturbed non-symmetric system. The conditions under which this holds will be investigated analytically as future work. Extensions to systems with higher order linear dynamics is another direction of continuing work.

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