A Simplified Approach to Analyze Complementary Sensitivity Trade-offs in Continuous-Time and Discrete-Time Systems

Neng Wan, Dapeng Li, and Naira Hovakimyan, *Fellow, IEEE*

Abstract—Continuous-time and discrete-time complementary sensitivity Bode integrals (CSBIs) are investigated via a simplified approach in this note. For continuous-time feedback systems with unbounded frequency domain, the CSBI weighted by $1/\omega^2$ is considered, where this simplified method reveals a more explicit relationship between the value of CSBI and the structure of the open-loop transfer function. With a minor modification of this method, the CSBI of discrete-time system is derived, and illustrative examples are provided. Compared with the existing results on CSBI, neither Cauchy integral theorem nor Poisson integral formula is used throughout the analysis, and the analytic constraint on the integrand is removed.

Index Terms—Complementary sensitivity Bode integral, simplified approach, control trade-offs, linear systems.

I. INTRODUCTION

This technical note extends the *simplified approach* for analysis of sensitivity Bode integrals from [1] to complementary sensitivity Bode integrals (CSBIs). Sensitivity function and complementary sensitivity function are two critical transfer functions that provide insights into the influences of external disturbance on the error signal and the measurement output, respectively. Freudenberg and Looze showed in [2], [3] that the integral over all frequencies of the logarithm of the absolute value of sensitivity function, $\ln|S(s)|$, is proportional to the sum of the unstable open-loop poles. Meanwhile, inspired by the well-known result on sensitivity function and complementary sensitivity functions [4], $S(s) + T(s) = 1$, it is natural to believe that similar trade-off should also exist for $\ln|T(s)|$. However, as $s \to \infty$, the integrand $\ln|T(s)|$ and the corresponding integral grow to infinity in continuous-time systems, which puzzled the researchers for a few years [5].

Several efforts have been made to tackle this issue of CSBI in continuous-time systems. One of the earliest results was contributed by Freudenberg and Looze [2], who exploited the harmonic property of $\ln|T(s)|$ by multiplying this function with a Poisson kernel and computed the CSBI with the help of Poisson integral formula and a limiting argument [4]. A more concise form of continuous-time CSBI was later

proposed by Middleton [6], who weighted $\ln|T(s)|$ by $1/\omega^2$ and adopted the Cauchy integral theorem to compute the integral. The employments of frequency inversion and Cauchy integral theorem in [6] require the inverse frequency function $\ln|T(1/s)|$ be analytic at $s = \infty$, such that it can be expanded as a Laurent series [4]. In [7], Yu *et al.* studied a type of CSBI weighted by $1/(s^2 + \alpha^2)^k$, where $\alpha \in \mathbb{R}$ and $k \geq 1$, and an information-theoretic approach to derive the CSBI of continuous-time stochastic system was presented in [8]. More previous results on discrete-time CSBI with a bounded frequency domain can be found in [5], [9], [10].

The simplified approach from [1] is extended here to analyze the CSBIs for both continuous-time and discrete-time systems. Compared with the prevailing results on CSBI of deterministic systems [2], [4]–[7], the salient feature of this method is that neither Cauchy integral theorem nor Poisson integral formula are invoked when deriving the CSBIs, which consequently allows to remove the analytic (harmonic) constraints on the integrands of CSBIs. In addition to a new approach to derive CSBI, this simplified approach also provides a more explicit explanation on how the complementary sensitivity trade-off is impacted by the structure of an open-loop transfer function, *e.g.* the distributions of zeros and poles, relative degree, number of pure integrators and leading coefficient. For continuous-time systems, we study the CSBI weighted by $1/\omega^2$ similar to [6], and with a slight modification, the simplified approach is applied to investigating the discrete-time CSBI. A few illustrative examples are given at the end of this note.

This note is organized as follows: Section II introduces the preliminaries; Section III studies the CSBI of continuous-time systems; Section IV investigates the CSBI of discrete-time systems; illustrative examples are shown in Section V, and Section VI draws the conclusions.

Notation: In this paper, we use $\ln(\cdot)$ to denote natural logarithm with the base of the mathematical constant e and $\log(\cdot)$ to denote the logarithm with base 2. For a complex number a, $|a|$ stands for the modulus. Complex variables are denoted as $s = j\omega$ and $z = e^{j\omega}$.

Neng Wan and Naira Hovakimyan are with the Department of Mechanical Science and Engineering and Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA. e-mail: {nengwan2, nhovakim}@illinois.edu.
Dapeng Li is a Principal Scientist with the JD.com Silicon Valley Research Center, Santa Clara, CA 95054, USA. e-mail: dapeng.li@jd.com.

II. PRELIMINARIES AND PROBLEM FORMULATION

Background knowledge and preliminary results on CSBI are stated in this section. Consider the block diagram of a general feedback system shown in Figure 1, which can be either a continuous-time system or a discrete-time system. Here L

denotes the open-loop transfer function; d is the external disturbance; e is the error signal, and y is the measurement output. The complementary sensitivity function $T(s)$ (or $T(z)$) is defined as the transfer function from external disturbance d to measurement output y.
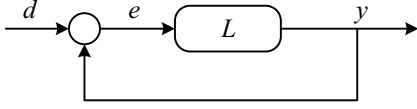

Fig. 1. General feedback system.

A. Continuous-Time System

For a continuous-time plant model $G(s)$ and a control mapping $C(s)$ the open-loop transfer function $L(s)$ can be generally expressed as

$$L(s) = G(s) \cdot C(s) = K \cdot \frac{\prod_{i=1}^{m}(s - z_i)}{s^{n-l} \cdot \prod_{i=1}^{l}(s - p_i)}, \quad (1)$$

where $K \in \mathbb{R}$, the relative degree is $\nu = n - m$ with $m \leq n$, $n - l \geq 0$ denotes the number of pure integrators, z_i and p_i respectively denote the zeros and poles of $L(s)$, and no z_i or p_i is at $s = 0$. When $\nu = 1$, the leading coefficient $K = \lim_{s \to \infty} sL(s)$. The complementary sensitivity function $T(s)$ for this continuous-time system is defined as

$$T(s) = L(s)/[1 + L(s)]. \quad (2)$$

In this note, we consider the continuous-time CSBI defined as follows:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \ln |T(s)| \, \frac{d\omega}{\omega^2}, \quad (3)$$

where a weighting function $1/\omega^2$ is involved [4], [6]. Consider the following frequency transformation

$$s = j\omega = (j\tilde{\omega})^{-1} = \tilde{s}^{-1}, \quad (4)$$

where the frequency variables satisfy $\omega = -(\tilde{\omega})^{-1}$. By change of variables, we can rewrite CSBI in (3) as follows

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \ln |\tilde{T}(\tilde{s})| \, d\tilde{\omega},$$

where $\tilde{T}(\tilde{s}) = T(s)$. The following lemma states an earlier result on CSBI, which was obtained by resorting to Cauchy integral theorem [4], [6].

Lemma 1. *Let z_{u_i}'s be the non-minimum phase zeros of open-loop transfer function $L(s)$, and suppose that $L(0) \neq 0$. Then, assuming closed-loop stability, if $L(s)$ is a proper rational function, then*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \ln \left| \frac{T(s)}{T(0)} \right| \frac{d\omega}{\omega^2} = \sum_i z_{u_i}^{-1} + \frac{1}{2 \cdot T(0)} \lim_{s \to 0} \frac{dT(s)}{ds}.$$

The theoretical basis of the simplified approach for continuous-time system is stated in the following lemma [1].

Lemma 2. *For complex numbers a and b, we have*

$$\int_{-\infty}^{\infty} \ln \left| \frac{(j\omega - a)}{(j\omega - b)} \right|^2 d\omega = 2\pi \cdot (|\mathrm{Re}\, a| - |\mathrm{Re}\, b|). \quad (5)$$

Remark 1. *The proof of Lemma 2 only requires some elementary techniques, such as integration by parts. Instead of weighting the left-hand side (LHS) of (5) by $1/\omega^2$ and deriving a new identity, we find that employing Lemma 2 to derive continuous-time CSBI can show more insights into the interactions between the value of CSBI and the structure of the open-loop transfer function $L(s)$.*

B. Discrete-Time System

With a discrete-time plant model $G(z)$ and control mapping $C(z)$, the open-loop transfer function $L(z)$ can generally be expressed as

$$L(z) = G(z) \cdot C(z) = K \cdot \frac{\prod_{i=1}^{m}(z - z_i)}{\prod_{i=1}^{n}(z - p_i)}, \quad (6)$$

where $K \in \mathbb{R}$, relative degree is $\nu = n - m \geq 0$, z_i and p_i are respectively the zeros and poles, and $z_i \neq 0$. Compared with the open-loop transfer function for continuous-time system (1), since frequency transformation is not involved when deriving the discrete-time CSBI, unit delays $1/z$ are not explicitly expressed in (6), and we allow $p_i = 0$ in discrete-time system. The discrete-time complementary sensitivity function $T(z)$ is then defined as

$$T(z) = L(z)/[1 + L(z)].$$

Since the frequency domain of discrete-time system is bounded, $\omega \in [-\pi, \pi]$, we consider the following type of CSBI without weighting function

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |T(z)| d\omega. \quad (7)$$

Previous result on the discrete-time CSBI is claimed in the following lemma, which was also derived on the basis of Cauchy integral theorem [5].

Lemma 3. *Let z_{u_i}'s be the strictly unstable zeros of open-loop transfer function $L(z)$. Then, assuming closed-loop stability, if $L(z)$ is a proper rational function, we have*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |T(z)| \, d\omega = \sum_i \log |z_{u_i}| + \log |K|,$$

where z_{u_i} denotes the unstable zeros in $L(z)$, and K is the leading coefficient of the numerator of $L(z)$, when the denominator is monic.

The fundamental tool for analyzing discrete-time CSBI is stated in the following lemma, whose proof only requires elementary techniques and is available in [1].

Lemma 4. *For a complex number a, we have*

$$\int_{-\pi}^{\pi} \log \left| e^{j\omega} - a \right|^2 d\omega = \begin{cases} 0, & \text{if } |a| \leq 1; \\ 2\pi \cdot \log |a|^2, & \text{if } |a| > 1. \end{cases}$$

III. CONTINUOUS-TIME COMPLEMENTARY SENSITIVITY BODE INTEGRAL

We investigate the continuous-time CSBI in this section. The results are stated in two categories, namely when relative degree $\nu \geq 1$ and $\nu = 0$. Under each category, we show how

the value of CSBI is related to the amount of pure integrators, as well as the leading coefficient of the open-loop transfer function. First, we consider a more general scenario when the open-loop transfer function is strictly proper, *i.e.* $\nu \geq 1$.

Theorem 1. *For an open-loop transfer function $L(s)$ with relative degree $\nu \geq 1$ and stable closed-loop system, the continuous-time CSBI satisfies*[*]

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln|T(s)|\frac{d\omega}{\omega^2}$$
$$= \begin{cases} \sum_i \operatorname{Re} z_{u_i}^{-1}, & \text{if } 0 \leq l \leq n-2; \\ \sum_i \operatorname{Re} z_{u_i}^{-1} - \frac{1}{2K}\cdot\frac{\prod_{i=1}^{n-1}(-p_i)}{\prod_{i=1}^{m}(-z_i)}, & \text{if } l = n-1; \\ \pm\infty, & \text{otherwise;} \end{cases}$$

where p_i, z_i and z_{u_i} respectively denote the poles, zeros, and non-minimum phase zeros in $L(s)$, and $n-l$ is the amount of pure integrators in $L(s)$.

Proof. When relative degree $\nu \geq 1$, the open-loop transfer function defined in (1) can be expressed as follows

$$L(s) = K \cdot \prod_{i=1}^{m}(s - z_i)/[s^{n-l}\cdot\prod_{i=1}^{l}(s - p_i)], \quad (8)$$

where $l \leq n$, $m+1 \leq n$, $K \in \mathbb{R}$, and no z_i or p_i is at $s = 0$. Substitute (8) into (2), and rewrite the complementary sensitivity function $T(s)$ in the following two equivalent forms

$$T_1(s) = \frac{K\cdot\prod_{i=1}^{m}(s - z_i)}{s^{n-l}\cdot\prod_{i=1}^{l}(s - p_i) + K\cdot\prod_{i=1}^{m}(s - z_i)}, \quad (9)$$

$$T_2(s) = K\cdot\prod_{i=1}^{m}(s - z_i)/\prod_{i=1}^{n}(s - r_i), \quad (10)$$

where r_i's denote the closed-loop poles with negative real parts, and $T_1(s) = T_2(s) = T(s)$. Applying frequency transformation (4) to complementary sensitivity functions (9) and (10) gives

$$\tilde{T}_1(\tilde{s}) = \frac{K\cdot\prod_{i=1}^{m}(\tilde{s}^{-1} - z_i)}{\tilde{s}^{l-n}\prod_{i=1}^{l}(\tilde{s}^{-1} - p_i) + K\cdot\prod_{i=1}^{m}(\tilde{s}^{-1} - z_i)}, \quad (11)$$

$$\tilde{T}_2(\tilde{s}) = K\cdot\prod_{i=1}^{m}(\tilde{s}^{-1} - z_i)/\prod_{i=1}^{n}(\tilde{s}^{-1} - r_i). \quad (12)$$

Multiplying the numerators and denominators of (11) and (12) by $\tilde{s}^n$ and with some algebraic manipulations, we have

$$\tilde{T}_1(\tilde{s}) = \frac{K\prod_{i=1}^{m}(-z_i)\tilde{s}^{n-m}\prod_{i=1}^{m}(\tilde{s}-z_i^{-1})}{\prod_{i=1}^{l}(-p_i)\prod_{i=1}^{l}(\tilde{s}-p_i^{-1}) + K\prod_{i=1}^{m}(-z_i)\tilde{s}^{n-m}\prod_{i=1}^{m}(\tilde{s}-z_i^{-1})} \quad (13)$$

$$\tilde{T}_2(\tilde{s}) = K\cdot\frac{\prod_{i=1}^{m}(-z_i)}{\prod_{i=1}^{n}(-r_i)}\cdot\frac{\tilde{s}^{n-m}\prod_{i=1}^{m}(\tilde{s}-z_i^{-1})}{\prod_{i=1}^{n}(\tilde{s}-r_i^{-1})}. \quad (14)$$

Some relationship among p_i, z_i, and r_i can be implied from (9)-(14). Equating the denominators of (9) and (10), we have

$$s^{n-l}\cdot\prod_{i=1}^{l}(s-p_i) + K\cdot\prod_{i=1}^{m}(s - z_i) = \prod_{i=1}^{n}(s-r_i). \quad (15)$$

Expanding both sides of (15) yields

$$s^n - \left(\sum_{i=1}^{l} p_i\right)s^{n-1} + \cdots + \prod_{i=1}^{l}(-p_i)s^{n-l}$$
$$+ K[s^m - (\sum_{i=1}^{m} z_i)\cdot s^{m-1} + \cdots + \prod_{i=1}^{m}(-z_i)] \quad (16)$$
$$= s^n - (\sum_{i=1}^{n} r_i)s^{n-1} + \cdots + \prod_{i=1}^{n}(-r_i).$$

Since complex roots always come in conjugate pairs, the products $\prod_{i=1}^{l}(-p_i)$, $\prod_{i=1}^{m}(-z_i)$ and $\prod_{i=1}^{n}(-r_i)$ in (16) are all real. Equating the denominators in (13) and (14) gives

$$\prod_{i=1}^{l}(-p_i)\prod_{i=1}^{l}(\tilde{s} - p_i^{-1}) + K\prod_{i=1}^{m}(-z_i)\tilde{s}^{n-m}$$
$$\cdot\prod_{i=1}^{m}(\tilde{s} - z_i^{-1}) = \prod_{i=1}^{n}(-r_i)\cdot\prod_{i=1}^{n}(\tilde{s} - r_i^{-1}). \quad (17)$$

Expanding both sides of (17) yields

$$\prod_{i=1}^{l}(-p_i)[\tilde{s}^l - (\sum_{i=1}^{l} p_i^{-1})\tilde{s}^{l-1} + \cdots + \prod_{i=1}^{l}(-p_i^{-1})]$$
$$+ K\prod_{i=1}^{m}(-z_i)[\tilde{s}^n - (\sum_{i=1}^{m} z_i^{-1})\tilde{s}^{n-1} + \cdots \quad (18)$$
$$+ \prod_{i=1}^{m}(-z_i^{-1})\tilde{s}^{n-m}] = \prod_{i=1}^{n}(-r_i)[\tilde{s}^n$$
$$- (\sum_{i=1}^{n} r_i^{-1})\tilde{s}^{n-1} + \cdots + \prod_{i=1}^{n}(-r_i^{-1})].$$

The value of CSBI varies depending on the amount of pure integrators in the open-loop transfer function $L(s)$.

Case 1: No pure integrator exists in $L(s)$, *i.e.* $l = n$.

When $l = n$, equating the constant terms in (16) gives the following identity

$$\prod_{i=1}^{n}(-p_i) + K\prod_{i=1}^{m}(-z_i) = \prod_{i=1}^{n}(-r_i). \quad (19)$$

Equating the coefficients of terms $\tilde{s}^{n-1}$ in (18) yields

$$\prod_{i=1}^{n}(-p_i)\left(\sum_{i=1}^{n} p_i^{-1}\right) + K\prod_{i=1}^{m}(-z_i)\left(\sum_{i=1}^{m} z_i^{-1}\right)$$
$$= \prod_{i=1}^{n}(-r_i)\left(\sum_{i=1}^{n} r_i^{-1}\right). \quad (20)$$

Applying Lemma 2 to $\tilde{T}_2(\tilde{s})$ in (14) and substituting (19) into the result, the CSBI satisfies

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln|T(s)|\frac{d\omega}{\omega^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\ln|\tilde{T}_2(\tilde{s})|d\tilde{\omega}$$
$$= \frac{1}{2}[\sum_{i=1}^{m}|\operatorname{Re} z_i^{-1}| - \sum_{i=1}^{n}|\operatorname{Re} r_i^{-1}|] \quad (21)$$
$$+ \frac{1}{\pi}\int_{-\infty}^{\infty}\ln\frac{|K\prod_{i=1}^{m}(-z_i)|}{|\prod_{i=1}^{n}(-p_i)+K\prod_{i=1}^{m}(-z_i)|}d\tilde{\omega}.$$

Since the first two terms on the right-hand side (RHS) of (21) are bounded, the CSBI is bounded if and only if $|K\prod_{i=1}^{m}(-z_i)| = |\prod_{i=1}^{n}(-p_i) + K\prod_{i=1}^{m}(-z_i)|$. This condition can be attained if: i) at least one $p_i = 0$, which contradicts the previous assumption that $p_i \neq 0$, and hence CSBI is undefined for this case; or ii) $\prod_{i=1}^{n}(-p_i) = -2K\prod_{i=1}^{m}(-z_i)$, which was not noticed in the preceding works and is omitted in the statement of Theorem 1, since this situation is trivial and can rarely be satisfied in practice[1]. With (20) and some further analysis, the CSBI satisfying condition ii) is $\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln|T(s)|/\omega^2\,d\omega = \sum_{i=1}^{n}\operatorname{Re} p_i^{-1} - \sum_i\operatorname{Re} z_{s_i}^{-1}$, where z_{s_i} denotes the minimum phase zeros of $L(s)$. For the other open-loop transfer functions without pure integrator, the corresponding CSBIs are unbounded. Specifically, when $|\prod_{i=1}^{n}(-p_i)+K\prod_{i=1}^{m}(-z_i)| < |K\prod_{i=1}^{m}(-z_i)|$, the CSBI is negative infinity; while when $|\prod_{i=1}^{n}(-p_i) + K\prod_{i=1}^{m}(-z_i)| > |K\prod_{i=1}^{m}(-z_i)|$, this integral is positive infinity.

Case 2: Single pure integrator exists in $L(s)$, *i.e.* $l = n-1$.

When $l = n-1$, equating the constant terms in (16) yields

$$K\prod_{i=1}^{m}(-z_i) = \prod_{i=1}^{n}(-r_i). \quad (22)$$

Equating the coefficients of terms $\tilde{s}^{n-1}$ in (18) gives

$$-\prod_{i=1}^{n-1}(-p_i) + K\prod_{i=1}^{m}(-z_i)\left(\sum_{i=1}^{m} z_i^{-1}\right)$$
$$= \prod_{i=1}^{n}(-r_i)\left(\sum_{i=1}^{n} r_i^{-1}\right). \quad (23)$$

Applying Lemma 2 to $\tilde{T}_2(\tilde{s})$ in (14) and substituting (22) into the result, the CSBI becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln|T(s)|\frac{d\omega}{\omega^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\ln|\tilde{T}_2(\tilde{s})|d\tilde{\omega}$$
$$= \frac{1}{2}\cdot[\sum_{i=1}^{m}|\operatorname{Re} z_i^{-1}| - \sum_{i=1}^{n}|\operatorname{Re} r_i^{-1}|]. \quad (24)$$

The first term on the RHS of (24) can be decomposed as

$$\sum_{i=1}^{m}\left|\operatorname{Re} z_i^{-1}\right| = \sum_i \operatorname{Re} z_{u_i}^{-1} - \sum_i \operatorname{Re} z_{s_i}^{-1}, \quad (25)$$

where z_{u_i}'s and z_{s_i}'s respectively denote the non-minimum phase zeros and minimum phase zeros in $L(s)$. Since the closed-loop system is stable and all the closed-loop poles r_i's have negative real parts, with identities (22) and (23), the second term on the RHS of (24) can be rewritten as

$$\sum_{i=1}^{n}|\operatorname{Re} r_i^{-1}| = \frac{\prod_{i=1}^{n-1}(-p_i) - K\prod_{i=1}^{m}(-z_i)\sum_{i=1}^{m}\operatorname{Re} z_i^{-1}}{\prod_{i=1}^{n}(-r_i)}$$
$$= -\sum_i \operatorname{Re} z_{s_i}^{-1} - \sum_i \operatorname{Re} z_{u_i}^{-1} + \frac{\prod_{i=1}^{n-1}(-p_i)}{K\prod_{i=1}^{m}(-z_i)}. \quad (26)$$

Combing the results in (24), (25) and (26), when single integrator exists in $L(s)$, the CSBI is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln|T(s)|\frac{d\omega}{\omega^2} = \sum_i \operatorname{Re} z_{u_i}^{-1} - \frac{1}{2K}\cdot\frac{\prod_{i=1}^{n-1}(-p_i)}{\prod_{i=1}^{m}(-z_i)}. \quad (27)$$

Case 3: Two or more pure integrators exist in $L(s)$, *i.e.* $0 \le l \le n-2$.

When $0 \le l \le n-2$, equating the constant terms in (16) gives the same identity as (22), $K\cdot\prod_{i=1}^{m}(-z_i) = \prod_{i=1}^{n}(-r_i)$. Equating the coefficients of terms $\tilde{s}^{n-1}$ in (18) yields

$$K\prod_{i=1}^{m}(-z_i)\left(\sum_{i=1}^{m} z_i^{-1}\right) = \prod_{i=1}^{n}(-r_i)\left(\sum_{i=1}^{n} r_i^{-1}\right). \quad (28)$$

Since all the closed-loop poles r_i's have negative real parts, with (22) and (28), we have

$$\sum_{i=1}^{n}\left|\operatorname{Re} r_i^{-1}\right| = \frac{K\prod_{i=1}^{m}(-z_i)\sum_{i=1}^{m}\operatorname{Re} z_i^{-1}}{\prod_{i=1}^{n}(-r_i)}$$
$$= -\sum_i \operatorname{Re} z_{u_i}^{-1} - \sum_i \operatorname{Re} z_{s_i}^{-1}. \quad (29)$$

Applying Lemma 2 to $\tilde{T}_2(\tilde{s})$ in (14) and substituting (22) and (29) into the result, the CSBI for $L(s)$ with two or more integrators is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln|T(s)|\frac{d\omega}{\omega^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\ln|\tilde{T}_2(\tilde{s})|d\tilde{\omega}$$
$$= \frac{1}{2}\cdot[\sum_{i=1}^{m}|\operatorname{Re} z_i^{-1}| - \sum_{i=1}^{n}|\operatorname{Re} r_i^{-1}|]$$
$$+ \frac{1}{\pi}\int_{-\infty}^{\infty}\ln\left|K\cdot\frac{\prod_{i=1}^{m}(-z_i)}{\prod_{i=1}^{n}(-r_i)}\right|d\tilde{\omega}$$
$$= \sum_i \operatorname{Re} z_{u_i}^{-1}. \quad (30)$$

Summarizing the results in (21), (27) and (30) leads to Theorem 1. This completes the proof. □

Next, we consider the scenario when the open-loop transfer function $L(s)$ is biproper, *i.e.* relative degree $\nu = n - m = 0$. The CSBI of this category is related to not only the number of pure integrators $n-l$, but also the value of leading coefficient K.

Corollary 2. *For an open-loop transfer function $L(s)$ with relative degree $\nu = 0$ and stable closed-loop system, the continuous-time CSBI satisfies*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln|T(s)|\frac{d\omega}{\omega^2}$$
$$= \begin{cases} \sum_i \operatorname{Re} z_{u_i}^{-1}, & \text{if } K \ne -1 \text{ and } 0 \le l \le n-2; \\ \sum_i \operatorname{Re} z_{u_i}^{-1} - \frac{\prod_{i=1}^{n-1}(-p_i)}{2K\prod_{i=1}^{n}(-z_i)}, & \text{if } K \ne -1 \text{ and } l = n-1; \\ \pm\infty, & \text{otherwise.} \end{cases}$$

Proof. When relative degree $\nu = 0$, the open-loop transfer function $L(s)$ can be expressed as follows

$$L(s) = K\cdot\prod_{i=1}^{n}(s-z_i)/[s^{n-l}\prod_{i=1}^{l}(s-p_i)], \quad (31)$$

where $K \in \mathbb{R}$, $l \le n$, and no zero z_i or pole p_i is at $s = 0$. When $K = -1$, since the term s^n vanishes in the denominator of complementary sensitivity function $T(s)$ in (2), applying the frequency transformation (4) and similar algebraic manipulations as in (14)-(31), the complementary sensitivity function $\tilde{T}(\tilde{s}) = T(s)$ satisfies

$$\tilde{T}(\tilde{s}) = \frac{\prod_{i=1}^{n}(-z_i)}{K'\prod_{i=1}^{q}(-r_i)}\cdot\frac{\prod_{i=1}^{n}(\tilde{s}-z_i^{-1})}{\tilde{s}^{n-q}\prod_{i=1}^{q}(\tilde{s}-r_i^{-1})}, \quad (32)$$

where r_i's are the closed-loop poles with negative real parts, $K' \in \mathbb{R}$ is the lumped coefficient, $q < n$, and the values of K' and q are determined by the distributions of poles p_i's and zeros z_i's in $L(s)$. In general, we do not have $\prod_{i=1}^{n}(-z_i) = K'\prod_{i=1}^{q}(-r_i)$, *i.e.* $T(0) = 1$; otherwise, one can derive the corresponding CSBI by applying the analysis in the proof of Theorem 1 to this specific set of p_i's and z_i's. Hence, applying Lemma 2 to $\tilde{T}(s)$ in (32) yields

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln|T(s)|\frac{d\omega}{\omega^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\ln|\tilde{T}(\tilde{s})|d\tilde{\omega}$$
$$= \frac{1}{2}[\sum_{i=1}^{n}|\operatorname{Re} z_i^{-1}| - \sum_{i=1}^{q}|\operatorname{Re} r_i^{-1}|] \quad (33)$$
$$+ \frac{1}{\pi}\int_{-\infty}^{\infty}\ln\left|\frac{\prod_{i=1}^{n}(-z_i)}{K'\prod_{i=1}^{q}(-r_i)}\right|d\tilde{\omega}.$$

Since the last term on the RHS of (33) is unbounded, in general, the CSBI is unbounded when $K = -1$.

When $K \ne -1$, the complementary sensitivity function $T(s)$ can be equivalently expressed as

$$T_1(s) = \frac{K\prod_{i=1}^{n}(s-z_i)}{s^{n-l}\prod_{i=1}^{l}(s-p_i) + K\prod_{i=1}^{n}(s-z_i)}, \quad (34)$$
$$T_2(s) = (K/K+1)\cdot\prod_{i=1}^{n}(s-z_i)/\prod_{i=1}^{n}(s-r_i), \quad (35)$$

where $T_1(s) = T_2(s) = T(s)$. Equating the denominators of (34) and (35) and expanding the polynomials give the following equation

$$s^n - (\sum_{i=1}^{l} p_i)s^{n-1} + \cdots + \prod_{i=1}^{l}(-p_i)s^{n-l}$$
$$+ K[s^n - (\sum_{i=1}^{n} z_i)s^{n-1} + \cdots + \prod_{i=1}^{n}(-z_i)] \quad (36)$$
$$= (K+1)\cdot[s^n - (\sum_{i=1}^{n} r_i)s^{n-1} + \cdots + \prod_{i=1}^{n}(-r_i)].$$

Adopting the frequency transformation (4) and some algebraic manipulations, (34) and (35) can be transformed into

$$\tilde{T}_1(\tilde{s}) = \frac{K\prod_{i=1}^{n}(-z_i)\prod_{i=1}^{n}(\tilde{s}-z_i^{-1})}{\prod_{i=1}^{l}(-p_i)\prod_{i=1}^{l}(\tilde{s}-p_i^{-1}) + K\prod_{i=1}^{n}(-z_i)\prod_{i=1}^{n}(\tilde{s}-z_i^{-1})}, \quad (37)$$

$$\tilde{T}_2(\tilde{s}) = \frac{K\prod_{i=1}^{n}(-z_i)}{(K+1)\prod_{i=1}^{n}(-r_i)}\cdot\frac{\prod_{i=1}^{n}(\tilde{s}-z_i^{-1})}{\prod_{i=1}^{n}(\tilde{s}-r_i^{-1})}. \quad (38)$$

Equating and expanding the denominators in (37) and (38), we have

$$\prod_{i=1}^{l}(-p_i)[\tilde{s}^l - (\sum_{i=1}^{l} p_i^{-1})\tilde{s}^{l-1} + \cdots + \prod_{i=1}^{l}(-p_i^{-1})]$$
$$+ K \prod_{i=1}^{n}(-z_i)[\tilde{s}^n - (\sum_{i=1}^{n} z_i^{-1})\tilde{s}^{n-1} + \cdots$$
$$+ \prod_{i=1}^{n}(-z_i^{-1})] = (K+1)\prod_{i=1}^{n}(-r_i)[\tilde{s}^n \quad (39)$$
$$- (\sum_{i=1}^{n} r_i^{-1})\tilde{s}^{n-1} + \cdots + \prod_{i=1}^{n}(-r_i^{-1})].$$

Similar to the proof of Theorem 1, we analyze the CSBIs for $L(s)$ with different amounts of pure integrators.

Case 1: No pure integrator exists in $L(s)$, i.e. $l = n$.

When $l = n$, equating the constant terms in (36) gives, $\prod_{i=1}^{n}(-p_i) + K \prod_{i=1}^{n}(-z_i) = (K+1)\prod_{i=1}^{n}(-r_i)$. Equating the coefficients of $\tilde{s}^{n-1}$ terms in (39) yields, $\prod_{i=1}^{n}(-p_i)(\sum_{i=1}^{n} p_i^{-1}) + K \prod_{i=1}^{n}(-z_i)(\sum_{i=1}^{n} z_i^{-1}) = (K+1)\prod_{i=1}^{n}(-r_i)(\sum_{i=1}^{n} r_i^{-1})$. Applying Lemma 2 to $\tilde{T}_2(\tilde{s})$ in (38), the complementary sensitivity Bode integral satisfies

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \ln|T(s)| \frac{d\omega}{\omega^2} = \frac{1}{2}[\sum_{i=1}^{n}|\text{Re } z_i^{-1}| - \sum_{i=1}^{n}|\text{Re } r_i^{-1}|]$$
$$+ \frac{1}{\pi}\int_{-\infty}^{\infty} \ln \frac{|K \prod_{i=1}^{n}(-z_i)|}{|\prod_{i=1}^{n}(-p_i) + K \prod_{i=1}^{n}(-z_i)|} d\tilde{\omega}. \quad (40)$$

Bode integral in (40) is bounded when at least one $p_i = 0$, which contradicts the fact that $p_i \neq 0$ when we defined $L(s)$ in (31). Hence, the integral is unbounded when $l = n$. Further analysis on this case refers to the comments after (21).

Case 2: Single pure integrator exists in $L(s)$, i.e. $l = n - 1$.

When $l = n - 1$, equating the constant terms in (36) gives $K \prod_{i=1}^{n}(-z_i) = (K+1)\prod_{i=1}^{n}(-r_i)$. Equating the coefficients of $\tilde{s}^{n-1}$ terms in (39) gives $-\prod_{i=1}^{n-1}(-p_i) + K \prod_{i=1}^{n}(-z_i)(\sum_{i=1}^{n} z_i^{-1}) = (K+1)\cdot\prod_{i=1}^{n}(-r_i)(\sum_{i=1}^{n} r_i^{-1})$. From (25) we have $\sum_{i=1}^{n}|\text{Re } z_i^{-1}| = \sum_i \text{Re } z_{u_i}^{-1} - \sum_i \text{Re } z_{s_i}^{-1}$. Since all the closed-loop poles r_i's are with negative real parts, the sum $\sum_{i=1}^{n}|\text{Re } r_i^{-1}| = \prod_{i=1}^{n-1}(-p_i)[K \prod_{i=1}^{n}(-z_i)] - \sum_{i=1}^{n} z_i^{-1}$. Applying Lemma 2 to $\tilde{T}_2(\tilde{s})$ in (38), the CSBI gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \ln|T(s)| \frac{d\omega}{\omega^2} = \sum_i \text{Re } z_{u_i}^{-1} - \frac{\prod_{i=1}^{n-1}(-p_i)}{2K \cdot \prod_{i=1}^{n}(-z_i)}. \quad (41)$$

Case 3: Two or more pure integrators exist in $L(s)$, i.e. $0 \leq l \leq n - 2$.

When $0 \leq l \leq n - 2$, equating the constant terms in (36) gives $K \prod_{i=1}^{n}(-z_i) = (K+1)\prod_{i=1}^{n}(-r_i)$. Equating the coefficients of $\tilde{s}^{n-1}$ terms in (39) gives $K \prod_{i=1}^{n}(-z_i)(\sum_{i=1}^{n} z_i^{-1}) = (K+1)\prod_{i=1}^{n}(-r_i)\cdot(\sum_{i=1}^{n} r_i^{-1})$. Meanwhile, we have $\sum_{i=1}^{n}|\text{Re } z_i^{-1}| = \sum_i \text{Re } z_{u_i}^{-1} - \sum_i \text{Re } z_{s_i}^{-1}$ and $\sum_{i=1}^{n}|\text{Re } r_i^{-1}| = -\sum_{i=1}^{n} z_i^{-1} = -\sum_i \text{Re } z_{u_i}^{-1} - \sum_i \text{Re } z_{s_i}^{-1}$. Hence, applying Lemma 2 to $\tilde{T}_2(\tilde{s})$ in (38), the complementary sensitivity Bode integral satisfies

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \ln|T(s)| \frac{d\omega}{\omega^2} = \sum_i \text{Re } z_{u_i}^{-1}. \quad (42)$$

Summarizing the results in (40-42) gives Corollary 2. This completes the proof. □

Remark 2. *The results on continuous-time CSBI have been presented in Theorem 1 for systems with $\nu \geq 1$ and Corollary 2 for systems with $\nu = 0$, respectively. In general, the results derived via the simplified approach, Theorem 1 and*

Corollary 2, match the earlier result, Lemma 1, obtained by employing Cauchy integral theorem. Nevertheless, more detailed relationship between CSBI and the features of $L(s)$ and more relaxed constraints on $\ln T(1/s)$ are attained by using the simplified approach. In both cases of $\nu \geq 1$ and $\nu = 0$, the values of CSBIs are mainly determined by the non-minimum phase zeros z_{u_i}, while the form of CSBI varies depending on the amount of pure integrators in $L(s)$. When only single pure integrator exists in $L(s)$, the value of CSBI is determined by the leading coefficient K, minimum phase zeros z_{s_i}, and the poles p_i in $L(s)$, which was not reported in the previous papers. Theorem 1 and Corollary 2 also show that the continuous-time CSBI defined in [6] is unbounded, when $L(s)$ does not contain any pure integrator, which is a limitation of this type of CSBI and did not receive enough attention in recent papers [4], [8]. Meanwhile, the analytic constraint on $\ln|T(1/s)|$, as well as the initial value constraint $T(0) \neq 0$, are not necessary in the proofs of Theorem 1 and Corollary 2, which can also be extended to the scenario when some of the closed-loop poles r_i's are on the imaginary axis via simplified approach.

IV. DISCRETE-TIME COMPLEMENTARY SENSITIVITY BODE INTEGRAL

Discrete-time CSBI is investigated in this section by using a simplified approach developed on the basis of Lemma 4. Compared with the continuous-time system, since the frequency domain of discrete-time system is bounded, we do not need to worry about the unboundedness of discrete-time CSBI, and hence neither weighting function nor frequency inversion is involved in this section. When the relative degree $\nu \geq 1$ in $L(z)$, we have the following result.

Theorem 3. *For an open-loop transfer function $L(z)$ with relative degree $\nu \geq 1$ and stable closed-loop system, the discrete-time CSBI satisfies*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|T(z)|d\omega = \sum_i \log|z_{u_i}| + \log|K|, \quad (43)$$

where z_{u_i}'s are the unstable zeros in $L(z)$, and K is the leading coefficient in (6).

Proof. When relative degree $\nu \geq 1$, the open-loop transfer function (6) can be expressed as

$$L(z) = K \cdot \prod_{i=1}^{m}(z - z_i)/\prod_{i=1}^{n}(z - p_i),$$

where $n \geq m + 1$. Then the discrete-time complementary sensitivity function takes the form

$$T(z) = \frac{K \cdot \prod_{i=1}^{m}(z-z_i)}{\prod_{i=1}^{n}(z-p_i) + K \cdot \prod_{i=1}^{m}(z-z_i)} = K \cdot \frac{\prod_{i=1}^{m}(z-z_i)}{\prod_{i=1}^{n}(z-r_i)}. \quad (44)$$

where r_i's denote the closed-loop poles. Since the closed-loop system is stable and all the closed-loop poles r_i's are within the unit circle, applying Lemma 4 to (44), the discrete-time CSBI satisfies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|T(z)|d\omega = \frac{1}{4\pi}\int_{-\pi}^{\pi} \log\left|K \frac{\prod_{i=1}^{m}(z-z_i)}{\prod_{i=1}^{n}(z-r_i)}\right|^2 d\omega$$
$$= \sum_i \log|z_{u_i}| + \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|K|d\omega, \quad (45)$$

which implies (43) in Theorem 3. □

We then consider the CSBI of a biproper open-loop system, *i.e.* when $\nu = n - m = 0$.

Corollary 4. *For an open-loop transfer function $L(z)$ with relative degree $\nu = 0$ and stable closed-loop system, the discrete-time CSBI satisfies*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|T(z)|d\omega = \sum_i \log|z_{u_i}| + \log\left|\frac{K}{1+K}\right|. \quad (46)$$

Proof. When relative degree $\nu = 0$, the discrete-time open-loop transfer function (6) can be expressed as follows

$$L(z) = K \cdot \prod_{i=1}^{n}(z-z_i)/\prod_{i=1}^{n}(z-p_i),$$

where $K = \lim_{z\to\infty} L(z)$. The complementary sensitivity function then becomes

$$T(z) = \frac{K\cdot\prod_{i=1}^{n}(z-z_i)}{\prod_{i=1}^{n}(z-p_i)+K\cdot\prod_{i=1}^{n}(z-z_i)} = \frac{K}{1+K}\frac{\prod_{i=1}^{n}(z-z_i)}{\prod_{i=1}^{n}(z-r_i)}. \quad (47)$$

When $K = -1$, the order of denominator in (47) will be less than n, *i.e.* at least one closed-loop pole r_i is out of unit circle and at infinity, which implies that the closed-loop system is not causal. Hence, in practice, the leading coefficient $K = -1$ is not allowed when $\nu = 0$ [1], though one can still obtain a bounded value by applying Lemma 4 to (47) or computing the integral (7) directly. When $K \neq -1$, since the closed-loop system is stable and all r_i's are inside the unit circle, applying Lemma 4 to (47), the discrete-time CSBI satisfies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|T(z)|d\omega = \frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\left|\frac{K\prod_{i=1}^{n}(z-z_i)}{(1+K)\prod_{i=1}^{n}(z-r_i)}\right|^2 d\omega$$
$$= \sum_i \log|z_{u_i}| + \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|\frac{K}{1+K}|d\omega, \quad (48)$$

which implies (46) in Corollary 4. $\qquad\square$

Remark 3. *The results on discrete-time CSBI have been presented in Theorem 3 for systems with $\nu \geq 1$ and in Corollary 4 for systems with $\nu = 0$, respectively. These results, derived by using the simplified approach, match the previous results in [5] generally. For both cases, $\nu \geq 1$ and $\nu = 0$, the CSBI is proportional to the sum of the logarithms of unstable or non-minimum phase zeros. However, the difference between the second terms on the RHS of (43) and (46) was not noted in the previous papers.*

V. ILLUSTRATIVE EXAMPLES

Illustrative examples that examine the previous theorems and corollaries are given in this section. First, we consider the following continuous-time transfer function $L_1(s)$ with two pure integrators $L_1(s) = -1.164 \times 10^{-4}(s-10)(s+0.0625)/[s^2\cdot(s+10)]$, where the relative degree $\nu = 1$, and a non-minimum phase zero is located at $s = 10$. The closed-loop complementary sensitivity function is $T_1(s) = -1.164 \times 10^{-4}\cdot(s-10)(s+0.0625)/[(s+10)(s^2+1.149\times 10^{-4}\cdot s+7.725\times 10^{-6})]$, which is a closed-loop stable plant with three closed-loop poles located at $s = 5.745\times 10^{-5}\pm 2.697\times 10^{-3}i$ and -10. The numerical computation of CSBI gives $\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln|T_1(s)|/\omega^2\,d\omega \approx 0.1000$. Applying Theorem 1 yields $\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln|T_1(s)|/\omega^2\,d\omega = \sum_i \mathrm{Re}\, z_{u_i}^{-1} = 1/10 = 0.1$, which matches the numerical result.

Next, we consider an open-loop transfer function $L_2(s)$ with only one pure integrator, *i.e.* $l = n - 1$. $L_2(s) =$ $-(s-1)/[s(s+3.5)]$, which, with relative degree $\nu = 1$, has a non-minimum phase zero at $s = 1$. The closed-loop complementary sensitivity function of $L_2(s)$ is $T_2(s) = -(s-1)/(s^2+2.5s+1)$, which is closed-loop stable with two closed-loop poles at $s = -0.5$ and -2. The numerical computation of CSBI gives $\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln|T_2(s)|/\omega^2 d\omega \approx -0.75$. Applying Theorem 1 yields $\frac{1}{2\pi}\int_{-\infty}^{\infty}\ln|T_2(s)|/\omega^2 d\omega = \sum_i \mathrm{Re}\, z_{u_i}^{-1} - (2K)^{-1}\prod_{i=1}^{n-1}(-p_i)/\prod_{i=1}^{m}(-z_i) = 1 - 1.75 = -0.75$, which also matches the numerical result.

Meanwhile, by following similar procedures as above, one can easily verify that the CSBI of the open-loop transfer function $L_3(s) = -2.0348\cdot(s-1)/(s^2+3s+2)$ is unbounded, which validates the last situation in Theorem 1. In the end, we use an example to examine Corollary 4, which was rarely noted before. Consider a biproper discrete-time system $L_4(z) = 2(z+2)/(z+0.5)$. By using numerical integration, the CSBI of $L_4(z)$ gives $(2\pi)^{-1}\int_{-\pi}^{\pi}\log|L_4(z)/(1+L_4(z))|d\omega \approx 0.4150 \approx \log 2 + \log(2/3)$, which justifies Corollary 4. For brevity, more examples that verify Corollary 2 and Theorem 3 are omitted in this note, and interested readers are referred to [1]–[10] and related references.

VI. CONCLUSION

A simplified approach for analyzing complementary sensitivity trade-offs in both continuous-time and discrete-time systems was proposed in this note. A comprehensive relationship between CSBIs and the features of open-loop transfer functions was interpreted by using this simplified approach. A few illustrative examples were presented to justify the results. However, similar to the result in [1], the simplified approach is unable to analyze non-rational transfer functions, such as transfer function with time delay [2]–[4]. This might be an interesting topic for future research on sensitivity analysis.

REFERENCES

[1] B. Wu and E. A. Jonckheere, "A simplified approach to Bode's theorem for continuous-time and discrete-time systems," *IEEE Trans. Autom. Control*, vol. 37, no. 11, pp. 1797–1802, 1992.

[2] J. S. Freudenberg and D. P. Looze, "Right half plane poles and zeros and design tradeoffs in feedback systems," *IEEE Trans. Autom. Control*, vol. 30, no. 6, pp. 555–565, 1985.

[3] ——, "A sensitivity tradeoff for plants with time delay," *IEEE Trans. Autom. Control*, vol. 32, no. 2, pp. 99–104, 1987.

[4] M. M. Seron, J. H. Braslavsky, and G. C. Goodwin, *Fundamental Limitations in Filtering and Control*. Springer, 2012.

[5] H. K. Sung and S. Hara, "Properties of complementary sensitivity function in SISO digital control systems," *International Journal of Control*, vol. 50, no. 4, pp. 1283–1295, 1989.

[6] R. H. Middleton, "Trade-offs in linear control system design," *Automatica*, vol. 27, no. 2, pp. 281–292, 1991.

[7] B. Yu, J. S. Freudenberg, R. B. Gillespie, and R. H. Middleton, "Beyond synchronization: String instability in coupled harmonic oscillator systems," *Int. J. Robust Nonlinear Control*, vol. 25, pp. 2745–2769, 2015.

[8] N. Wan, D. Li, and N. Hovakimyan, "Sensitivity analysis of continuous-time systems based on power spectral density," in *Proc. 57th IEEE Conf. Decision Contr.*, Miami Beach, FL, USA, 2018.

[9] K. Okano, S. Hara, and H. Ishii, "Characterization of a complementary sensitivity property in feedback control: An information theoretic approach," *Automatica*, vol. 45, pp. 504–509, 2009.

[10] H. Ishii, K. Okano, and S. Hara, "Achievable sensitivity bounds for mimo control systems via an information theoretic approach," *Systems & Control Letters*, vol. 60, no. 2, pp. 111–118, 2011.