Augmented Consensus Algorithm for Discrete-time Dynamical Systems

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Abstract: We propose a novel state estimation algorithm for consensus dynamics subject to measurement error. We first demonstrate that with properly tuned parameters, our algorithm attains the same equilibrium value that would be attained using the traditional algorithm based on local state feedback (nominal consensus). We then show that our approach improves consensus performance in a particular class of problems by reducing the state error (i.e., the difference between the agent states and the consensus value). A numerical example compares the performance of the distributed algorithm we propose to that of the traditional local feedback scheme. The results show that the proposed algorithm significantly reduces the state error.

Keywords: Distributed Control, Networked Dynamical Systems, Discrete-time Systems, Consensus Algorithm, Measurement Error

1. INTRODUCTION

The need to coordinate a network of dynamical systems to attain a common state arises in a number of applications. Examples of these so-called consensus problems include frequency regulation in power systems (Paganini and Mallada, 2017), and time synchronization of computer systems (Mallada et al., 2015). Platoons of vehicles traveling in formation at a common velocity represent another important class of consensus problems, see e.g., Wang et al. (2012).

Consensus problems for linear dynamical systems are typically defined in terms of static local state feedback laws in which connected neighboring agents share their state information, see e.g., Jadbabaie et al. (2003), Olfati-Saber and Murray (2004), and Ren et al. (2005a). This local feedback law, which we refer to as the Traditional Consensus Algorithm (TCA), leads to a distributed algorithm that enables each agent's state to asymptotically converge to an average value under ideal conditions, i.e. with no external disturbances or state errors, see e.g Bullo (2017).

Given that the network attains consensus, it is important to evaluate the relative quality of the algorithm. This consensus performance can be quantified in a number of ways, including the time to reach consensus, the steady-state error of the state values, and the variance of fluctuations around the equilibrium, i.e., the consensus state, in the presence of white stochastic noise. In fact, it is known that the TCA drives the expected value of the states to consensus in the presence of white stochastic additive noise, but the resulting state fluctuations can lead to the states drifting away from their consensus value (Xiao et al., 2007). In the steady-state error analysis context, the introduction of global information in the form of absolute state measurements relative to a common datum has been shown to be critical in maintaining finite fluctuations (Bamieh et al., 2012). Ji et al. (2018) aimed to quantify this effect by studying how the relative proportion of this global versus local feedback affects the steady-state error in vehicle platoons. That work supports previous findings indicating that even that a small amount global feedback is enough to maintain finite steady-state error. The introduction of a leader agent, see e.g., Lin et al. (2012), and Fitch and Leonard (2016), provides another type of global information that has been shown to improve performance in vehicular platoons (Lin et al., 2012).

Consensus performance for networks subject to noise can also be improved by altering the network interconnection structure see e.g., Ellens et al. (2011), Summers et al. (2015), Abbas and Egerstedt (2012), and Jadbabaie and Olshevsky (2019). For example, Ellens et al. (2011) show that the state errors for agents subject to additive noise is smaller for systems connected over a complete graph versus a line graph. Sarkar et al. (2018) demonstrate that graph structure can also affect the convergence rate.

Performance has also been improved by optimizing the graph edge weights given a fixed interconnection structure. Xiao et al. (2007) used this approach to minimize the state variance from its equilibrium. Similarly, Ghosh et al. (2008) optimized the steady-state variance of the distance between adjacent nodes.

Changes to the graph structure or edge weighs can be computationally intensive or difficult to implement in practice. In engineering applications, the network structure or graph weights may be predetermined by physical constraints or costly to change. For example, the structure of a power system network is typically fixed by transmission lines and associated infrastructure, with topological changes limited to contingency events. Changing the power system graph edge weights requires replacing wires, which may be infeasible in practice.

In this work, we instead aim to improve consensus performance in systems subject to measurement errors through a state estimation algorithm that acts as a consensus filter. Our approach differs from Kalman filter based algorithms that rely on measurement based iterative updates of the state error co-variance to obtain state estimates, see e.g., Ren et al. (2005b), Alighanbari and How (2006), Olfati-Saber (2007), and Li et al. (2016). We instead focus on systems subject to measurement errors, and use the state estimation to construct a feedback law that drives the expected value of the states to consensus. We refer to our approach as an augmented consensus algorithm (ACA). The ACA provides a distributed feedback law, and is therefore also distinguished from typical Jacobi based methods, e.g. Barooah and Hespanha (2005), in that it does not require global information.

The introduction of state estimation alters the dynamics of the system, which can result in a different equilibrium value (consensus state); a problem common to the approach of adding global feedback. Our first result characterizes the conditions under which the ACA attains the same consensus state (equilibrium) as the TCA. We then focus on the known problem of the buildup measurement errors that can result in unbounded state errors in systems controlled by TCA. We provide conditions under which the ACA reduces this error buildup as well as the amplitude of the fluctuations around the consensus state. In other words, we provide a distributed consensus algorithm for a particular class of network of linear dynamical systems subject to measurement errors. Our algorithm improves performance without requiring global information (e.g. absolute feedback, a leader or the system state error co-variance), centralized coordination, or topological changes. Numerical results support our analysis and show that our algorithm leads to smaller state errors than the TCA.

The reminder of this paper is organized as follows. In section 2, we introduce the problem setting and outline the problem of measure error accumulation in systems controlled by the TCA. Section 3 details the steps of the ACA. In section 4, we analytically study the behavior of a network of linear dynamical systems subject to measurement error operating under the ACA. Conditions under which the expected value of the states reach the same equilibrium value as the TCA (nominal consensus) but with reduced state errors are provided. Numerical analysis and conclusions are provided in sections 5 and 6, respectively.

2. BACKGROUND AND PROBLEM SETTING

In this section, we first introduce the mathematical notation and terminology used throughout this paper. We then describe the behavior of a network of linear dynamical systems subject to measurement errors updated according to the TCA. In particular, we show how measurement errors can lead to unbounded state error in order to motivate the ACA proposed in this work.

2.1 Notations

We use $\mathbb{E}()$ to denote the expected value, and $\mathcal{N}\{\mu, \sigma\}$ to denote a normal distribution with mean μ and standard deviation σ. $\mathbf{0}_{n\times n} \in \mathbb{R}^{n\times n}$ denotes a matrix with all elements equal to zero, and $I_{n\times n} \in \mathbb{R}^{n\times n}$ indicates an $n \times n$ identity matrix. $\mathbf{1}_n \in \mathbb{R}^n$ is a column vector with all elements equal to 1 and span$\{\mathbf{1}_n\}$ indicates the span of $\mathbf{1}_n$. $\mathbf{0}_n \in \mathbb{R}^n$ is a column vector of zeros.

Given a matrix $A \in \mathbb{R}^{n\times n}$, A^{-1} is the inverse of A such that $AA^{-1} = I_{n\times n}$. A^T denotes the transpose of A. null$\{A\}$ denotes the nullspace of the matrix A. $A > 0$ indicates that A is positive definite. We use $\rho(A)$ to denote the spectral radius of A, and $\lambda(A)$ to denote an eigenvalue of A.

Definition 1. A non-negative matrix $A \in \mathbb{R}^n$ with unit row sum, i.e., $\sum_{j=1}^{n} a_{ij} = 1$ is said to be a row-stochastic matrix, see e.g., Horn and Johnson (1986).

A weighted digraph is denoted by a triplet $\mathcal{G} = (\mathcal{N}, \mathcal{E}, W)$, where $\mathcal{N}$ is the set of nodes and $\mathcal{E}$ is a set of ordered pairs (i, j) of nodes $i, j \in N$ called edges. W is a set of nonnegative weights associated with each ordered node pair; for the ordered pair $(i, j) \in \mathcal{E}$, $w_{(i,j)} > 0$ is the directed edge weight associated with the ordered node pair. If $(i, j) \notin \mathcal{E}$, then $w_{(i,j)} = 0$.

2.2 Traditional Consensus Algorithm

Given a networked dynamical system with underlying graph $G(\mathcal{N}, \mathcal{E}, W)$, the dynamics are given by,

$$x^k = x^{k-1} + u^{k-1}. \tag{1}$$

Here $x^k = [x_1^k, \cdots, x_n^k]^T \in \mathbb{R}^n$ is the vector of states x_i^k, where $i \in \mathbb{N}^+$ represents the i^{th} node, $k \in \mathbb{N}^+$ is the time step and $u^k = [u_1^k, \cdots, u_n^k]^T \in \mathbb{R}^n$ is the corresponding vector of control inputs.

If we consider a static control strategy, $u^k = Fx^k$, then the discrete system (1) can be represented by,

$$x^k = Ax^{k-1}, \tag{2}$$

where $A = I + F \in \mathbb{R}^{n\times n}$.

Definition 2. We refer to the algorithm (2) as the Traditional Consensus Algorithm (TCA) if A is a row-stochastic matrix with a simple eigenvalue [1] equal to 1 and all other eigenvalues $\lambda^*(A)$ satisfy $|\lambda^*(A)| < 1$. The consensus state (equilibrium) of such a system is given by,

$$\lim_{k\to+\infty} x(k) = \lim_{k\to+\infty} A^k x_0 = w^T x_0 \mathbf{1}_n, \tag{3}$$

where w is the left eigenvector of A associated with the simple eigenvalue 1 (Bullo, 2017).

Remark 3. It is easy to verify that the static feedback matrix F also has a simple zero eigenvalue, and w in (3) is its corresponding left eigenvector using Definition 2. All eigenvalues $\lambda^*(F)$ of F other than 0 have negative real parts, i.e., $Re(\lambda^*(F)) < 0$.

[1] Recall that an eigenvalue is simple if and only if its algebraic and geometric multiplicities are 1s.

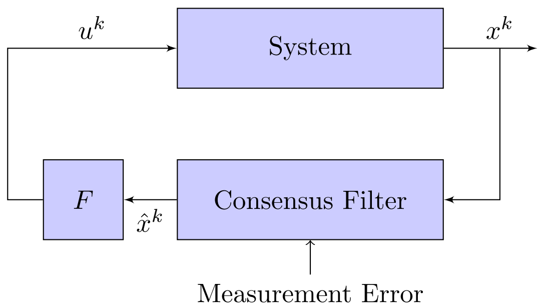

Fig. 1. Augmented Consensus Algorithm Block-diagram.

Definition 4. (*Nominal Consensus*). We refer to consensus state attained by the TCA in equation (3) as nominal consensus.

We next describe how measurement error affects the behavior of the TCA. If we assume additive measurement error v^k, then the noisy state dynamics are given by

$$x^k = x^{k-1} + F(x^{k-1} + v^k), \tag{4}$$

and the state at step k is given by,

$$x^k = A^k x_0 + \sum_{i=1}^{k} A^{k-i} F v^i. \tag{5}$$

If the measurement error is white noise, i.e., $\mathbb{E}(v^k) = 0$, then the expected value of the states are given by

$$\lim_{k\to+\infty} \mathbb{E}(x^k) = A^k x_0 = w^T x_0 \mathbf{1}_n,$$

i.e., the expected value of the system states reaches the nominal consensus value. Such behavior is referred to as consensus in the average sense.

However, equation (5) also indicates that the measurement errors accumulate through the system evolution. Hence the state may become unbounded or fail to reach consensus, e.g., fluctuate around consensus, as $k \to \infty$, unless $\sum_{i=0}^{k} A^{k-i} F v^i = 0$.

In the next section, we propose an augmented consensus algorithm (ACA) that aims to eliminate this build-up of errors and thereby improve the performance of the noisy consensus dynamics.

3. AUGMENTED CONSENSUS ALGORITHM

The Augmented Consensus Algorithm (ACA) comprises a three-step consensus filter and associated feedback law, as depicted in Fig. 1. In particular, at time step k, the three parts of the algorithm are given by

$$\text{Measurement:} \quad z^k = Hx^k + v^k, \tag{6a}$$

$$\text{Prediction:} \quad \hat{x}^{k|k-1} = \hat{x}^{k-1} + F\hat{x}^{k-1}, \tag{6b}$$

$$\text{Correction:} \quad \hat{x}^k = \operatorname{argmin} J(\bullet)\big|_{(\hat{x}^{k|k-1}, z^k)}, \tag{6c}$$

where $z^k \in \mathbb{R}^n$ and $v^k \in \mathbb{R}^n$ are the respective measurement and measurement error at step k. $H \in \mathbb{R}^{n\times n}$ is the projection of x^k onto z^k, i.e. $H = I$ when the measurements correspond to the states.

In (6b), $\hat{x}^{k|k-1}$ is the prediction of the state at step k based on the estimate of the previous state $\hat{x}^{k-1}$, and F is the same static feedback matrix as in equation (2).

The Correction step (6c) involves a quadratic optimization problem with objective function $J(\bullet)$ that is constrained by the dynamics in (6a) and (6b).

Specifically, the optimization problem in the correction step for linear consensus dynamics with measurement errors is defined as,

$$\begin{aligned} \min \quad & J(\hat{x}^k) = \frac{1}{2}(\hat{x}^k - \hat{x}^{k|k-1})^T(\hat{x}^k - \hat{x}^{k|k-1}) \\ & + \frac{1}{2}(z^k - H\hat{x}^k)^T R^{-1}(z^k - H\hat{x}^k), \\ \text{s.t.} \quad & \text{(6a), (6b)}, \end{aligned} \tag{7}$$

where $R = R^T > 0$ is a system parameter that defines the relative weighting of the filter versus the measurement error.

Remark 5. Note that this algorithm can be generalized to deal with a larger class of system errors by redefining the objective function in the optimization problem (7).

Finally, the estimation is used as feedback,

$$\text{Feedback:} \quad u^k = F\hat{x}^k, \tag{8}$$

where F is the same as in the TCA (2).

The solution to the optimization problem (7) is given by,

$$\hat{x}^k = (I + H^T R^{-1} H)^{-1} A\hat{x}^{k-1} + H^T R^{-1} z^k. \tag{9}$$

Grouping the dynamics of the real and estimated state we obtain,

$$\begin{bmatrix} x^k \\ \hat{x}^k \end{bmatrix} = \begin{bmatrix} I & F \\ 0 & (I+Q)^{-1}A \end{bmatrix} \begin{bmatrix} x^{k-1} \\ \hat{x}^{k-1} \end{bmatrix} + \begin{bmatrix} 0_{n\times n} \\ H^T R^{-1} \end{bmatrix} z^k,$$

where $Q = H^T R^{-1} H$ and $A = I + F$. The full dynamics of the real and estimated states with respect to measurement error can be represented by,

$$\begin{bmatrix} x^k \\ \hat{x}^k \end{bmatrix} = \hat{A} \begin{bmatrix} x^{k-1} \\ \hat{x}^{k-1} \end{bmatrix} + \begin{bmatrix} 0 \\ H^T R^{-1} \end{bmatrix} v^k \tag{10}$$

where

$$\hat{A} = \begin{bmatrix} I & F \\ Q & QF + (I+Q)^{-1}A \end{bmatrix}$$

is the new state matrix. The real and estimated states at step k can then be written as

$$\begin{bmatrix} x^k \\ \hat{x}^k \end{bmatrix} = (\hat{A})^k \begin{bmatrix} x_0 \\ \hat{x}_0 \end{bmatrix} + \sum_{i=1}^{k} (\hat{A})^{k-i} \begin{bmatrix} 0 \\ H^T R^{-1} \end{bmatrix} v^i \tag{11}$$

Equation (11) indicates that at any time instant $k > 1$, the measurement errors accumulate in both the real and estimated states. In the next section, we provide analytical results that provide conditions under which the ACA attenuates this accumulated noise and filters out the measurement errors leading to an overall reduction in the state errors.

4. PERFORMANCE ANALYSIS

As expected the introduction of state estimation alters the dynamics of the system. In this section we first show that the ACA reaches some consensus state in the average sense as defined in section 2. We then demonstrate that the objective function defined in (7) enables the ACA to attain nominal consensus in expectation through judicious selection of the quadratic parameters. We then compare the performance of the TCA and ACA.

In all of the analysis below we employ the following assumption regarding the initial state.

Assumption 1. The initial state is accurate, i.e., $x_0 = \hat{x}_0$.

4.1 Consensus in the Mean Sense

Consider measurement error with zero mean, i.e., $\mathbb{E}(v^k) = 0$. The dynamics of the expected value of the states are given by,

$$\mathbb{E}\left(\begin{bmatrix} x^k \\ \hat{x}^k \end{bmatrix}\right) = \hat{A}\mathbb{E}\left(\begin{bmatrix} x^{k-1} \\ \hat{x}^{k-1} \end{bmatrix}\right).$$

Theorem 6. Given a networked system who updates according to the ACA, if the state matrix $\hat{A}$ is row-stochastic with a simple 1 eigenvalue and all other eigenvalues $|\lambda^*(\hat{A})| < 1$, then the expected value of the system states achieves consensus.

Theorem 6 only provides the conditions under which the expected value of the states reach consensus but it does not ensure that this equilibrium is the same as that of the unfiltered dynamics. The following theorem provides conditions under which the ACA drives the expected value of the system states to nominal consensus.

Theorem 7. For a networked system that updates according to the ACA. If F and $Q > 0$ are simultaneously diagonalizable and the eigenvalues, λ_i and θ_i, of F and Q that correspond to the same eigenvector satisfy

$$\rho\left(\begin{bmatrix} 1 & \lambda_i \\ \theta_i & \lambda_i\theta_i + \dfrac{1+\lambda_i}{1+\theta_i} \end{bmatrix}\right) < 1, \ \forall i \neq 1, \tag{12}$$

then the expected value of the system states reach nominal consensus.

Proof. If the static feedback matrix F is diagonalizable, i.e., $F = V\Lambda V^{-1}$, then the matrices V, V^{-1} and Λ can be sorted as

$$V = [\mathbf{1}_n, v_r], \ \Lambda = \begin{bmatrix} 0 & \\ & \Lambda_r \end{bmatrix}, \ V^{-1} = \begin{bmatrix} w^T \\ w_r^T \end{bmatrix}, \tag{13}$$

where $\mathbf{1}_n$ and ω_r are the right and left eigenvectors associated with the simple zero eigenvalue 0. $v_r \in \mathbb{C}^{n\times(n-1)}$ and $w_r \in \mathbb{C}^{n\times(n-1)}$ are matrices that are composed of the rest of the right and left eigenvectors. $\Lambda_r = \mathrm{diag}(\lambda_i(F)) \in \mathbb{C}^{(n-1)\times(n-1)}$ is a diagonal matrix with the rest eigenvalues of F on its main diagonal, note that the order of the rest eigenvalues is uniquely determined by v_r.

Furthermore, if F and Q are simultaneously diagonalizable, then, $Q = V\Theta V^{-1}$ and $\Theta = \begin{bmatrix} \theta_1 & \\ & \Theta_r \end{bmatrix}$, where θ_1 is the eigenvalue of Q corresponding to the eigenvector $\mathbf{1}_n$.

Denote $e_i = [0, \cdots, 1, \cdots, 0]^T \in \mathbb{R}^{2n}$ as a column vector with 1 in its ith position and the rest are 0s. We can define a permutation matrix

$$E = [e_1, \ e_{n+1}, \ e_2, \ e_{n+2}, \cdots, \ e_n, \ e_{2n}]. \tag{14}$$

From previous analysis, $\hat{A}$ can be decomposed by using E and V as

$$\hat{A} = \begin{bmatrix} V & \\ & V \end{bmatrix} EME^T \begin{bmatrix} V^{-1} & \\ & V^{-1} \end{bmatrix}, \tag{15}$$

where $M = \mathrm{diag}(M_i)$ is a block diagonal matrix, and each block is $M_i = \begin{bmatrix} 1 & \lambda_i \\ \theta_i & \lambda_i\theta_i + \dfrac{1+\lambda_i}{1+\theta_i} \end{bmatrix}$. In particular, the first block is $M_1 = \begin{bmatrix} 1 & 0 \\ \theta_1 & \dfrac{1}{1+\theta_1} \end{bmatrix}$, with $Q > 0$, $|\dfrac{1}{1+\theta_1}| < 1$. Therefore, when condition (12) holds,

$$\lim_{k\to+\infty} x^k = [I \ 0]\, \hat{A}^k x_0 = [I \ 0] \begin{bmatrix} \mathbf{1}_n \\ 0 \end{bmatrix} [w^T \ 0]\, x_0 = w^T x_0 \mathbf{1}_n.$$

Remark 8. We note that the condition of simultaneously diagonalizable F and $Q > 0$ holds for at least one common class of problems, those in which each state is measured, i.e., $H = I$, and $R = \eta I$).

For the reminder of this paper, we assume the ACA drives the expected value of the states to nominal consensus, i.e., the conditions listed in Theorem 7 hold.

4.2 Real and Estimated State Deviation

In this section, we first provide analytical tools to compare the performance of the TCA and ACA in terms of error accumulation. Then we show that under certain conditions, the ACA completely eliminates the effect of the measurement error.

The following theorem quantifies two types of error accumulation, (a) the *steady-state variance* of the deviation of each state from the average state (consensus), and (b) the state deviation of either the real or estimated state from the nominal consensus value at a time step of interest, p.

Theorem 9. Consider two networks of linear dynamical systems, G_{ACA} and G_{TCA}, that are respectively governed by the ACA and TCA. The output at the k^{th} step for both systems is defined as,

$$y^k = \left(I - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T\right) x^k, \quad \forall k > 0. \tag{16}$$

(a) The steady-state variance of deviation of G_{ACA} is lower than that of G_{TCA} if,

$$||G_{\mathrm{ACA}}||_{\mathcal{H}_2} < ||G_{\mathrm{TCA}}||_{\mathcal{H}_2}, \tag{17}$$

where $||G||_{\mathcal{H}_2}$ denotes the $\mathcal{H}_2$ norm of the networked system G.

(b) The state deviations from the nominal consensus state at the p^{th} step of G_{ACA} are lower than those of G_{TCA} if,

$$\left|\left|[I \ 0]\sum_{i=1}^{p} \hat{A}^{p-i} \begin{bmatrix} 0_{n\times n} \\ H^T R^{-1} \end{bmatrix} v^i\right|\right| < \left|\left|\sum_{i=1}^{p} A^{p-i} F v^i\right|\right|.$$

Proof.

(a) The system output defined in equation (16) is analogous to the **deviation from average** problem in (Bamieh et al., 2012; Oral and Gayme, 2019). Specifically, the condition (17) can be expanded as,

$$\mathrm{trace}\left\{ \begin{bmatrix} 0_{n\times n} \\ H^T R^{-1} \end{bmatrix}^T \sum_{k=0}^{+\infty} \left((\hat{A}^T)^k \begin{bmatrix} C^T \\ 0_{n\times n} \end{bmatrix} [C \ 0_{n\times n}]\, \hat{A}^k\right) \begin{bmatrix} 0_{n\times n} \\ H^T R^{-1} \end{bmatrix}\right\} < \mathrm{trace}\left\{F^T \sum_{k=0}^{+\infty} (A^T)^k C^T C A^k F\right\},$$

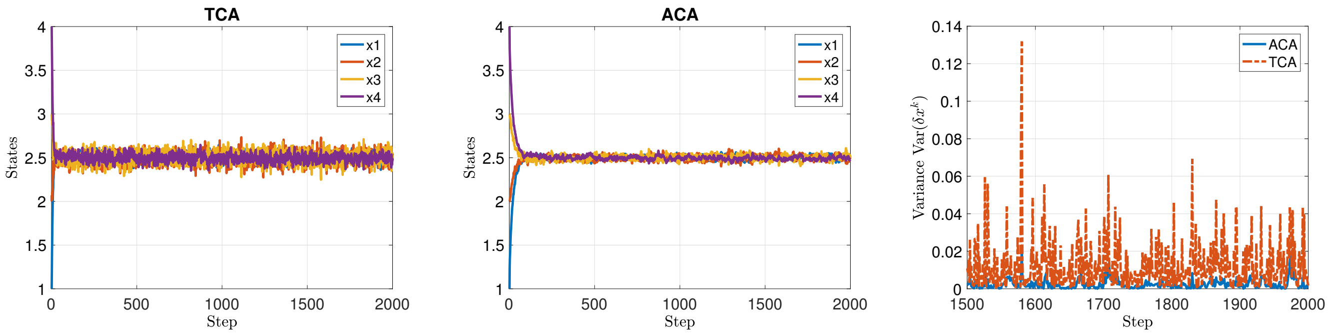

Fig. 2. The left and middle panel show the evolution of the states from the initial condition $x_0 = [1\ 2\ 3\ 4]^T$. Both the TCA (left) and ACA (middle) enable the system's expected state values to converge to the nominal consensus state ($\frac{5}{2}[1\ 1\ 1\ 1]^T$), but the fluctuations around this value are noticeably smaller under the ACA. The variance of state deviation $\mathrm{Var}(\delta x)$ in both the ACA and TCA from step 1500 to step 2000 is depicted in the right panel.

where $C = I - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^T$, i.e., the comparison of steady-state variance of the deviation of the TCA and ACA can be computed analytically without requiring information about the measurement error at each step.

(b) The deviation of the real and estimated states from nominal consensus can be represented as,

$$\delta x^k = x^k - \mathbb{E}(x^k), \quad \delta\hat{x}^k = \hat{x}^k - \mathbb{E}(\hat{x}^k),$$

and the dynamics of the deviations are,

$$\begin{bmatrix}\delta x^k\\ \delta\hat{x}^k\end{bmatrix} = \hat{A}\begin{bmatrix}\delta x^{k-1}\\ \delta\hat{x}^{k-1}\end{bmatrix} + \begin{bmatrix}0_{n\times n}\\ H^TR^{-1}\end{bmatrix}v^k.$$

The corresponding deviation at the p^{th} time for the TCA and ACA at each step can be computed directly under the given conditions to complete the proof.

Remark 10. Theorem 9 indicates that by properly tuning the algorithm parameters R and H, the measurement error accumulation can be reduced compared with the TCA approach, leading to a reduction in the state error.

In addition to attenuating the error accumulation under the conditions given in Theorem 9, the ACA can directly cancel the contribution of the measurement error given certain conditions. Specifically, for a system operating under the TCA, the measurement error v^h can be eliminated at the step $k > h$ if and only if $v^h \in \mathrm{span}\{\mathbf{1}_n\}$. In the following discussion, we show how the ACA can *bypass* this strong condition by tuning the algorithm parameters.

In order to facilitate this discussion, we first present the decomposition of the jth power of the ACA state matrix defined in (10), i.e., $(\hat{A})^j$. Based on the analysis in Section 4.1, $(\hat{A})^j$ can be decomposed in the following manner,

$$(\hat{A})^j = \begin{bmatrix}V & \\ & V\end{bmatrix}\begin{bmatrix}\mathcal{M}_{1,j} & \mathcal{M}_{2,j}\\ \mathcal{M}_{3,j} & \mathcal{M}_{4,j}\end{bmatrix}\begin{bmatrix}V^{-1} & \\ & V^{-1}\end{bmatrix}, \qquad (18)$$

where for any partition $p \in \{1, \cdots, 4\}$, $\mathcal{M}_{p,j}$ is a diagonal matrix.

Theorem 11. If the expected equilibrium value (consensus state) attained by ACA is the expected value of the nominal consensus, and the measurement error at step h satisfies,

$$v^h \in \mathrm{null}\{\mathcal{M}_{2,(k-h)}V^{-1}H^TR^{-1}\}, \quad k > h, \qquad (19)$$

where $\mathcal{M}_{2,(k-h)}$ is the upper-right block of based on the decomposition of $(\hat{A})^{(k-h)}$ given in (18), then this measurement error v^h can be eliminated and will not contribute to the overall state error after the kth step.

Proof. According to equation (11), the expression governing the measurement error build up at step k is given by,

$$[I\ 0_{n\times n}](\hat{A})^{(k-h)}\begin{bmatrix}0_{n\times n}\\ H^TR^{-1}\end{bmatrix}v^h = V\mathcal{M}_{2,(k-h)}V^{-1}H^TR^{-1}v^h.$$

Corollary 12. If the expected equilibrium value (consensus state) attained by ACA is the expected value of the nominal consensus, and the measurement error at step h satisfies, $H^TR^{-1}v^h \in \mathrm{span}\{\mathbf{1}_n\}$, then for any step $k > h$, there is no additional buildup of measurement error.

Proof. As F and Q are simultaneously diagonalizable, $(I+Q)^{-1}$ and F commute. Since $F\mathbf{1}_n = \mathbf{0}_n$, and $A\mathbf{1}_n = \mathbf{1}_n$, $\forall j \geq 0$,

$$\begin{aligned}
&[I\ 0_{n\times n}]\,(\hat{A})^j\begin{bmatrix}0_{n\times n}\\ H^TR^{-1}\end{bmatrix}v^h = [I\ 0_{n\times n}]\,(\hat{A})^j\begin{bmatrix}\mathbf{0}_n\\ \eta\mathbf{1}_n\end{bmatrix}\\
&= [I\ 0_{n\times n}](\hat{A})^{(j-2)}\begin{bmatrix}\eta F\mathbf{1}_n\\ \eta\left(QF(I+Q)^{-1}+(I+Q)^{-2}\right)\mathbf{1}_n\end{bmatrix}\\
&= [I\ 0_{n\times n}](\hat{A})^{(j-2)}\begin{bmatrix}\mathbf{0}_n\\ \eta(I+Q)^{-2}\ \ \mathbf{1}_n\end{bmatrix}\\
&\ \vdots\\
&= [I\ 0_{n\times n}]\begin{bmatrix}\mathbf{0}_n\\ \eta(I+Q)^{-j}\mathbf{1}_n\end{bmatrix} = 0_n,
\end{aligned}$$

where $H^TR^{-1}v^i = \eta\mathbf{1}_n$, $\eta \in \mathbb{R}$.

5. NUMERICAL RESULTS

We now simulate a networked system with four nodes connected over line graph in the presence of measurement error to evaluate the performance our new algorithm. The measurement error satisfies a normal distribution, $v^k \sim \mathcal{N}\{0, 0.1\}$. We assume that each individual state is measurable, i.e., $H = I$, and set $R = \frac{1}{4}I$. The static feedback matrix is $F = -L_{sym} = -\frac{1}{4}\times\begin{bmatrix}1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1\\ & & -1 & 1\end{bmatrix}$.

As L_{sym} is a symmetric Laplacian matrix, the left eigenvector associated with the simple zero eigenvalue of L_{sym} is $0.25 \times [1\ 1\ 1\ 1]^T$. We simulate the system for 2000 steps. The state evolution with both the TCA and ACA are respectively plotted in the left and middle panels of Fig. 2. We further quantify the performance through the variance of the state deviation, $\mathrm{Var}(\delta x^k) = \delta x^T \delta x$, which is plotted in the right panel of Fig. 2. The figures illustrate that although the TCA converges faster, the ACA significantly reduces the state fluctuations. As expected, the variance the of state deviation resulting from the ACA is much lower than the variance in the TCA. In particular, the average variance of the state deviation for TCA is 0.02 whereas the ACA reduces the average variance to 0.005.

6. CONCLUSION

In this paper, we present a consensus-filter for systems subject to measurement noise. The filter enables the system to reach consensus with lower state errors (state deviations from consensus value) than a system operating under the TCA. Theoretical analysis shows that the ACA drives the expected values of the system states to consensus. We then provide a method to ensure that the ACA attains the same consensus state as the original TCA dynamics. In both cases the state errors are reduced versus the TCA. Simulation results confirm that our algorithm reduces the state errors and drives the expected value of the system states to the original TCA equilibrium value. Future work includes providing a distributed algorithm to obtain the system parameters R and H. Deriving explicit conditions under which the ACA can lead to lower deviation from nominal consensus is another topic of on-going work.

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