Using asymptotic embedding methods for dynamic estimation of spatial fields with mobile sensors

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***Abstract*— This paper proposes an asymptotic embedding method for the dynamic reconstruction of spatially varying fields. By assuming that the spatial field is the solution to an elliptic partial differential equation, then the elliptic PDE is embedded into a parabolic PDE which represents the time-varying estimator. An important advantage of the dynamic estimation scheme is the significant reduction in the use of sensing devices needed to reconstruct the spatial field. Static estimation schemes impose stringent conditions on the regularity of a regression matrix, which links the basis functions to the number of measurements. To further improve the performance of the dynamic estimator, a guidance scheme is proposed that repositions mobile sensors within the spatial field, which is linked to the performance of the dynamic estimator. Extensions to collaborative estimation and optimization of the placement of static sensors are also summarized to provide an integrated account on all facets of optimal dynamic estimation of spatial fields. Numerical simulations for spatial fields in one and two spatial dimensions are included along with a comparison of static reconstruction as quantified by the number of sensing devices required and the relative error.**

***Index Terms*— Distributed parameter systems; spatial fields; asymptotic embedding; dynamic estimation; mobile sensors.**

I. Introduction

This work considers the dynamic estimation of unknown spatial fields as described by spatially varying functions. The static estimation approach requires that the number of measurements be at least the same as the dimension of the basis set, i.e. the number of basis functions. Even in that case, one must assume that the resulting regressor matrix is not rank deficient [1]. A least square or a minimum norm solution results when the number of measurements differ from the number of basis functions. As mentioned in the earlier effort on the use of asymptotic embedding methods for the dynamic reconstruction of spatial fields [2], other approaches were considered for the estimation of spatial fields, see [3], [4], [5], [6], [7].

To remove this dependence of the measurements (i.e. sensing devices) to the accuracy of the unknown field estimates, a *dynamic* estimation scheme is proposed and which embeds the elliptic PDE whose solution is the unknown spatial field, into a parabolic PDE representing the dynamic estimator. Asymptotic embedding (regularization) methods have been employed in the past for parameter identification of elliptic PDEs [8], [9], [10], [11], but not for state reconstruction of spatial fields till recently, [2].

The contribution of this work is many-fold; first it expands the dynamic estimator structure of a Luenberger observer presented in [2] to include a Kalman-based filter. Second, it summarizes earlier results in [12] on the optimal sensor location, in the context of estimation schemes with asymptotic embedding. Third, it unifies the Lyapunov-based guidance in [13] to the gradient-based guidance in [14]. Finally, it provides a framework for collaborative estimation of decentralized filters with mobile sensors in the context of dynamic estimation of spatial fields.

To provide an insight on aspects of the proposed asymptotic embedding methods for the dynamic estimation of spatial fields with mobile and static sensors, both an 1D and 2D elliptic PDEs are considered.

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Research was partially supported through NSF-CMMI grant # 1825546 .

II. Problem motivation

To motivate the proposed work on the use of static and mobile sensors for the dynamic reconstruction of static fields, that is, spatial fields that are spatially varying but time invariant, consider the following 2D spatial function

$$x(\xi,\zeta) = 2\sin\left(\frac{\pi\xi}{L_\xi}\right)\sin\left(\frac{\pi\zeta}{L_\zeta}\right), \quad 0 \le \xi \le L_\xi,\; 0 \le \zeta \le L_\zeta. \quad (1)$$

It is assumed that there are m measurements at the spatial coordinates (ξ_i,ζ_i), $i = 1,\dots,m$, as obtained by m (pointwise) sensors. It is desired to estimate (or reconstruct) the spatial field in (1). Using a least-squares estimation scheme [1], one can use the m measurements to obtain the *static* estimate $\widehat{x}(\xi,\zeta)$ of $x(\xi,\zeta)$. Assuming that the basis functions are $\{\phi_1,\phi_2,\phi_3,\phi_4\} = \{\sin\left(\frac{\pi\xi}{L_\xi}\right)\sin\left(\frac{\pi\zeta}{L_\zeta}\right), \sin\left(\frac{\pi\xi}{L_\xi}\right), \sin\left(\frac{\pi\zeta}{L_\zeta}\right), 1\}$ then the static estimate of $x(\xi,\zeta)$ admits the following expansion

$$\widehat{x}(\xi,\zeta) = \sum_{i=1}^{4} \widehat{\alpha}_i \phi_i(\xi,\zeta) = \Phi(\xi,\zeta)\widehat{\boldsymbol{\alpha}} \quad (2)$$

where $\boldsymbol{\alpha} = \left[\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \right]^T$, $\widehat{\boldsymbol{\alpha}}$ is the *static* estimate of $\boldsymbol{\alpha}$, and $\Phi(\xi,\zeta) = \left[\ \phi_1(\xi,\zeta)\quad \phi_2(\xi,\zeta)\quad \phi_3(\xi,\zeta)\quad \phi_4(\xi,\zeta)\ \right]$. Evaluating the expansion in (2) at the m measurements

$$Y_m = \begin{bmatrix} x(\xi_1,\zeta_1) \\ \vdots \\ x(\xi_m,\zeta_m) \end{bmatrix}, \quad (3)$$

we have

$$Y_m = \Phi_m \widehat{\boldsymbol{\alpha}},$$

where the *regressor matrix* $\Phi_m = \{\phi_j\}_{i=1}^m$ is given by

$$\Phi_m = \begin{bmatrix} \phi_1(\xi_1,\zeta_1) & \phi_2(\xi_1,\zeta_1) & \phi_3(\xi_1,\zeta_1) & \phi_4(\xi_1,\zeta_1) \\ \vdots & \vdots & \vdots & \vdots \\ \phi_1(\xi_m,\zeta_m) & \phi_2(\xi_m,\zeta_m) & \phi_3(\xi_m,\zeta_m) & \phi_4(\xi_m,\zeta_m) \end{bmatrix}.$$

Uniqueness of the solution requires the regressor matrix be square and non-singular. The necessary condition requires that $m \geq 4$ and sufficient condition is that $\text{rank}(\Phi) = 4$. To ensure the rank condition, the measurement points must be at spatial locations that result in linearly independent columns of the regressor matrix. A necessary condition is that they should not be placed at the zeros of the basis functions. An estimate of $x(\xi,\zeta)$ can be obtained even when the rank condition is not satisfied, or when $m < 4$. One results in a least squares solution and the other produces a minimum norm solution [1]. The estimate of the coefficient vector is

$$\widehat{\alpha} = (\Phi_m)^{\dagger} Y_m, \tag{4}$$

and the estimate of the spatial field is given by (2). The matrix $(\Phi_m)^{\dagger}$ is the Moore-Penrose inverse of Φ_m [15].

Three different sets of grid points locating the measurements in the spatial domain $\Omega = [0,L_\xi] \times [0,L_\zeta]$ were considered and are given in Table I. Table II summarizes the

TABLE I
GRID POINTS.

Grid 1	Grid 2	Grid 3
$(0.25L_\xi, 0.25L_\zeta)$	$(0.2L_\xi, 0.2L_\zeta)$	$(0.1481L_\xi, 0.1667L_\zeta)$
$(0.25L_\xi, 0.75L_\zeta)$	$(0.4L_\xi, 0.4L_\zeta)$	$(0.2593L_\xi, 0.2778L_\zeta)$
$(0.75L_\xi, 0.25L_\zeta)$	$(0.6L_\xi, 0.6L_\zeta)$	$(0.3704L_\xi, 0.3889L_\zeta)$
$(0.75L_\xi, 0.75L_\zeta)$	$(0.8L_\xi, 0.8L_\zeta)$	$(0.4815L_\xi, 0.5000L_\zeta)$

static estimation of the coefficient vector $\widehat{\alpha}$ and the estimate $\widehat{x}(\xi,\zeta)$ for the three grid choices. The relative error is

$$100\frac{|x-\widehat{x}|_{L_2}}{|x|_{L_2}} = 100\sqrt{\frac{\int_0^{L_\xi}\int_0^{L_\zeta}\left(x(\xi,\zeta)-\widehat{x}(\xi,\zeta)\right)^2 \mathrm{d}\zeta \mathrm{d}\xi}{\int_0^{L_\xi}\int_0^{L_\zeta} x^2(\xi,\zeta)\,\mathrm{d}\zeta \mathrm{d}\xi}}$$

Exact reconstruction of the spatial field is achieved in the case of perfect (noise-free) measurements with the rank of the regressor matrix being equal to the dimension of the basis. Grid 1 offers the largest estimation error since the grid points (measurement locations) resulted in rank deficient Φ_m.

TABLE II
EFFECTS OF MEASUREMENTS ON SOLUTION ACCURACY.

	Grid 1	Grid 2	Grid 3
$\text{rank}(\Phi_m)$	1	2	4
rel. error	32.1%	9.34%	0%
$\widehat{\alpha}$	$\begin{pmatrix}0.2222\\0.3143\\0.3143\\0.4444\end{pmatrix}$	$\begin{pmatrix}1.1989\\0.6164\\0.6164\\-0.4478\end{pmatrix}$	$\begin{pmatrix}2\\0\\0\\0\end{pmatrix}$

It is obvious that even in the unlikely case of noise-free measurements, one may not be able to estimate the true coefficient vector α exactly, unless the sensors are placed at the right locations.

III. MAIN RESULTS

Using appropriate continuity and differentiability conditions for the unknown spatial function $x(\xi)$ (spatial field), such as the one in (1), it is assumed that it is the solution to an elliptic PDE written in abstract form

$$0 = \mathcal{A}x + f, \tag{5}$$

and defined over a Hilbert space H. Such an assumption can be justified from a priori qualitative knowledge of the spatial field in question. The state space H has inner product $\langle\cdot,\cdot\rangle$ and corresponding induced norm $|\cdot|$. Let V be a reflexive Banach space with norm $\|\cdot\|_V$, and assume V is embedded densely and continuously in H with its dual denoted by V^* and with norm $\|\cdot\|_*$. We have $V \hookrightarrow H \hookrightarrow V^*$ with the embeddings dense and continuous and $|\phi| \leq k\|\phi\|$ for $\phi \in V$.

We can now characterize the operator $\mathcal{A}$ in (5). Define the elliptic operator $\mathcal{A} : V \to V^*$ that satisfies the boundedness ($\exists a_b > 0$, $|\langle\mathcal{A}\phi,\psi\rangle| \leq a_b\|\phi\|\|\psi\|$ for $\phi,\psi \in V$) and V-coercivity ($\exists a_c > 0$, $\text{Re}\langle-\mathcal{A}\phi,\phi\rangle \geq a_c\|\phi\|^2$ for $\phi \in V$). A possibly third condition may be imposed, that of symmetry ($\langle\mathcal{A}\phi,\psi\rangle = \langle\phi,\mathcal{A}\psi\rangle$). Assuming that $f \in V^*$, then (5) admits a unique solution, [16], [17].

A representative example of a system that has the abstract representation in (5) is the Poisson equation (elliptic PDE)

$$-\text{div}(\alpha\,\text{grad}x) = f \quad \text{in } \Omega, \qquad x|_{\partial\Omega} = 0, \tag{6}$$

where the spatial domain Ω is an open bounded and connected subset of $\mathbb{R}^n$ having Lipschitz boundary $\partial\Omega$, and the non-homogeneous term $f \in H^{-1}(\Omega)$, $\alpha \in L^\infty(\Omega)$. As mentioned in [2], equation (6) should not be restricted to Dirichlet boundary conditions; this is used to motivate the proposed asymptotic embedding scheme. In fact, (6) can be furnished with mixed boundary conditions, as for example

$$x(\xi)\Big|_{\Gamma_1} = 0, \qquad \frac{\partial x(\xi)}{\partial n} = g(\xi), \text{ in } \Gamma_2, \quad \partial\Omega = \Gamma_1 \cup \Gamma_2.$$

Additionally, (6) is not necessarily restricted to inhomogeneous problems and the case $f \equiv 0$ can also be considered.

Associated with (6), or its abstract representation (5), are the process measurements obtained by m sensing devices

$$y = \int_\Omega c(\xi)x(\xi)\,\mathrm{d}\xi. \tag{7}$$

The m-dimensional output vector y is constant as it obtains partial state observation of the time-invariant state of (5). However, when measurement noise is included or when the sensor is attached to a mobile platform, then the output vector y becomes time varying. The observations in (7) can include interior and/or boundary measurements. The abstract form of (7) is written in terms of the observation operator

$$y = Cx, \qquad C : V^* \to \mathbb{R}^m. \tag{8}$$

The spatial function $c(\xi)$ denotes the sensor model and describes the manner it averages (spatially) the state $x(\xi)$ over its spatial support. In the case of pointwise measurements, the function $c(\xi)$ is simply the spatial delta function.

Objective: The estimation and control objective is to employ estimation schemes for the *dynamic* reconstruction of the state of a time invariant equation (5) using a reduced number of fixed or mobile sensors that are independent of the dimension of the basis functions.

The various components of the contribution are

1) *Filter gain structure:* Luenberger vs Kalman filter, with the former assigning the structure of the filter kernel and in the latter solving an Operator Riccati Equation to obtain the expression for the filter kernel.
2) *Estimation structure:* centralized vs decentralized structure; single estimator with m observations or m separate estimators utilizing a single measurement and with a consensus protocol in each penalizing differ-

ences in their state estimates.

3) *Sensor location selection:* for static sensors, find the optimal spatial locations with respect to a filter performance metric.
4) *Mobile sensor guidance:* Propose the sensors' spatial relocation, assumed to be onboard mobile platforms.

A. Filter gain structure

Using asymptotic embedding methods [2], the estimator for the elliptic PDE (equiv. time invariant equation (5)) is

$$\begin{aligned}\dot{\widehat{x}}(t,\xi) &= \operatorname{div}(\alpha \operatorname{grad}\widehat{x}(t,\xi)) + f(\xi)\\ &\quad + \lambda(\xi)\big(y(t) - \int_\Omega c(\xi)\widehat{x}(t,\xi)\,d\xi\big)\\ \widehat{x}(0,\xi) &= 0.\end{aligned} \tag{9}$$

The spatial function $\lambda(\xi)$ is the *filter gain kernel* and its computation depends on the estimator design.

1) Luenberger observer-based kernel design: A simple choice of $\lambda(\xi)$ when the Luenberger observer design is used, is the weighted adjoint of the observation operator in (8)

$$\lambda(\xi) = c(\xi)\Gamma, \tag{10}$$

where $0 < \Gamma = \Gamma^T$ is an $m \times m$ gain matrix. The estimator is

$$\begin{aligned}\dot{\widehat{x}}(t,\xi) &= \operatorname{div}(\alpha \operatorname{grad}\widehat{x}(t,\xi)) + f(\xi)\\ &\quad + c(\xi)\Gamma\big(y(t) - \int_\Omega c(\xi)\widehat{x}(t,\xi)\,d\xi\big)\\ \widehat{x}(0,\xi) &= 0.\end{aligned} \tag{11}$$

With regards to the abstract form of (9), the estimator is

$$\begin{aligned}\dot{\widehat{x}}(t) &= \mathcal{A}\widehat{x}(t) + f + L\,(y(t) - \mathcal{C}\widehat{x}(t)),\\ \widehat{x}(0) &= 0.\end{aligned} \tag{12}$$

2) Kalman filter-based kernel design: The filter kernel is derived from the solution to an operator Riccati equation corresponding to the abstract form (5), (8), Using the state and output operators, and assuming that the process and measurement noise covariance operators are given by Q and R, respectively, the requisite algebraic Riccati equation is

$$0 = \mathcal{A}\Sigma + \Sigma\mathcal{A}^* + \Sigma\mathcal{C}^* R^{-1}\mathcal{C}\Sigma + Q, \qquad \text{in } D(\mathcal{A}^*), \tag{13}$$

and the associated filter operator is

$$L = \Sigma\mathcal{C}^* R^{-1}. \tag{14}$$

The function $\lambda(\xi)$ in (9) is the kernel representation of L^*.

B. Estimation structure

The m measurements in (7) can be used for a single centralized filter such as the one in (12) with the filter gain based on Kalman (using (12), (14)) or Luenberger observer design (using (11)). The alternative, is to use decentralized estimation with collaboration. A single estimator is associated with each of the m measurements and (12) now becomes

$$\begin{aligned}\dot{\widehat{x}}_i(t) &= \mathcal{A}\widehat{x}_i(t) + f + L_i\,(y_i(t) - \mathcal{C}_i\widehat{x}_i(t))\\ &\quad - \mathcal{P}_i \sum_{j\in\mathbb{N}_i} \big(\widehat{x}_i(t) - \widehat{x}_j(t)\big)\\ \widehat{x}_i(0) &= 0,\end{aligned} \tag{15}$$

where $\mathbb{N}_i$ denotes the set of neighbors of the i^{th} sensor/estimator and the term $-\mathcal{P}_i \sum_{j\in\mathbb{N}_i} \big(\widehat{x}_i(t) - \widehat{x}_j(t)\big)$ denotes the consensus protocol implemented by each of the distributed estimators in (15). The details of the derivation of the consensus operators $\mathcal{P}_i$, $i = 1,\dots,m$ are given in [18].

C. Sensor location selection

To improve the estimation scheme, given by the centralized estimator in (9) or (15), one can improve the performance by optimizing the sensor locations. The basic idea behind this is to parameterize the filter kernel by candidate sensor locations within the spatial domain and optimize an appropriate estimation performance metric, see [12].

D. Mobile sensor guidance

In a similar fashion to the previous location optimization, to improve the estimator performance, one can assume a significantly reduced number of sensors are now affixed on mobile platforms and can move freely within the domain.

Specifically, for a physical problem described by (6), it is assumed that a single mobile sensor is used to provide process measurements. Thus, (7) is now re-written as

$$y(t;\xi_0(t)) = \int_\Omega c(\xi;\xi_0(t))x(\xi)\,d\xi \tag{16}$$

where $\xi_0(t) \in \mathbb{R}^n$ is the mobile sensor centroid and $c(\xi;\xi_0(t))$ denotes the centroid-parameterized sensor distribution function. For the case of pointwise measurements, the sensor function is the spatial Delta function with $c(\xi;\xi_0(t)) = \delta(\xi - \xi_0(t))$ and the mobile sensor measurement (16) becomes

$$y(t;\xi_0(t)) = x(\xi_0(t)). \tag{17}$$

There are two performance-based approaches for the mobile sensor guidance. The first one utilizes the Luenberger design and parameterizes the filter kernel $\lambda(\xi) = c(\xi_0(t))\Gamma$ by the time varying sensor centroid to arrive at a modified version of the estimator (11). When the sensor model in (17) is assumed, then the state estimator in (11) becomes

$$\begin{aligned}\dot{\widehat{x}}(t,\xi) &= \operatorname{div}(\alpha \operatorname{grad}\widehat{x}(t,\xi)) + f(\xi)\\ &\quad + \delta(\xi - \xi_0(t))\Gamma\big(x(\xi_0(t)) - \widehat{x}(t,\xi_0(t))\big)\\ \widehat{x}(0,\xi) &= 0.\end{aligned} \tag{18}$$

The sensor guidance, as described by the time derivative of the sensor centroid, is extracted from Lyapunov design methods for the state error $e(t,\xi) = x(\xi) - \widehat{x}(t,\xi)$. The state error, through the difference of (6), (18) is given by

$$\begin{aligned}\dot{e}(t,\xi) &= \operatorname{div}(\alpha \operatorname{grad} e(t,\xi)) - \delta(\xi - \xi_0(t))\Gamma e(t,\xi_0(t))\\ e(0,\xi) &= x(\xi).\end{aligned} \tag{19}$$

Following [13], the Lyapunov-based sensor guidance is $\dot{\xi}_0(t) = e(t,\xi_0(t))e_{\xi}(t,\xi_0(t))$. To include aspects of the mobile platforms, it is assumed that the speed of the platforms is fixed and only the direction is to be provided by the estimator performance. In this case the guidance takes the form $\dot{\xi}_0(t) = e(t,\xi_0(t))e_{\xi}(t,\xi_0(t))\upsilon$ where υ is the assumed constant speed of the mobile platform. To link the Lyapunov-based guidance to the gradient based guidance in [14], we provide a unifying guidance that can be used regardless of the estimator design and is given by

$$\boxed{\dot{\xi}_0(t) = \frac{e(t,\xi_0(t))}{|e(t,\xi_0(t))|}\,\frac{\nabla e(t,\xi_0(t))}{|\nabla et,\xi_0(t))|}\,\upsilon,} \tag{20}$$

TABLE III

INTEGRATED ESTIMATOR DESIGN AND MOBILE SENSOR GUIDANCE.

filter gain structure	observer	guidance
Luenberger	(18)	(20)
Kalman	(21), (22), (23)	(20)

where $\nabla e(t,\boldsymbol{\xi}_0(t))$ is the gradient vector ($\in \mathbb{R}^n$) of the state error evaluated at the current location of the sensor centroid.

Equipped with the above unifying guidance, we summarize the selection of the filter gain for each of the two cases of gain structure. For Luenberger observer with a mobile sensor, the integrated estimator and sensor guidance are given by (18) and (20). To implement a Kalman-based design, the estimator is written in abstract form (similar to (12))

$$\begin{aligned}\dot{\widehat{x}}(t) &= \mathcal{A}\widehat{x}(t)+f+L(\boldsymbol{\xi}_0(t))\left(y(t)-C(\boldsymbol{\xi}_0(t))\widehat{x}(t)\right)\\ \widehat{x}(0) &= 0\end{aligned} \tag{21}$$

where the centroid-dependent filter operator $L(\boldsymbol{\xi}_0(t))$ is given via the solution to the differential operator Riccati equation

$$\begin{aligned}\dot{\Sigma}(\boldsymbol{\xi}_0(t)) &= \mathcal{A}\Sigma(\boldsymbol{\xi}_0(t))+\Sigma(\boldsymbol{\xi}_0(t))\mathcal{A}^*\\ &\quad +\Sigma(\boldsymbol{\xi}_0(t))C^*(\boldsymbol{\xi}_0(t))R^{-1}C(\boldsymbol{\xi}_0(t))\Sigma(\boldsymbol{\xi}_0(t))+Q,\end{aligned} \tag{22}$$

and the associated filter operator is

$$L(\boldsymbol{\xi}_0(t)) = \Sigma(\boldsymbol{\xi}_0(t))C^*(\boldsymbol{\xi}_0(t))R^{-1}. \tag{23}$$

The filter kernel $\lambda(\xi_0(t))$ is the kernel representation of $L^*(\boldsymbol{\xi}_0(t))$ and it is used in (9) to implement the Kalman-based design with mobile sensor. The associated state error equation is governed by

$$\begin{aligned}\dot{e}(t) &= \mathcal{A}e(t)-L(\boldsymbol{\xi}_0(t))\left(y(t)-C(\boldsymbol{\xi}_0(t))\widehat{x}(t)\right),\\ e(0) &= x.\end{aligned} \tag{24}$$

Remark 1: The nonhomogeneous term f in (5) was assumed known and used in (18) or (21). When f is not known, then one can follow the approach presented in [2]. One approach would incorporate the adaptive-like estimate of the nonhomogeneous term by defining it as

$$\widehat{f}(t) = -\mathcal{A}\widehat{x}(t), \tag{25}$$

whereas the other approach would ignore it in the estimator expressions (18) or (21). However, the unknown term would still appear in the error equation (19) or (24). In this case, one can still achieve convergence of the state error that is polynomial in t; i.e. the norm of the state error converges to a residual st whose bound is dictated by the norm of f [2].

IV. NUMERICAL EXAMPLES AND CONCLUSIONS

A. Elliptic PDE embedded in Diffusion-Reaction PDE in 1D

The following elliptic PDE in considered

$$0 = 0.05\frac{\partial^2 x}{\partial\xi}+0.05\pi^2 x(\xi), \quad 0\le\xi\le 1, \quad x(0)=x(1)=0,$$

with solution $x(\xi)=\sin(\pi\xi)$. The spatial operator in (5) is

$$\mathcal{A}\phi = 0.05\phi''+0.05\pi^2\phi, \quad \phi\in H_0^1(0,1).$$

A single mobile sensor with centroid $\xi_0(t)$ is assumed to provide process information

$$\begin{aligned}y(t;\xi_0(t)) &= \int_0^1 \delta(\xi-\xi_0(t))x(\xi)\,\mathrm{d}\xi\\ &= x(\xi_0(t))\end{aligned}$$

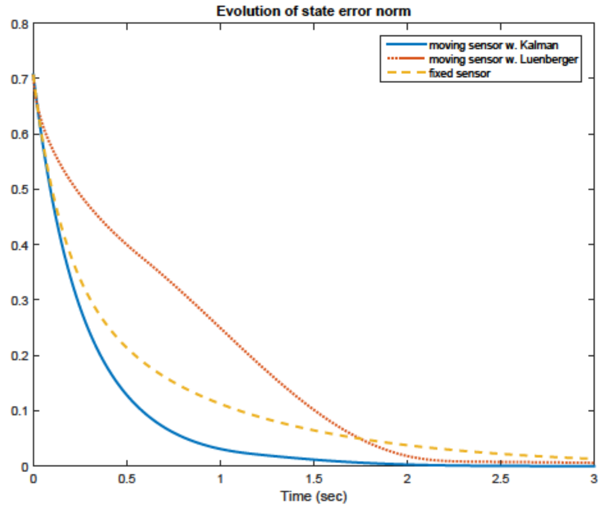

Fig. 1. Evolution of error L_2 norm.

If the state $x(\xi)$ were known to the user, then the mobile sensor would measure $y(t;\xi_0(t)) = \sin(\pi\xi_0(t))$. The state estimator is

$$\begin{aligned}\dot{\widehat{x}}(t,\xi) &= 0.05\left(\widehat{x}''(t,\xi)+\pi^2\widehat{x}(t,\xi)\right)\\ &\quad +\lambda(\xi_0(t))\left(y(t;\xi_0(t))-\widehat{x}(t,\xi_0(t))\right)\\ \widehat{x}(t,0) &= \widehat{x}(t,1)=0\end{aligned}$$

The guidance (20) was used for both types of the filter gain structure summarized in Table III.

To simulate the estimator, a finite dimensional approximation of the abstract form (5) and (12) was implemented. A finite element based Galerkin scheme with 100 linear elements was used to obtain the matrix representations of the system operators. The requisite spatial integrals were computed with the aid of a composite two-point Gauss Legendre quadrature rule [19]. The finite element scheme resulted in a semi-discrete system (spatial discretization) of differential equations that are numerically integrated in the time interval $[0,3]$s using the stiff differential equation solver `ode23s` from the Matlab® ODE suite and which is based on the 4th order Runge-Kutta scheme [19].

For the implementation of the Kalman-based filter gain structure, the finite dimensional approximation of the differential Riccati equation (22) was propagated in time using a modification of the method in [20][ch. 4], that included time sub-cycling in the time loop of the time integration.

Three different cases were considered: (a) a fixed sensor placed at $\xi = 0.65$ using the Kalman-based filter gain with $R=10^{-4}$ and $Q=10I$, (b) a moving sensor with initial location $\xi_0(0)=0.65$ and a Kalman-based filter gain with $R=10^{-4}$ and $Q=10I$, and finally, (c) a moving sensor with initial location $\xi_0(0)=0.65$ with a Luenberger-based filter gain with $\lambda(\xi)=10\delta(\xi-\xi_0(t))$ in (18). The last two cases used a speed of $\upsilon=0.25$.

The $L_2(\Omega)$ norm of the state estimation error for the above three cases is depicted in Figure 1 where it is observed that the case of a Kalman filter with a mobile sensor can perform better than the Kalman filter with a fixed sensor.

The sensor trajectory for the two filter structures is depicted in Figure 2 and the spatial distribution of the state

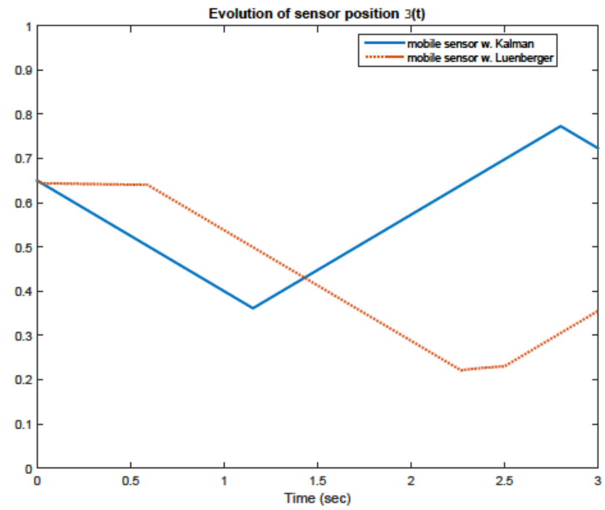

Fig. 2. Mobile sensor trajectory.

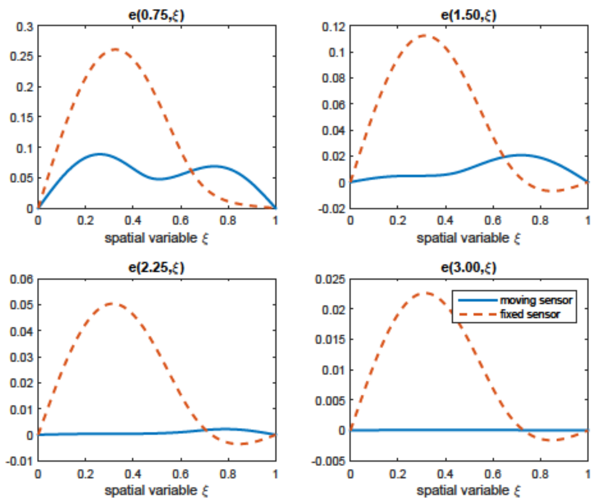

Fig. 3. Spatial distribution of state error $e(t,\xi)$ at four time instances.

error at four time instances is presented in Figure 3.

B. Elliptic PDE embedded in Diffusion-Reaction PDE in 2D

The 2D elliptic PDE is defined over the spatial domain $[0,L_X]\times[0,L_Y]=[0,1]\times[0,1]$

$$0=\alpha\frac{\partial^2 x(\xi,\zeta)}{\partial\xi^2}+\alpha\frac{\partial x^2(\xi,\zeta)}{\partial\zeta^2}+\alpha\gamma x(\xi,\zeta),$$

where

$$\gamma=\left(\frac{\pi}{L_\xi}\right)^2+\left(\frac{2\pi}{L_\zeta}\right)^2,\quad \alpha=10^{-2},\quad L_\xi=L_\zeta=1.$$

The boundary conditions were Dirichlet BCs $x(0,\zeta)=x(L_\xi,\zeta)=0$ for $0<\zeta<L_\zeta$, and $x(\xi,0)=x(\xi,L_\zeta)=0$, $0<\xi<L_\xi$, and the solution is $x(\xi,\zeta)=2\sin(\pi\xi)\sin(2\pi\zeta)$. The model representing the sensing device, is the pointwise measurement described by the spatial Delta function at the sensor centroid $\theta_s(t)=(\xi_s(t),\zeta_s(t))$ and thus

$$y(t;\theta_s(t))=\int_0^{L_\xi}\int_0^{L_\zeta}\delta(\xi-\xi_s(t))\delta(\zeta-\zeta_s(t))x(\xi,\zeta)\,d\xi\,d\zeta.$$

The sensor guidance (20) was used with $(\xi_s(0),\zeta_s(0))=(0.125L_\xi,0.875L_\zeta)$ and speed $\upsilon=5$. The corresponding state

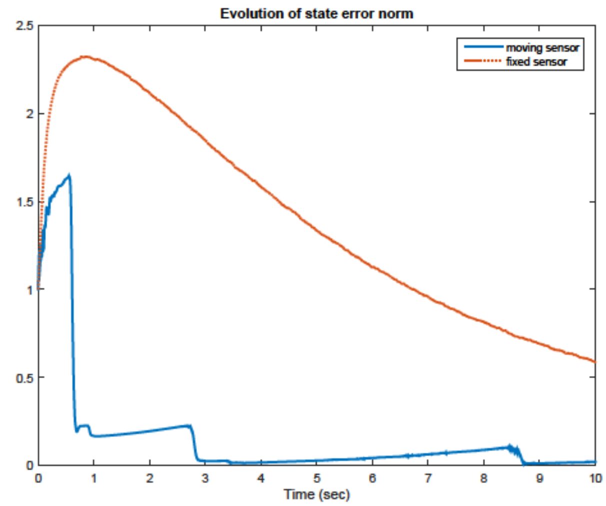

Fig. 4. Evolution of error L_2 norm.

estimator with the mobile sensor is similarly given by

$$\frac{\partial\widehat{x}(t,\xi,\zeta)}{\partial t}=\alpha\frac{\partial^2\widehat{x}(t,\xi,\zeta)}{\partial\xi^2}+\alpha\frac{\partial\widehat{x}^2(t,\xi,\zeta)}{\partial\zeta^2}+\alpha\gamma\widehat{x}(t,\xi,\zeta)$$

$$+\lambda(\xi_s(t),\zeta_s(t))\Big(y(t)-\widehat{x}(t,\xi_s(t),\zeta_s(t))\Big)$$

where once again the centroid-dependent filter kernel $\lambda(\xi_s(t),\zeta_s(t))$ is found via the time propagation of the finite dimensional approximation of the differential Riccati equation with $R=10^{-3}$, $Q=100I$.

The semidiscrete system of differential equations for the state estimator was derived using the approximation

$$\widehat{x}(t,\xi,\zeta)=\sum_{i=1}^{n_x}\sum_{j=1}^{n_y}\widehat{\alpha}_{ij}(t)\phi_i(\xi)\psi_j(\zeta)$$

where $\phi_i(\xi)$, $\psi_j(\zeta)$ are the 1D linear functions.

Similar to the 1D case, the evolution of the $L_2(\Omega)$ norm of the state estimation error is depicted in Figure 4 for the case of a moving sensor and a fixed sensor. The spatial distribution of the state estimation error at the final time is depicted in Figure 5 where it is observed that the state error with a moving sensor converges to zero (pointwise) faster than the case with a single sensor at fixed spatial location. The trajectory of the mobile sensor is presented in Figure 6.

To have a further comparison between the static estimation scheme summarized in (2), (4), we now use noisy measurements of the true value of the spatial field $x(\xi,\zeta)=2\sin(\pi\xi)\sin(2\pi\zeta)$ using the same noise statistics as for the sensor measurements for the dynamic estimator, i.e. use a noise with covariance $R=10^{-3}$ in

$$y_i(t)=x(\xi_i,\zeta_i)+n_i,\qquad n_i\sim N(0,\sqrt{R})$$

Using the same trial functions used in the Galerkin approximation for the dynamic estimation scheme, the resulting 100×100 regression matrix Φ_m had a rank of 64. The static estimate of the spatial field using 100 noisy measurements is compared to the final value of the dynamic estimate with a single mobile sensor. The resulting state estimation errors for both cases are presented in Figure 7. While the dynamic estimator (Kalman-based) with the mobile sensor results in smaller spatially distributed error compared to the

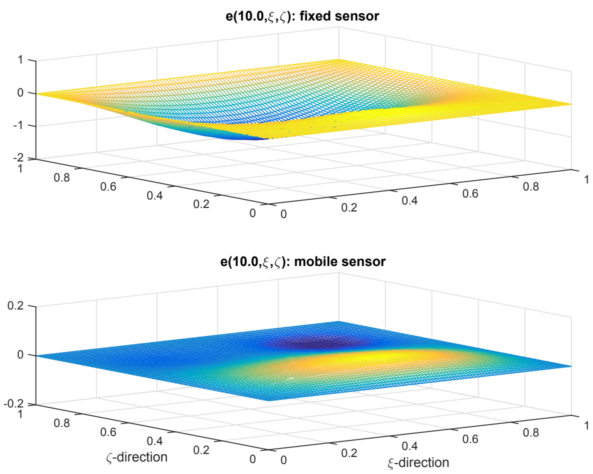

Fig. 5. Error distribution at the final time.

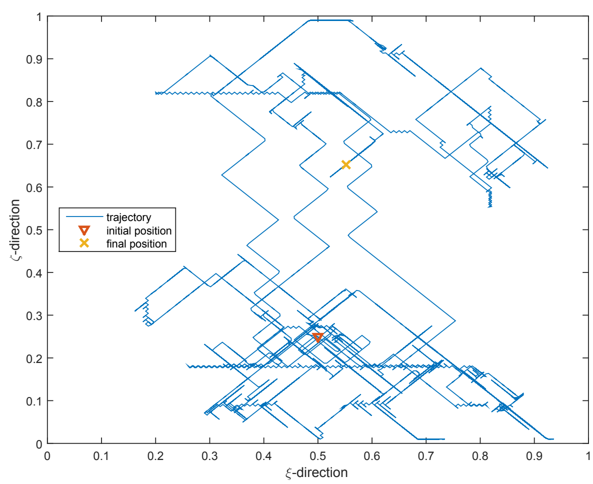

Fig. 6. Mobile sensor trajectory.

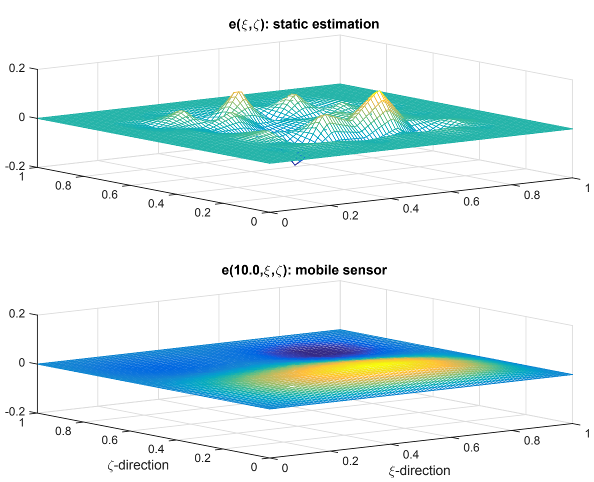

Fig. 7. Mobile sensor trajectory.

static estimation scheme, the most important advantage of the proposed asymptotic embedding method for the estimation of spatially distributed fields is the significant reduction in required hardware. That is, the reduction from 100 static sensors used in the static estimation scheme compared to a single mobile sensor in the dynamic estimation scheme. The economic impact on the acquisition and maintenance of hardware is immediately obvious.

REFERENCES

[1] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge University Press, Cambridge, 2004. [Online]. Available: https://doi.org/10.1017/CBO9780511804441

[2] M. A. Demetriou and F. Fahroo, "Estimation of spatial fields using asymptotic embedding methods and lagrangian sensing," in *Proc. of the IEEE Conf. on Decision and Control*, 10-13 Dec. 2012, pp. 955–960.

[3] R. Graham and J. Cortes, "Adaptive information collection by robotic sensor networks for spatial estimation," *IEEE Trans. on Automatic Control*, vol. 57, no. 6, pp. 1404–1419, December 2012.

[4] ——, "Asymptotic optimality of multicenter voronoi configurations for random field estimation," *IEEE Trans. on Automatic Control*, vol. 54, no. 1, pp. 153–158, January 2009.

[5] S. I. Azuma, M. S. Sakar, and G. J. Pappas, "Nonholonomic source seeking in switching random fields," in *Proc. of the IEEE Conf. on Decision and Control (CDC)*, Dec. 15-17 2010, pp. 6337–6342.

[6] J. Cochran, A. Siranosian, N. Ghods, and M. Krstic, "3-d source seeking for underactuated vehicles without position measurement," *IEEE Transactions on Robotics*, vol. 25, no. 1, pp. 117–129, Feb 2009.

[7] J. Cochran and M. Krstic, "Nonholonomic source seeking with tuning of angular velocity," *IEEE Transactions on Automatic Control*, vol. 54, no. 4, pp. 717 –731, April 2009.

[8] H. W. Alt, K. H. Hoffmann, and J. Sprekels, "A numerical procedure to solve certain identification problems," *Intern. Series Numer. Math*, vol. 68, pp. 11–43, 1984.

[9] K. H. Hoffmann and J. Sprekels, "The method of asymptotic regularization and restricted parameter identification problems in variational inequalities," *in Free Boundary Problems: Application and Theory*, vol. IV, pp. 508–513, Maubuisson, 1984.

[10] ——, "On the identification of coefficients of elliptic problems by asymptotic regularization," *Numer. Funct. Anal. and Optimiz.*, vol. 7, pp. 157–177, 1984-85.

[11] ——, "On the identification of parameters in general variational inequalities by asymptotic regularization," *SIAM J. Math. Anal.*, vol. 17, pp. 1198–1217, 1986.

[12] M. A. Demetriou and F. Fahroo, "Optimal sensor selection for optimal filtering of spatially distributed processes," in *Proc. of the International Symposium on Mathematical Theory of Networks and Systems*, University of Minnesota, MN, USA, July 12-15 2016.

[13] M. A. Demetriou, "Guidance of mobile actuator-plus-sensor networks for improved control and estimation of distributed parameter systems," *IEEE Trans. Automat. Control*, vol. 55, no. 7, pp. 1570–1584, 2010.

[14] ——, "Closed-loop guidance of mobile sensors for the estimation of spatially distributed processes," in *Proc. of the American Control Conference*, Milwaukee, Wisconsin, June 27-29 2018.

[15] R. A. Horn and C. R. Johnson, *Matrix analysis*, 2nd ed. Cambridge University Press, Cambridge, 2013.

[16] L. C. Evans, *Partial differential equations*, 2nd ed., ser. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010, vol. 19.

[17] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*. Berlin-Heidelberg: Springer, 2001.

[18] M. A. Demetriou, "Design of consensus and adaptive consensus filters for distributed parameter systems," *Automatica J. IFAC*, vol. 46, no. 2, pp. 300–311, 2010.

[19] M. A. Celia and W. G. Gray, *Numerical methods for differential equations*. Englewood Cliffs, NJ: Prentice Hall Inc., 1992.

[20] P. Benner, M. Bollhöfer, D. Kressner, C. Mehl, and T. Stykel, Eds., *Numerical algebra, matrix theory, differential-algebraic equations and control theory*. Springer, 2015, festschrift in honor of Volker Mehrmann.