Incorporating impact of hazardous and toxic environments on the guidance of mobile sensor networks used for the cooperative estimation of spatially distributed processes

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Abstract— **This work incorporates the effects that hazardous environments have on sensing devices, in the guidance of mobile platforms with onboard sensors. Mobile sensors are utilized in the state reconstruction of spatiotemporally varying processes, often described by advection-diffusion PDEs. A typical sensor guidance policy is based on a gradient ascent scheme which repositions the sensors to spatial regions that have larger state estimation errors. If the cumulative measurements of the spatial process are used as a means to represent the effects of hazardous environments on the sensors, then the sensors are considered inoperable the instance the cumulative measurements exceed a device-specific tolerance level. A binary guidance policy considered earlier repositioned the sensors to regions of larger values of the state estimation errors thus implementing an information-sensitive policy. The policy switched to an information-averse guidance the instance the cumulative effects exceeded a certain tolerance level. Such a binary policy switches the sensor velocity abruptly from a positive to a negative value. To alleviate these discontinuity effects, a ternary guidance policy is considered and which inserts a third guidance policy, the information-neutral policy, that smooths out the transitions from information-sensitive to information-averse guidance. A novelty in this ternary guidance has to do with the level-set approach which changes from a guidance towards large values of the state estimation error towards level sets of the state estimation error and eventually towards reduced values of the state estimation error. An example on an advection-diffusion PDE in 2D employing a single interior mobile sensor using both the binary and ternary guidance policies is used to demonstrate the effects of hazardous environments on both the sensor life expectancy and the performance of the state estimator.**

Index Terms— **Distributed parameter systems; Level-set guidance; hazardous environments; sensor life; mobile sensors.**

I. INTRODUCTION

Various approaches exist for the use of mobile sensors in source localization and state reconstruction. Some have a priori defined paths for the mobile sensors whereas others associate their guidance with the estimation cost. Very few tie the guidance and associated formation motion, to the performance of the state estimator. Two estimation approaches emerge: (i) use of an optimal filter (Kalman) which formulates the sensor motion and state estimation as an optimal control problem [?], [?], (ii) use of a Luenberger-type observer which utilizes a Lyapunov-like method to extract the guidance of the sensors [?], [?]. In the former, while optimal, reveals many computational challenges. The covariance is propagated forward in time while the guidance must be integrated backward in time. In the latter, there is no optimality, other than the inferred one from an inverse optimality point of view for the stability based guidance. A somewhat compromising solution emerges as the one that uses a spatial gradient approach for the sensor guidance and couples it to the solution of a location-dependent Riccati equation for the covariance. In other words, the location-dependent covariance is propagated forward in time via the solution to a location-dependent Riccati equation. The sensor guidance is based on a gradient ascent policy [?].

Modification of the sensor guidance include collision avoidance and interagent communication constraints. Coupling of the guidance of the mobile sensors can occur at different levels. Couplings can come through Lyapunov-based guidance or through collaborative estimation [?], [?].

The effects and impact of the environment on equipment and the estimator performance have not been dealt with the same detail. In particular, the effects of toxic environments on the health status and operation of the sensing devices has not been considered in the literature other than some scant works. Prolonged exposure of the sensor onboard a mobile platform to the field, or plume, that is tasked with monitoring and detection can have detrimental effects on both the equipment and the state estimator learning capabilities.

This work takes into account the detrimental effects of the environment on the sensing devices and extends the earlier work [?] on the use of a *binary guidance policy* that supervised a transition from an *information-sensitive* to an *information-averse* guidance. This binary policy attempted to resolve the following <u>dilemma:</u> *guide the sensor to spatial regions of larger values (concentration) of the process state thereby increasing the accumulated measurements or guide the sensor to spatial regions of smaller values of the process state thereby decreasing the accumulated measurements*. Larger values of the process state lead to faster convergence of the state estimation error whereas smaller values of the process state lead to slower convergence of the state estimation error. Increasing the accumulated measurements accelerate the demise of a sensing device whereas low values of the accumulated measurements prolong the life of a sensing device. The binary policy essentially changes abruptly the direction that the mobile sensor must have whenever the accumulated measurements exceed a certain user-defined threshold value. To alleviate sudden changes in the commanded direction of the mobile platforms, a *ternary*

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policy is introduced here. This policy is termed *information-neutral* and is instituted whenever the accumulated measurements are within a neighbourhood of the threshold value.

The contributions of this work are as follows: the speed at which the mobile platforms with the onboard sensors can traverse within the spatial domain is now extended to *measurement-dependent* vehicle speeds; the gradient-ascent guidance policy that switches the commanded directions of the mobile sensor(s) now includes concepts from Level-set methods [**?**] to introduce the third policy which steers the mobile sensors to directions that are normal to the gradient of the associated state estimation error. The level-set extension is conceptually similar to the one presented in [**?**], [**?**].

II. Problem formulation

To demonstrate the proposed policy modifications, we consider the 2D spatial process considered in [**?**]. Different from the 1D case, the 2D case allows mobile sensors to implement guidance modifications to account for collision avoidance. Consider an advection-diffusion PDE in its general form

$$\begin{aligned} \frac{\partial u(t,\boldsymbol{\xi})}{\partial t} &= \mathcal{A}(\boldsymbol{\xi})u(t,\boldsymbol{\xi}) + b(t,\boldsymbol{\xi})f_d(t) && \text{in } Q \\ \frac{\partial u(t,\boldsymbol{\xi})}{\partial n} &= -g(t,\boldsymbol{\xi})u(t,\boldsymbol{\xi}) + F(t,\boldsymbol{\xi}) && \text{in } \Sigma \\ u(0,\boldsymbol{\xi}) &= u_0(\boldsymbol{\xi}) && \text{in } \Omega. \end{aligned} \tag{1}$$

where $\boldsymbol{\xi} = \{\xi_1,\dots,\xi_n\}$ denote the spatial variable in a fixed spatial domain Ω assumed to be an open bounded subset of an n-dimensional space $\mathbb{R}^n$ and t denote the time. The boundary of the spatial domain Ω is denoted by Γ which is assumed to be an infinitely differentiable $(n-1)$ dim. variety. The regularity of the boundary is assumed to help establishing existence and uniqueness of the solutions to the physical process. The process (**??**) is considered in a time interval denoted by $I = (0,T)$. The system representing the process is considered in the cylinder $Q = I \times \Omega$ and $\Sigma = I \times \partial\Omega$ with Q_∞, Σ_∞ is defined in a similar fashion.

Fig. 1. 2D PDE with in-domain and boundary controls and observations.

The derivative at the boundary $\frac{\partial u(t,\boldsymbol{\xi})}{\partial n}$ denotes the outward normal derivative and the spatial function g is continuous function of $(t,\boldsymbol{\xi}) \in \Sigma$. The spatial distribution of the control function in the interior $b(t,\boldsymbol{\xi})$ is a continuous function of $\boldsymbol{\xi} \in Q$ and $f_d(t)$ is the corresponding control signal. The boundary control function $F(t,\boldsymbol{\xi})$ represents the forcing function at the boundary, which may be spatially distributed over a small region at the boundary, pointwise-in-space at the boundary, or scanning at the boundary. The spatial process operator is given by

$$\mathcal{A} = \sum_{i,j}^{n} \alpha_{ij}(t,\boldsymbol{\xi}) \frac{\partial^2}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{n} \beta_i(t,\boldsymbol{\xi}) \frac{\partial}{\partial \xi_i} + \gamma(t,\boldsymbol{\xi}), \tag{2}$$

where the coefficients are defined in $I \times \overline{\Omega}$.

With regards to the process in Fig. **??**, the advection-diffusion PDE is

$$\begin{aligned} &\frac{\partial u(t,x,y)}{\partial t} = \mathcal{A}(x,y)u(t,x,y) + b_I(x,y)f_I(t) \\ &u(0,x,y) = u_0(x,y) \qquad \text{in } [0,L_X]\times[0,L_Y], \\ &u(t,0,y) = 0, \quad 0 \le y \le L_Y, \\ &u(t,x,0) = 0, \quad 0 \le x \le L_X, \\ &u_y(t,x,L_Y) = b_T(x)f_T(t) \\ &\qquad\qquad + c_T(x)u(t,x,L_Y), \quad \text{in } 0 \le x \le L_X, \\ &u_x(t,L_X,y) = b_R(y)f_R(t) \\ &\qquad\qquad + c_R(y)u(t,L_X,y), \quad \text{in } 0 \le y \le L_Y, \end{aligned} \tag{3}$$

where now the spatial coordinates are $\boldsymbol{\xi} = (\xi_1,\xi_2) = (x,y)$.

Along with the process state equation, one considers the expressions for the process measurements. These are characterized by the measurement models representing the sensing devices used, and by the spatial location of the sensors onboard the mobile platforms. For fixed-in-space (immobile) sensors, the measurements are given by sensors that are positioned in both the interior and the boundaries

$$Z(t) = \begin{bmatrix} Z_I(t) \\ Z_B(t) \end{bmatrix} = \begin{bmatrix} \int_{\Omega} c_I(t,\boldsymbol{\xi})u(t,\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \\ \int_{\partial\Omega} c_B(t,\boldsymbol{\xi})u(t,\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \end{bmatrix}, \tag{4}$$

where the spatial functions $c_I(t,\boldsymbol{\xi})$, $c_B(t,\boldsymbol{\xi})$ denote the sensor models. To account for the dependence on the spatial location, the two spatial functions are now parameterized by the time varying centroids $\boldsymbol{\xi}_I(t)$ and $\boldsymbol{\xi}_B(t)$ and are given

$$\begin{aligned} Z(t;\boldsymbol{\xi}(t)) &= \begin{bmatrix} Z_I(t;\boldsymbol{\xi}_I(t)) \\ Z_B(t;\boldsymbol{\xi}_B(t)) \end{bmatrix} \\ &= \begin{bmatrix} \int_{\Omega} c_I(t,\boldsymbol{\xi};\boldsymbol{\xi}_I(t))u(t,\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \\ \int_{\partial\Omega} c_B(t,\boldsymbol{\xi};\boldsymbol{\xi}_B(t))u(t,\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \end{bmatrix}. \end{aligned} \tag{5}$$

The expression (**??**) is used to represent both mobile and immobile sensors; for the latter replace $\boldsymbol{\xi}_I(t), \boldsymbol{\xi}_B(t)$ by $\boldsymbol{\xi}_I, \boldsymbol{\xi}_B$.

It now remains to provide the sensor model, described by the sensor model functions. In the remainder the sensor models are assumed to be given by the Dirac delta functions. For example, when a single interior sensor with centroid $(x_I(t), y_I(t))$, a single boundary sensor at the top with centroid $(x_T(t), L_Y)$, and a single boundary sensor at the right boundary with centroid $(L_X, y_R(t))$, are considered in Fig. **??**, then (**??**) becomes

$$Z(t;\boldsymbol{\xi}(t)) = \begin{bmatrix} Z_I(t;x_I(t),y_I(t)) \\ Z_T(t;x_T(t)) \\ Z_R(t;y_R(t)) \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^{L_X}\int_0^{L_Y} \delta(x-x_I(t))\delta(y-y_I(t))u(t,x,y)\,\mathrm{d}y\,\mathrm{d}x \\ \int_0^{L_X} \delta(x-x_T(t))u(t,x,L_Y)\,\mathrm{d}x \\ \int_0^{L_Y} \delta(y-y_R(t))u(t,L_X,y)\,\mathrm{d}y \end{bmatrix}$$

$$= \begin{bmatrix} u(t;x_I(t),y_I(t)) \\ u(t;x_T(t),L_Y) \\ u(t;L_X,y_R(t)) \end{bmatrix}.$$

Using the mobile sensor centroid-parameterized measurements along with the process dynamics (**??**), one must now provide the expression for the estimate $\widehat{u}(t,x,y)$ of $u(t,x,y)$ and the sensor repositioning as defined by the derivative $\dot{\boldsymbol{\xi}}(t)$.

III. Summary of state estimation with gradient-ascent sensor guidance policy

At this stage, one can proceed to the design of the combined centralized estimator and sensor guidance. Different from the traditional approach of formulating the integrated design as an optimal control problem, for simplicity, we define the state estimator parameterized by the location-dependent filter kernels. The design of the filter kernels may come from a Kalman filter design or from a Luenberger observer design. For simplicity, we define the centralized filter and postpone the filter kernel selection for later on. Such a centralized estimator is given by an exact copy of the process (**??**) augmented by output injection terms appropriately weighted by the filter kernels; it is given by

$$\begin{aligned} \frac{\partial \widehat{u}(t,x,y)}{\partial t} &= \mathcal{A}(x,y)\widehat{u}(t,x,y) + b_I(x,y)f_I(t) \\ &\quad +\lambda_I(t,x,y;x_I(t),y_I(t))\times \\ &\qquad \big(u(t,x_I(t),y_I(t)) - \widehat{u}(t,x_I(t),y_I(t))\big) \\ \widehat{u}(0,x,y) &= \widehat{u}_0(x,y) \qquad \text{in } [0,L_X]\times[0,L_Y], \\ \widehat{u}(t,0,y) &= 0, \qquad 0\le y\le L_Y, \\ \widehat{u}(t,x,0) &= 0, \qquad 0\le x\le L_X, \\ \widehat{u}_y(t,x,L_Y) &= b_T(x)f_T(t) + c_T(x)\widehat{u}(t,x,L_Y) \\ &\quad +\lambda_T(x;x_T(t))\big(u(t,x_T(t),L_Y) - \widehat{u}(t,x_T(t),L_Y)\big), \\ \widehat{u}_x(t,L_X,y) &= b_R(y)f_R(t) + c_R(y)\widehat{u}(t,L_X,y) \\ &\quad +\lambda_R(y;y_R(t))\big(u(t,L_X,y_R(t)) - \widehat{u}(t,L_X,y_R(t))\big). \end{aligned} \tag{6}$$

The three filter kernels $\lambda_I(t,x,y;x_I(t),y_I(t))$ (interior), $\lambda_T(x;x_T(t))$ (top boundary) and $\lambda_R(y;y_R(t))$ (right boundary) depend on both the spatial variables (x,y) and the corresponding time-varying centroids $(x_I(t),y_I(t))$, $x_T(t)$, $y_R(t)$.

The derivation of the spatial repositioning of the centroids (guidance) depends on the state estimation error $e(t,x,y) = u(t,x,y) - \widehat{u}(t,x,y)$ and is given by

$$\begin{aligned} \frac{\partial e(t,x,y)}{\partial t} &= \mathcal{A}(x,y)e(t,x,y) \\ &\quad -\lambda_I(t,x,y;x_I(t),y_I(t))e(t,x_I(t),y_I(t)) \\ e(0,x,y) &= e_0(x,y) \qquad \text{in } [0,L_X]\times[0,L_Y], \\ e(t,0,y) &= 0, \qquad 0\le y\le L_Y, \\ e(t,x,0) &= 0, \qquad 0\le x\le L_X, \\ e_y(t,x,L_Y) &= -\lambda_T(x;x_T(t))e(t,x_T(t),L_Y), \\ e_x(t,L_X,y) &= -\lambda_R(y;y_R(t))e(t,L_X,y_R(t)). \end{aligned} \tag{7}$$

It now remains to define the sensor guidance. Assuming that the mobile platform carrying the i^{th} sensor can move with a prescribed maximum speed υ_i^{max} and the direction is the only motion control signal, and denoting the gradient of the state error at the current sensor location by the unit outward normal vector

$$\overrightarrow{N}(t,\boldsymbol{\xi}_i(t)) \triangleq \frac{\nabla e(t,\boldsymbol{\xi})}{|\nabla e(t,\boldsymbol{\xi})|}\bigg|_{\boldsymbol{\xi}=\boldsymbol{\xi}_i(t)} = \frac{\nabla e(t,\boldsymbol{\xi}_i(t))}{|\nabla e(t,\boldsymbol{\xi}_i(t))|}, \tag{8}$$

for all sensors $i=1,\dots,N$, then the *unconstrained* gradient ascent guidance policy is given by

$$\boxed{\dot{\boldsymbol{\xi}}_i(t) = \overrightarrow{N}(t,\boldsymbol{\xi}_i(t))\,\upsilon_i^{max},\ \ i=1,\dots,N.} \tag{9}$$

IV. Process effects on sensor measurements and performance

The process is assumed to have a cumulative effect on the sensor life expectancy and performance. A measure of these effects is taken to be the accumulated measurements as described by the past measurements. While in general processes this would involve the summation over time of the absolute value of the measurements, in the case of processes representing physical quantities such as concentration or temperature, the cumulative effects will be described via the summation over time of the measurements; this is because these physical quantities are associated with positive systems. Following [**?**], define the *accumulated mass* $m_i(t)$ of the i^{th} sensor as the total amount of the measurements that the sensor has been exposed. This leads to the following.

Definition 1: [**?**][*accumulated mass*] The accumulated mass of the i^{th} mobile sensor is the amount of the measurements obtained by the sensor up to the current time t

$$m_i(t) = \int_0^t Z_i(\tau;\boldsymbol{\xi}_i(\tau))\,\mathrm{d}\tau, \quad i=1,\dots,N. \tag{10}$$

Remark 1: The accumulated mass $m_i(t)$ provides information on the cumulative effects of exposure of the i^{th} sensor to the spatial field over a time period.

One would expect to associate the accumulated mass to a tolerance level, beyond of which the device onboard the mobile platform will cease to operate. Such a quantity is the *maximum mass* m_i^{max} and is defined below.

Definition 2: [**?**][*maximum mass*] The maximum mass m_i^{max} is defined as the limit of the exposure to the process state beyond of which the i^{th} sensor becomes inoperative.

Remark 2: When a sensor exceeds the maximum mass, i.e. $m_i(t) \ge m_i^{max}$, it is an indication that the device may be useless or saturated and, no longer reading, or no longer transmitting readings, or, transmitting incoherent readings.

To keep track of the instance the accumulated mass $m_i(t)$ of the i^{th} sensor exceeds its maximum mass m_i^{max}, is via the use of the i^{th} sensor *indicator function*.

$$\mathbf{1}_{m_i}(t) = \begin{cases} 1 & \text{if } m_i(t) < m_i^{max} \\ 0 & \text{if } m_i(t) \geq m_i^{max} \end{cases} \qquad i = 1, \ldots, N. \tag{11}$$

To introduce a change in the guidance policy, from an *information-sensitive* to an *information-averse*, another quantity needs to be defined and that is the *threshold mass*.

Definition 3: [**?**][*threshold mass*] The threshold mass of the i^{th} sensor, denoted by $m_i^{thresh} \leq m_i^{max}$ is a *user-defined* threshold that the guidance policy employs to switch from an *information-sensitive* to an *information-averse* motion in order to prolong the life expectancy of the sensor.

This definition is pivotal in defining a third guidance policy, termed the *information-neutral* policy. In the binary guidance policy, the switching from information-sensitive to information-averse occurs the instance $m_i(t^*) \geq m_i^{thresh}$, i.e. $t^* = \min\arg\{\mathbf{1}_{m_i(t)} = 0\}$. Similar to (**??**), define the function

$$\begin{aligned} \sigma(m_i(t)) &\triangleq 1 - 2H(m_i(t) - m_i^{thresh}) \\ &= \text{sign}\big(m_i^{thresh} - m_i(t)\big), \end{aligned} \tag{12}$$

also depicted in Fig. **??**. Whenever $\sigma(m_i(t)) = 1$, meaning $m_i(t) < m^{thresh}$, the guidance implemented is information-sensitive. In the other case $\sigma(m_i(t)) = -1$, meaning $m_i(t) \geq m^{thresh}$, and the guidance implemented is information-averse. From the graph in Fig. **??**, it is evident that the guidance policy changes abruptly at the instance $m_i(t) = m^{thresh}$. To avoid this discontinuity, which affects the motion of the mobile platforms, the function $\sigma(m_i(t))$ can be made continuous at the discontinuity through an approximation.

The binary guidance policy used in [**?**] includes the function $\sigma(m_i(t))$ as a means to switch the direction of the sensor from going towards spatial regions of positive spatial gradients of the state error, meaning also to spatial regions of increasing state estimation error, to spatial regions of negative spatial gradients of the state error, meaning also to spatial regions of decreasing state estimation error. Incorporating the indicator function (**??**) into the sensor vehicle speed to result in a time-varying and in particular, *measurement-dependent speed*, we define this speed via

$$\boxed{\upsilon_i(t) \triangleq \mathbf{1}_{m_i(t)} \left(1 - \frac{m_i(t)}{m_i^{max}}\right) \upsilon_i^{max}, \quad i = 1, \ldots, N.} \tag{13}$$

Remark 3: The time varying vehicle speed $\upsilon_i(t)$ will decrease as the accumulated mass $m_i(t)$ approaches the maximum mass m_i^{max}. As a safety precaution, using the indicator function (**??**) the vehicle speed becomes zero the instant the accumulated mass exceeds the maximum mass.

We present the *binary policy* from [**?**], but in a general form,

$$\boxed{\dot{\boldsymbol{\xi}}_i(t) = \sigma(m_i(t))\,\overrightarrow{N}(t, \boldsymbol{\xi}_i(t))\,\upsilon_i(t), \;\; i = 1, \ldots, N.} \tag{14}$$

The above will stop the sensor vehicle the moment the accumulated mass exceeds the maximum mass, i.e. at $t^* = \min\arg\{\mathbf{1}_{m_i(t)} = 0\}$. It will guide the mobile sensor to the direction of positive gradient of the state error thus implementing an information-sensitive policy. The instance the accumulated mass exceeds the threshold mass, it switches direction, via the function $\sigma(m_i(t))$, and now implements an information-averse policy and repositioning the mobile sensor to the direction of negative gradient of the state error. Once the indicator function becomes zero, the state estimator must now switch to a naïve observer as no process information is available. To reflect this, a redefinition of the measurements needs to be used in order to account for the unavailability of process measurements. Instead, the indicator function is inserted in front of the output injection terms and the state estimator (**??**) is now given by

$$\begin{aligned} \frac{\partial \widehat{u}(t,x,y)}{\partial t} &= \mathcal{A}(x,y)\widehat{u}(t,x,y) + b_I(x,y) f_I(t) \\ &\quad + \lambda_I(t,x,y;x_I(t),y_I(t)) \times \\ &\qquad \mathbf{1}_{m_I(t)}\big(u(t,x_I(t),y_I(t)) - \widehat{u}(t,x_I(t),y_I(t))\big) \\ \widehat{u}(0,x,y) &= \widehat{u}_0(x,y) \qquad \text{in } [0,L_X]\times[0,L_Y], \\ \widehat{u}(t,0,y) &= 0, \qquad 0 \leq y \leq L_Y, \\ \widehat{u}(t,x,0) &= 0, \qquad 0 \leq x \leq L_X, \\ \widehat{u}_y(t,x,L_Y) &= b_T(x) f_T(t) + c_T(x)\widehat{u}(t,x,L_Y) + \\ &\lambda_T(x;x_T(t))\mathbf{1}_{m_T(t)}\big(u(t,x_T(t),L_Y) - \widehat{u}(t,x_T(t),L_Y)\big), \\ \widehat{u}_x(t,L_X,y) &= b_R(y) f_R(t) + c_R(y)\widehat{u}(t,L_X,y) + \\ &\lambda_R(y;y_R(t))\mathbf{1}_{m_R(t)}\big(u(t,L_X,y_R(t)) - \widehat{u}(t,L_X,y_R(t))\big). \end{aligned} \tag{15}$$

At this stage, the introduction to the ternary guidance policy can be made by any of the proposed choices for the approximation $\sigma^{approx}(m_i(t))$ of the function $\sigma(m_i(t))$, or using a Level-set based modification of the sensor direction.

A. *Approximations of* $\sigma(m_i(t))$

Using the approximation of the function $\sigma(m_i(t))$, the resulting ternary guidance is given by

$$\boxed{\dot{\boldsymbol{\xi}}_i(t) = \sigma^{approx}(m_i(t))\,\overrightarrow{N}(t, \boldsymbol{\xi}_i(t))\,\upsilon(t)_i, \;\; i = 1, \ldots, N.} \tag{16}$$

where $\sigma^{approx}(m_i(t))$ can be chosen from the following:

1) Analytic approximation: use the previous binary policy with a correction of the function given by

$$\sigma^a(q) = -\frac{(q - m^{thresh})}{\sqrt{(q - m^{thresh})^2 + \varepsilon^2}}, \quad \varepsilon > 0 \tag{17}$$

which is depicted in Fig. **??**. In this case, the ternary policy based on the analytic approximation of $\sigma(m_i(t))$ is given by (**??**) with $\sigma^{approx}(m_i(t)) = \sigma^a(m_i(t))$.

2) Polynomial approximation: implement a variation of the above, by generating a band around the threshold mass, as for example given by a percentage of m_i^{thresh} and modify the speed whenever the accumulated mass is inside this band; for example using a linear approximation in the interval $[m_i^{thresh} - \varepsilon, m_i^{thresh} + \varepsilon]$, one can

Fig. 2. Graph of $\sigma(m_i(t))$.

Fig. 3. Analytic approximation of $\sigma(m_i(t))$.

have an approximation of $\sigma(m_i(t))$ given by

$$\sigma_1^p(q) = \begin{cases} 1 & \text{if } q < m_i^{thresh} - \varepsilon \\ -\frac{(q-m_i^{thresh})}{\varepsilon} & \text{if } |m_i^{thresh} - q| \le \varepsilon \\ -1 & \text{if } q > m_i^{thresh} + \varepsilon \end{cases} \quad (18)$$

and depicted in Fig. **??**. Similarly, the ternary policy based on the linear approximation of $\sigma(m_i(t))$ is given by (**??**) with $\sigma^{approx}(m_i(t)) = \sigma_1^p(m_i(t))$. A variant is to use a piecewise approximation (0^{th} order approximation) for the interval $[m_i^{thresh} - \varepsilon, m_i^{thresh} + \varepsilon)$

$$\sigma_0^p(q) = \begin{cases} 1 & \text{if } q < m_i^{thresh} - \varepsilon \\ k_1 & \text{if } m_i^{thresh} - \varepsilon \le q \le m_i^{thresh} \\ k_2 & \text{if } m_i^{thresh} < q \le m_i^{thresh} + \varepsilon \\ -1 & \text{if } q > m_i^{thresh} + \varepsilon \end{cases} \quad (19)$$

with some choices for k_1, k_2: (i) $k_1 = 0.5$, $k_2 = -k_1$, (ii) $k_1 = k_2 = 0$, (iii) $k_1 = k_2 = 0.5$. The ternary policy based on a 0^{th} order approximation of $\sigma(m_i(t))$ is given by (**??**) with $\sigma^{approx}(m_i(t)) = \sigma_0^p(m_i(t))$.

B. Level-set approach

Use a modification that does not make changes directly to the function $\sigma(m_i(t))$, but instead uses the state error to change the mobile sensor path in the direction where the state error is neither increasing nor decreasing. Specifically, the sensor is guided along the contour lines of the state error (zero level-set of $e(\cdot,x,y) =$ const.) whenever $|m_i^{thresh} - m_i(t)| \le \varepsilon$. Define the direction along the instantaneous isolines of $e(\cdot.x.y)$ and normal to the gradient $\overrightarrow{N}(t, \boldsymbol{\xi}_i(t))$

$$\overrightarrow{P}(t, \boldsymbol{\xi}_i(t)) \ \ : \ \ \overrightarrow{P}(t, \boldsymbol{\xi}_i(t)) \cdot \overrightarrow{N}(t, \boldsymbol{\xi}_i(t)) = 0 \quad (20)$$

Then the complete guidance for all ranges of the accumulated mass $m_i(t)$ is given by

$$\dot{\boldsymbol{\xi}}_i(t) = \begin{cases} \overrightarrow{P}(t, \boldsymbol{\xi}_i(t)) \upsilon_i(t) & \text{if } |m_i^{thresh} - m_i(t)| \le \varepsilon \\ \sigma(m_i(t)) \overrightarrow{N}(t, \boldsymbol{\xi}_i(t)) \upsilon_i(t) & \text{otherwise.} \end{cases} \quad (21)$$

Table **??** summarizes all options for the proposed policy.

TABLE I

OVERVIEW OF GRADIENT ASCENT GUIDANCE POLICIES.

policy	guidance
unconstrained	(**??**)
binary	(**??**) with (**??**)
ternary	{(**??**) with (**??**) or (**??**) or (**??**)}, or {(**??**), (**??**) }

Fig. 4. 0^{th} and 1^{st} order polynomial approximation of $\sigma(m_i(t))$.

Fig. 5. Evolution of state estimation error $L_2(\Omega)$ norm.

Fig. 6. Accumulated mass.

V. EXAMPLE

The 2D diffusion PDE considered in [**?**] defined in the spatial domain $[0, L_X] \times [0, L_Y] = [0,1] \times [0,1]$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\big(\alpha(x)\frac{\partial u}{\partial x}\big) + \frac{1}{2}\frac{\partial}{\partial y}\big(\alpha(y)\frac{\partial u}{\partial y}\big) + 5\frac{\partial u}{\partial x} - 6\frac{\partial u}{\partial y} - 10^{-2}\kappa u + b(x,y) f_d(t)$$

is also considered here where the spatial coefficient is

$$\alpha(x) = 0.01\Big[1 + 0.3\sin(2\pi x)\big(\sin^3(x^3) + \sin^3(L-x)^3\big)\Big],$$

with $\kappa = (\pi/L_X)^2 + (\pi/L_Y)^2$. The initial conditions were chosen as $u(0,x,y) = \sin(\pi x)\big(2 + \sin(2\pi y)\big)$ and Dirichlet boundary conditions were considered. The system was simulated using a Galerkin-based finite element approximation with 20 linear splines in each spatial direction. The estimator (**??**) was also approximated with a Galerkin-based finite element scheme and the initial condition was $\widehat{u}(0,x,y) = 0$. An interior mobile sensor was used to provide measurements

$$Z_I(t) = \int_0^{L_X}\int_0^{L_Y} \delta(x - x_I(t))\delta(y - y_I(t)) u(t,x,y)\, \mathrm{d}y\, \mathrm{d}x,$$

and the filter kernel was set equal to the weighted adjoint of the output operator, $\lambda(x,y) = 63\delta(x - x_I(t))\delta(y - y_I(t))$. The maximum mass was $m^{max} = 4$ with the threshold mass set to $m^{thresh} = 2.5$ and the length of the band was $\varepsilon = \frac{1}{4}m^{max} = 1$, thus implementing the ternary policy for $1.5 \le m_I(t) \le 3.5$.

Fig. **??** depicts the evolution of the $L_2(\Omega)$ state error norm for the case of no limitations on the accumulated mass, the case of the binary policy (**??**) and the proposed ternary policy given by the piecewise option. Fig. **??** shows the evolution of the accumulated mass $m(t)$ for the unconstrained policy, the binary policy and the proposed ternary policy. The unconstrained policy crosses the maximum mass at $t = 2.59$s meaning that without any precautions, the sensor would stop operating. The binary policy prolongs the sensor life expectancy by 0.82s when it crosses the maximum mass at $t = 3.41$s. The ternary policy extends the sensor life further by 6.04s when it crosses the maximum mass at $t = 8.63$s.

TABLE II

EFFECTS OF GUIDANCE ON SENSOR LIFE DURATION AND EXTENSION.

policy	life duration	life extension
unconstrained	2.59s	NA
binary	3.41s	0.82s
ternary	8.63s	6.04s

VI. CONCLUSIONS

This paper provided a modification to the gradient-ascent guidance policy for mobile sensors employed in the state reconstruction of spatiotemporally varying processes. Incorporating the detrimental effects of the spatial process on the life expectancy and operation of the sensing devices, two policies emerged: the information-sensitive and the information-averse policies. The former would guide the mobile sensors to spatial regions with larger values of the process state and thereby accelerating the exposure of the sensor to the spatial process. The latter would guide the mobile sensors to spatial regions with smaller values of the process state and thereby minimizing the exposure of the sensor to the spatial process. With the information-sensitive policy the mobile sensor has more value of information of the process state where with the information-averse policy the mobile sensor has less value of information of the process state. The switch from information-sensitive to information-averse occurred the instance the accumulated past histories of the measurement exceeded a device-specific threshold. To alleviate the abrupt changes in the mobile sensor motion as instituted by the binary guidance policy, a modification was proposed and which added an intermediate policy termed the information-neutral. Part of the modification utilized level-set methods to steer the mobile sensors to a direction along the level sets of the state estimation error and normal to the spatial gradient. This was demonstrated in a 2D advection-diffusion PDE with an interior mobile sensor and showed that a significant life extension of the sensing device can be achieved with the proposed ternary policy.