SAMPLING OF GRAPH SIGNALS WITH BLUE NOISE DITHERING

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ABSTRACT

This paper discusses the generalization of the concept of blue noise sampling from traditional halftoning to signal processing on graphs. Making use of the spatial properties of blue noise, we generate sampling patterns that provide reconstruction errors that are similar to the ones obtained with state of the art approaches. This sampling scheme presents an alternative to those techniques that require spectral decompositions.

Index Terms— Graph signal processing, sampling, blue noise dithering.

1. INTRODUCTION

The everyday growing datasets that describe connectivity and interaction in multiple applications have been successfully modeled on graphs [1], promoting research interests about how to process quantities that are interrelated, and these applications range from social networks [2], brain signals analysis [3] up to financial systems [4]. Many efforts have been done in order to obtain or translate well established results in traditional signal processing to signal processing on graphs [1]. At the very core of all these applications resides the problem of *graph signal sampling* where the interest relies on identifying those nodes of the graph that would guarantee a unique representation of a graph signal with a given bandwidth [1, 5–7]. Up to now, most of the research about sampling signals on graphs relied on the use of the spectral decomposition of the adjacency matrix or the Laplacian operator [7, 8], however those eigen-decompositions are not always available.

This work offers a substantially different approach from previous works, considering that the spectral decomposition of the graph matrices is not available. The sampling patterns obtained with the proposed approach have common characteristics on the spectral domain, as they are characterized by a high frequency energy and represent the extension to graphs of what is called in the dithering literature as blue noise [9]. Additionally, the vertex-domain properties of the obtained sampling patterns resemble the spatial characteristics of traditional blue noise, showing an homogeneous distribution of the sampling nodes where the nodes are as far as possible from each other.

This paper is organized as follows. In Section 2 the notation and the problem of sampling on graphs is described including previous approaches. In Section 3 blue noise sampling on graphs is defined, while an algorithm for the generation of blue noise on graphs is introduced in Section 4. In Section 5 a set of experiments is presented showing a clear comparison between blue noise sampling and other techniques. Finally in Section 6 a set of conclusions is presented.

2. SAMPLING THEORY ON GRAPHS

2.1. Background and notation

Let $G = (\mathrm{V}, \mathrm{E})$ be a weighted graph with nodes V, and edges E. $\mathbf{W}$ is the adjacency matrix, with $\mathbf{W}(u, v)$ the weight connecting the nodes u and v. Let $\mathbf{D}$ be the diagonal matrix whose entries are given by $\mathbf{D}(u, u) = \sum_{v=u} \mathbf{W}(v, u)$. The combinatorial Laplacian matrix associated to G is defined as $\mathbf{L} = \mathbf{D} - \mathbf{W}$, whose eigenvalues are $0 = \mu_1 \leq \mu_2 \leq \ldots \leq \mu_N$, $N = |\mathrm{V}|$[10]. If G is connected, $\mu_1 = 0$ and $\mu_f > 0$ for all $f > 1$ [10]. A signal, $\boldsymbol{x}$, on the graph is then defined as the function $\boldsymbol{x} : \mathrm{V} \longrightarrow \mathrm{R}$ represented by the vector $\boldsymbol{x} \in \mathrm{R}^N$ where $\boldsymbol{x}(v)$ is the value of the signal on $v \in \mathrm{V}$. The eigenvector decomposition of $\mathbf{L}$ is represented as $\mathbf{L} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\mathrm{T}}$, with $\mathbf{U}$ being the matrix of eigenvectors. The Graph Fourier Transform of the signal, $\boldsymbol{x}$, is given by $\hat{\boldsymbol{x}} = \mathbf{U}^{\mathrm{T}}\boldsymbol{x}$. On the spectral axes, it is said that the signal $\boldsymbol{x}$ has bandwidth ω, if $\hat{\boldsymbol{x}}(k) = 0$ for all $\mu_k \geq \omega$, on the discrete axes the bandwidth of $\boldsymbol{x}$ is given by k. The set of signals of bandwidth ω is represented as $PW_\omega(G) = \mathrm{span}\{\mathbf{U}_k : \mu_k \leq \omega\}$ which is the so called Paley-Wiener space of bandwidth ω [5], where $\mathbf{U}_k$ is the matrix whose columns are the first k column vectors in $\mathbf{U}$.

The vector of samples of a signal $\boldsymbol{x}$ on $\mathrm{S} \subset \mathrm{V}$ are given by $\boldsymbol{x}_\mathrm{S} = \mathbf{M}\boldsymbol{x}$ where $\mathbf{M} = [\boldsymbol{\delta}_{s_1}, \ldots, \boldsymbol{\delta}_{s_m}]^{\mathrm{T}}$, $s_i \in \mathrm{S}\ \forall\ i = 1, \ldots, m$ and $\boldsymbol{\delta}_v$ is the $N-$ dimensional Kronecker column vector centered at $v \in \mathrm{V}$. If $\boldsymbol{x}$ is known to be bandlimited,

This work was supported in part by the National Science Foundation, grant NSF #1815992, by the UDRF foundation strategic initiative award, and by a University of Delaware Dissertation Fellowship Award.

from $\boldsymbol{x}_S$ a reconstructed version of $\boldsymbol{x}$ can be obtained as:

$$\boldsymbol{x}_{rec} = \underset{\boldsymbol{z}\in span(\mathbf{U}_k)}{\operatorname{argmin}} \|\mathbf{M}\boldsymbol{z} - \boldsymbol{x}_S\|_2^2 = \mathbf{U}_k(\mathbf{M}\mathbf{U}_k)^{\dagger}\boldsymbol{x}_S \quad (1)$$

with $(\mathbf{M}\mathbf{U}_k)^{\dagger}$ being the pseudo-inverse of $\mathbf{M}\mathbf{U}_k$ [8]. Then the central challenge in the sampling problem relies on finding the set $S \subset V$ that leads to the minimum error between $\boldsymbol{x}$ and $\boldsymbol{x}_{rec}$. In what follows we will refer to the ratio m/N between the number of sampling nodes and N as the sampling density d. The function $\boldsymbol{s} : V \rightarrow \{0, 1\}^N$ represents the sampling pattern associated to $S \subset V$, where $\boldsymbol{s}_S = 1$ and $\boldsymbol{s}_{S^c} = 0$.

2.2. Previous approaches

Previous approaches faced the sampling problem by the use of greedy techniques that build the sampling set selecting node by node according to a specific criteria that involve the calculations of eigenvalues and singular values of submatrices of $\mathbf{U}_k$. In [11] the optimal sampling set, S^{opt}, is found as $S^{opt} = \arg\max_{|S|=m} \sigma_1^2$, whereas in [12] the optimal set is given by $S^{opt} = \arg\max_{|S|=m} \prod_{i=1}^{k} \sigma_i^{-2}$ and recently in [8] $S^{opt} = \arg\max_{|S|=m} = \sum_{i=1}^{k} \sigma_i^2$; where σ_i is the i^{th} singular value of $\mathbf{M}\mathbf{U}_k$. In [12] the so called *graph spectral proxies* of order q, $\Omega_q(S) = (\sigma_{1,q})^{\frac{1}{2q}}$, are introduced to facilitate the calculation of the optimal sampling set, where $\sigma_{1,q}$ is the smallest eigenvalue of $(\mathbf{L}^{\mathrm{T}}_{S^c,S^c})^q \mathbf{L}^q_{S^c,S^c}$, with $\mathbf{L}^q$ being the q^{th} power of $\mathbf{L}$ and $\mathbf{L}_{S^c,S^c}$ represents the matrix obtained from $\mathbf{L}$ deleting the columns and rows indexed by S. It can be proved that for any q and any $S \subset V$, zero error reconstruction is possible when $\omega < \Omega_q(S)$. Then, the optimal sampling set of nodes is given by

$$S^{opt} = \arg\max_{|S|=m} \Omega_q(S). \quad (2)$$

The solution of (2) is achieved using a heuristic rule, that consists in calculating the first eigenvector of $\mathbf{L}^q_{S^c,S^c}$ while the nodes are aggregated one at a time. In particular, a node is added to the sampling set according to the index location of the component with the highest absolute value for the first eigenvector of $\mathbf{L}^q_{S^c,S^c}$. The value of q should be selected as large as possible for high accuracy.

Other approaches do not require spectral decompositions. In [13] an approach is proposed, where the nodes are selected according to a random matrix operator $\mathbf{P}$ which is designed jointly with the sampling matrix $\mathbf{M}$, and the reconstruction of the signal is given by

$$\boldsymbol{x}_{rec} = \arg\min_{\boldsymbol{z}\in\mathbb{R}^N} \left(\|\mathbf{P}^{-1/2}(\mathbf{M}\boldsymbol{z} - \boldsymbol{x}_S)\|_2^2 + \tau \boldsymbol{z}^{\mathrm{T}} g(\mathbf{L}) \boldsymbol{z} \right), \quad (3)$$

where $\tau \in \mathbb{R}^+$ and $g(\cdot)$ is a polynomial function, both of which are selected empirically. The optimal way to determine $\mathbf{P}$ requires the eigenvector decomposition of $\mathbf{L}$, however in [13] an eigen decomposition free estimation of $\mathbf{P}$ is proposed allowing an approximate solution of (3), with this approach it can be proven that $\mathbf{M}\mathbf{P}^{-1/2}$ satisfies the restricted isometry property, but only for a number of samples on the order of $O(k \log k)$. Most recently [8] proposed the use of *determinantal point processes (DPP)* to obtain an estimate of $\mathbf{P}$ including the case where eigen decompositions are not available and the reconstruction of the signal is obtained solving eqn. (3).

In [14] the problem of graph signal sampling is considered in a scenario in which only one node is used for the sampling of the product between the shifting operator and the signal, considering several powers of the shifting operator.

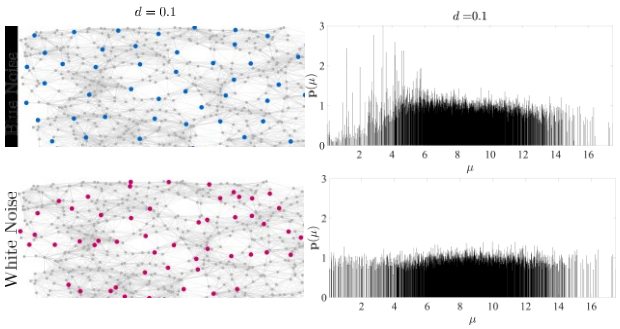

Fig. 1. Bottom-left: Random(uniform) sampling pattern distribution on the nodes of the graph. Bottom-right: spectral characteristics of 100 averaged spectrums. Top-left: Blue noise sampling pattern distribution on the nodes of the graph. Top-right: spectral characteristics of 100 averaged spectrums.

3. BLUE NOISE DITHERING ON GRAPHS

The concept of blue noise dithering has played a central role in applications involving representation and printing of images, where a gray scale image has to be represented with a binary pattern in such a way that this representation preserves essential properties of the original image. In particular, the new representation looks to the human eye as an image free of artifacts that represents the range of gray scale tones of the original one [9,15,16]. The spatial and spectral characteristics of blue noise sampling patterns allow the generation of such patterns either considering the spectral shaping of its Fourier transform or the spreading of the samples in the spatial domain following simple and intuitive principles.

3.1. Spectral Statistics

On the spectral axes, blue noise sampling patterns are characterized by a low energy content on the low frequencies. In order to quantify the behavior of a pattern regarding its spectral properties, we average the power spectrum of several of its realizations, to this end the idea of periodograms is extended to signals on graphs as follows. Let q be the number of realizations, $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_q$, of a stochastic signal, its power

spectrum can be calculated as

$$\mathbf{p}(f) = \frac{N}{q} \frac{\sum_{i=1}^{q} \hat{\mathbf{x}}_i(f)^2}{\sum_{i=1}^{q} \|\hat{\mathbf{x}}_i\|^2} \quad f = 2, \ldots, N. \tag{4}$$

where the f^{th} component of $\mathbf{p}$ is associated to the f^{th} Fourier coefficient. Notice that $\mathbf{p}(f)$ is the average of what is called in [10] the amplitude spectrum of the signals $\mathbf{x}_q$. It is important to point out that for a uniform random sampling pattern the behavior of $\mathbf{p}$ is expected to be flat. In Fig. 1 an estimate of the power spectra of a random (uniform) sampling pattern and a blue noise sampling pattern is depicted on a random sensor network considering a density of $d = 0.1$. Then, approximate versions of blue noise sampling patterns can be defined as the minimizers of the cost function $R_{\mathbf{s}}$, given by

$$R_{\mathbf{s}} = \frac{1}{m} \sum_{=2}^{N} \frac{\hat{\mathbf{s}}(f)^2}{\mu}. \tag{5}$$

$R_{\mathbf{s}}$ can be considered as a measure of *redness* for the sampling pattern $\mathbf{s}$, and basically penalizes the low frequency content.

3.2. Spatial characteristics of Blue-Noise

As can be observed in Fig. 1, the vertex distribution of a blue noise sampling pattern on the graph displays a *spread uniformity* in the sense that nodes where the sampling pattern is equal to 1 are located as far as possible from each other. In the following section, this property is exploited in order to generate blue noise patterns.

4. GENERATING BLUE-NOISE SAMPLING SETS

In order to generate blue noise sampling patterns we propose an adapted version of the void and cluster algorithm originally developed in digital halftoning applications [17]. The adapted version of this algorithm for the generation of blue noise sampling patterns on graphs can be appreciated in the Algorithm 1. The spreading of the sampling nodes is performed considering the geodesic distances on the graph, $\mathbf{\Gamma}$, evaluated on a Gaussian kernel $\mathbf{K}(i,j) = e^{-\frac{\mathbf{\Gamma}(i,j)^2}{\sigma}}$, where the value of σ is related with the average distance between sampling points and should be selected accordingly.

The distances between sampling nodes using $\mathbf{K}$ are iteratively computed in $\mathbf{c}(\text{supp}(\mathbf{s})) = \mathbf{K}(\text{supp}(\mathbf{s}), \text{supp}(\mathbf{s}))$, where the support of the sampling pattern is used to update the components of $\mathbf{c}$ and the distance between sampling nodes and the complement set is computed in $\mathbf{c}(\text{supp}(\mathbf{s})^c) = \mathbf{K}(\text{supp}(\mathbf{s}), \text{supp}(\mathbf{s})^c) - \tau$, where $\tau \geq N$. The use of the kernel is meant to emphasize the interaction between sampling nodes that are close. In this way, the ones in $\mathbf{s}$, are relocated to reduce the average distances between sampling points moving nodes from clustered regions to voids

Algorithm 1 Void and cluster algorithm for graphs

Input: m: number of samples, σ.
Output: $\mathbf{s}$: sampling pattern

Initialisation : $\mathbf{s} = \mathbf{0}$, IndA=-1, IndB=-1.
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Calculate $\mathbf{K}(i,j) = e^{-\frac{\mathbf{\Gamma}(i,j)}{\sigma^2}}$ for all $1 \leq i,j \leq N$.
$\mathbf{c} = \mathbf{K}\mathbf{1}_{N\times 1}$.
Get M as m nodes selected at random.
$\mathbf{s}(\mathrm{M}) = 1$.
for $r = 1:1:N$ **do**
  $\mathbf{c}(\text{supp}(\mathbf{s})) = \mathbf{K}(\text{supp}(\mathbf{s}), \text{supp}(\mathbf{s}))$.
  $\mathbf{c}(\text{supp}(\mathbf{s})^c) = \mathbf{K}(\text{supp}(\mathbf{s}), \text{supp}(\mathbf{s})^c) - \tau$.
  { }
  $\mathbf{s}(\arg\max_i \mathbf{c}(i)) = 0$.
  $\mathbf{s}(\arg\min_i \mathbf{c}(i)) = 1$.
  if IndA=$\arg\max_i\{\mathbf{c}(i)\}$ and IndB=$\arg\min_i\{\mathbf{c}(i)\}$ **then**
    break
  else
    IndA=$\arg\min_i\{\mathbf{c}(i)\}$.
    IndB=$\arg\max_i\{\mathbf{c}(i)\}$.

end if
16: **end for**
return $\mathcal{S}$

on the vertex domain. The results of the void and cluster algorithm can be appreciated in Fig. 2, where blue noise sam- pling patterns are displayed considering several densities d. The vertex-domain spreading and uniformity can be clearly appreciated, as there are not cluttered sampling nodes. Ad- ditionally the spectral characteristics of blue noise are also depicted.

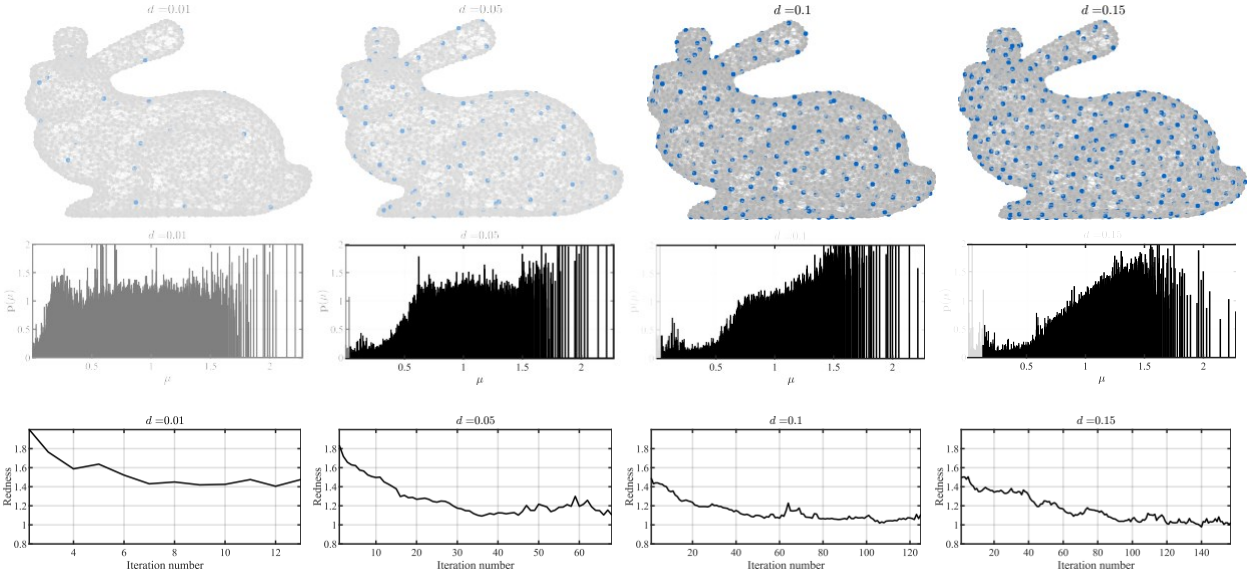
5. EXPERIMENTS

Blue noise sampling patterns are tested against state of the art techniques considering different graphs and signal mod- els. Given a graph model and a signal model, a set of 100 signals is generated, sampled and reconstructed considering several sampling rates. The mean squared error (MSE) is cal- culated for each reconstruction and averaged over the number of signals used. The schemes of sampling considered for the experiment are: Random (uniform) sampling, blue noise sam- pling by void and cluster, the sampling scheme proposed by Chen et. al. [11] and the sampling scheme proposed by Anis et. al. [12]. The signal models used are:

- SM1: A random signal of bandwidth $k = 50$, where the Fourier coefficients are generated from the Gaussian distri- bution $(0,\ 0.5^2)$. The samples captured are contaminated with additive Gaussian noise such that the Signal to Noise Ratio is $SNR = 20$dB.

- SM2: A random signal of approximate bandwidth $k = 50$. The Fourier coefficients are generated from the Gaussian

N



(a) (b) (c) (d)

Fig. 2. Void and cluster: blue-noise sampling patterns for different intensities d. First row: Localization on the graph of the nodes selected in a blue-noise sampling pattern. Second row: Power spectral density for the different blue-noise sampling patterns. Third row: the redness R_s of the sampling pattern vs the iteration number using void and cluster.

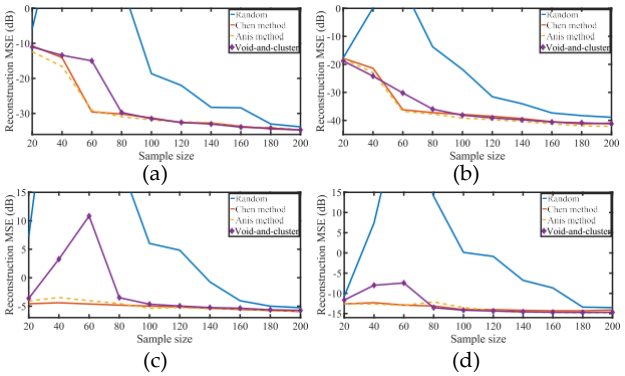
distribution $N(0, 0.5^2)$ and modulated by $h(\mu)$, where

$$h(\mu) = \begin{cases} 1 & \text{If} \quad \mu \leq \mu_{50} \\ e^{-4(\mu-\mu_{50})} & \text{If} \quad \mu > \mu_{50} \end{cases} \tag{6}$$

The graphs used are:

- Graph G_1: A bunny graph with $N = 2503$ nodes, where the edges weights are given by the Euclidean distance between vertices.
- Graph G_2: A sensor network with $N = 500$ nodes, where the edges weights are given by the Euclidean distance between vertices.

In Fig. 3 the reconstruction error of different signals from a set of samples is depicted considering different graphs, and several signal models. The results show consistently that blue noise sampling leads to competitive results with respect to the state of the art techniques. In [18] some theoretical insights are provided about why blue noise sampling promotes good sampling sets on graphs.



(a) (b) (c) (d)

Fig. 3. The averaged mean squared error of the reconstructed signals from a sampled representation: (a) The graph G_1 and the signal model SM1. (b) The graph G_1 and the signal model SM2. (c) The graph G_2 and the signal model SM1. (d) The graph G_2 and the signal model SM2.

6. CONCLUSIONS

The concept of blue noise sampling on graphs is discussed and its performance against other techniques has been evaluated. Taking into account the characteristics of these patterns on the vertex domain, an algorithm for its generation has been also discussed. This maximization of the distance between sampling nodes obeys the intuitive idea that the samples of the signals should not be too close to each other as they will be highly correlated and would lead to poor reconstructions.

The calculation of geodesic distances on the graph necessary for the use of void and cluster could be in some cases computationally expensive. However, efficient implementations can be obtained exploiting the simple principle of spreading the sampling nodes homogeneously and as far as possible from each other. This efficient implementation for the generation of blue noise is part of our current research.

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