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On the Growth of L^2-Invariants of Locally Symmetric Spaces, II: Exotic Invariant Random Subgroups in Rank One

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In the 1st paper of this series we studied the asymptotic behavior of Betti numbers, twisted torsion, and other spectral invariants for sequences of lattices in Lie groups G. A key element of our work was the study of *invariant random subgroups* (IRSs) of G. Any sequence of lattices has a subsequence converging to an IRS, and when G has higher rank, the Nevo–Stuck–Zimmer theorem classifies all IRSs of G. Using the classification, one can deduce asymptotic statements about spectral invariants of lattices. When G has real rank one, the space of IRSs is more complicated. We construct here several uncountable families of IRSs in the groups $\mathrm{SO}(n, 1)$, $n \geq 2$. We give dimension-specific constructions when $n = 2, 3$, and also describe a general gluing construction that works for every n. Part of the latter construction is inspired by Gromov and Piatetski-Shapiro's construction of non-arithmetic lattices in $\mathrm{SO}(n, 1)$.

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Introduction

This is the 2nd half of our paper *On the growth of L^2-invariants of locally symmetric spaces*, which was posted on the arXiv in 2012 as [2] and which we have split in two for publication (the 1st part appeared as [1]). With the exception of this added introduction, all the sections of this paper appeared in the earlier preprint, and we have preserved the original section numbers so as to not break existing citations.

Invariant random subgroups. Let G be a locally compact 2nd countable group, and Sub_G the set of closed subgroups of G. We consider Sub_G with the Chabauty topology, see [21].

Definition. An *invariant random subgroup* (IRS) of G is a random element of Sub_G whose law μ is a Borel probability measure on Sub_G invariant under the conjugation action of $G \circlearrowright \mathrm{Sub}_G$. Often, we will abusively call μ an IRS as well.

The term IRS was introduced by Abért–Glasner–Virág in [4] for discrete groups, although they were also studied by Vershik [56] under a different name, and we introduced IRSs to Lie groups in our earlier paper [1]. If G acts by measure preserving transformations on a standard probability space X, then almost every stabilizer G_x, where $x \in X$, is a closed subgroup of G, see [55, Theorem 3.2]. Hence, the stabilizer of a random $x \in X$ is an IRS of G. In fact, by [1, Theorem 2.6], all IRSs can be obtained as random stabilizers from probability measure preserving (p.m.p) actions.

Concrete examples of IRSs include normal subgroups of G, as well as random conjugates $g\Gamma g^{-1}$ of a lattice $\Gamma < G$, where the conjugate is chosen by selecting Γg randomly against the given finite measure on $\Gamma\backslash G$. More generally, any IRS H of a lattice $\Lambda < G$ *induces* an IRS of G, obtained by conjugating H by a random element of $\Gamma\backslash G$. We describe this in more detail in Section 11.1 below.

Though the definition of an IRS may seem rather unassuming, even in a very general context there are restrictions on which groups can actually appear in the support of one. For example, an invariant random subgroup supported on amenable subgroups almost surely lies in the amenable radical [9], and in the context of nonpositive curvature there is the stronger statement that an IRS is almost surely "geometrically dense" [23], which is a generalization of the "Borel density theorem" proven in [1]. There is a growing literature on IRSs on more specific groups, in particular construction of examples (see, e.g., [13, 17, 26, 51]) and their applications, see especially [15, 27, 32, 54]. A still open question is whether there exists a simple, non-discrete locally compact group that does not have any IRSs beyond itself and the trivial subgroup; candidates

are the "Neretin groups", which are known not to contain any lattice [8].

Invariant random subgroups in semisimple Lie groups. Suppose now that G is a simple, noncompact center-free Lie group. In [1], we proved the following strong rigidity result for higher-rank IRSs using the Nevo–Stück–Zimmer Theorem [46, 50] and Kazhdan's property (T).

Theorem. If $\text{rank}_{\mathbb{R}} G \geq 2$, the ergodic IRSs of G are exactly $\{e\}$, G, and IRSs constructed as random conjugates of a lattice $\Gamma < G$. Moreover, if the laws of these IRSs are denoted μ_{id}, μ_G, and μ_Γ, then for any sequence of pairwise non-conjugate lattices $\Gamma_n < G$, the measures μ_{Γ_n} weakly converge to μ_{id}.

This completely describes the topological spaces of IRSs in simple higher-rank Lie groups. On the other hand, if G is a group of real rank one then there are always more IRSs than those described in this theorem. For example, every cocompact lattice in G is a Gromov-hyperbolic group and hence contains plenty of normal infinite subgroups of infinite index, see Theorem 17.2.1 of [45]. Taking a random conjugate of one of the latter we obtain an IRS, which violates the conclusion of the theorem above. More generally we can induce from IRSs in lattices that yields further examples as many lattices in $\text{SO}(n, 1)$ (and a few in $\text{SU}(2, 1)$) surject onto nonabelian free groups, which have plenty of IRSs by [14]. It turns out that in the case where $G = \text{SO}(n, 1)$ the wealth of available IRSs goes well beyond these examples, as hyperbolic geometry yields constructions of ergodic IRSs, which are not induced from a lattice. These constructions are the main object of this paper and we will describe them in some detail in the next section.

First, though, let us say a few words about further restrictions on the groups appearing in IRSs of rank one Lie groups. Making the Borel and geometric density theorems referenced above more precise, we will prove the following in Section 11:

Theorem 1. (See Proposition 11.3 for a more general statement) Suppose that μ is an IRS of $SO(n, 1)$ that does not have an atom at $\{id\} \in \text{Sub}_G$. Then μ-a.e. $H \in \text{Sub}_G$ has full limit set, that is, $\Lambda(H) = \partial\mathbb{H}^n$.

In particular, the probability that an IRS is geometrically finite and infinite covolume is zero. In the later work [3], Abért and Biringer prove a much more precise result in dimension 3; any ergodic torsion-free IRS of $\text{SO}(3, 1)$ that is almost surely finitely generated must be either a lattice IRS or be supported on doubly degenerate subgroup (one of our constructions shows that the latter case contains a wealth of interesting examples). In dimension 2 any non-lattice ergodic torsion-free IRS must be supported on infinite-rank-free subgroups; the topology of the corresponding infinite-type surfaces

is very restricted, as described in [12].

Our main motivation in [1] to study IRSs was their applications to the "limit multiplicity problem". As an example, a corollary of our classification of IRSs in higher rank is that for any sequence of cocompact lattices in an irreducible higher-rank Lie group G with property (T), which are in addition "uniformly discrete" (not intersecting a fixed identity neighbourhood in G—conjecturally this is always true if assuming torsion-freeness), the Betti numbers are always sublinear in the covolume except in the middle dimension if G has discrete series.

This result is proven through the notion of *Benjamini–Schramm convergence*, which gives a geometric meaning to the convergence of IRSs in the weak topology, and the generalization of the Lück Approximation Theorem to this setting. Since the space of IRSs is compact there are always limit points; thus, understanding the space of IRSs might in principle give information about the geometry and topology of locally symmetric spaces of large volume.

Constructions of IRSs in real hyperbolic spaces. To describe these examples, it will be more useful to interpret discrete (by a corollary of Borel's density theorem [2, Theorem 2.9], any IRS of a simple Lie group G that does not have an atom at $G \in \mathrm{Sub}_G$ is discrete almost surely) IRSs of $SO(n, 1)$ as *random framed hyperbolic n-orbifolds*.

Whenever Γ is a discrete subgroup of $SO(n, 1)$, the quotient $M_\Gamma = \Gamma\backslash\mathbb{H}^n$ is a hyperbolic n-orbifold. If a baseframe is fixed in $\mathbb{H}^n$, its projection gives a canonical baseframe for M_Γ, and the map $\Gamma \mapsto M_\Gamma$ is a bijection from the set of discrete subgroups of G to the set of isometry classes of framed hyperbolic n-orbifolds. A random Γ then gives a random framed orbifold. Intuitively, a random framed hyperbolic n-orbifold represents an IRS if whenever a particular (unframed) orbifold is chosen, the base frames for that orbifold are distributed according to the natural Riemannian volume on the frame bundle. This is made precise in Abert–Biringer [3], but here we will always just work with the IRS directly.

Example 2. (IRSs from shift spaces, see Section 13) Let N_0, N_1 be two hyperbolic n-manifolds with totally geodesic boundary, such that each boundary is the disjoint union of two isometric copies of some fixed hyperbolic $(n-1)$-manifold Σ. Given a sequence $\alpha = (\alpha_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, construct a hyperbolic n-manifold N_α by gluing copies of N_0, N_1 according to the pattern prescribed by α.

If ν is a shift invariant measure on $\{0,1\}^{\mathbb{Z}}$, construct a random framed hyperbolic n-manifold by picking α randomly with respect to ν, and then picking a vol-random base frame from the frame bundle of the "center" block, that is, the copy of N_0 or N_1

corresponding to $0 \in \mathbb{Z}$. This defines an IRS of $SO(n, 1)$. Moreover, we show that if N_0, N_1 embedded in non-commensurable compact arithmetic n-manifolds M_0, M_1, and α is not supported on a periodic orbit, then the IRS is not induced by a lattice.

In some sense, these shift examples are still quite "finite", in that the manifolds involved are constructed as gluings of compact pieces. So for instance, there is a universal upper bound for the injectivity radius at every point of every manifold in the support of such an IRS. Here is an example that is more truly "infinite type", and where the injectivity radius is not bounded above. Note: one could also construct IRSs with unbounded injectivity radius by summing an appropriate sequence of lattice IRSs against a geometric series, but the following examples are ergodic:

Example 3. (Random trees of pants, see Section 12.1) Let S be a topological surface constructed by gluing together pairs of pants glued together in the pattern dictated by an infinite 3-regular tree. If the set of simple closed curves on S along which two pairs of pants are glued is written $\mathcal{C}$, then one can produce a hyperbolic metric on S by specifying *Fenchel–Nielsen coordinates*, which consist of a choice of length and twist parameter $(l_c, t_c) \in (0, \infty) \times S^1$ for each curve $c \in C$.

Fix a probability measure on $(0, \infty)$, consider S^1 with the Lebesgue probability measure and fix a "center" pair of pants on S. Construct a random framed hyperbolic surface by choosing (l_c, t_c) independently and randomly from $(0, \infty) \times S^1$, equipped with the product measure, and choosing a base frame vol-randomly from the frame bundle of the center pair of pants. The result is a random framed hyperbolic surface corresponding to an ergodic IRS of $SO(2, 1)$. When ν is chosen appropriately, this IRS is not induced from an IRS of a lattice, and when ν has unbounded support, the injectivity radius at the base frame is not bounded above.

In three dimensions, there is a canonical example of a nontrivial finitely generated IRS with infinite covolume; the IRS induced by the infinite cyclic cover $\hat{M}$ of a hyperbolic 3-manifold fibering M over the circle. Namely, Thurston's Hyperbolization Theorem [53] gives many examples of closed hyperbolic 3-manifolds M equipped with a fibration

$$\Sigma \longrightarrow M \longrightarrow S^1,$$

where Σ is a closed surface. On the level of fundamental groups, this becomes

$$1 \longrightarrow \pi_1 \Sigma \longrightarrow M \longrightarrow \mathbb{Z}.$$

Identify $\pi_1 M$ with a lattice in $SO(3, 1)$. Then $\pi_1 \Sigma$ is a normal subgroup of $\pi_1 M$, and hence can be *induced* to an IRS of $SO(3, 1)$, as explained above and in Section 11.1. The resulting IRS is a.e. isomorphic to $\pi_1 \Sigma$, and the quotient manifolds are all isometric to the corresponding $\mathbb{Z}$-cover of $\pi_1 M$, which has infinite volume.

In Section 12.5 we generalize this example by showing that the basic idea behind the shift IRS construction can be performed in three dimensions without producing infinitely generated fundamental group. Namely,

Example 4. (IRSs supported on surface groups, see Section 12.5) Let Σ be a closed, orientable surface and $\phi_1, \dots, \phi_n : \Sigma \longrightarrow \Sigma$ be pseudo-Anosov homeomorphisms that freely generate a Schottky subgroup of $\mathrm{Mod}(\Sigma)$. Choose a sequence (w_i) of words in the letters $\phi_1, \dots, \phi_n$ with $|w_i| \to \infty$ and let M_i be the hyperbolic mapping torus with monodromy w_i. After passing to a subsequence, the measures (μ_{M_i}) weakly converge to an IRS μ of $SO(3, 1)$ that is supported on uniformly thick, doubly degenerate hyperbolic 3-manifolds homeomorphic to $\Sigma \times \mathbb{R}$. Moreover, if the words (w_i) are chosen appropriately, no manifold in the support of μ covers a finite volume manifold, and in particular μ is not induced from an IRS of any lattice.

The homeomorphisms $\phi_1, \dots, \phi_n$ play the role of the building blocks N_0, N_1 in the shift examples, in the sense that the geometry of the mapping torus of each ϕ_k roughly appears as a "block" in the manifold M_n whenever ϕ_k is a letter in w_n, and these blocks will persist in the limit IRS. Making this intuition precise requires us to use Minsky's model manifold machinery for understanding ends of hyperbolic 3-manifolds up to bilipschitz equivalence [40, 44], although in our setting we only need his earliest version for thick manifolds. Although it seems plausible that one could use his machinery to construct the limit IRS directly, it is much easier to proceed as we do above through a limiting argument.

Further questions. The topology of the space of IRSs of $SO(n, 1)$ is quite rich—indeed, it is built on the already rich structure of the Chabauty topology. While a complete understanding of it (as in the higher-rank case) is certainly intractable, we are interested in the following question, which we phrase more generally:

Question 5. Is every ergodic IRS of a simple Lie group G other than μ_G a weak limit of IRSs corresponding to lattices of G?

It is easy to see that all the examples in $SO(n, 1)$ that we have described are weak limits of lattice IRSs, and this is trivially true for higher-rank G. Question 5 is a

continuous analog of a question of Aldous–Lyons [5, Question 10.1] that asks whether every unimodular random graph is a weak limit of finite graphs and that is in turn a generalization of the question of whether all groups are sofic.

While the doubly degerenate examples in dimension 3 are finitely generated, and there are no such examples in dimension 2, the following question is open:

Question 6. Are there non-lattice IRSs in $\mathrm{SO}(n, 1)$, $n \geq 4$ that are finitely generated with positive probability?

Note that it is already unknown whether lattices in $SO(n, 1)$ may have infinite, infinite index finitely generated normal subgroups.

For some groups there are no constructions of IRSs known beyond lattices and their normal subgroups. We may ask the following questions:

- Are there non-lattice, almost surely irreducible ergodic IRSs in product groups such as $\mathrm{PSL}_2(\mathcal{R}) \times \mathrm{PSL}_2(\mathcal{R})$?
- Are there examples of ergodic IRSs in $\mathrm{SU}(n, 1)$, $n \geq 1$, which are not induced from lattices?

11 Limit Sets and Induced IRSs

11.1. Induction

To begin with, let Γ be a lattice in a Lie group G and suppose that Γ contains a normal subgroup Λ; we construct an IRS supported on the conjugacy class of Λ as follows. The map $G \ni g \mapsto g^{-1}\Lambda g \in \mathrm{Sub}_G$ factors as

$$G \longrightarrow \Gamma\backslash G \longrightarrow \mathrm{Sub}_G,$$

and the 2nd arrow pushes forward Haar measure on $\Gamma\backslash G$ to an IRS μ_Λ of G. Geometrically, if $D \subset G$ is a fundamental domain for Γ and $\overline{D}$ is its image in G/Λ, then a μ_Λ-random subgroup is the stabilizer in G of a random point in $\overline{D}$. If Λ is a sublattice of Γ this is coherent with the previous definition.

This construction produces new examples of invariant random subgroups of rank one Lie groups. Namely, lattices in $\mathbb{R}$-rank one simple Lie groups are Gromov hyperbolic and it follows from [29, Theorem 5.5.A] that they contain infinite, infinite index normal subgroups. In other words, the Margulis normal subgroup theorem fails for these groups. Note that if $G = \mathrm{SO}(1, n)$ or $\mathrm{SU}(1, n)$, this can be seen for example

because in any dimension there are compact real or complex hyperbolic manifolds with positive 1st Betti number. Now any such infinite, infinite index normal subgroup Λ of a lattice in one of these groups G gives an IRS supported on the conjugacy class of Λ. In particular, Theorem fails for these G.

The above is a special case of *induction* of an IRS of a lattice Γ to an IRSs of G. To define this more carefully, note that if Γ is a finitely generated group, then an invariant random subgroup of Γ is just a probability measure $\mu \in \mathcal{P}(\mathrm{Sub}_\Gamma)$ that is invariant under conjugation. Examples include the Dirac mass at a normal subgroup, and the mean over the (finitely many) conjugates of a finite index subgroup of a normal subgroup. Less trivially, Bowen [16] has shown that there is a wealth of invariant random subgroups of free groups.

So, let μ be an IRS of a lattice Γ in a Lie group G. Define the IRS of G induced from μ to be the random subgroup obtained by taking a random conjugate of Γ and then a μ-random subgroup in this conjugate (which is well defined because of the invariance of μ). Formally, the natural map

$$G \times \mathrm{Sub}_\Gamma \ni (g, \Lambda) \longrightarrow g\Lambda g^{-1} \in \mathrm{Sub}_G$$

factors through the quotient of $G \times \mathrm{Sub}_\Gamma$ by the Γ-action $(g, \Lambda)\gamma = (g\gamma, \gamma^{-1}\Lambda\gamma)$. This quotient has a natural G-invariant probability measure, and we define our IRS to be the push forward of this measure by the factored map $(G \times \mathrm{Sub}_\Gamma)/\Gamma \to \mathrm{Sub}_G$.

11.2. Limit sets of rank one IRSs

We show in this section that IRSs in rank one groups have either full or empty limit set. This is a trivial application of Poincaré recurrence that works whenever one has a reasonable definition of limit set.

Let G be a simple Lie group with $\mathrm{rank}_{\mathbb{R}}(G) = 1$. The symmetric space G/K is a Riemannian manifold with pinched negative curvature and therefore has a natural Gromov boundary $\partial_\infty X$. The limit set $\Lambda(H)$ of a subgroup $H < G$ is the set of accumulation points on $\partial_\infty X$ of some (any) orbit Hx, where $x \in X$. We say that H is nonelementary if $\Lambda(H)$ contains at least three points.

11.3. Proposition Suppose that G is a simple Lie group with $\mathrm{rank}_{\mathbb{R}}(G) = 1$ and that $H < G$ is a closed, nonelementary subgroup. Let $A \trianglelefteq H$ be the compact, normal subgroup consisting of all elements that fix pointwise the union of all axes of hyperbolic elements of H. Then if μ is an IRS of H, either

(1) the limit set of a μ-random $\Gamma < H$ is equal to $\Lambda(H)$, or
(2) $\mu(\mathrm{Sub}_A) > 0$.

In particular, any IRS of G without an atom at id has limit set $\partial_\infty X$. As a further example explaining condition (2), note that $H = \mathrm{SO}(2, 1) \times \mathrm{SO}(3)$ embeds in $\mathrm{SO}(5, 1)$ and any IRS of $A = \mathrm{SO}(3)$ induces an IRS of H with empty limit set.

Proof. Assume that with positive μ-probability, the limit set of a subgroup $\Gamma < H$ is smaller than $\Lambda(H)$. As $\Lambda(H)$ is 2nd countable, there exists an open set $U \subset \Lambda(H)$ such that $\Lambda(\Gamma) \cap U = \emptyset$ with μ-probability $\epsilon > 0$. Since H is nonelementary, there is a hyperbolic element $h \in H$ with repelling fixed point $\lambda_- \in U$ (see [31, Theorem 1.1]). The element h acts on $\partial_\infty X$ with North–South dynamics [31, Lemma 4.4], and we let $\lambda_+ \in \partial_\infty X$ be its attracting fixed point. Then for each i, the μ-probability that $\Lambda(\Gamma) \cap h^i(U) = \emptyset$ is also ϵ, by H-invariance of μ.

Passing to a subsequence, we may assume that the sets $h^i(U)$ form a nested increasing chain with union $\partial_\infty X \setminus \lambda_+$. Therefore, passing to the limit we have that the μ-probability that $\Lambda(\Gamma) \subset \{\lambda_+\}$ is ϵ. But the μ-probability that $\Lambda(\Gamma) = \{\lambda_+\}$ is zero, since as Γ is nonelementary, the H-orbit of λ_+ is infinite [31, Theorem 1.1] and there is equal probability of having limit set any translate of λ_+. Therefore, $\Lambda(\Gamma) = \emptyset$ with positive μ-probability.

Suppose now that (2) does not hold; then with positive μ-probability we have $\Lambda(\Gamma) = \emptyset$ and $\Gamma \notin A$. Pick some $\Gamma < H$ that satisfies these two conditions such that in every neighborhood of Γ the μ-probability of satisfying the two conditions is positive. Since Γ has empty limit set, Γ must be finite, and therefore has a nonempty fixed set $F_\Gamma \subset X$, which is a totally geodesic hyperplane in X. Since Γ is not contained in A, there is some hyperbolic element $h \in H$ whose axis is not contained in F_Γ. Then fixing $x \in X$,

$$\max_{\gamma \in \Gamma} \mathrm{dist}\left(\gamma \circ h^i(x), h^i(x)\right) \longrightarrow \infty$$

as i increases. Moreover, this is true uniformly over some neighborhood $\mathcal{U}$ of $\Gamma \in \mathrm{Sub}_H$. Namely, for sufficiently small $\mathcal{U}$, we have that as i increases,

$$\inf_{\text{finite } \Gamma' \in \mathcal{U}} \max_{\gamma \in \Gamma'} \mathrm{dist}\left(\gamma \circ h^i(x), h^i(x)\right) \longrightarrow \infty.$$

We can rephrase this by saying that as i increases,

$$\inf_{\text{finite } \Gamma' \in h^{-i}\mathcal{U}h^i} \max_{\gamma \in \Gamma'} \text{dist}\,(\gamma(x), x) \longrightarrow \infty.$$

Thus, for an infinite collection of indices i, the subsets $h^{-i}\mathcal{U}h^i$ are pairwise disjoint in Sub_H. This is a contradiction, since they all have the same positive μ-measure. ■

If $\Gamma < G$ is a subgroup whose limit set is not the full boundary $\partial_\infty G$, then there is no upper bound for the local injectivity radius $inj_{\Gamma\backslash X}(x)$ at points $x \in \Gamma\backslash X$. Suppose that μ is an ergodic IRS of G. One can ask whether the function

$$\text{Sub}_G \to \mathbb{R}, \quad \Gamma \mapsto inj_{\Gamma\backslash X}([id])$$

necessarily have finite μ-expected value. Here, $[id]$ is the projection of the identity element under $G \longrightarrow \Gamma\backslash X = \Gamma\backslash G/K$. Dropping the ergodicity condition one can easily construct convex combinations of lattice IRSs with infinite expected injectivity radius. Also, in Section 12.1 we construct ergodic IRSs that have unbounded injectivity radius.

12 Exotic IRSs in Dimensions 2 and 3

12.1. Random trees of pants—examples in $G = \text{SO}(2, 1)$

The idea for this construction was suggested by Lewis Bowen.

Suppose that S is a topological surface obtained by gluing together pairs of pants in the pattern dictated by a 3-valent graph X, that is, properly embed X in $\mathbb{R}^3$ and let S be the boundary of some regular neighborhood of it. Let $\mathcal{C}$ be the set of simple closed curves on S corresponding to the boundary components of these pants. Given a function

$$\mathcal{C} \to (0, \infty) \times S^1, \quad c \mapsto (l_c, t_c),$$

called *Fenchel–Nielsen coordinates*, we can construct a hyperbolic structure on S by gluing together hyperbolic pairs of pants whose boundary curves have lengths l_c, and where the two pairs of pants adjacent to a boundary curve are glued with twisting parameter t_c. The resulting structure is well defined up to an isometry that fixes the homotopy class of every curve in $\mathcal{C}$. See [39] for finite-type surfaces, and [6] for a discussion of issues in the infinite-type case.

Pick a Borel probability measure ν on $(0,\infty)$ and consider S^1 with Lebesgue probability measure λ. We then have a probability measure $(\nu\times\lambda)^{\mathcal{C}}$ on the moduli space $\mathcal{M}(S)$. Fix some pair of pants $P\subset S$ bounded by curves in $\mathcal{C}$. We create an IRS μ of G as follows. Randomly select an element $[d]\in\mathcal{M}(S)$, represented by a hyperbolic metric d on S. Let $P_d\subset S$ be the totally d-geodesic pair of pants in the homotopy class of P and pick a base frame f on P_d randomly with respect to its Haar probability measure m_d. The stabilizer $\mathrm{Stab}(d,f)$ of f under the G-action on the frame bundle of (S,d) is a μ-random subgroup of G.

More formally, if $A\subset \mathrm{Sub}_G$ is a Borel subset, let

$$\mu(A)=\int_{d\in\mathcal{M}(S)} m_d\left(\begin{smallmatrix}\text{frames } f \text{ on } P_d\\ \text{with } \mathrm{Stab}(d,f)\in A\end{smallmatrix}\right)\, d(\nu\times\lambda)^{\mathcal{C}}.$$

12.2. Proposition If the 3-valent graph X is vertex transitive, then μ is G-invariant. Moreover, if X is also infinite then μ is ergodic.

Proof. Fix $g\in G$ and some Borel subset $A\subset \mathrm{Sub}_G$. Subdividing A if necessary, we may assume that there is some pair of pants $P'\subset S$ bounded by curves in $\mathcal{C}$ such that if d is a hyperbolic metric on X as above and f is a frame on P_d, then

$$\mathrm{Stab}(d,f)\in A \implies gf\in P'_d,$$

where P'_d is the totally d-geodesic pair of pants homotopic to P'. Then we have

$$\begin{aligned}\mu(A)&=\int_{d\in\mathcal{M}(S)} m_d\left(\begin{smallmatrix}\text{frames } f \text{ on } P'_d\\ \text{with } \mathrm{Stab}(d,f)\in gA\end{smallmatrix}\right)\, d(\nu\times\lambda)^{\mathcal{C}}\\ &=\int_{d'\in\mathcal{M}(S)} m_{d'}\left(\begin{smallmatrix}\text{frames } f \text{ on } P_{d'}\\ \text{with } \mathrm{Stab}(d',f)\in gA\end{smallmatrix}\right)\, d(\nu\times\lambda)^{\mathcal{C}}\\ &=\mu(gA).\end{aligned}$$

Here, the 1st equality follows from our assumption on A and the fact that g acts as a measure preserving homeomorphism on the frame bundle of (S,d). For the 2nd, let $v\mapsto v'$ be a graph isomorphism of X taking the vertex corresponding to P to that corresponding to P'. Then there is an induced map $d\mapsto d'$ on $\mathcal{M}(S)$; this map preserves the measure $(\nu\times\lambda)^{\mathcal{C}}$, so the 2nd inequality follows.

Now suppose that X is infinite and $A\subset \mathrm{Sub}_G$ is a G-invariant set. Define

$$\overline{A}=\{\, d\in\mathcal{M}(S)\mid \exists \text{ a frame } f \text{ on } P_d \text{ with } \mathrm{Stab}(d,f)\in A\,\}.$$

The set $\overline{A}$ is invariant under the action on $\mathcal{M}(S)$ corresponding under Fenchel–Nielsen coordinates to the subgroup of $\mathcal{C}$-permutations arising from graph automorphisms of X. As X is infinite and vertex transitive, for every finite subset F of $\mathcal{C}$ there is such a permutation such that F is disjoint from its image. It follows from a standard argument that the $(\nu\times\lambda)^{\mathcal{C}}$-measure of $\overline{A}$ is either 0 or 1. As A is G-invariant, it can be recovered from $\overline{A}$ as the set of stabilizers of all frames on (S, d) where d ranges through $\overline{A}$. Therefore, $\mu(A)$ is either 0 or 1. ■

If X is infinite and vertex transitive, and ν is nonatomic and supported within $(0, \epsilon)$, where ϵ is less than the Margulis constant, the measure μ cannot be induced from a lattice. For then with full $(\nu \times \lambda)^{\mathcal{C}}$-probability, the length parameters of the Fenchel–Nielsen coordinates of a point in $\mathcal{M}(S)$ cannot be partitioned into finitely many rational commensurability classes. In a finite volume hyperbolic surface, there are only finitely many closed geodesics with length less than ϵ. If a hyperbolic surface isometrically covers a finite volume hyperbolic surface, then the lengths of its closed geodesics that are shorter than ϵ can be partitioned into finitely many rational commensurability classes. Therefore, at most a measure zero set of Fenchel–Nielsen coordinates give hyperbolic structures on S that isometrically cover finite volume hyperbolic surfaces. This shows that μ cannot be induced from a lattice.

There is one additional feature of this example that is of interest.

12.3. Proposition If X is an infinite 3-valent tree and ν has unbounded support, then the injectivity radius at the base frame of a framed hyperbolic surface has infinite μ-essential supremum.

To prove the proposition, we need the following lemma.

12.4. Lemma Suppose $l > 0$ and P is a hyperbolic pair of pants with geodesic boundary all of whose boundary components have length in $[l, l + 1]$. Let γ be a geodesic segment in P that has endpoints on ∂P but is not contained in ∂P. Then

$$\text{length}(\gamma) \geq \sinh\left(\frac{1}{\sinh(l)}\right).$$

Also, if the endpoints of γ lie on the same component of ∂P then $\text{length}(\gamma) \geq \frac{l-1}{2}$.

Proof. It suffices to prove the lemma when γ is a simple closed curve. For the 1st part, double P to obtain a closed hyperbolic surface of genus 2. There is a closed geodesic $\bar{\gamma}$ on this surface homotopic to the double of γ; this has length at most twice that of γ. The Collar Lemma [24, Lemma 12.6] states that the radius $\sinh^{-1}\left(1/\sinh\left(\frac{1}{2}\text{length}\bar{\gamma}\right)\right)$-neighborhood of $\bar{\gamma}$ is an annulus; as some boundary curve of P intersects $\bar{\gamma}$, this radius is at most l. This gives the 1st inequality.

If the endpoints of γ lie on the same component of ∂P, they part that component into two arcs α and β. Without loss of generality, $\text{length}(\alpha) \leq \frac{l+1}{2}$. But the concatenation $\gamma \cdot \alpha$ is homotopic to one of the other two boundary components of P, so it must have length at least l. The lemma follows. ∎

Proof of Proposition 12.3. Recall that to pick a μ-random framed hyperbolic surface, we choose length and twist parameters for each edge of X, produce from these a hyperbolic metric d on the surface S and then choose a base frame randomly from a totally geodesic pair of pants P_d on S corresponding to some root of X. Fix some large l, $R > 0$ such that $\nu([l, l+1]) > 0$. Then with positive probability the length parameters for every edge in an R-ball around the root in X are within $[l, l+1]$. It follows that the injectivity radius at any point $p \in P_d$ is at least

$$\star = \min\left\{\frac{l-1}{2},\ R\sinh\left(\frac{1}{\sinh(l)}\right)\right\}.$$

To see this, note that the injectivity radius is realized as the length of a geodesic segment that starts and terminates at p. Either this geodesic enters and leaves the same boundary component of some pair of pants with boundary lengths in $[l, l+1]$, in which case the 1st estimate applies, or it passes through at least R such pants and the 2nd applies. As $\star$ can be made arbitrarily large, the proposition follows.

∎

12.5. IRSs of $SO(3,1) \cong \mathrm{PSL}_2\mathbb{C}$ supported on thick surface groups

Suppose that Σ is a closed, orientable surface of genus g. In this section we construct a large family of IRSs of $\mathrm{PSL}_2\mathbb{C}$ that are supported on subgroups Γ with $\mathbb{H}^3/\Gamma$ homeomorphic to $\Sigma \times \mathbb{R}$. These examples are similar in spirit to those constructed—in any dimensions—by gluings in the following section, but have the added feature that they are supported on finitely generated subgroups of $\mathrm{PSL}_2\mathbb{C}$.b

12.6.

The construction makes use of the action of the mapping class group $\mathrm{Mod}(\Sigma)$ on the Teichmuller space $\mathbb{T}(\Sigma)$; we refer to [24], [33], [39] for the general theory. We identify $\mathbb{T}(\Sigma)$ as the space of equivalence classes of hyperbolic metrics on Σ, where two metrics d_0, d_1 are equivalent if there is an isometry $(\Sigma, d_0) \to (\Sigma, d_1)$ homotopic to the identity map. We will sometimes denote elements of Teichmuller space as (Σ, d) and sometimes as X, depending on context. The group $\mathrm{Mod}(\Sigma)$ acts properly discontinuously on $\mathbb{T}(\Sigma)$ and the quotient is the moduli space of all hyperbolic metrics on Σ. Teichmuller space admits a natural *Teichmuller metric* (see [33]), with respect to which $\mathrm{Mod}(\Sigma)$ acts by isometries. Thurston has shown [52] how to give the union of $\mathbb{T}(\Sigma)$ with the space of projective measured lamination space $\mathcal{PML}(\Sigma)$ a natural topology so that the resulting space is homeomorphic to a ball of dimension $6g - 6$, with $\mathbb{T}(\Sigma)$ as the interior and $\mathcal{PML}(\Sigma)$ as the boundary. This topology is natural, in the sense that the action of $\mathrm{Mod}(\Sigma)$ on $\mathbb{T}(\Sigma)$ extends continuously to the natural action of $\mathrm{Mod}(\Sigma)$ on $\mathcal{PML}(\Sigma)$.

Our construction of IRSs of $\mathrm{PSL}_2\mathbb{C}$ relies on the following definition.

Definition 12.7. (Farb–Mosher) A finitely generated, free subgroup $F \subset \mathrm{Mod}(\Sigma)$ is *Schottky* if any orbit of the action of F on $\mathbb{T}(\Sigma)$ is *quasi-convex*, that is, after fixing $X \in \mathbb{T}(\Sigma)$, there are some $C > 0$ such that any Teichmuller geodesic segment that joins two points from the orbit $F(X)$ lies in a C-neighborhood of $F(X)$.

Remark. Farb and Mosher [25] have shown that if $\phi_1, \dots, \phi_n$ are pseudo-Anosov elements of $\mathrm{Mod}(\Sigma)$ with pairwise distinct attracting and repelling laminations, then for all choices of sufficiently large exponents the elements $\phi_1^{e_1}, \dots, \phi_n^{e_n}$ freely generate a purely pseudo-Anosov Schottky subgroup of $\mathrm{Mod}(\Sigma)$.

Suppose from now on that $\phi_1, \dots, \phi_n \in \mathrm{Mod}(\Sigma)$ freely generate a Schottky subgroup $F \subset \mathrm{Mod}(\Sigma)$. Choose a sequence of finite strings

$$e^1 = \left(e^1_1, \dots, e^1_{n_1}\right), \quad e^2 = \left(e^2_1, \dots, e^2_{n_2}\right), \quad \dots$$

with entries in $\{0, \dots, n\}$ and let C be the sub-shift of $\{0, \dots, n\}^{\mathbb{Z}}$ consisting of strings all of whose finite substrings are contained in e^i for some i. Set

$$f_i : \Sigma \to \Sigma, \quad f_i = \phi_{e^i_{n_i}} \circ \cdots \circ \phi_{e^i_1}.$$

A celebrated theorem of Thurston [53] then implies that each *mapping torus*

$$M_{f_i} = \Sigma \times [0,1]/(x,0) \sim (f_i(x),1)$$

admits a (unique) hyperbolic metric. We let μ_i be the corresponding IRS of $\mathrm{PSL}_2\mathbb{C}$.

Theorem 12.8. Any weak limit of a subsequence of (μ_i) is an IRS μ of $\mathrm{PSL}_2\mathbb{C}$ that is supported on subgroups $\Gamma < \mathrm{PSL}_2\mathbb{C}$ with $\mathbb{H}^3/\Gamma$ homeomorphic to $\Sigma \times \mathbb{R}$. Moreover, if the shift space C does not contain periodic sequences, no subgroup $\Gamma < \mathrm{PSL}_2\mathbb{C}$ in the support of μ is contained in a lattice of $\mathrm{PSL}_2\mathbb{C}$.

Before beginning the proof in earnest, we give a motivational outline. The idea is to use the Schottky group $F := \langle \phi_1, \dots, \phi_n \rangle$ to associate to every element $\gamma \in \{0, \dots, n\}^{\mathbb{Z}}$ of the shift space a pair consisting of the following elements:

(1) a hyperbolic 3-manifold N_γ homeomorphic to $\Sigma \times \mathbb{R}$;
(2) a "coarse base point" P_γ, that is, a subset of N_γ with universally bounded diameter.

Shifting a string γ corresponds to shifting the base point of N_γ and convergence of $\gamma_i \in \{0, \dots, n\}^{\mathbb{Z}}$ corresponds to based Gromov–Hausdorff convergence of the associated pairs $(N_{\gamma_i}, P_{\gamma_i})$. A periodic string with period $(e_1, \dots, e_k)$ corresponds to the infinite cyclic cover of the mapping torus $M_{\phi_{e_1},\dots,\phi_{e_k}}$, where the placement of the base point depends on the particular shift of the periodic string; moreover, no aperiodic string produces a hyperbolic 3-manifold that covers a finite volume manifold. If the IRSs μ_i limit to μ as in the statement of the theorem, then the support of μ consists of subgroups $\Gamma < \mathrm{PSL}_2\mathbb{C}$ with $\mathbb{H}^3/\Gamma$ a based Gromov–Hausdorff limit of the mapping tori M_{f_i} (in fact, of their infinite cyclic covers) under some choice of base points. Using the correspondence above, we see that such $\mathbb{H}^3/\Gamma$ arise from elements of $\{0, \dots, n\}^{\mathbb{Z}}$ that are limits of periodic sequences used in producing the mapping tori M_{f_i}. Varying the base points chosen on M_{f_i} gives Gromov–Hausdorff limits corresponding exactly to the elements of the subshift $C \subset \{0, \dots, n\}^{\mathbb{Z}}$. Therefore, as long as C does not contain periodic sequences, no such Gromov–Hausdorff limit can cover a finite volume hyperbolic 3-manifold.

The correspondence between elements of $\{0, \dots, n\}^{\mathbb{Z}}$ and coarsely based hyperbolic 3-manifolds occupies most of the exposition to follow. It will be convenient to embed the shift space inside of an auxiliary space, consisting of geodesics in the

Schottky group $F = \langle \phi_1, \dots, \phi_n \rangle$. Namely, we consider F with its *word metric*

$$\operatorname{dist}(g, h) = \min \left\{ k \mid h^{-1} g = \phi_{i_1} \dots \phi_{i_k} \right\},$$

and define the *space of geodesics* in F as the set

$$\mathcal{G}(F) := \left\{ \gamma : \mathbb{Z} \longrightarrow F \mid \gamma \text{ word-isometric embedding} \right\},$$

which we consider with the compact-open topology. The space of geodesics $\mathcal{G}(F)$ has a natural *shift operator* defined by the formula

$$S : \mathcal{G}(F) \longrightarrow \mathcal{G}(F), \quad S(\gamma)(i) = \gamma(i-1).$$

The group F acts on $\mathcal{G}(F)$ via $(g \cdot \gamma)(x) = g\gamma(x)$ and the quotient $\mathcal{G}(F)/F$ can be identified with the space of geodesics $\gamma : \mathbb{Z} \to F$ with $\gamma(0) = 1$. Note that the shift operator S descends to another "shift operator", also called S, on $\mathcal{G}(F)/F$. Finally, there is then a natural shift invariant embedding

$$\{0, \dots, n\}^{\mathbb{Z}} \longrightarrow \mathcal{G}(F)/F, \quad e \mapsto [\gamma_e]$$

determined by the constraint $\gamma_e(i)^{-1} \gamma_e(i+1) = \phi_{e_i}$.

It now remains to relate elements of $\mathcal{G}(F)/F$ to hyperbolic 3-manifolds. The key to this is Minsky's theorem [41] that ϵ-thick doubly degenerate hyperbolic 3-manifolds homeomorphic to $\Sigma \times \mathbb{R}$ are modeled on geodesics in the Teichmuller space $\mathbb{T}(\Sigma)$. The plan for the rest of the section is as follows. In Section 12.9, we show how to go from elements of $\mathcal{G}(F)/F$ to Teichmuller geodesics. Section 12.13 describes Minsky's theorem above, and the following section completes the relationship between elements of $\mathcal{G}(F)/F$ and hyperbolic 3-manifolds. Section 12.20 shows that convergence in $\mathcal{G}(F)/F$ translates to based Gromov–Hausdorff convergence of hyperbolic 3-manifolds. We then indicate how shift-periodic elements of $\mathcal{G}(F)/F$ correspond to infinite cyclic covers of mapping tori and end with a section devoted to the proof of Theorem 12.8.

12.9. Geodesics in F and thick Teichmuller geodesics

Let $O : F \longrightarrow \mathbb{T}(\Sigma)$ be some orbit of the action $F \circlearrowright \mathbb{T}(\Sigma)$. By [25, Theorem 1.1], the map O is a quasi-isometric embedding that extends continuously to an F-equivariant embedding $O : \partial_\infty F \longrightarrow \mathcal{PML}(\Sigma)$. The following lemma is implicit in [25].

Lemma 12.10. For sufficiently large $C > 0$, if $\gamma : \mathbb{Z} \to F$ is any geodesic then there is a Teichmuller geodesic $\alpha_\gamma : \mathbb{R} \to \mathbb{T}(\Sigma)$ such that the Hausdorff distance between $O \circ \gamma(\mathbb{Z})$ and $\alpha_\gamma(\mathbb{R})$ is at most C. If we require

$$\lim_{t\to\infty} \alpha_\gamma(t) = \lim_{i\to\infty} O \circ \gamma(i), \quad \lim_{t\to-\infty} \alpha_\gamma(t) = \lim_{i\to-\infty} O \circ \gamma(i),$$

then α_γ is unique up to orientation preserving reparameterization.

Proof. By quasi-convexity of $O(F) \subset \mathbb{T}(\Sigma)$, each of the geodesic segments

$$\alpha_i = [O \circ \gamma(-i), O \circ \gamma(i)] \subset \mathbb{T}(\Sigma)$$

is contained in the ϵ-thick part

$$\mathbb{T}_\epsilon(\Sigma) = \{(\Sigma, d) \in \mathbb{T}(\Sigma) \mid inj(\Sigma, d) \geq \epsilon\}$$

of $\mathbb{T}(\Sigma)$ for some universal $\epsilon = \epsilon(F, O)$. Therefore, Theorem 4.2 from [43] implies that there exists some $C = C(F, O) > 0$ such that $\{O \circ \gamma(-i), \ldots, O \circ \gamma(i)\}$ is contained in a C-neighborhood of α_i for all i. This means that all α_i pass within a bounded distance of $O \circ \gamma(0)$, so Arzela–Ascoli's theorem guarantees that after passing to a subsequence (α_i) converges to a geodesic $\alpha_\gamma : \mathbb{R} \to \mathbb{T}(\Sigma)$. Then $O \circ \gamma(\mathbb{Z})$ lies in a C-neighborhood of $\alpha_\gamma(\mathbb{R})$, and thus also $\alpha_\gamma(\mathbb{R})$ lies in a C'-neighborhood of $O \circ \gamma(\mathbb{Z})$ for some C' depending on C and the distortion constants of the quasi-isometric embedding O. The uniqueness is [25, Lemma 2.4]. ■

Note in particular that α_γ has image contained in the ϵ-thick part $\mathbb{T}_\epsilon(\Sigma)$ for some universal $\epsilon = \epsilon(F, O)$.

Remark 12.11. Here one word of caution about Thurston's boundary of Teichmuller space is in order; if $\gamma : [0, \infty) \to \mathbb{T}(\Sigma)$ is a geodesic ray, then it is not always true that γ converges to a single point in the boundary $\mathcal{PML}(\Sigma)$ [35]. It is, however, true if γ is contained in $\mathbb{T}_\epsilon(\Sigma)$, the ϵ-thick part of Teichmuller space, as follows from three theorems of Masur [36], [37], [38, Theorem 1.1]. We state these as a lemma.

12.12. Lemma (Masur) Fix $\epsilon > 0$ and let $\gamma : [0, \infty) \to \mathbb{T}_\epsilon(\Sigma)$ be a geodesic ray. Then γ converges to a point $\lambda \in \mathcal{PML}(\Sigma)$, and the lamination λ is filling and uniquely ergodic.

Furthermore, if two geodesic rays $\gamma, \gamma' : [0, \infty) \to \mathbb{T}_\epsilon(\Sigma)$ converge to the same point of $\mathcal{PML}(\Sigma)$, then they are asymptotic.

12.13. Doubly degenerate hyperbolic structures on $\Sigma \times \mathbb{R}$

In this section we review some well-known facts about the geometry of hyperbolic structures on $\Sigma \times \mathbb{R}$ and present Minsky's theorem that thick doubly degenerate hyperbolic structures are modeled on Teichmuller geodesics. To begin, consider the deformation space

$$AH(\Sigma) = \{\rho : \pi_1\Sigma \to \mathrm{PSL}_2\mathbb{C} \mid \rho \text{ discrete, faithful}\}/\mathrm{PSL}_2\mathbb{C},$$

where the quotient is with respect to the conjugation action of $\mathrm{PSL}_2\mathbb{C}$. We consider $AH(\Sigma)$ with the topology induced from the compact-open topology on the representation variety $\mathrm{Hom}(\pi_1\Sigma, \mathrm{PSL}_2\mathbb{C})$. The space $AH(\Sigma)$ has a geometric interpretation as the set of "marked isometry types" of hyperbolic structures on $\Sigma \times \mathbb{R}$:

$$AH(\Sigma) \cong \left\{(N, \mu) \;\middle|\; \begin{array}{c} N \text{ is a hyperbolic 3-manifold} \\ \text{homeomorphic to } \Sigma\times\mathbb{R} \end{array}, \begin{array}{c} \mu:\Sigma\longrightarrow N \text{ is a} \\ \text{homotopy equivalence} \end{array}\right\} \Big/ \sim,$$

where $(N_1, \mu_1) \sim (N_2, \mu_2)$ when there is an isometry $\phi : N_1 \to N_2$ with $\phi \circ \mu_1$ homotopic to μ_2. Here, the correspondence assigns to a representation ρ the quotient manifold $N = \mathbb{H}^3/\rho(\pi_1\Sigma)$ and a marking $\mu : \Sigma \to N$ such that composing $\mu_* : \pi_1\Sigma \to \pi_1 N$ with the holonomy map gives ρ up to conjugacy. We will use these two descriptions of $AH(\Sigma)$ interchangeably.

Suppose now that N is a complete hyperbolic 3-manifold with no cusps that is homeomorphic to $\Sigma \times \mathbb{R}$. The two ends of N admit a geometric classification; very loosely, an end of N is *convex cocompact* if level surfaces increase in the area as they exit the end, while an end is *degenerate* if the areas of level surfaces stay bounded. We refer the reader to [39] for actual definitions and further exposition.

One says N is *doubly degenerate* if both its ends are degenerate. We write

$$\mathcal{DD}(\Sigma) = \{\,(N, \mu) \in AH(\Sigma) \mid N \text{ doubly degenerate}\}$$

for the space of all doubly degenerate elements of the deformation space $AH(\Sigma)$.

If (N, μ) is doubly degenerate, each of its ends has an *ending lamination*, a geodesic lamination on Σ that captures the geometric degeneration of level surfaces exiting that end (see [39]). Ending laminations are always filling and they always admit

a transverse measure of full support [42, Lemmas 2.3 and 2.4]. It is therefore convenient to define the space $\mathcal{EL}(\Sigma)$ as the set of supports of filling measured laminations on Σ; in particular, the weak-* topology on (filling) measured laminations descends to a natural topology on $\mathcal{EL}(\Sigma)$. We then have a function

$$\mathcal{E} : \mathcal{DD}(\Sigma) \longrightarrow \mathcal{EL}(\Sigma) \times \mathcal{EL}(\Sigma)$$

that takes a doubly degenerate hyperbolic 3-manifold to its two ending laminations.

The following theorem is well known; the injectivity is Thurston's Ending Lamination Conjecture, recently resolved by Brock–Canary–Minsky [18], and the topological content follows from standard arguments and Brock's proof of the continuity of Thurston's length function [19]. However, a nicely written proof of the full statement was recorded by Leininger and Schleimer in [34].

12.14. Theorem (Theorem 6.5, [34]) The map above gives a homeomorphism

$$\mathcal{E} : \mathcal{DD}(\Sigma) \longrightarrow \mathcal{EL}(\Sigma) \times \mathcal{EL}(\Sigma) - \Delta$$

onto the space of distinct pairs of elements of $\mathcal{EL}(\Sigma)$.

A fundamental result of Minsky [42] states that doubly degenerate hyperbolic 3-manifolds with injectivity radius bounded away from zero are "modeled" on universal curves over bi-infinite geodesics in Teichmuller space. This was an early part of Minsky's program to prove Thurston's Ending Lamination Conjecture, which as noted above was established by Brock–Canary–Minsky in [18]. The following theorem is a major step in this work.

12.15. Theorem (Minsky [41]) If $(N, \mu) \in \mathcal{DD}(\Sigma)$ is a doubly degenerate hyperbolic 3-manifold and $inj(N) \geq \epsilon > 0$, there is some $C = C(\epsilon, \Sigma)$ and a bi-infinite Teichmuller geodesic $\alpha : \mathbb{R} \longrightarrow \mathbb{T}(\Sigma)$ with the following properties:

(1) If (Σ, d) is a point on the geodesic $\alpha(\mathbb{R}) \subset \mathbb{T}(\Sigma)$, there is a point $(\Sigma, d') \in \mathbb{T}(\Sigma)$ at distance at most C from (Σ, d) and a pleated surface $f_d : (\Sigma, d') \longrightarrow N$ in the homotopy class of μ.

(2) If $f : (\Sigma, d') \longrightarrow N$ is a pleated surface in the homotopy class of μ, then the distance in $\mathbb{T}(\Sigma)$ between (Σ, d') and the image $\alpha(\mathbb{R})$ is at most C.

Moreover, α limits to exactly two points on $\mathcal{PML}(\Sigma) = \partial\mathbb{T}(\Sigma)$, the two projective measured laminations supported on the two ending laminations $\mathcal{E}(N,\mu)$. If

$$\mathcal{E}(N,\mu) = \left(\lim_{t\to\infty}\alpha(t), \lim_{t\to-\infty}\alpha(t)\right),$$

then α is unique up to orientation preserving reparameterization. In this case, we say that (N,μ) is modeled on *the geodesic* $\alpha \subset \mathbb{T}(\Sigma)$.

Here, a *pleated surface* $f : (\Sigma, d') \longrightarrow N$ is a map that is an isometric embedding on the complement of some geodesic lamination on (Σ, d'). Note that in particular any pleated surface is 1-lipschitz. We refer the reader to [39] for more details.

Remark. We should mention that the statement of Theorem 12.15 given here is slightly different than that of [41]. First of all, his theorem is more general since it deals with arbitrary hyperbolic 3-manifolds rather than doubly degenerate structures on $\Sigma \times \mathbb{R}$. Also, his (1) states that (Σ, d) can be mapped into N by a map with bounded *energy*, rather than giving a pleated map from a nearby point in $\mathbb{T}(\Sigma)$. The version of (1) above can be deduced from his (1) and Proposition 6.2 from his paper. Next, Minsky's statement does not reference uniqueness of α or its endpoints in $\mathcal{PML}(\Sigma)$. However, if a geodesic α satisfies (1) then it follows from Corollary 9.3 of [42] all accumulation points of α in $\mathcal{PML}(\Sigma)$ are supported on one of the ending laminations of (N,μ). Furthermore, (1) implies immediately that α is contained in some ϵ'-thick part of Teichmuller space, so Theorem 12.12 shows that the ending laminations of (N,μ) support unique projective measured laminations and that these are the endpoints of α in $\mathcal{PML}(\Sigma)$. Theorem 12.12 shows that any two such α lie at bounded Hausdorff distance, so as any such α is contained in the thick part of Teichmuller space the uniqueness follows from [25, Lemma 2.4].

12.16. Proposition Suppose that $\alpha : \mathbb{R} \longrightarrow \mathbb{T}(\Sigma)$ is a geodesic in the ϵ-thick part of $\mathbb{T}(\Sigma)$. Then there is a unique $(N,\mu) \in \mathcal{DD}(\Sigma)$ that is modeled on α, in the sense of Theorem 12.15. Moreover, $inj(N) \geq \epsilon'$ for some $\epsilon' > 0$ depending only on ϵ.

Proof. Let $\lambda_+, \lambda_- \in \mathcal{PML}(\Sigma)$ be the endpoints of α given by Theorem 12.12. Then by Theorem 12.14, there is a doubly degenerate hyperbolic 3-manifold $(N,\mu) \in \mathcal{DD}(\Sigma)$ with ending laminations $\{|\lambda_+|, |\lambda_-|\}$. Combining theorems of Minsky [44] and Rafi [47], we see that as α lies in the ϵ-thick part of $\mathbb{T}(\Sigma)$ there is a lower bound $inj(N) \geq \epsilon' > 0$ for

the injectivity radius of N. Then Theorem 12.15 shows that there is some Teichmuller geodesic $\alpha' \subset \mathbb{T}(\Sigma)$ on which (N, μ) is modeled. But the uniqueness statement in Theorem 12.12 then implies that $\alpha' = \alpha$. ■

12.17. Geodesics in F and hyperbolic 3-manifolds

Now suppose that $\gamma \in \mathcal{G}(F)$ and let $\alpha_\gamma : \mathbb{R} \to \mathbb{T}(\Sigma)$ represent the oriented Teichmuller geodesic at bounded Hausdorff distance from $O \circ \gamma(\mathbb{Z})$ given by Lemma 12.10. Let

$$(N_\gamma, \mu) \in \mathcal{DD}(\Sigma)$$

be the unique doubly degenerate hyperbolic 3-manifold modeled on the geodesic α_γ, as given by Proposition 12.16. Then recalling that $\gamma(0) \in \mathrm{Mod}(\Sigma)$, we define

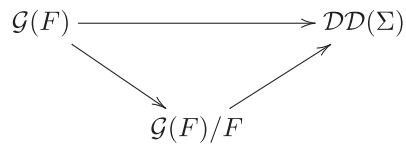
$$\mu_\gamma = \mu \circ \gamma(0) : \Sigma \longrightarrow N_\gamma.$$

This gives a map $\mathcal{G}(F) \longrightarrow \mathcal{DD}(\Sigma)$ defined by $\gamma \mapsto (N_\gamma, \mu_\gamma)$.

Remark. It follows from Proposition 12.16 that the manifolds N_γ above all have injectivity radius $inj(N_\gamma) \geq \epsilon$ for some $\epsilon > 0$ depending only on F and O.

12.18.

The map : $\mathcal{G}(F) \longrightarrow \mathcal{DD}(\Sigma)$ factors as



To see this, suppose that $\gamma \in \mathcal{G}(F)$ and $g \in F$. Then $O(g \cdot \gamma) = gO(\gamma)$, so the nearby Teichmuller geodesics satisfy $\alpha_{g\gamma} = g\alpha_\gamma$. However, if (N_γ, μ) is modeled on α_γ then it follows immediately from the conditions in Theorem 12.15 that $(N_\gamma, \mu \circ g^{-1})$ is modeled on $\alpha_{g\gamma}$. So, $N_\gamma = N_{g\gamma}$ and $\mu_{g\gamma} = \mu \circ g^{-1} \circ (g \cdot \gamma(0)) = \mu \circ \gamma(0) = \mu_\gamma$.

12.19. Proposition The map $\mathcal{G}(F)/F \to \mathcal{DD}(\Sigma)$, $[\gamma] \mapsto (N_\gamma, \mu_\gamma)$ is an embedding.

Proof. As mentioned above, the quotient map $\mathcal{G}(F) \longrightarrow \mathcal{G}(F)/F$ restricts to a homeomorphism on the subset $\mathcal{G}_0(F)$ consisting of geodesic $\gamma \in \mathcal{G}(F)$ with $\gamma(0) = 1$. Using the

notation of Section 12.17, for $\gamma \in \mathcal{G}_0(F)$ we have $\mu = \mu_\gamma$. Therefore,

$$\begin{array}{ccc} \mathcal{G}_0(F) & \xrightarrow{\gamma \mapsto (N_\gamma, \mu_\gamma)} & \mathcal{DD}(\Sigma) \\ & {\scriptstyle E} \searrow \quad \nearrow {\scriptstyle \mathcal{E}^{-1}} & \\ & \mathcal{EL}(\Sigma) \times \mathcal{EL}(\Sigma) - \Delta & \end{array}$$

commutes, where E takes a geodesic $\gamma : \mathbb{Z} \to F$ to the pair of supports of

$$\left(O\left(\lim_{t \to \infty} \gamma(t) \right), O\left(\lim_{t \to -\infty} \gamma(t) \right) \right) \in \mathcal{PML}(\Sigma) \times \mathcal{PML}(\Sigma) - \Delta$$

and $\mathcal{E}$ is the map that takes a doubly degenerate hyperbolic 3-manifold to its pair of ending laminations (see Theorem 12.14). The maps E and $\mathcal{E}^{-1}$ are continuous and injective by Lemma 12.10 and Theorem 12.14, respectively. As the domain $\mathcal{G}_0(F)$ is compact and the co-domain $\mathcal{DD}(\Sigma)$ is Hausdorff, our map is an embedding. ■

12.20. The topologies of $\mathcal{G}(F)/F$ and $\mathcal{DD}(\Sigma)$

Let $(\Sigma, d_0) = O(1)$ be the point in Teichmuller space that is the image of the identity $1 \in F$. We then have the following:

12.21. Lemma For all sufficiently large $C = C(F, O) > 0$, there is some $D > 0$ as follows. If $\gamma \in \mathcal{G}(F)$ then the map $\mu_\gamma : \Sigma \to N_\gamma$ is homotopic to a C-lipschitz map

$$m : (\Sigma, d_0) \longrightarrow N_\gamma.$$

Moreover, after fixing γ, the union $P_\gamma^C = \bigcup_m m(\Sigma)$ of the images of all such maps is a subset of N_γ with diameter at most D.

Proof. With the notation of Section 12.17, the point $O(\gamma(0))$ lies at a bounded distance from the geodesic α_γ. By Theorem 12.15 (1), there is a pleated surface $f' : (\Sigma, d') \to N_\gamma$ in the homotopy class of μ such that

$$\operatorname{dist}\big((\Sigma, d'),\, O(\gamma(0)) \big) \leq C', \text{ for some } C' = C'(F, O).$$

But then there is a pleated surface $f : \left(\Sigma, \gamma(0)_*^{-1} d\right) \to N_\gamma$ given by $f = f' \circ \gamma(0)$ homotopic to μ_γ. Since the Teichmuller distance between $\left(\Sigma, \gamma(0)_*^{-1} d\right)$ and (Σ, d_0) is bounded by C',

there is a C-lipschitz map $(\Sigma, d_0) \to (\Sigma, \gamma(0)_*^{-1} d)$ homotopic to the identity map [22]. Composing this with f yields the map m desired.

For the diameter bound on P_γ, fix an essential closed curve a on Σ. Then if

$$m : (\Sigma, d_0) \longrightarrow N_\gamma$$

is a C-lipschitz map homotopic to μ_γ, the image $m(a)$ is a closed curve in N_γ with length at most $C \operatorname{length}_{d_0}(a)$. As the geodesic representative of $\mu_\gamma(a)$ has length at least $inj(N_\gamma) \geq \epsilon = \epsilon(F) > 0$, its distance to $m(a)$ is at most some $D' = D'(F)$. Therefore, $D = 2D' + C\operatorname{diam}(\Sigma, d_0)$ is a bound for the diameter of the union P_γ. ■

One way to interpret Lemma 12.21 is that one can regard the image of $\mathcal{G}(F)$ as a space of hyperbolic 3-manifolds with preferred coarsely defined base points. The base point for (N_γ, μ_γ) is just any point contained in P_γ; these base points are then well defined up to a universally bounded error. This viewpoint motivates the following lemma that should be interpreted as saying that $\mathcal{G}(F)$ maps continuously when the co-domain is given the topology of based Gromov–Hausdorff convergence.

12.22. Lemma Suppose that $\gamma_i \to \gamma$ in $\mathcal{G}(F)$ and let C be as in Lemma 12.21. If $p_i \in P^C_{\gamma_i}$ and (N_{γ_i}, p_i) converges in the Gromov–Hausdorff topology to a based hyperbolic 3-manifold (N, p), then N is isometric to N_γ.

Proof. By Proposition 12.19, the pairs $(N_{\gamma_i}, \mu_{\gamma_i})$ converge to (N_γ, μ_γ) in $\mathcal{DD}(\Sigma)$. Let $\rho_i : \pi_1\Sigma \longrightarrow \mathrm{PSL}_2\mathbb{C}$ be representations with $N_{\gamma_i} \cong \mathbb{H}^3/\rho_i(\pi_1\Sigma)$ in such a way that the composition of $\mu_* : \pi_1\Sigma \to \pi_1 N_{\gamma_i}$ with the holonomy map gives ρ_i, and assume by conjugation that ρ_i converges to some ρ_γ similarly associated with (N_γ, μ_γ). As there is a universal lower bound for the injectivity radii $inj(N_{\gamma_i})$ by Lemma 12.10 and Proposition 12.16, the image $\rho_\gamma(\pi_1\Sigma)$ does not contain parabolic elements. So, [7, Theorem F] implies that $\rho_i(\pi_1\Sigma)$ converges to $\rho_\gamma(\pi_1\Sigma)$ in the Chabauty topology.

Fix some point $p \in \mathbb{H}^3$ and denote its projection to quotients by $\bar{p}$. By [10, Theorem E.1.13] the manifolds $(N_{\gamma_i}, \bar{p})$ converge in the Gromov–Hausdorff topology to a based hyperbolic 3-manifold isometric to N_γ. As noted in the proof of the lemma above, the points p_i lie at bounded distance from the geodesic representatives in N_{γ_i} of the loops $\mu_{\gamma_i}(a)$, where $a \subset \Sigma$ is any essential closed curve. However, the same is true of the points $\bar{p}$ since these geodesic representatives are the projections of the axes of elements $\rho_i([a]) \in \mathrm{PSL}_2\mathbb{C}$, which converge as $i \to \infty$. Therefore, the distances $\operatorname{dist}(\bar{p}, p_i)$

are universally bounded above, which implies immediately that the Gromov–Hausdorff limit N is isometric to N_γ. ■

It is clear from the definitions that shifting a geodesic $\gamma \in \mathcal{G}(F)$ does not change the isometry type of the manifold N_γ. Adopting the viewpoint suggested by Lemma 12.21, shifting γ just changes the coarsely defined base point P_γ for N_γ. The following lemma shows that the base points $P_{S^n(\gamma)}$ of shifts of γ coarsely cover N_γ.

12.23. Lemma There exists some $C = C(F, O) > 0$ such that if $\gamma \in \mathcal{G}(F)$ and $p \in N_\gamma$, then there exists some shift $S^n(\gamma)$ and an isometry

$$i : N_\gamma \longrightarrow N_{S^n(\gamma)}$$

such that the distance between $i(p)$ and $P^C_{S^n(\gamma)}$ is at most C.

Proof. As in Section 12.17, assume that (N_γ, μ) is the element of $\mathcal{DD}(\Sigma)$ modeled on the Teichmuller geodesic at bounded Hausdorff distance from $\gamma(\mathbb{Z})$. To prove the lemma, we need to show that for large C and some $i \in \mathbb{Z}$, there is a C-lipschitz map $(\Sigma, d_0) \longrightarrow N_\gamma$ in the homotopy class of $\mu \circ \gamma(i)$ whose image is within C of p.

As N_γ is doubly degenerate, it follows from work of Thurston [52] that through the point $p \in N_\gamma$ there is some essential closed curve with length at most some $C(\Sigma)$. Again by [52], this closed curve is geodesically realized by a pleated surface $f : (\Sigma, d) \longrightarrow N_\gamma$ in the homotopy class of μ. Since $inj(N_\gamma) \geq \epsilon = \epsilon(F, O)$, the distance from p to $f(\Sigma)$ is at most some $C(C', \epsilon) = C(F, O)$.

By Theorem 12.15 and Lemma 12.10, there is a point $\gamma(i) \in \mathbb{T}(\Sigma)$ at Teichmuller distance at most some $C(F, O)$ from (Σ, d). Therefore, there is a $C(F, O)$-lipschitz map $g : O(\gamma(i)) \to (\Sigma, d)$ homotopic to the identity map [22]. Then

$$(\Sigma, d_0) = O(\gamma(0)) \xrightarrow{\gamma(i)} O(\gamma(i)) \xrightarrow{g \sim id} (\Sigma, d) \xrightarrow{f \sim \mu} N_\gamma$$

composes to a C-lipschitz map homotopic to $\mu \circ \gamma(i)$ as desired. ■

We use this lemma to prove the following proposition, which loosely states that Gromov–Hausdorff limits can always be realized as limits in $\mathcal{G}(F)/F$ up to shifts.

12.24. Proposition Suppose γ_i is a sequence in $\mathcal{G}(F)$, $p_i \in N_{\gamma_i}$ and that (N_{γ_i}, p_i) converges in the Gromov–Hausdorff topology to (N, p). Then after passing to a subsequence, there is a sequence (n_i) in $\mathbb{Z}$ and some $\gamma \in \mathcal{G}(F)$ such that

$$\left[S^{n_i}(\gamma_i)\right] \longrightarrow [\gamma] \in \mathcal{G}(F)/F$$

and N_γ is isometric to N.

Proof. By Lemma 12.23, we may assume after shifting each γ_i that the base points p_i lie in $P^{\mathcal{C}}_{\gamma_i}$. Without changing their images in $\mathcal{G}(F)/F$, we may translate the geodesics γ_i so that $\gamma_i(0) = 1$ for all i. Then after passing to a subsequence, (γ_i) converges to some $\gamma \in \mathcal{G}(F)$, and the proposition follows from Lemma 12.22. ■

12.25. Shift periodicity and cyclic covers of mapping tori

If $g : \Sigma \to \Sigma$ is a pseudo-Anosov homeomorphism, we define $\widehat{M}_g$ to be the infinite cyclic cover of

$$M_g = \Sigma \times [0,1]/(x,0) \sim (g(x),1)$$

corresponding to the fundamental group of any of the level surfaces $\Sigma \times \{t\}$. Note that the map g is necessary to determine $\widehat{M}_g$—it is not sufficient to know only the homeomorphism class of M_g. The main result here is the following:

12.26. Proposition Suppose that $\gamma \in \mathcal{G}(F)$. If $[\gamma] \in \mathcal{G}(F)/F$ is shift-periodic, then N_γ is isometric to $\widehat{M}_g$ for some $g : \Sigma \to \Sigma$. On the other hand, if $[\gamma]$ is not shift-periodic then N_γ does not cover a finite volume hyperbolic 3-orbifold.

Proof. Since γ is shift-periodic in $\mathcal{G}(F)/F$, one can easily check that it is the *axis* of some element $g \in F$, that is, $\gamma(\mathbb{Z})$ is invariant under the action of g on F by left translation and the restriction of g to $\gamma(\mathbb{Z})$ is a nontrivial translation.

Let (N_γ, μ) be the doubly degenerate hyperbolic 3-manifold modeled on the Teichmuller geodesic at bounded Hausdorff distance from $O \circ \gamma(\mathbb{Z})$, as in Definition 12.17. Then $\mu_\gamma = \mu \circ \gamma(0)$ and $\mu_{g\gamma} = \mu \circ g\gamma(0)$. Since (N_γ, μ_γ) and $(N_g\gamma, \mu_{g\gamma})$ represent the same point of $\mathcal{TD}(\Sigma)$, this means that there is an isometry $i : N_\gamma \to N_\gamma$ with $i \circ \mu = \mu \circ g$. As g has infinite order, the isometry i cannot have fixed points. Therefore, the quotient is a hyperbolic 3-manifold M. But $\pi_1 M$ is isomorphic to $\pi_1 M_g$, so a theorem

of Waldhausen [57] implies that they are homeomorphic, in which case N_γ is isometric to $\widehat{M}_g$. This finishes the 1st part of the proposition.

For the 2nd statement, suppose that $N_\gamma = N$ covers a finite volume hyperbolic 3-orbifold. Thurston's Covering Theorem (as Thurston's proof is not readily available, we refer the reader to [20] for a proof by Canary of a more general result. Note that although Canary's statement does not deal with orbifold covers, the proof works just as well) implies that N_γ is isometric to the fiber subgroup of a mapping torus of Σ. Therefore, there is some isometry $i : N_\gamma \longrightarrow N_\gamma$ with $i \circ \mu = \mu \circ f$ for some pseudo-Anosov homeomorphism $f : \Sigma \longrightarrow \Sigma$. Here, $\mu : \Sigma \longrightarrow N_\gamma$ is the map given in Section 12.17. Then it follows from Theorem 12.15 that the action of f on Teichmuller space leaves the geodesic $\alpha \subset \mathbb{T}(\Sigma)$ on which (N_γ, μ) is modeled invariant.

Fix some point X on $\alpha \subset \mathbb{T}(\Sigma)$. As $O \circ \gamma(\mathbb{Z})$ lies at bounded Hausdorff distance from α, for each $i \in \mathbb{Z}$ there is some $j_i \in \mathbb{Z}$ with

$$\sup_i \operatorname{dist}\Big(f^i(X),\, O \circ \gamma(j_i)\Big) < \infty.$$

Therefore, by the equivariance of O we have that

$$\sup_i \operatorname{dist}\Big(\gamma(j_i)^{-1} f^i(X),\, O \circ \gamma(0)\Big) < \infty.$$

Since the action of $\mathrm{Mod}(\Sigma)$ on $\mathbb{T}(\Sigma)$ is properly discontinuous, this means that the set $\{\gamma(j_i)^{-1} f^i \mid i \in \mathbb{Z}\}$ is finite. In other words, we have some $i \neq k$ with

$$g := f^{k-i} = \gamma(j_i)\gamma(j_k)^{-1} \in F.$$

This means that there is some element $g \in F$ that acts as a nontrivial translation along the Teichmuller geodesic α. But recall from Lemma 12.10 that the extension $O : \partial_\infty F \longrightarrow \mathcal{PML}(\Sigma)$ is an embedding. Therefore, as $O \circ \gamma(\mathbb{Z})$ and $O(\mathrm{Axis}(g))$ accumulate to the same points of $\mathcal{PML}(\Sigma)$ we must have $\gamma(\mathbb{Z}) = \mathrm{Axis}(g)$. Now if $\gamma(\mathbb{Z})$ is the axis of $g \in F$, then for some k we have

$$g\gamma(i) = \gamma(i + k) = S^{-k}\gamma(i).$$

Then γ and $S^{-k}(\gamma)$ have the same projection in $\mathcal{G}(F)/F$, so $[\gamma]$ is periodic. ■

12.27. The proof of Theorem 12.8

For easy reading, we briefly recall the relevant notation. We have pseudo-Anosov maps $\phi_0, \dots, \phi_n \in \mathrm{Mod}(\Sigma)$ that generate a Schottky subgroup $F < \mathrm{Mod}(\Sigma)$. The group F acts on the space $\mathcal{G}(F)$ of its geodesics and there is a shift-invariant embedding

$$\{0, \dots, n\}^{\mathbb{Z}} \longrightarrow \mathcal{G}(F)/F, \qquad e \mapsto [\gamma_e]$$

determined by the constraints $\gamma_e(i)^{-1}\gamma_e(i+1) = \phi_{e_i}$ and $\gamma_e(0) = 1$. Note that any γ_e satisfying the 1st property has the same projection in $\mathcal{G}(F)/F$, so the condition that $\gamma_e(0) = 1$ is necessary only to uniquely specify γ_e within its equivalence class.

As in the beginning of the section, choose a sequence of finite strings

$$e^1 = \left(e^1_1, \dots, e^1_{n_1}\right), \quad e^2 = \left(e^2_1, \dots, e^2_{n_2}\right), \quad \dots$$

with entries in $\{0, \dots, n\}$ and let C be the sub-shift of $\{0, \dots, n\}^{\mathbb{Z}}$ consisting of strings all of whose finite substrings are contained in e^i for some i.

12.28. Lemma Let $\bar{e}^i \in \{0, \dots, n\}^{\mathbb{Z}}$ be a bi-infinite string obtained by concatenating copies of e^i. Then the sub-shift C consists of all the accumulation points of sequences $S^{n_j}(\bar{e}^{i_j})$, where $n_j \in \mathbb{Z}$ and $i_j \in \mathcal{N}$ is increasing. Consequently, if

$$\bar{e}^i \mapsto [\gamma_i] \in \mathcal{G}(F)/F \quad \text{and} \quad C \mapsto \Gamma_C \subset \mathcal{G}(F)/F$$

then Γ_C consists of all accumulation points in $\mathcal{G}(F)/F$ of sequences $S^{n_j}(\gamma_{i_j})$, where $n_j \in \mathbb{Z}$ and $i_j \in \mathcal{N}$ is increasing.

Write N_i for the manifolds N_{γ_i} modeled on the geodesics in F corresponding to the string e^i, where γ_i is from the lemma above and N_{γ_i} is defined in Section 12.17. The mapping classes $f_i = \phi_{e^i_{n_i}} \circ \cdots \circ \phi_{e^i_1} \in \mathrm{Mod}(\Sigma)$ all pseudo-Anosov, so each

$$M_i = \Sigma \times [0,1]/(x,0) \sim (f_i(x),1) \tag{12.28.1}$$

has a hyperbolic metric, unique up to isometry. By Proposition 12.26, we identify N_i with the infinite cyclic cover $\widehat{M}_g$ corresponding to $\pi_1(\Sigma \times \{t\})$. We prove the following:

12.29. Theorem Assume $p_i \in M_i$ and that some subsequence of (M_i, p_i) converges in the Gromov–Hausdorff topology to a hyperbolic 3-manifold (N, p). If N covers a finite volume hyperbolic 3-orbifold, then C contains a shift-periodic point.

Proof. Pick lifts $\hat{p}_i$ of p_i in the cyclic covers $N_i \longrightarrow M_i$. We claim that a subsequence of $(N_i, \hat{p}_i)$ converges in the Gromov–Hausdorff topology to (N, p). To see this, note that the projection onto the 2nd factor in Equation 12.28.1 gives a map $M_i \to S^1$; define the *circumference* of M_i to be the length of the shortest loop that projects to a nontrivial element of $\pi_1(S^1)$. In [11, Proposition 5.1], it is shown that there are only finitely many ϵ-thick mapping tori with circumference less than a given constant. The proof only uses that the covers N_i are ϵ-thick, though, which we know to be the case by Lemma 12.10 and Proposition 12.16. So, it follows that the circumferences of M_i go to infinity. In other words, there is an increasing sequence $r_i \in \mathbb{R}$ such that the ball $B_{M_i}(p_i, r_i)$ is isometric to $B_{N_i}(\hat{p}_i, r_i)$ for all i. It follows immediately that a subsequence of $(N_i, \hat{p}_i)$ converges to (N, p).

Recall that $N_i = N_{\gamma_i}$, the manifold associated to $\gamma_i \in \mathcal{G}(F)$. By Proposition 12.24, there is a sequence (n_i) in $\mathbb{Z}$ and some geodesic $\gamma \in \mathcal{G}(F)$ with

$$[S^{n_i}(\gamma_i)] \longrightarrow [\gamma] \in \mathcal{G}(F)/F$$

and N_γ isometric to N. Since N covers a finite volume hyperbolic 3-orbifold, $[\gamma]$ is shift-periodic in $\mathcal{G}(F)/F$ by Proposition 12.26. Lemma 12.28 implies that $\gamma \in \Gamma_C$, but then as the map $\{0, \dots, n\}^{\mathbb{Z}} \longrightarrow \mathcal{G}(F)/F$ is a shift-invariant embedding, γ must be the image of some shift-periodic point of C. This finishes the proof. ■

To conclude this section, we derive the statement of Theorem 12.8 given at the beginning of the section. Suppose that μ_i is the IRS of $\mathrm{PSL}_2\mathbb{C}$ corresponding to the hyperbolic 3-manifold M_i. If we write $M_i = \mathbb{H}^3/\Gamma_i$, then μ_i is supported on the set of conjugates of $\Gamma_i < \mathrm{PSL}_2\mathbb{C}$. Suppose that μ is the weak limit of some subsequence of μ_i. Then μ is supported within the set of accumulation points in $\mathrm{Sub}_{\mathrm{PSL}_2\mathbb{C}}$ of sequences $g_i\Gamma_i g_i^{-1}$, where $g_i \in \mathrm{PSL}_2\mathbb{C}$. But if

$$g_i\Gamma_i g_i^{-1} \longrightarrow \Gamma < \mathrm{PSL}_2\mathbb{C}$$

then there are base points $p_i \in M_i$ and $p \in \mathbb{H}^3/\Gamma$ such that (M_i, p_i) converges in the Gromov-Hausdorff topology to $(\mathbb{H}^3/\Gamma, p)$. Therefore, Theorem 12.8 follows from Theorem 12.29.

13 A General Gluing Construction in SO(n, 1)

The analysis of this section has been an inspiration for some later works regarding counting manifolds, which have already been published [28, 48].

13.1. Notation Let N_0, N_1 be two real hyperbolic n-manifolds such that each have totally geodesic boundary and each boundary is the disjoint union of two copies of some fixed hyperbolic $(n-1)$-manifold Σ. Label for each N_a a component Σ_a^+ of ∂N_a, and denote the other one by Σ_a^-; we call $i_a^\pm$ the corresponding embeddings of Σ in ∂N_a. Given a sequence $\alpha = (\alpha_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ we let N_α denote the manifold obtained by gluing copies of N_0, N_1 according to the pattern prescribed by α:

$$N_\alpha = \left(\bigsqcup_{i\in\mathbb{Z}} N_{\alpha_i} \times \{i\}\right) \Big/ \left(i^+_{\alpha_i} x, i\right) \sim \left(i^-_{\alpha_{i+1}} x, i+1\right) \quad (i \in \mathbb{Z}, x \in \Sigma).$$

For $i \in \mathbb{Z}$, we shall denote by $N_{\alpha,i}$ the image of $N_{\alpha_i} \times \{i\}$ in N_α. More generally, for an interval $I \subset \mathbb{Z}$ set $N_{\alpha,I} = \cup_{i\in I} N_{\alpha,i}$.

13.2. Construction of the IRS

Let ν be a Borel probability measure on the Cantor set $\{0,1\}^{\mathbb{Z}}$. We define a measure μ_ν on the set of framed hyperbolic n-manifolds, and the IRS will be the corresponding measure on the set of discrete subgroups of SO(1, n) discussed in the introduction.

Let ν' be the measure on $\{0,1\}^{\mathbb{Z}}$, defined for Borel sets $A \subset \{0,1\}^{\mathbb{Z}}$ by

$$\nu'(A) = \frac{\int_A \operatorname{vol}(N_{\alpha_0})\, d\nu(\alpha)}{\int_{\{0,1\}^{\mathbb{Z}}} \operatorname{vol}(N_{\alpha_0})\, d\nu(\alpha)}.$$

By definition, we obtain a μ_ν-random framed hyperbolic n-manifold by first choosing α randomly against ν', and then choosing a random base frame from $N_{\alpha,0}$.

13.3. Example Let σ be the shift map on $\{0,1\}^{\mathbb{Z}}$ and suppose that ν is a σ-invariant probability measure on $\{0,1\}^{\mathbb{Z}}$ that is supported on a periodic orbit, that is, there is some $\alpha \in \{0,1\}^{\mathbb{Z}}$ and $k \in \mathbb{Z}$ with $\sigma^k(\alpha) = \alpha$ and $\nu = \frac{1}{k}\sum_{i=0}^{k-1} \delta_{\sigma^i(\alpha)}$. We can construct a

closed manifold M from α:

$$M = \left(\bigsqcup_{i \in \mathbb{Z}/k\mathbb{Z}} N_{\alpha_i} \times \{i\} \right) \Big/ \left(i^+_{\alpha_i} x, i \right) \sim \left(i^-_{\alpha_{i+1}} x, i+1 \right) \quad (i \in \mathbb{Z}/k\mathbb{Z}).$$

Then each N_β, with β in the support of ν, is an infinite cyclic cover of M and the random subgroup μ_ν is the ergodic IRS obtained—as in Section 11 above—from the normal subgroup $\pi_1(N_\alpha)$ of the lattice $\pi_1(M)$.

More generally, we have the following result.

13.4. Lemma Let ν be a shift-invariant ergodic measure on $\{0,1\}^{\mathbb{Z}}$. Then the random subgroup μ_ν constructed above is an ergodic IRS.

Proof. Fix $g \in G = \mathrm{SO}(1, n)$. Translated through the correspondence between framed manifolds and discrete subgroups of G, the conjugation action of $g \circlearrowright \mathrm{Sub}_G$ restricts to the right action of g on the frame bundle $\mathcal{F}M = \Gamma \backslash G$ of any hyperbolic n-manifold $M = \Gamma \backslash \mathbb{H}^n$. Note that this action preserves the Haar measure.

Let $\alpha \in \{0,1\}^{\mathbb{Z}}$ and let U be an open-bounded set of frames on N_α. Then U and gU are contained in some submanifold $N_{\alpha,I}$, where $I \subset \mathbb{Z}$ is an interval. Because ν is shift-invariant, we get the same random framed manifold by selecting a random frame from $N_{\beta,I}$ for a ν'-random β.

Let V be some neighborhood of α containing all $\beta \in \{0,1\}^{\mathbb{Z}}$ such that for all $i = k, \dots, k+l$ we have $\beta_i = \alpha_i$; for $\beta \in V$ let U_β be the image of U in N_β. Since g preserves the Haar measure on $\mathcal{F}N_\alpha$, when taking a random frame in $N_{\beta,I}$ we have the same probability to land in U_β or gU_β, that is, if we set

$$W = W(U, V) = \{y \in \mathcal{F}U_\beta,\ \beta \in V\}$$

then we get

$$\mu_\nu(W) = \mu_\nu(\{y \in gU_\beta,\ \beta \in V\}) = \mu_\nu(gW).$$

The G-invariance follows since the sets $\{y \in U_\beta,\ \beta \in V\}$ form a basis for Borel sets in the support of μ_ν.

To show ergodicity, note that the group G acts transitively on the frame bundle of any connected hyperbolic n-manifold. So if a G-invariant set S of frames contains a frame on some N_α it contains all frames on N_α. It follows that $\{\alpha \ : \ \Lambda_\alpha \in S\}$ is a

shift-invariant set. Since ν is ergodic, it follows that this set has full or zero measure. Therefore, S has full or zero measure for μ_ν.

■

Remarks.

1) The IRSs we have constructed above are always limits of lattice IRSs since shift-invariant measures are limits of measures supported on finite orbits.
2) We could have made the construction with more general graphs. If a group Ω acts freely on a locally finite graph T, D is a connected fundamental domain for Ω and N_0, N_1 are manifolds with totally geodesic boundary whose boundary components are all isometric and indexed by ∂D then for any $\alpha \in \{0,1\}^\Omega$ we can glue them along T in the manner prescribed by α to get a hyperbolic manifold. We can then construct ergodic IRSs in the same manner as above from Ω-ergodic probability measures on $\{0,1\}^\Omega$.

13.2. Exoticity

We now show that after choosing suitable N_0 and N_1, the construction above yields IRSs that are not induced from a lattice.

13.6. Theorem Suppose that $n \geq 3$ and that N_0 (resp. N_1) is isometrically embedded in a compact arithmetic manifold M_0 (resp. M_1). If M_0, M_1 are non-commensurable then for any sequence $\alpha \in \{0,1\}^\mathbb{Z}$ that is not periodic the manifold N_α does not cover any finite volume hyperbolic manifold.

13.7. Corollary Under the same hypotheses as in the above theorem, if the ergodic shift-invariant measure ν is not supported on a periodic orbit then the support of the IRS μ_ν is disjoint from the set of all subgroups of all lattices of G (in particular it follows that μ_ν cannot be induced from a lattice).

The proof of this theorem occupies the rest of this section. In Section 13.8, we recall how to construct (non-commensurable pairs of) arithmetic manifolds with totally geodesic hypersurfaces. These will be the manifolds M_0, M_1 above, and cutting along the hypersurfaces will give the desired N_0, N_1. The reason we use arithmetic manifolds here is the very strong disjointness criterion in Proposition 13.10, which says that isometric immersions of N_0, N_1 into a common manifold cannot have overlapping images. Using

this, we then show that if there is a covering map from N_α to a finite volume manifold, then α is periodic.

13.8. Constructing arithmetic manifolds

The standard way to construct arithmetic hyperbolic manifolds that contain totally geodesic hypersurfaces is as follows. Let F be a totally real number field and q a quadratic form in $n+1$ variables over F such that q is definite positive at all real places of F but one, where it has signature $(1, n)$. Then the group of integer points $\Gamma_q = \mathrm{SO}(q, \mathcal{O}_F)$ is a lattice in $\mathrm{SO}(1, n)$. If q is written as $a_1x_1^2 + \ldots + a_{n+1}x_{n+1}^2$ where $a_1, \ldots, a_n$ are totally positive and a_{n+1} is negative at exactly one real place, then Γ_q contains the subgroup associated to the quadratic form in n variables $a_2x_2^2 + \ldots + a_{n+1}x_{n+1}^2$ that gives rise to an imbedded totally geodesic hypersurface. It follows from work of Millson that there exists an ideal $\mathfrak{p}$ such that this hypersurface is actually embedded in the manifold associated to the principal congruence subgroup of level $\mathfrak{p}$, that is, $\Gamma \cap \ker(\mathrm{SL}(n+1, \mathcal{O}_F) \to \mathrm{SL}(n+1, \mathcal{O}_F/\mathfrak{p}))$. Moreover, we can choose $\mathfrak{p}$ so that this hypersurface S is non-separating. In this case, $M - S$ is the interior of a compact manifold N that has two boundary components, both isometric to S. Note also that the isometry type of S depends only on $a_2, \ldots, a_{n+1}$ and $\mathfrak{p}$.

The simplest example of the previous procedure is when $F = \mathbb{Q}$ and $a_1, \ldots, a_n > 0$, $a_{n+1} < 0$ but then the manifolds obtained are noncompact for $n \geq 4$. However, if $F = \mathbb{Q}(\sqrt{d})$ for a square-free rational integer $d > 0$, $a_1, \ldots, a_n \in \mathbb{Q}_+^*$ and $a_{n+1}/\sqrt{d} \in \mathbb{Q}^*$ then q is anisotropic over F so that $\Gamma_q \backslash \mathbb{H}^3$ is compact.

Now we want to find $a_1, \ldots, a_{n+1}$ and a_1' such that

- both $a_1, \ldots, a_{n+1}$ and $a_1', \ldots, a_{n+1}$ satisfy the conditions above;
- the lattices obtained from $q = a_1x_1^2 + \ldots + a_{n+1}x_{n+1}^2$ and $q' = a_1'x_1^2 + \ldots + a_{n+1}x_{n+1}^2$ are non-commensurable.

By [30, 2.6] it suffices to show that q' and λq are not isometric for any $\lambda \in F^*$. For n odd, since the discriminants of q' and λq are equal for all λ it suffices that $a_1/a_1' \notin F^2$ since then the discriminants of λq and q' are never the same (as noted in [30]). For example, we can take $F = \mathbb{Q}(\sqrt{2})$ and

$$q = x_1^2 + \ldots + x_n^2 - 3\sqrt{2}x_{n+1}^2,\ q' = 7x_1^2 + \ldots + x_n^2 - 3\sqrt{2}x_{n+1}^2.$$

For n even we need to consider another invariant. Let k be any field; for $u, v \in k^*$ the Hilbert symbol (u, v) is defined in [49, III, 1.1] as 1 if $1 = uv^2 + vy^2$ for some $x, y \in k$ and -1 otherwise. Then it is shown in [49, IV, Théorème 2] that

$$\varepsilon(q) = \prod_{i<j}(a_i, a_j)$$

is an isometry invariant of q over k. Now suppose that $k = \mathbb{Q}_p$ for a prime $p > 2$. Then for $a, b \in \mathbb{Z}_p$, we have $(a, b) = -1$ if and only if either

(1) p divides a (resp. b) and b (resp. a) is a nonsquare unit (modulo squares), or
(2) a, b have the same p-valuation mod 2 and $-a^{-1}b$ is a square unit (see [49, III, Théorème 2]).

Now let q and q' be as above and $\lambda \in \mathbb{Q}_7^*$. Since $7 = 3 \pmod 4$, -1 is not a square mod 7 and it follows that $(\lambda, \lambda) = 1$, so that $\varepsilon(\lambda q) = (\lambda, -\lambda\sqrt{2})^n = 1$ since n is even. On the other hand, we have $-3\sqrt{2} = 5 \pmod 7$, which is not a square, so that $\varepsilon(q') = (7, -3\sqrt{2}) = -1$. It follows that q' and λq are not isometric over $\mathbb{Q}_7$ for any $\lambda \in \mathbb{Q}_7$.

In conclusion, this shows that we can find non-commensurable compact arithmetic n-manifolds M_0, M_1 that both contain a totally geodesic hypersurface isometric to some fixed S, and then we can cut M_0, M_1 along to produce manifolds N_0, N_1 as required in the statement of Theorem 13.6.

13.9. The proof of Theorem 13.6

Suppose that $n \geq 3$ and $N_0 \subset M_0$, $N_1 \subset M_1$ are hyperbolic n-manifolds as in the statement of the theorem. The reason we require M_0, M_1 to be non-commensurable arithmetic manifolds is the following.

13.10. Proposition Suppose that M is another hyperbolic n-manifold and $i_0 : N_0 \longrightarrow M$ and $i_1 : N_1 \longrightarrow M$ are isometric immersions. Then the images of i_0, i_1 are disjoint, except possibly along their boundaries.

To prove this, recall the following commensurability criterion (see [30, 1.6]).

13.11. Lemma If Γ, Γ' are two arithmetic subgroups in SO(1, n) such that the intersection $\Gamma \cap \Gamma'$ is Zariski-dense in SO(1, n), then this intersection has finite index in both of them (so that in particular Γ, Γ' are commensurable).

Proof of Proposition 13.10. Hoping for a contradiction, assume that the images of the interiors intersect. To begin with, assume also that there are components $\Sigma_0 \subset \partial N_0$ and $\Sigma_1 \subset \partial N_1$ such that $i_0(\Sigma_0) \cap i_1(int(N_1)) \neq \emptyset$ and $i_1(\Sigma_1) \cap i_0(int(N_0)) \neq \emptyset$. Then the preimages $\Sigma_0' = i_1^{-1}(i_0(\Sigma_0))$ and $\Sigma_1' = i_0^{-1}(i_1(\Sigma_1))$ are properly immersed totally geodesic hypersurface in N_1, N_0, respectively.

Fixing a monodromy map, identify $\pi_1(M)$ with a discrete subgroup of SO(1, n). By choosing a base point within $N_0 \cap N_1$, we can also select subgroups in $\pi_1(M)$ that represent the fundamental groups of all the other manifolds and hypersurfaces above, such that *both* $\pi_1\Sigma_0'$ *and* $\pi_1\Sigma_1'$ *are contained in* $\pi_1 N_1 \cap \pi_1 N_0$.

By Corollary 1.7.B of [30], for instance, the Zariski closure of $\pi_1\Sigma_0'$ in SO(1, n) is isomorphic to $SO(1, n-1)$. Similarly, the Zariski closure of $\pi_1\Sigma_1'$ is also isomorphic to $SO(1, n-1)$. These two copies of SO(1, $n-1$) embedded in SO(1, n) are different, since they stabilize the preimages in $\mathbb{H}^3$ of distinct closed, immersed hypersurfaces in M. So, the group $\langle \pi_1\Sigma_0', \pi_1\Sigma_1' \rangle$ is Zariski dense in SO(1, n). But

$$\langle \pi_1\Sigma_0', \pi_1\Sigma_1' \rangle \subset \pi_1 N_0 \cap \pi_1 N_1,$$

so $\pi_1 M_0, \pi_1 M_1$ can be represented by lattices in SO(1, n) with Zariski dense intersection. By Lemma 13.11, this contradicts that M_0, M_1 are not commensurable.

The only remaining case is that, say, $i_0(\partial N_0)$ does not intersect $i_1(int(N_1))$, so that $i_1(N_1) \subset i_0(int(N_0))$. In this case, though, $i_0^{-1}(i_1(N_1))$ is a compact, immersed submanifold of N_1. So, some finite index subgroup of $\pi_1 N_1$ injects into $\pi_1 N_0$, and as $\pi_1 N_1$ is Zariski dense in SO(1, n), we get a contradiction just as before. ∎

The appeal to Corollary 1.7.B of [30] is the part of the argument above that uses $n \geq 3$. If $n = 2$, then Σ_0' could be a geodesic segment, in which case its (trivial) fundamental group is certainly not Zariski dense in SO(1, 1).

We are now ready to finish the proof of Theorem 13.6, which we encapsulate in the following proposition.

13.12. Proposition Suppose that $\alpha = (\alpha_k) \in \{0,1\}^{\mathbb{Z}}$ and that $f : N_\alpha \longrightarrow M$ is a covering map, where M has finite volume. Then α is periodic.

Proof. For convenience, assume throughout the following that α is not a constant sequence. Recall that $N_{\alpha,i} \cong N_{\alpha_i}$ is the i^{th} block in the gluing representing N_α. Let Σ_i be the hypersurface that is the common boundary of $N_{\alpha,i}$ and $N_{\alpha,i+1}$. All the Σ_i are isometric to a fixed hyperbolic $(n-1)$-manifold Σ.

By Proposition 13.10, we have that if $\alpha_i = 0$ and $\alpha_j = 1$, then

$$int(f(N_{\alpha,i}) \cap f(N_{\alpha j})) = \emptyset. \tag{13.12.1}$$

Hence, $M = I_0 \cup I_1$, where $I_0 = f(\cup_{i,\alpha_i=0} N_{\alpha,i})$ and I_1 is defined similarly. It follows that $I_0 \cap I_1$ is a set of totally geodesic hypersurfaces in M, each of which is covered by Σ. A priori, you might imagine that the common boundary of I_0 and I_1 has corners, remembering that the surfaces $f(\Sigma_i)$ are only immersed in M. However, if $\alpha_i = 0$ and

$\alpha_{i+1} = 1$, say, then any transverse self intersection of the image of $f(\Sigma_i)$ would create interior in the intersection $f(N_{\alpha_i}) \cap f(N_{\alpha_{i+1}})$, contradicting (13.12.1).

Let's call a connected submanifold of N_α that is a maximal union of consecutive blocks isometric to N_0 a *0-chunk*, and define a *1-chunk* similarly. The restriction of f to any 0-chunk is a covering map onto some component $C \subset I_0$, and the degree of this covering is $2\mathrm{vol}(\Sigma)/\mathrm{vol}(\partial C)$. By volume considerations, the number of blocks in any 0-chunk that covers C must then be

$$\frac{2\mathrm{vol}(\Sigma) \cdot \mathrm{vol}(C)}{\mathrm{vol}(N_0) \cdot \mathrm{vol}(\partial C)}. \tag{13.12.2}$$

Of course, all the same statements hold for 1-chunks covering components of I_1.

From the covering property, every component $C \subset I_0$ has either one or two boundary components. It follows that either

(1) the (finitely many) components of I_0 and I_1 all have two boundary components, and are arranged in M end-to-end in a circle, or
(2) the components of I_0 and I_1 are arranged in a line segment, with one-boundary-component C's at the extremities.

When a component C has two boundary components, the two boundary components of a chunk covering C cover distinct components of ∂C. From this, it follows that α is periodic. Namely, a cyclic word representing α can be obtained from the circle in case (1) by using (13.12.2) to determine the number of 0's and 1's to associate to each component of I_0 and I_1. Case (2) is similar, except that the cyclic word is produced by traversing the line segment twice, first forward and then backward, but only counting the endpoints once each. ■

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