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ON THE EXPANSION OF GROUP-BASED LIFTS*

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Abstract. A k-lift of an n-vertex base graph G is a graph H on $n \times k$ vertices, where each vertex v of G is replaced by k vertices $v_1, \dots, v_k$ and each edge uv in G is replaced by a matching representing a bijection π_{uv} so that the edges of H are of the form $(u_i, v_{\pi_{uv}(i)})$. Lifts have been investigated as a means to efficiently construct expanders. In this work, we study lifts obtained from *groups and group actions*. We derive the spectrum of such lifts via the representation theory principles of the underlying group. Our main results are 1. a uniform random lift by a cyclic group of order k of any n-vertex d-regular base graph G, with the nontrivial eigenvalues of the adjacency matrix of G bounded by λ in magnitude, has the new nontrivial eigenvalues bounded by $\lambda + \mathcal{O}(\sqrt{d})$ in magnitude with probability $1 - ke^{-\Omega(n/d^2)}$. The probability bounds as well as the dependency on λ are almost optimal. As a special case, we obtain that there is a constant c_1 such that for every $k \leq 2^{c_1 n/d^2}$, there exists a lift H of every Ramanujan graph by a cyclic group of order k such that H is *almost Ramanujan* (nontrivial eigenvalues of the adjacency matrix at most $O(\sqrt{d})$ in magnitude). This result leads to a quasi-polynomial time deterministic algorithm to construct almost Ramanujan expanders; 2. there is a constant c_2 such that for every $k \geq 2^{c_2 nd}$, there *does not* exist an *abelian k-lift* H of any n-vertex d-regular base graph such that H is almost Ramanujan. This can be viewed as an analogue of the well-known nonexpansion result for constant degree abelian Cayley graphs. Suppose k_0 is the order of the largest abelian group that produces expanding lifts. Our two results highlight lower and upper bounds on k_0 that are tight up to a factor of d^3 in the exponent, thus suggesting a threshold phenomenon.

Key words. expanders, Ramanujan graphs, random lifts

AMS subject classifications. 05C50, 05C25

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1. Introduction.

Expander graphs have spawned research in pure and applied mathematics during the last several years, with applications in multiple fields including complexity theory, robust computer networks, error-correcting codes, derandomization, compressed sensing, and metric embeddings [28, 15]. Informally, an expander is a graph in which every small subset of vertices has a relatively large edge boundary. Most applications are concerned with d-regular graphs. The largest eigenvalue of the adjacency matrix of d-regular graphs is d and we call this a *trivial eigenvalue*. In the case of bipartite d-regular graphs, the largest and smallest eigenvalues of their adjacency matrix are d and $-d$ and we refer to these as trivial eigenvalues. The expansion of d-regular graphs is controlled by the difference between d and the largest (in magnitude) nontrivial eigenvalue of the adjacency matrix. We will denote the largest (in magnitude) nontrivial eigenvalue of the adjacency matrix by λ. Roughly,

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the smaller λ is, the better the graph expansion. The Alon–Boppana bound [25] states that $\lambda \geq 2\sqrt{d-1} - o(1)$ (here, $o(1)$ is with respect to n). Graphs with $\lambda \leq 2\sqrt{d-1}$ are known as *Ramanujan graphs* [19].

A simple probabilistic argument shows the existence of infinite families of expander graphs [26]. However, constructing such infinite families explicitly has proven to be a challenging task. It is easy to construct Ramanujan graphs with a small number of vertices: d-regular complete graphs and complete bipartite graphs are Ramanujan. The challenge is to construct an infinite family of d-regular graphs that are all Ramanujan, which was first achieved by Lubotzky, Phillips, and Sarnak [19] and Margulis [23]. They built Ramanujan graphs from Cayley graphs. For every prime p, they built Ramanujan graphs of degree $p+1$ by relying on deep number-theoretic facts. In two breakthrough papers, Marcus, Spielman, and Srivastava showed the existence of bipartite Ramanujan graphs of all degrees [21, 22]. However they do not provide an efficient algorithm to construct those graphs. Cohen [6] adapted the techniques of [22] to design an efficient algorithm to construct Ramanujan multigraphs. A striking result of Friedman [9, 3] and a slightly weaker but more general result of Puder [27] show that almost every d-regular graph on n vertices is very close to being Ramanujan, i.e., for every $\epsilon > 0$, asymptotically almost surely, $\lambda < 2\sqrt{d-1} + \epsilon$. It is still unknown whether the event that a random d-regular graph is exactly Ramanujan happens with constant probability. Despite a large body of work on the topic, all attempts to efficiently construct large Ramanujan expander (simple) graphs of any given degree have failed, and exhibiting such a construction remains an intriguing open problem.

A combinatorial approach to constructing expanders, initiated by Friedman [8], is to obtain new (larger) Ramanujan graphs from smaller ones. In this approach, we start with a base graph which is "lifted" to obtain a larger graph. Concretely, a k-lift of an n-vertex base graph G is a graph H on kn vertices, where each vertex u of G is replaced by k vertices $u_1, \ldots, u_k$ and each edge uv in G is replaced by a matching between $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$. In other words, for each edge uv of G there is a permutation π_{uv} of k elements so that the corresponding k edges of H are of the form $u_i v_{\pi_{uv}(i)}$. The graph H is a (uniformly) *random* lift of G if for every edge uv the bijection π_{uv} is chosen uniformly at random from the set S_k of permutations of k elements.

Since we are focusing on Ramanujan graphs, we will restrict our attention to lifts of d-regular graphs. It is easy to see that any lift H of a d-regular base graph G is itself d-regular and inherits all the eigenvalues of G. We will refer to the inherited eigenvalues as "old" eigenvalues and the rest of the eigenvalues as "new" eigenvalues. In order to use the lifts approach for constructing expanders, it is necessary that the lift also inherits the expansion properties of the base graph. Naturally, one hopes that a random lift of a Ramanujan graph will also be (almost) Ramanujan with high probability.

Friedman [8] first studied the eigenvalues of random k-lifts of regular graphs and proved that every new eigenvalue of H is $O(d^{3/4})$ with high probability. He conjectured a bound of $2\sqrt{d-1}+o(1)$, which would be tight (see, e.g., [13]). Linial and Puder [16] improved Friedman's bound to $O(d^{2/3})$. Lubetzky, Sudakov, and Vu [18] showed that the magnitude of every nontrivial eigenvalue of the lift is $O(\lambda \log d)$, where λ is the largest (in magnitude) nontrivial eigenvalue of the base graph, thus improving on the previous results when G is significantly expanding. Adarrio-Berry and Griffiths [1] further improved the bounds above by showing that every new eigenvalue of H is $O(\sqrt{d})$, and Puder [27] proved the nearly optimal bound of $2\sqrt{d-1}+1$. All those results hold with probability tending to 1 as $k \to \infty$, thus the order k of the lift in question needs to be large. Nearly no results were known in the regime where k is bounded with respect to the number of nodes n of the graph. A "relativized" version of the Alon–Boppana conjecture regarding lower bounding the new eigenvalues of lifts was also recently shown in [11] and [3].

Bilu and Linial [2] were the first to study k-lifts of graphs with bounded k, and suggested constructing Ramanujan graphs through a sequence of 2-lifts of a base graph: start with a small d-regular Ramanujan graph on some finite number of nodes (e.g., K_{d+1}). Every 2-lift operation doubles the number of vertices in the graph. If there is a way to preserve expansion after lifting, then repeating this operation will give large good expanders of the same bounded degree d. Bilu and Linial [2] showed that if the starting graph G is significantly expanding so that $\lambda(G) = O(\sqrt{d \log d})$, then there exists a random 2-lift of G that has all its new eigenvalues upper bounded in magnitude by $O(\sqrt{d \log^3 d})$. In a recent breakthrough work, Marcus, Spielman, and Srivastava [21] showed that for every bipartite d-regular graph G, there exists a 2-lift of G, such that the new eigenvalues achieve the Ramanujan bound of $2\sqrt{d-1}$. But their result still does not provide an efficient algorithm to find such lifts.

1.1. Our contributions. In this work, we study the lifts approach to efficiently construct almost Ramanujan expanders of all degrees. We derive these lifts from groups. This is a natural generalization of Cayley graphs.

DEFINITION 1.1 (Γ-lift). *Let Γ be a group of order k with $\cdot$ denoting the group operation. A Γ-lift of an n-vertex base graph $G = (V, E)$ is a graph $H = (V \times \Gamma, E')$ obtained as follows: it has kn vertices, where each vertex u of G is replaced by k vertices $\{u\} \times \Gamma$. For each edge uv of G, we choose an element $g_{uv} \in \Gamma$ and replace that edge by a perfect matching between $\{u\} \times \Gamma$ and $\{v\} \times \Gamma$ that is given by the edges $u_i v_j$ for which $g_{uv} \cdot i = j$.*

We denote the order k of the group Γ to be the order of the lift. We refer to Γ-lifts obtained using $\Gamma = \mathbb{Z}/k\mathbb{Z}$, the additive group of integers modulo k, as shift k-lifts. Since every cyclic group of order k is isomorphic to $\mathbb{Z}/k\mathbb{Z}$, we have that Γ-lifts are shift k-lifts whenever Γ is a cyclic group of order k.

A tight connection between the spectrum of Γ-lifts and the representation theory of the underlying group Γ is known [24, 7]. This connection tells us that the lift incurs the eigenvalues of the base graph, while its new eigenvalues are the union of eigenvalues of a collection of matrices arising from the group elements assigned to the edges and the irreducible representations of the group. We note that this connection has also been recently used in [14] in the context of expansion of lifts, aiming to generalize the results in [22]. In this work, we address the expansion of Γ-lifts obtained from cyclic groups and abelian groups.

In order to understand the expansion properties of lifts, it suffices to focus on the new eigenvalues of the lifted graph by the above-mentioned connection. We present a high probability bound on the expansion of random shift k-lifts for bounded k.

THEOREM 1.2. *Let G be a d-regular n-vertex graph, where $2 \le d \le \sqrt{n/(3\ln n)}$, with largest (in magnitude) nontrivial eigenvalue λ. Let H be a random shift k-lift of G with λ_{new} being the largest (in magnitude) new eigenvalue of H. Then*

$$\lambda_{\text{new}} = O(\lambda)$$

with probability $1 - k \cdot e^{-\Omega(n/d^2)}$. Moreover, if $\lambda \le d/\log d$, then

$$\lambda_{\text{new}} - \lambda = O(\sqrt{d})$$

with probability $1 - k \cdot e^{-\Omega(n/d^2)}$.

We say that a graph is *almost Ramanujan* if all its nontrivial eigenvalues are bounded by $O(\sqrt{d})$ in magnitude. By the above result, if the base graph G is Ramanujan, then the random shift k-lift will be almost Ramanujan with high probability.

Remark 1. In contrast to lifts of order k, where $k \to \infty$ when $n \to \infty$, the dependency of λ_{new} on λ is necessary for the case of bounded k. This has previously been observed by the authors in [2] who gave the following example: Let G be a disconnected graph on n vertices that consists of $n/(d+1)$ copies of K_{d+1}, and let H be a random 2-lift of G. Then the largest nontrivial eigenvalue of G is $\lambda = d$ and it can be shown that, with high probability, $\lambda_{\text{new}} = \lambda = d$. Therefore, our eigenvalue bounds are nearly tight.

Remark 2. Theorem 1.2 involves a setting distinct from that of [10, 18, 1] with some significant differences. To prove Theorem 1.2, in the case of $k = 2$ (or 2-lifts), one seeks to estimate a Rayleigh quotient $(y^T By)/(y^T y)$, where y is an arbitrary vertex function and B is a random symmetric matrix which is ± 1 at the entries where A is 1, and 0 otherwise. There are two extreme cases that illustrate the crux of the argument, namely, (1) where $y \in \{0, \pm 1\}^n$ and y has a large support, and (2) the same except that y has a small support (which we call mini support). In the case of large support, standard counting arguments suffice to bound the Rayleigh quotient; this is analogous to [10, 18, 1], but the context is different because the randomness here comes from the many random ± 1 entries of B. The case of mini support is entirely new here: the Rayleigh quotient estimates here hold *with probability one*, using the fact that the base graph (with n vertices, n being large) is an expander; this argument can be found in Claim 4.3, whose proof is very short. The mini support situation in [10, 18, 1] holds only with high probability, and requires much more involved and subtle calculations. In addition, just as in [10, 18, 1], more work is required to analyze a general y, which is dyadically expanded to, roughly speaking, reduce the analysis of its Rayleigh quotient to cases at either extreme.

Specializing Theorem 1.2 for the case of 2-lifts gives the following corollary which improves upon the multiplicative $\log d$ factor in the eigenvalue bound that is present in the result of Bilu and Linial [2].

COROLLARY 1.3. *Let G be a d-regular n-vertex graph, where $2 \le d \le \sqrt{n/(3\ln n)}$, with largest (in magnitude) nontrivial eigenvalue λ. Let H be a random 2-lift of G with λ_{new} being the largest (in magnitude) new eigenvalue of H. Then*

$$\lambda_{\text{new}} = O(\lambda)$$

with probability $1 - e^{-\Omega(n/d^2)}$. Moreover, if $\lambda \le d/\log d$, then

$$\lambda_{\text{new}} - \lambda = O(\sqrt{d})$$

with probability $1 - e^{-\Omega(n/d^2)}$.

Remark 3. The multiplicative $\log d$ factor in the eigenvalue bound present in the result of Bilu and Linial [2] arises due to the use of the converse of the expander mixing lemma along with an epsilon-net style argument in their analysis. The converse of the expander mixing lemma is provably tight, so straightforward use of the converse will indeed incur the $\log d$ factor. We are able to improve the eigenvalue bound by performing a fine-grained analysis of the epsilon-net argument, avoiding direct use of the converse.

Lifts based on groups immediately suggest an algorithm towards building d-regular n-vertex Ramanujan expanders. In order to describe this algorithm, we first describe the brute-force algorithm that follows from the existential result of [21]. The approach is to start with the complete bipartite graph $K_{d,d}$ and lift the graph $\log_2(n/2d)$ times. At each stage, we do a brute-force search over the space of all possible 2-lifts and pick the best one (i.e., one with the smallest new maximum eigenvalue in magnitude). However, since a graph (V, E) has $2^{|E|}$ possible 2-lifts, it follows that the final lift will be chosen from among $2^{nd/4}$ possible 2-lifts, which means that the brute-force algorithm will run in time exponential in nd.

Next, suppose that for every $k \geq 2$, we are guaranteed the existence of a group Γ of order k such that for every base graph there exists a Γ-lift that has all its new eigenvalues at most $2\sqrt{d-1}$ in magnitude. For example, [4] suggests the possibility that for every k and for every base graph, there exists a shift k-lift that has all new eigenvalues with magnitude at most $2\sqrt{d-1}$. Then a brute-force algorithm similar to the one above, would perform only one lift operation of the base graph $K_{d,d}$ to create a Γ-lift with $n = 2dk$ vertices. This algorithm would only have to choose the best among k^{d^2} possibilities (k different choices of group element per edge of the base graph), which is polynomial in n, the size of the constructed graph (here we have assumed that d is a constant). This motivates the following question: what is the largest possible group Γ that might produce expanding Γ-lifts? Our next result rules out the existence of large abelian groups that might lead to (even slightly) expanding lifts.

THEOREM 1.4. *For every n-vertex d-regular graph G, every real value $\epsilon \in (0, 1/e)$, and every abelian group Γ of size at least*

$$k = \exp\left(\frac{nd \log \frac{1}{\epsilon} + \log n}{\log \frac{1}{e\epsilon}}\right),$$

all Γ-lifts H of G have λ_H at least ϵd in magnitude. In particular, when $k = 2^{\Omega(nd)}$, there is no Γ-lift H of any n-vertex d-regular graph G all of whose nontrivial eigenvalues are bounded by $O(\sqrt{d})$ in magnitude whenever Γ is an abelian group of order k.

Theorem 1.4 shows that we cannot expect to have arbitrarily large abelian groups with expanding lifts as suggested in [4].

Remark 4. The first and only known efficient construction of Ramanujan expander simple graphs are Cayley graphs of certain groups [19]. We observe that a Cayley graph for a group Γ with generator set S can be obtained as a Γ-lift of the bouquet graph (a graph that consists of one vertex with multiple self-loops) [20]. Our nonexpansion result for abelian groups complements the known result on nonexpansion of abelian Cayley graphs [12].

Remark 5. Our Theorems 1.4 and 1.2 can be viewed as lower and upper bounds on the largest order k_0 of an abelian group Γ such that for every n-vertex graph, there exists a Γ-lift for which all new eigenvalues are $O(\sqrt{d})$. On the one hand, Theorem 1.2 shows that, for $k = 2^{O(n/d^2)}$, most of the shift k-lifts of a Ramanujan graph have their new eigenvalues to be $O(\sqrt{d})$. On the other hand, Theorem 1.4 shows that for $k = 2^{\Omega(nd)}$, there is no shift k-lift that achieves such eigenvalue bounds. This suggests a threshold behavior for k_0.

We observe that Theorem 1.2 leads to a deterministic quasi-polynomial time algorithm for constructing almost Ramanujan families of graphs.

THEOREM 1.5. *There exists an algorithm that runs in time $2^{O(d^4 \log^2 n)}$ to construct a d-regular n-vertex graph such that all its nontrivial eigenvalues are $O(\sqrt{d})$ in magnitude.*

Algorithm 1 Quasi-polynomial time algorithm to construct expanders of arbitrary size n.

1: Pick an r such that $r2^{cr/d^2} = n$ for a constant c that appears in the eigenvalue bound in Theorem 1.2. Do an exhaustive search to find a d-regular graph G' on r vertices with $\lambda = O(\sqrt{d})$.

2: For $k = 2^{cr/d^2}$, do an exhaustive search to find a shift k-lift G of the base graph G' with minimum new eigenvalue (in magnitude).

Proof. We use Algorithm 1. We note that the choice of r in the first step ensures that $r = O(d^2 \log n)$. By Theorem 1.2, there exists a lift G of the base graph G' such that $\lambda(G) = O(\sqrt{d})$. Thus, the exhaustive search in the second step gives a graph G whose nontrivial eigenvalues are $O(\sqrt{d})$ in magnitude.

In order to bound the running time, we note that the first step can be implemented to run in time $2^{O(r^2)} = 2^{O(d^4 \log^2 n)}$. To bound the running time of the second step, we observe that for each edge in G', there are k possible choices. Therefore, the size of the search space is at most $k^{rd/2} = 2^{cr^2/2d} = 2^{O(d^3 \log^2 n)}$ and for each k-lift, it takes $poly(n)$ time to compute $\lambda(G)$. Thus, the overall running time of the algorithm is $2^{O(d^4 \log^2 n)}$. ∎

Organization. We give some preliminary definitions, notations, facts, and lemmas in section 2. We prove Theorem 1.4 in section 3. We illustrate the techniques behind proving Theorem 1.2 by presenting and proving a slightly weaker version of Theorem 1.2 (see Theorem 4.1) in section 4. We prove the concentration inequality (Lemma 4.2) needed for the weaker version in section 5. We use a stronger version of the concentration inequality and prove Theorem 1.2 in section 6.

2. Preliminaries.

In this section, we define certain notations and present the needed combinatorial inequalities and facts.

Notations. Let $G := (V, E)$ be a d-regular graph with n vertices. If G is d-regular bipartite, we will assume that the bipartition of the vertex set is given by $(\{1, \ldots, n/2\}, \{n/2 + 1, \ldots, n\})$. Let A be the adjacency matrix of G. Since A is a real symmetric matrix, its eigenvalues are also real. Let the eigenvalues of A be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For a d-regular graph G, it is well known that $\lambda_1 = d$. If G is bipartite, then $\lambda_n = -d$ and we define $\lambda_G := \max\{|\lambda_i| : i \in \{2, 3, \ldots, n-1\}\}$. If G is nonbipartite, we define $\lambda_G := \max\{|\lambda_i| : i \in \{2, 3, \ldots, n\}\}$. Thus, λ_G denotes the largest (in magnitude) nontrivial eigenvalue of G. When G is clear from the context, we will drop the subscript and simply write λ. For subsets $S, T \subseteq V$, let $E(S, T)$ be the number of edges $uv \in E$ with $u \in S$ and $v \in T$. We denote the largest eigenvalue of a matrix M by $\|M\|$ and the support of a vector x by $S(x)$. We define log() to be the log function with base 2. We represent e^x by $exp(x)$. Given a vector x whose coordinates are from $\{0, \pm 2^{-1}, \pm 2^{-2}, \ldots, \pm 2^{-i}, \ldots\}$ we define the *dyadic decomposition* of x as the collection of vectors $\{2^{-i} u_i\}_{i \in \mathbb{Z}}$, where each u_i is a vector whose jth coordinate is defined as

$$[u_i]_j := \begin{cases} 1 & \text{if } x_j = 2^{-i}, \\ -1 & \text{if } x_j = -2^{-i}, \\ 0 & \text{otherwise.} \end{cases}$$

2.1. Combinatorial inequalities. We will use the following combinatorial identity which was also used in earlier epsilon-net style arguments to bound eigenvalues [10, 18, 1]. We present its proof for the sake of completeness.

LEMMA 2.1 (discretization lemma). *Let $M \in \mathbb{R}^{n\times n}$ be a matrix with diagonal entries being* 0*.*

1. *For every $x \in \mathbb{R}^n$ with $||x||_\infty \le 1/2$ there exists $y \in \{0, \pm 2^{-1}, \pm 2^{-2}, \dots, \pm 2^{-i}, \dots\}^n$ such that $|x^T Mx| \le |y^T My|$ and $\|y\|^2 \le 4\|x\|^2$. Moreover, each coordinate of x between 2^{-i} and $2^{-(i-1)}$ is rounded to either 2^{-i} or $2^{-(i-1)}$ and between -2^{-i} and $-2^{-(i-1)}$ is rounded to either -2^{-i} or $-2^{-(i-1)}$ in y.*
2. *For every $x_1, x_2 \in \mathbb{R}^n$ with $\|x_1\|_\infty, \|x_2\|_\infty \le 1/2$, there exist $y_1, y_2 \in \{0, \pm 2^{-1}, \pm 2^{-2}, \dots, \pm 2^{-i}, \dots\}^n$ such that $|x_1^T M x_2| \le |y_1^T M y_2|$, $\|y_1\|^2 \le 4\|x_1\|^2$, $\|y_2\|^2 \le 4\|x_2\|^2$ and for $b \in \{1, 2\}$ each coordinate of x_b between 2^{-i} and $2^{-(i-1)}$ is rounded to either 2^{-i} or $2^{-(i-1)}$ and between -2^{-i} and $-2^{-(i-1)}$ is rounded to either -2^{-i} or $-2^{-(i-1)}$ in y_b.*

Proof. Let $x \in \mathbb{R}^n$ with $||x||_\infty \le 1/2$. We first give a randomized rounding rule that rounds each nonzero coordinate of x independently. Consider a nonzero coordinate x_j. Suppose $|x_j| \in [2^{-i}, 2^{-(i-1)})$. Let $\delta_j \in [0, 1)$ be a real value such that $x_j = sign(x_j)(1 + \delta_j)2^{-i}$. We round x_j to $sign(x_j)2^{-(i-1)}$ with probability δ_j and $sign(x_j) \cdot 2^{-i}$ with probability $1 - \delta_j$. Let the resulting rounded vector be z.

We note that $E[z_j] = x_j$ for every $j \in [n]$. Since each coordinate is rounded independently and the diagonal entries of M are 0, we get that $E[z^T Mz] = x^T Mx$. Hence, there exists a vector $y \in \{\pm 2^{-1}, \pm 2^{-2}, \dots, \pm 2^{-i}, \dots\}^n$ that can be generated by derandomizing this rounding procedure such that $|x^T Mx| \le |y^T My|$. Furthermore, $\|y\|^2 \le 4\|x\|^2$. By definition every coordinate in x with value between 2^{-i} and $2^{-(i-1)}$ is rounded to either 2^{-i} or $2^{-(i-1)}$ and between -2^{-i} and $-2^{-(i-1)}$ is rounded to either -2^{-i} or $-2^{-(i-1)}$ in y.

The proof of the second part of the lemma is the same as the first part. Here we obtain z_1 and z_2 by the same procedure and follow the same argument to get y_1 and y_2. ∎

We will use the following inequality.

LEMMA 2.2. *Let $r \ge 2, x > 1/2, z > 0$ be real values and t be a nonnegative integer such that $r^t \le z/2$. Then,*

$$\sum_{i=0}^{t} (r^i \log(z/r^i))^x \le c(r)(r^t \log(z/r^t))^x$$

for a value $c(r)$ that depends only on r. Moreover, $c(2) < 9$.

Proof. Let $a_i := (r^i \log(z/r^i))^x$ for all $i \in \{0, 1, \dots, t\}$. Let us consider the ratio of consecutive terms a_{i+1}/a_i for $i \in \{0, 1, \dots, t-1\}$:

$$\begin{aligned}\frac{a_{i+1}}{a_i} &= \left(\frac{r^{i+1}\log(z/r^{i+1})}{r^i \log(z/r^i)}\right)^x \\ &= \left(r\left(1 - \frac{\log(r)}{\log(z) - i\log(r)}\right)\right)^x \\ &\ge \left(r\left(1 - \frac{\log(r)}{1 + (t-i)\log(r)}\right)\right)^x \qquad \text{(since } r^t \le z/2\text{).}\end{aligned}$$

If $i \le t-2$, we get that $a_{i+1}/a_i \ge r^x(\frac{1+\log(r)}{1+2\log(r)})^x$. Let $\alpha(r) := r^x(\frac{1+\log(r)}{1+2\log(r)})^x$. Then $\alpha(r) > \frac{2}{\sqrt{3}} > 1$ for $r \ge 2$. Also for $i = t-1$, we get that $a_{i+1}/a_i \ge (r/(1+\log(r)))^x \ge 1$. Now consider

$$S_{-1} := a_0 + a_1 + \cdots + a_{t-1}.$$

Multiplying both sides by $\alpha(r) - 1$, we get

$$\begin{aligned}(\alpha(r)-1)S_{-1} &= (\alpha(r)-1)(a_0 + a_1 + \cdots + a_{t-1})\\ &= -a_0 + (\alpha(r)a_0 - a_1) + (\alpha(r)a_1 - a_2) + \cdots + a_{t-1}\alpha(r).\end{aligned}$$

Since, $a_{i+1} \ge \alpha(r)a_i$ for $i \in \{0, 1, \dots, t-2\}$ and $a_0 \ge 0$, we get

$$(\alpha(r)-1)S_{-1} \le a_{t-1}\alpha(r).$$

Hence,

$$S_{-1} \le a_{t-1}\left(\frac{\alpha(r)}{\alpha(r)-1}\right).$$

Therefore

$$\sum_{i\in[t]} a_i \le S_{-1} + a_t \le \left(1 + \left(\frac{\alpha(r)}{\alpha(r)-1}\right)\right) a_t.$$

Setting $c(r) = (1 + (\frac{\alpha(r)}{\alpha(r)-1}))$ we get the identity. We observe that $\alpha(2)$ is greater than $\frac{2}{\sqrt{3}}$ which implies that $c(2) < 9$. ∎

We also need the following lemma.

Lemma 2.3. *For every $c_1 > 0$, there exists c_2 s.t. $\sqrt{\sqrt{x}\log\frac{1}{x}} \le c_1 + c_2 x$, where $0 \le x \le 1$.*

To prove this lemma, we first observe that $\lim_{x\to 0}\sqrt{\sqrt{x}\log\frac{1}{x}} = 0$. Hence, there exists $x_0 > 0$ such that for $x < x_0$, we have $\sqrt{\sqrt{x}\log\frac{1}{x}} < c_1$. By setting $c_2 = \sqrt{\frac{\log 1/x_0}{x_0^{3/2}}}$, we get the desired result.

We will use the following Hoeffding inequality for concentration bounds.

Lemma 2.4 (Hoeffding inequality). *Let $X_1, \dots, X_n$ be independent random variables such that X_i is strictly bounded within the interval $[a_i, b_i]$, then*

$$\mathbb{P}\left(\left|\sum_{i=1}^n X_i - \sum_{i=1}^n E[X_i]\right| \ge t\right) \le 2e^{-\frac{2t^2}{\sum_{i=1}^n (b_i-a_i)^2}}.$$

2.2. Spectral graph theory basics. Expander graphs have certain desirable properties that are also present in random graphs. This intuition is quantified by the following well-known fact known as the expander mixing lemma which bounds the deviation between the number of edges between two subsets and the expected number in a random graph.

Theorem 2.5 (expander-mixing lemma; see [17, Lemma 2.5]). *Let $G = (V, E)$ be a nonbipartite graph. Then*

$$\left|E(S,T) - \frac{d|S||T|}{n}\right| \le \lambda_G\sqrt{|S||T|} \quad \forall\ S, T \subseteq V.$$

We also have an analogue for bipartite graphs (by proceeding along the lines of the proof of the expander mixing lemma). The following theorem states the general bound.

THEOREM 2.6. *Let $G = (V, E)$ be a graph. Then*

$$E(S,T) \leq 2\frac{d|S||T|}{n} + \lambda_G\sqrt{|S||T|} \quad \forall\ S,T \subseteq V.$$

We need the following theorem showing that expanders have a small diameter in order to show nonexpansion of large abelian lifts.

THEOREM 2.7 (see [5, Theorem 1]). *The diameter of a d-regular graph G with n vertices is at most* $(\log n)/\log(d/\lambda_G)$.

2.3. Lifts. In this section we define lifts of graphs and state some of their properties.

DEFINITION 2.8 ($(\Gamma, S, \cdot)$-lift). *Let Γ be a group, S be a set of size k and $\cdot$ be a group action of Γ on S. A $(\Gamma, S, \cdot)$-lift of an n-vertex base graph $G = (V, E)$ is a graph $H = (V \times S, E')$ obtained as follows: it has $k \times n$ vertices, where each vertex u of G is replaced by k vertices $\{u\} \times S$. For each edge uv of G, we choose an element $g_{uv} \in \Gamma$ and replace that edge by a perfect matching between $\{u\} \times S$ and $\{v\} \times S$ that is given by the edges u_iv_j for which $g_{uv} \cdot i = j$. We define the* order *of the lift to be k.*

We note that if $S = \Gamma$ and the group action $\cdot$ is the left group multiplication itself, then $(\Gamma, S, \cdot)$-lifts are just Γ-lifts.

Remark 6 (group elements as permutations). An action of a group Γ on a set S induces an embedding from Γ to $Sym(S)$, where $Sym(S)$ is the symmetric group of S (group of all permutations of S). Thus, we can identify group elements with permutations of $|S| = k$ objects. By this perspective, the set of edges of the lift H can be rewritten as $E' = \{u_iv_j | uv \in E, \pi_{uv}(i) = j\}$, where π_{uv} is the permutation corresponding to the group element that we choose for edge uv.

Besides Γ-lifts another interesting case of $(\Gamma, S, \cdot)$-lifts is when $\Gamma = Sym([k])$ (the symmetric group on k elements), $S = [k]$, and the group action $\cdot : \Gamma \times S \to S$ is defined by $\sigma \cdot t = \sigma(t)$, i.e., the action of the permutation on the corresponding element. Such lifts are known as *general lifts* or simply *k-lifts*. Recall that shift k-lifts are Γ-lifts where the group Γ is a cyclic group. We will use the term *abelian lifts* to refer to Γ-lifts where the group Γ is an abelian group.

DEFINITION 2.9 (generalized signing). *Let $G = (V, E)$ be a base graph. Let E^f denote an arbitrary orientation of the edges of G (replace every edge $uv \in E$ with a directed edge $u \to v$ or $v \to u$) and E^r denote the reverse orientation. Given a group Γ, a set S, and an action $\cdot$ of Γ on S as in Definition* 2.8, *we define a generalized signing of G as a function $s : E^f \cup E^r \to \Gamma$ with the property that if $s(u, v) = g$, then $s(v, u) = g^{-1}$.*

We observe that there is a bijection between generalized signings and $(\Gamma, S, \cdot)$-lifts.

2.3.1. Spectrum of shift lifts. In this section, we characterize the spectrum of shift k-lifts as a union of the spectrum of certain matrices. We will denote the cyclic group of order k as C_k. For a shift k-lift of a graph $G = (V, E)$ with adjacency matrix A, which is given by the signing $(s(i, j) = g_{i,j})_{(i,j)\in E}$, define the following family of Hermitian matrices $A_s(\omega)$ parameterized by ω, where ω is a primitive kth root of unity:

$$[A_s(\omega)]_{ij} := \begin{cases} 0 & \text{if } A_{ij} = 0, \textit{and} \\ \omega^{g_{i,j}} & \text{if } A_{ij} = 1. \end{cases}$$

We have the following lemma regarding the spectrum of shift k-lifts.

LEMMA 2.10. *Let $G = (V, E)$ be a graph and H be a shift k-lift of G with the corresponding signing of the edges $(s(i,j) = g_{i,j})_{(i,j)\in E}$, where $g_{i,j} \in C_k$. Then the set of eigenvalues of H are given by*

$$\bigcup_{\omega:\ \omega \text{ is a primitive } k\text{th root of unity}} \textit{eigenvalues}\,(A_s(\omega))\,.$$

Proof. Let $n := |V|$ and let A_H be the adjacency matrix of H. Then,

$$A_H = \begin{bmatrix} A^{11} & A^{12} & \cdots & A^{1k} \\ A^{21} & A^{22} & \cdots & A^{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A^{k1} & A^{k2} & \cdots & A^{kk} \end{bmatrix},$$

where $A^{ab} \in \{0,1\}^{n\times n}$ with $A^{ab}_{ij} = 1$ if $A_{ij} = 1$ and $b \equiv a+g_{ij} \mod k$ and 0 otherwise. Let v be an eigenvector of $A_s(\omega)$ with eigenvalue λ, where ω is a primitive kth root of unity. That is, $A_s(\omega)v = \lambda v$. We will show that

$$z := \begin{bmatrix} v \\ \omega v \\ \vdots \\ \omega^{k-1} v \end{bmatrix}$$

is an eigenvector of A_H with eigenvalue λ and, moreover, the collection of eigenvectors created this way are orthogonal.

We first show that z is an eigenvector of A_H with eigenvalue λ. For this, we observe that

$$A_H z = \begin{bmatrix} \sum_{b=1}^k \omega^{b-1} A^{1b} v \\ \sum_{b=1}^k \omega^{b-1} A^{2b} v \\ \vdots \\ \sum_{b=1}^k \omega^{b-1} A^{kb} v \end{bmatrix}.$$

Hence, it is enough to show that for every $a \in [k]$,

$$\sum_{b=1}^{k} \omega^{b-1} A^{ab} = \omega^{a-1} A_s(\omega).$$

If A_{ij} is 0, then the ijth entry on both the left and right sides is 0. Suppose A_{ij} is 1. Then the ijth entry of A^{ab} is 1 if $b \equiv a + g_{ij} \mod k$ and zero otherwise. So, the ijth entry on the left side is equal to $\omega^{a+g_{ij}-1}$. Moreover, the ijth entry of $A_s(\omega)$ is equal to $\omega^{g_{ij}}$. So, the ijth entry on the right side is equal to $\omega^{a+g_{ij}-1}$. Thus, the matrices in the left and right sides are equal. Hence, any eigenvalue of $A_s(\omega)$ is an eigenvalue of A_H, where ω is a primitive kth root of unity.

Next, we show that these are the only eigenvalues of A_H. To prove this, it is enough to show that these nk eigenvectors are orthogonal to each other. Let ω_1, ω_2 be primitive kth root of unity and v_1, v_2 be eigenvectors of $A_s(\omega_1), A_s(\omega_2)$, respectively. Let z_1, z_2 be defined as follows:

$$z_1 := \begin{bmatrix} v_1 \\ \omega_1 v_1 \\ \vdots \\ \omega_1^{k-1} v_1 \end{bmatrix}, z_2 := \begin{bmatrix} v_2 \\ \omega_2 v_2 \\ \vdots \\ \omega_2^{k-1} v_2 \end{bmatrix}.$$

We show that $\langle z_1, z_2 \rangle = 0$:

$$\langle z_1, z_2 \rangle = \sum_{i=0}^{k-1} \langle \omega_1^i v_1, \omega_2^i v_2 \rangle = \sum_{i=0}^{k} \omega_1^i \omega_2^{-i} \langle v_1, v_2 \rangle.$$

The last equality follows from the fact that for n-dimensional complex vectors a, b, the inner product $\langle a, b \rangle$ is $\sum_{t=1}^{n} a_t b_t^*$, where b_i^* is the complex conjugate of b_i. If $\omega_1 = \omega_2$, then v_1, v_2 are orthogonal. Hence, $\langle v_1, v_2 \rangle = 0$ which also implies that $\langle z_1, z_2 \rangle = 0$. If $\omega_1 \neq \omega_2$, then $\omega_1 \cdot \omega_2^{-1}$ is a primitive kth root of unity that is not equal to 1. Hence,

$$\langle z_1, z_2 \rangle = \frac{(\omega_1 \omega_2^{-1})^k - 1}{\omega_1 \omega_2^{-1} - 1} \langle v_1, v_2 \rangle = 0. \qquad \square$$

The above lemma simplifies significantly for 2-lifts as noted in the corollary below.

Corollary 2.11. *When $k = 2$, the set of eigenvalues of a 2-lift H is given by the eigenvalues of A and the eigenvalues of A_s, where A_s is the signed adjacency matrix corresponding to the signing s with entries from $\{0, 1, -1\}$.*

3. Nonexpansion of abelian lifts.

In this section we show that it is impossible to find (even slightly) expanding graphs using lifts in large abelian groups Γ and thus prove Theorem 1.4. By Theorem 2.7, we know that if a graph is an expander, then it has small diameter. We show that if the size of the (abelian) group Γ is large, then *all* Γ-lifts of any base graph have a large diameter and, hence, they cannot be expanders. We restate Theorem 1.4 for convenience.

Theorem 1.4. *For every n-vertex d-regular graph G, every real value $\epsilon \in (0, 1/e)$, and every abelian group Γ of size at least*

$$k = \exp\left(\frac{nd \log \frac{1}{\epsilon} + \log n}{\log \frac{1}{e\epsilon}}\right),$$

all Γ-lifts H of G have λ_H at least ϵd in magnitude. In particular, when $k = 2^{\Omega(nd)}$, there is no Γ-lift H of any n-vertex d-regular graph G all of whose nontrivial eigenvalues are bounded by $O(\sqrt{d})$ in magnitude whenever Γ is an abelian group of order k.

Proof. We prove the contrapositive. Let Γ be an abelian group of order k and $G = (V, E)$ be a base graph on n-vertices that is d-regular. Let $e_1, \ldots, e_{nd/2}$ be an arbitrarily chosen ordering of the edges E. Let H be a lift graph obtained using a Γ-lift. Recall that the signing of the edges of the base graph correspond to group elements, which in turn correspond to permutations of k elements. Let these signings of the edges be $(\sigma_e)_{e \in E(G)}$. Let us define a layer L_i of H to be the set of vertices $\{v_i : v \in V\}$. We note that H has k layers.

Let us fix an arbitrary vertex v in G. Let Δ denote the diameter of H. Then, for every $j \in \{2, \ldots, k\}$ there exists a path of length at most Δ in H from v_1 to a vertex in L_j. A layer j is reachable within distance Δ in H iff there exists a walk $e_1, e_2, \ldots, e_t$ from v of length $t \leq \Delta$ in G such that $\sigma_{e_t} \sigma_{e_{t-1}} \ldots \sigma_{e_2} \sigma_{e_1}(1) = j$. Thus the set of layers reachable within distance Δ in H is contained in the set $S := \{\sigma_{e_t} \ldots \sigma_{e_1}(1) : e_1, \ldots, e_t \text{ is a walk from } v \text{ in } G \text{ of length } t \leq \Delta\}$. Since the group Γ is abelian, $S \subseteq \{\sigma_{e_1}^{a_1} \sigma_{e_2}^{a_2} \ldots \sigma_{e_{nd/2}}^{a_{nd/2}}(1) \mid \sum_{i=1}^{nd/2} |a_i| \leq \Delta\} =: T$. Since H has k layers, the cardinality of S is at least k.

The number of integral a_i's satisfying $\sum_{i=1}^{nd/2} |a_i| \leq \Delta$ is at most $\binom{(nd/2)+\Delta}{(nd/2)} \cdot 2^{(nd/2)}$. Therefore,

$$k \leq |T| \leq \binom{\frac{nd}{2}+\Delta}{\frac{nd}{2}} 2^{\frac{nd}{2}} \leq \left(2e\left(1+\frac{2\Delta}{nd}\right)\right)^{\frac{nd}{2}} \leq (2e)^{\frac{nd}{2}} e^{\Delta}.$$

Since H has nk vertices, using Theorem 2.7, we have $\Delta \leq (\log nk)/\log(d/\lambda(H))$. Thus, if $\lambda(H) \leq \epsilon d$, then $\Delta \leq (\log nk)/\log(1/\epsilon)$ and, consequently,

$$k \leq (2e)^{\frac{nd}{2}} e^{\frac{\log nk}{\log \frac{1}{\epsilon}}}.$$

Rearranging the terms, we obtain that

$$k \leq (2e)^{\frac{nd}{2\left(1-\frac{1}{\log \frac{1}{\epsilon}}\right)}} \exp\left(\frac{\log n}{\left(\log \frac{1}{\epsilon}\right)\left(1-\frac{1}{\log \frac{1}{\epsilon}}\right)}\right) \leq \exp\left(\frac{nd \log \frac{1}{\epsilon} + \log n}{\log \frac{1}{e\epsilon}}\right). \qquad \square$$

4. Expansion of random 2-lifts: Overview. In this section, we illustrate the main techniques involved in proving Theorem 1.2 by stating and proving a slightly weaker version, namely, Theorem 4.1. It focuses only on 2-lifts akin to Corollary 1.3 and is weaker in comparison to the eigenvalue bound in Corollary 1.3 by a multiplicative factor of four. The proof of this weaker result captures the main ideas involved in the proof of Theorem 1.2.

THEOREM 4.1. *Let G be a d-regular n-vertex graph, where $2 \leq d \leq \sqrt{n/(3\ln n)}$ with largest (in magnitude) nontrivial eigenvalue λ. Let H be a random 2-lift of G with λ_{new} being the largest (in magnitude) new eigenvalue of H. Then,*

$$\lambda_{\text{new}} \leq 4\lambda + 10^{14} \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)$$

with probability at least $1 - e^{-n/d^2}$.

In order to prove this theorem, we use the concentration inequality in Lemma 4.2 (recall that for a vector x, its support is denoted by $S(x)$).

LEMMA 4.2. *Let G be a d-regular n-vertex graph, where $2 \leq d \leq \sqrt{n/(3\ln n)}$, with largest (in magnitude) nontrivial eigenvalue λ. Let H be a random 2-lift of G with corresponding signed adjacency matrix A_s. The following statements hold with probability at least $1 - e^{-n/d^2}$:*

1. *For all $u_1, \ldots, u_r \in \{0, \pm 1\}^n$, and $v_1, \ldots, v_\ell \in \{0, \pm 1\}^n$ satisfying*
   - (I) *$S(u_i) \cap S(u_j) = \emptyset$ for every $i, j \in [r]$ and $S(v_i) \cap S(v_j) = \emptyset$ for every $i, j \in [\ell]$, and*
   - (II) *either $|S(u_i)| > n/d^2$ for every $i \in [r]$ with nonzero u_i, or $|S(v_i)| > n/d^2$ for every $i \in [\ell]$ with nonzero v_i,*

   we have

$$\left|\sum_{i \leq j} (2^{-i} u_i^T) A_s (2^{-j} v_j)\right| \leq 377 \max(\sqrt{\lambda \log d}, \sqrt{d}) \sum_{i=1}^{r} |S(u_i)| 2^{-2i} + \left(\frac{\lambda}{5} + 10^{12}\sqrt{d}\right) \sum_{j=1}^{\ell} |S(v_j)| 2^{-2j}.$$

2. *For all* $u_1, \ldots, u_r \in \{0, \pm 1\}^n$ *and* $v_1, \ldots, v_\ell \in \{0, \pm 1\}^n$ *satisfying* (I), (II), *and*
 (III) $|S(u_i)| > |S(v_j)|$ *for every* $i \in [r], j \in [\ell]$ *with nonzero* u_i,
 we have

$$\left| \sum_{i \leq j} (2^{-i} u_i^T) A_s (2^{-j} v_j) \right|$$
$$\leq 31 \max \left(\sqrt{\lambda \log d}, \sqrt{d} \right) \left(\sum_{i=1}^{r} |S(u_i)| 2^{-2i} + \sum_{j=1}^{\ell} |S(v_j)| 2^{-2i} \right).$$

We show the concentration inequality in Lemma 4.2 from Hoeffding's inequality by taking a suitable union bound (see section 5 for a complete proof). We will now prove Theorem 4.1 using the lemma above.

Proof of Theorem 4.1. Let s denote the signing corresponding to H and A_s denote the signed adjacency matrix. By Corollary 2.11, the largest (in magnitude) new eigenvalue of the lift is $\lambda_{new} = \max_{x \in \mathbb{R}^n} |x^T A_s x| / x^T x$. To prove an upper bound on λ_{new}, we will bound $|x^T A_s x| / x^T x$ for all x with high probability. In particular, assuming that the events given by Lemma 4.2 hold, we will show that

$$\left| x^T A_s x \right| \leq 4 \left(\lambda + 10^{13} \sqrt{d} \right) \|x\|^2.$$

By rescaling we may assume that the maximum entry of x is less than $1/2$ in absolute value. By Lemma 2.1, there exists a vector $y \in \{0, \pm 2^{-1}, \pm 2^{-2}, \ldots, \pm 2^{-i}, \ldots\}^n$ such that $|x^T A_s x| \leq |y^T A_s y|$ and $\|y\|^2 \leq 4 \|x\|^2$. We will prove a bound on $|y^T A_s y|$ for every $y \in \{0, \pm 2^{-1}, \pm 2^{-2}, \ldots, \pm 2^{-i}, \ldots\}^n$, which in turn will imply the desired bound on $|x^T A_s x|$. Let us consider the dyadic decomposition of $y = \sum_{i=1}^{\infty} 2^{-i} u_i$ obtained as follows: a coordinate of u_i is 1 if the corresponding coordinate of y is 2^{-i}, it is -1 if the corresponding coordinate of y is -2^{-i}, and is zero otherwise. We note that $S(u_i) \cap S(u_j) = \emptyset$ for every pair $i, j \in \mathbb{N}$.

Next, we partition the set of vectors u_i's based on their support sizes. Let $M := \{i \in \mathbb{N} : |S(u_i)| \leq n/d^2\}$ and $L := \{i \in \mathbb{N} : |S(u_i)| > n/d^2\}$ (we abbreviate M and L for mini and large supports, respectively). Correspondingly, define $y_M := \sum_{i \in M} 2^{-i} u_i$ and $y_L = \sum_{i \in L} 2^{-i} u_i$. We note that $y = y_M + y_L$, $\|y\|^2 = \|y_M\|^2 + \|y_L\|^2 = \sum_{i \in \mathbb{N}} |S(u_i)| 2^{-2i}$, and

$$|y^T A_s y| \leq |y_M^T A_s y_M| + 2 |y_M^T A_s y_L| + |y_L^T A_s y_L|.$$

We next bound each term in the right-hand side (RHS) using the following three claims.

CLAIM 4.3.

$$|y_M^T A_s y_M| \leq \left(\lambda + \frac{8}{d} \right) \|y_M\|^2.$$

Proof. Let y'_M be a vector obtained from y_M by taking the absolute values of each entry. Then $\|y_M\|^2 = \|y'_M\|^2$ and $|y_M^T A_s y_M| \leq y'^T_M A y'_M$. Let $J = vv^T$ and $J' = v'v'^T$, where v is the all-ones vector and v' is defined as follows: $v'_i = 1$ for $1 \leq i \leq n/2$ and $v'_i = -1$ for $n/2 + 1 \leq i \leq n$. For nonbipartite graph G, we have

$$y'^T_M A y'_M = y'^T_M \left(A - \frac{d}{n} J \right) y'_M + y'^T_M \left(\frac{d}{n} J \right) y'_M \leq \lambda \|y'_M\|^2 + y'^T_M \left(\frac{d}{n} J \right) y'_M.$$

Above, we have used the fact that $A - \frac{d}{n}J$ has the same set of eigenvalues as A except for one—the eigenvalue d for the matrix A is translated to zero for the matrix $A - \frac{d}{n}J$. Similarly, for bipartite graphs, we have

$$\begin{aligned} y_M'^T A y_M' &= y_M'^T \left(A - \frac{d}{n}J + \frac{d}{n}J'\right) y_M' + y_M'^T \left(\frac{d}{n}J\right) y_M' - y_M'^T \left(\frac{d}{n}J'\right) y_M' \\ &\le \lambda \|y_M'\|^2 + y_M'^T \left(\frac{d}{n}J\right) y_M' - y_M'^T \left(\frac{d}{n}J'\right) y_M'. \end{aligned}$$

Above, we have used the fact that $A - \frac{d}{n}J + \frac{d}{n}J'$ has the same set of eigenvalues as A except for two—the largest (in magnitude) two eigenvalues d for the matrix A are translated to zero for the matrix $A - \frac{d}{n}J + \frac{d}{n}J'$. It remains to bound $|y_M'^T\left(\frac{d}{n}J\right)y_M'|$ and $|y_M'^T\left(\frac{d}{n}J'\right)y_M'|$. Consider the dyadic decomposition of $y_M' = \sum_{i\in M} 2^{-i}u_i'$, where the coordinates of u_i' are the absolute values of the coordinates of u_i.

$$\begin{aligned} &\left|y_M'^T \left(\frac{d}{n}J\right) y_M'\right|, \left|y_M'^T \left(\frac{d}{n}J'\right) y_M'\right| \\ &\quad \le 2\sum_{i\in M}\sum_{j\in M: j\ge i} \frac{d}{n} 2^{-i}|S(u_i)| 2^{-j}|S(u_j)| \\ &\quad \le 2\sum_{i\in M} \frac{1}{d} 2^{-2i}|S(u_i)| \sum_{j\in M: j\ge i} 2^{i-j} \qquad (\text{since } |S(u_j)| \le n/d^2 \ \forall\ j \in M) \\ &\quad \le \frac{4}{d}\|y_M'\|^2. \end{aligned}$$

□

Remark 7. The bound in Claim 4.3 does not rely on the high probability event given by Lemma 4.2. Hence, in contrast to [10, 18, 1] our bound for the mini support case is valid with probability one and follows directly from the expansion of the base graph.

CLAIM 4.4.

$$|y_L^T A_s y_L| \le \left(\frac{2\lambda}{5} + (3\cdot 10^{12}) \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)\right) \|y_L\|^2.$$

Proof. By the triangle inequality,

$$\begin{aligned} |y_L^T A_s y_L| &= \left|\sum_{i,j\in L} (2^{-i}u_i^T) A_s (2^{-j}u_j)\right| \\ &\le \left|\sum_{i,j\in L: i\le j} (2^{-i}u_i) A_s (2^{-j}u_j)\right| + \left|\sum_{i,j\in L: i>j} (2^{-i}u_i) A_s (2^{-j}u_j)\right|. \end{aligned}$$

We bound each term using the first part of Lemma 4.2. We now clarify our choice of parameters to apply Lemma 4.2. For both terms, our choice is $r \leftarrow \max\{i \in L\}$, $\ell = r$, $u_i \leftarrow u_i$ if $i \in L$ and $u_i \leftarrow \overline{0}$ if $i \notin L$, $v_i = u_i$ for every $i \in [r]$, where $\overline{0}$ is the all-zeroes vector. We note that the conditions (I) and (II) of Lemma 4.2 are satisfied by this choice since every pair $S(u_i), S(u_j)$ is mutually disjoint and $|S(u_i)| > n/d^2$

for all $i \in L$. Consequently,

$$|y_L^T A_s y_L| \le 754 \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right) \sum_{i \in L} |S(u_i)| 2^{-2i} + \left(\frac{\lambda}{5} + 2 \cdot 10^{12} \sqrt{d}\right) \sum_{j \in L} |S(u_j)| 2^{-2j} \le \left(\frac{2\lambda}{5} + (2 \cdot 10^{12} + 754) \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)\right) \|y_L\|^2. \qquad \square$$

Claim 4.5.

$$|y_M^T A_s y_L| \le 408 \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right) \|y_M\|^2 + \left(\frac{\lambda}{5} + (2 \cdot 10^{12}) \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)\right) \|y_L\|^2.$$

Proof. By the triangle inequality,

$$|y_M^T A_s y_L| = \left| \sum_{i \in M, j \in L} (2^{-i} u_i^T) A_s (2^{-j} u_j) \right| \le \left| \sum_{i \in M, j \in L : i \le j} (2^{-i} u_i) A_s (2^{-j} u_j) \right| + \left| \sum_{i \in M, j \in L : i > j} (2^{-i} u_i) A_s (2^{-j} u_j) \right|.$$

We bound the first and second terms by the first and second parts of Lemma 4.2, respectively. Let $\overline{0}$ be the all-zeroes vector. We now clarify our choice of parameters to apply Lemma 4.2. For the first term, our choice is $r \leftarrow \max\{i \in M\}$, $\ell \leftarrow \max\{i \in L\}$, $u_i \leftarrow u_i$ if $i \in M$ and $u_i \leftarrow \overline{0}$ if $i \notin M$, and $v_i \leftarrow u_i$ if $i \in L$ and $v_i \leftarrow \overline{0}$ if $i \notin L$. For the second term, our choice is $r \leftarrow \max\{i \in L\}$, $\ell \leftarrow \max\{i \in M\}$, $u_i \leftarrow u_i$ if $i \in L$ and $u_i \leftarrow \overline{0}$ if $i \notin L$, and $v_i \leftarrow u_i$ if $i \in M$ and $v_i \leftarrow \overline{0}$ if $i \notin M$. The conditions (I), (II), and (III) of Lemma 4.2 are satisfied for the respective choices since every pair $S(u_i), S(u_j)$ is mutually disjoint, $|S(u_i)| > n/d^2$ for all $i \in L$, and $|S(u_i)| > n/d^2 \ge |S(u_j)|$ for every $i \in L, j \in M$. Consequently,

$$|y_M^T A_s y_L| \le 377 \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right) \sum_{i \in M} |S(u_i)| 2^{-2i} + \left(\frac{\lambda}{5} + 10^{12} \sqrt{d}\right) \sum_{j \in L} |S(u_j)| 2^{-2j} + 31 \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right) \left(\sum_{j \in L} |S(u_j)| 2^{-2j} + \sum_{j \in M} |S(u_j)| 2^{-2j} \right) \le 408 \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right) \|y_M\|^2 + \left(\frac{\lambda}{5} + (10^{12} + 31) \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)\right) \|y_L\|^2. \qquad \square$$

From the above three claims, we have

$$|y^T A_s y| \le \left(\lambda + 817 \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)\right) \|y_M\|^2 + \left(\frac{4\lambda}{5} + (7 \cdot 10^{12}) \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)\right) \|y_L\|^2 \le \left(\lambda + 8 \cdot 10^{12} \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)\right) \|y\|^2.$$

Therefore, we have

$$\begin{aligned}|x^T A_s x| &\le |y^T A_s y|\\ &\le \left(\lambda + 8\cdot 10^{12}\max\left(\sqrt{\lambda\log d}, \sqrt{d}\right)\right)\|y\|^2\\ &\le 4\left(\lambda + 8\cdot 10^{12}\max\left(\sqrt{\lambda\log d}, \sqrt{d}\right)\right)\|x\|^2. \qquad \square\end{aligned}$$

We note that in the above proof, the multiplicative factor of 4 is a by product of the discretization of x. This can be avoided if we do not discretize x straightaway, but instead "push" the discretization a little deeper into the proof. Indeed, we can see that the proof of Claim 4.3 where we bound $|y_M^T(A-(d/n)J)y_M|$ by $\lambda\|y_M\|^2$ does not require y_M to be a discretized vector. This is how we are able to prevent the multiplicative factor loss so as to obtain Theorem 1.2.

5. Concentration inequality. In this section, we prove Lemma 4.2. Our proof based on case analysis to bound the quadratic form of a random matrix closely resembles a similar proof given in [10]. Our random matrix distribution is different from the one that was of interest in [10] and moreover our probability bounds are stronger.

In order to prove Lemma 4.2 we need to upper bound

$$\left|\sum_{i\le j} 2^{-i-j} u_i^T A_s v_j\right|$$

for all sets of vectors $\{u_1,\dots,u_r\}$, $\{v_1,\dots,v_\ell\}$ satisfying the assumptions of the lemma over random choices of A_s. A natural approach is to use the triangle inequality and upper bound each term $|u_i^T A_s v_j|$ separately for each i, j. We note that $u_i^T A_s v_j$ is a sum of $|E(S(u_i), S(v_j))|$ independent and identically ditributed random variables with mean zero (one for each edge between $S(u_i)$ and $S(v_j)$). By the expander mixing lemma (Theorem 2.6), we may upper bound the size of $E(S(u_i), S(v_j))$ by $2d|S(u_i)||S(v_j)|/n + \lambda\sqrt{|S(u_i)||S(v_j)|}$. Depending on which of these two terms in the RHS dominates, we have two cases. For each case, we use a different concentration bound (Lemma 5.1 and Corollary 5.3). We begin with the needed concentration bounds.

5.1. Concentration bounds.

LEMMA 5.1. *Let G be a d-regular, n-vertex graph, where $2 \le d \le \sqrt{n/3\ln n}$, with largest (in magnitude) nontrivial eigenvalue λ. Let H be a uniformly random 2-lift of G with corresponding signed adjacency matrix A_s. The following property holds with probability at least $1 - e^{-(n\log d)/\sqrt{d}}$ (over the random choice of signings).*

For every $r \in \{0,1,\dots,(1/2)\log d\}$, every $a, b_0, b_1, \dots, b_r \in \{0,\pm1\}^n$ satisfying

(i) *$S(b_i)\cap S(b_j) = \emptyset$ for all $i, j \in [r], i \ne j$,*

(ii) *$|S(a)| \ge 2^{2i}|S(b_i)|$ for all $i \in [r]$, and*

(iii) *$\frac{d}{\lambda}\sqrt{|S(b_i)||S(a)|} \ge n$ for all $i \in [r]$ with nonzero b_i,*

we have

$$\left|a^T A_s\left(\sum_{i=0}^r 2^i b_i\right)\right| \le 14\sqrt{\frac{d}{n}|S(a)|^2\left(\sum_{i=0}^r |S(b_i)|2^{2i}\right)\log\left(\frac{2n}{|S(a)|}\right)}.$$

Proof. For notational convenience, let $b = \sum_{i=0}^r 2^i b_i$. Fix $a, b_1, b_2, \dots, b_r \in \{0,\pm1\}^n$. Then $a^T A_s b$ is a sum of independent random variables (one for each edge between $S(a)$ and $S(b_i)$) with mean 0. This is because the intersection between the support of any two vectors b_i and b_j is empty. The sum of squares of the difference between the maximum and the minimum values of these variables is at most $\sum_{i=1}^{r} 4E(S(b_i), S(a))2^{2i}$. For vectors $a, b_1, \ldots, b_r$ satisfying (ii) and (iii), by the expander mixing lemma, we have $E(S(b_i), S(a)) \leq 3\frac{d|S(b_i)||S(a)|}{n}$. We note that this inequality holds even if b_i is a zero vector.

By Lemma 2.4,

$$Pr\left(|a^T A_s b| > 14\sqrt{\frac{d}{n}|S(a)|^2\left(\sum_i |S(b_i)|2^{2i}\right)\log\left(\frac{2n}{|S(a)|}\right)}\right) \leq 2\exp\left(-\frac{98|S(a)|}{3}\log\left(\frac{2n}{|S(a)|}\right)\right).$$

Now fixing the values of the support sizes $\alpha = |S(a)|, \beta_i = |S(b_i)|$, the number of possible choices for a is at most $\binom{n}{\alpha} \cdot 2^{\alpha} \leq \exp(3\alpha \log(\frac{2n}{\alpha}))$. Similarly the number of possible choices for each b_i is at most $\exp(3\beta_i \log(\frac{2n}{\beta_i}))$. Therefore the total number of choices for $b_1, \ldots, b_r$ is at most $\exp(\sum_{i=1}^{r} 3\beta_i \log(\frac{2n}{\beta_i}))$. Since each $\alpha, \beta_i \leq n$, we can replace each β_i by its upper bound $\alpha 2^{-2i}$. Hence, using Lemma 2.2,

$$\exp\left(\sum_{i=1}^{r} 3\beta_i \log\left(\frac{2n}{\beta_i}\right)\right) \leq \exp\left(3\sum_{i=1}^{r} \alpha 2^{-2i} \log\left(\frac{2n}{\alpha 2^{-2i}}\right)\right) \leq \exp\left(27\alpha \log\left(\frac{2n}{\alpha}\right)\right).$$

Therefore, the total number of choices of $a, b_1, \ldots, b_r$ of sizes $\alpha, \beta_1, \ldots \beta_r$, respectively, is at most

$$\exp\left(30\alpha \log\left(\frac{2n}{\alpha}\right)\right).$$

By taking a union bound over the choices of vectors with the fixed support sizes, the probability of the existence of a set of vectors $a, b_1, \ldots, b_r$ with sizes $\alpha, \beta_1, \ldots, \beta_r$, respectively, satisfying (i), (ii), and (iii), but violating the conclusion is bounded by

$$2\exp\left(-\frac{8\alpha}{3}\log\left(\frac{2n}{\alpha}\right)\right) \leq 2\exp\left(-\frac{8n}{3\sqrt{d}}\log\left(2\sqrt{d}\right)\right).$$

Above, we have used that $\alpha \geq n/\sqrt{d}$ which follows since $\alpha = |S(a)| \geq n\lambda/d \geq n/\sqrt{d}$ by (ii) and (iii). Next, let us bound the number of choices for the support sizes of the vectors $a, b_1, \ldots, b_r$. The number of choices for the support sizes is at most $n^{2+(1/2)\log d}$. Therefore taking the union bound over the choice of the support sizes, we get that the total probability is at most

$$2\exp\left(\left(2 + \frac{1}{2}\log d\right)\ln n\right)\exp\left(-\frac{8n}{3\sqrt{d}}\log(2\sqrt{d})\right) \leq \exp\left(-\frac{n\log d}{\sqrt{d}}\right). \qquad \square$$

In the next lemma, let $N_G(S) := \{v : \exists u \in S \text{ with } (u, v) \in E(G)\}$ denote the set of neighbors for a set S of nodes in G.

LEMMA 5.2. *Let G be a d-regular, n-vertex graph, where $2 \leq d \leq \sqrt{n/3\ln n}$ with largest (in magnitude) nontrivial eigenvalue λ and H be a uniformly random 2-lift of G with corresponding signed adjacency matrix A_s. The following property holds with probability at least $1 - e^{-3n/d^2}$ (over the random choice of signings).*

For every $a, b \in \{0, \pm 1\}^n, q, w \in \{1, \ldots, n\}$ *satisfying*

(i) $|S(a)| \leq q$, $|S(b)| \leq w$, $S(b) \subset N_G(S(a))$,

(ii) $q \leq w \leq dq$,

(iii) $w > \frac{n}{d^2}$, *and*

(iv) $\frac{d}{\lambda}\sqrt{qw} < n$,

we have

$$|a^T A_s b| \leq 10\sqrt{\lambda\sqrt{qw^3}\log\left(\frac{2dq}{w}\right)}. \tag{5.1}$$

Proof. For a pair of vectors $a, b \in \{0, \pm 1\}^n$ and $q, w \in \{1, \ldots, n\}$, let $Bad(a, b, q, w)$ denote the event that inequality (5.1) is violated. We need to upper bound the probability that there exists (a, b, q, w) satisfying (i), (ii), (iii), and (iv) such that $Bad(a, b, q, w)$ happens. We note that the sum $a^T A_s b$ over random choices of A_s is a sum of independent random variables chosen from $\{\pm 2, \pm 1\}$, all of which have mean 0. The number of such random variables being summed is at most $E(S(a), S(b))$, i.e., the number of edges between $S(a)$ and $S(b)$.

Therefore for a fixed a, b, q, w, by applying the Hoeffding inequality (Lemma 2.4), we get that

$$P(Bad(a, b, q, w)) \leq 2\exp\left(-\frac{50\lambda\sqrt{qw^3}\log\left(\frac{2dq}{w}\right)}{E(S(a), S(b))}\right).$$

Now using (iv) and the expander mixing lemma (Theorem 2.6), we have

$$E(S(a), S(b)) \leq 2d|S(a)||S(b)|/n + \lambda\sqrt{|S(a)||S(b)|} \leq 2dqw/n + \lambda\sqrt{qw} \leq 3\lambda\sqrt{qw}.$$

Substituting this into the previous expression, we obtain

$$P(Bad(a, b, q, w)) \leq 2\exp\left(-(50/3)w\log\left(\frac{2dq}{w}\right)\right).$$

We will use the union bound now. For this purpose, we will first fix q, w, and the size of the support of a and b. We take a union bound over all possible choices of a, b of that fixed size, and then take a union bound over all choices of the support sizes. For fixed support sizes $\alpha = |S(a)|, \beta = |S(b)|$, we observe that the total number of choices for the support sets for a are $\binom{n}{\alpha}$. Now, since $S(b)$ is a subset of $N_G(S(a))$, the number of choices of $S(b)$ is bounded by $\binom{d\alpha}{\beta}$. Also, since each entry in a, b is 0 or ± 1 the total number of choices for a and b is at most

$$\binom{n}{\alpha}2^\alpha\binom{d\alpha}{\beta}2^\beta \leq \exp\left(3\alpha\log\left(\frac{2n}{\alpha}\right)\right)\exp\left(3\beta\log\left(\frac{2d\alpha}{\beta}\right)\right).$$

We will first show upper bounds on each of these terms. Since $w \geq \frac{n}{d^2}$, by (ii), we have $q \geq \frac{n}{d^3}$. Also, $\alpha = |S(a)| \leq q, \beta = |S(b)| \leq w$. Therefore,

$$\begin{aligned}
\exp\left(3\alpha\log\left(\frac{2n}{\alpha}\right)\right) &\leq \exp\left(3q\log\left(\frac{2n}{q}\right)\right) \\
&\leq \exp\left(9q\log(2d)\right) \\
&= \exp\left(9\frac{\frac{q}{w}\log(2d)}{\log\left(2d\frac{q}{w}\right)}\cdot w\log\left(2d\frac{q}{w}\right)\right) \\
&\leq \exp\left(9w\log\left(2d\frac{q}{w}\right)\right).
\end{aligned}$$

The last line follows from the fact that $x\log(d)/\log(2dx)$ is bounded by 1 for $x \in [1/d, 1]$ and that $\frac{q}{w} \in [1/d, 1]$. Further,

$$\exp\left(3\beta\log\left(\frac{2d\alpha}{\beta}\right)\right) \le \exp\left(3\beta\log\left(\frac{2dq}{\beta}\right)\right) \le \exp\left(3w\log\left(\frac{2dq}{w}\right)\right).$$

The last inequality follows by the fact that $x\log\frac{2c}{x}$ is an increasing function if $x < c$. Therefore, by the union bound we get that the probability of a bad event for fixed q, w and support sizes $\alpha = |S(a)|, \beta = |S(b)|$ is at most

$$2\exp\left(-(14/3)w\log\frac{4dq}{w}\right) \le 2\exp\left(-\frac{14n}{3d^2}\log\frac{4dq}{w}\right) \le 2\exp\left(-\frac{14n}{3d^2}\log 2\right).$$

Now the number of choices of the support sizes of a and b is at most n^2, and the number of choices for q and w is at most n^2 and, therefore,

$$P\left(\exists(a,b,q,w) \text{ satisfying (i), (ii), (iii), and (iv): } Bad(a,b,q,w)\right)$$
$$\le 2n^4\exp\left(-\frac{14n}{3d^2}\log 2\right) \le \exp\left(-\frac{3n}{d^2}\right). \qquad \square$$

COROLLARY 5.3. *Let G be a d-regular, n-vertex graph, where $2 \le d \le \sqrt{n/3\ln n}$, with largest (in magnitude) nontrivial eigenvalue λ and H be a uniformly random 2-lift of G with corresponding signed adjacency matrix A_s. The following property holds with probability at least $1 - e^{-3n/d^2}$ (over the random choice of signings).*

For every $a, b \in \{0, \pm 1\}^n$ satisfying

(i) $|S(a)| \le |S(b)| \le d|S(a)|$,

(ii) $|S(b)| > \frac{n}{d^2}$, *and*

(iii) $\frac{d}{\lambda}\sqrt{|S(a)||S(b)|} < n$,

we have

$$|a^T A_s b| \le 10\sqrt{\lambda\sqrt{|S(a)||S(b)|}|S(b)|\log\left(\frac{2d|S(a)|}{|S(b)|}\right)}. \tag{5.2}$$

Proof. For every a, b, we apply the bound from Lemma 5.2 on $|a^T A_s b'|$ with $q = |S(a)|, w = |S(b)|$, where b' is the same as b restricted to the coordinates in $S(b) \cap N_G(S(a))$. We observe that $|a^T A_s b| = |a^T A_s b'|$ and hence the corollary. $\square$

5.2. Proof of Lemma 4.2. Next, we use Corollary 5.3 and Lemma 5.1 to prove Lemma 4.2. We restate the lemma below for the benefit of the reader.

LEMMA 4.2. *Let G be a d-regular n-vertex graph, where $2 \le d \le \sqrt{n/(3\ln n)}$, with largest (in magnitude) nontrivial eigenvalue λ. Let H be a random 2-lift of G with corresponding signed adjacency matrix A_s. The following statements hold with probability at least $1 - e^{-n/d^2}$:*

1. *For all $u_1, \ldots, u_r \in \{0, \pm 1\}^n$, and $v_1, \ldots, v_\ell \in \{0, \pm 1\}^n$ satisfying*
   (I) *$S(u_i) \cap S(u_j) = \emptyset$ for every $i, j \in [r]$ and $S(v_i) \cap S(v_j) = \emptyset$ for every $i, j \in [\ell]$, and*
   (II) *either $|S(u_i)| > n/d^2$ for every $i \in [r]$ with nonzero u_i, or $|S(v_i)| > n/d^2$ for every $i \in [\ell]$ with nonzero v_i,*

we have

$$\left|\sum_{i\le j}(2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \le 377\max(\sqrt{\lambda\log d},\sqrt{d})\sum_{i=1}^{r}|S(u_i)|2^{-2i}$$
$$+\left(\frac{\lambda}{5}+10^{12}\sqrt{d}\right)\sum_{j=1}^{\ell}|S(v_j)|2^{-2j}.$$

2. *For all* $u_1,\dots,u_r\in\{0,\pm1\}^n$ *and* $v_1,\dots,v_\ell\in\{0,\pm1\}^n$ *satisfying* (I), (II), *and*
(III) $|S(u_i)|>|S(v_j)|$ *for every* $i\in[r], j\in[\ell]$ *with nonzero* u_i,
we have

$$\left|\sum_{i\le j}(2^{-i}u_i^T)A_s(2^{-j}v_j)\right|$$
$$\le 31\max\left(\sqrt{\lambda\log d},\sqrt{d}\right)\left(\sum_{i=1}^{r}|S(u_i)|2^{-2i}+\sum_{j=1}^{\ell}|S(v_j)|2^{-2i}\right).$$

Proof. For notational convenience, we will replace $|S(u_i)|$ by s_i and $|S(v_j)|$ by t_j. We split the sum

$$\sum_{i\le j}(2^{-i}u_i^T)A_s(2^{-j}v_j)$$

into several subcases depending on i,j and the sizes of $S(u_i)$ and $S(v_j)$. Figure 1 summarizes the splitting of (i,j) into various terms depending on the various values of i, j, s_i, and t_j. Next, we bound each of the terms separately. By Lemma 5.1 and Corollary 5.3, we know that A_s satisfies the property mentioned in both of them with

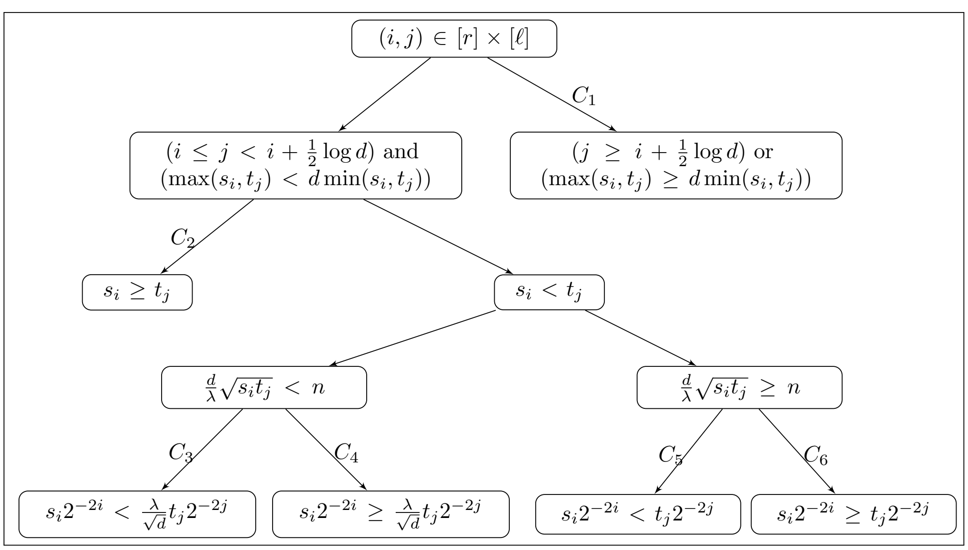

Fig. 1.

probability at least $1 - 2e^{-3n/d^2}$. We bound the terms assuming that A_s satisfies the property mentioned in Lemma 5.1 and Corollary 5.3.

CLAIM 5.4.

$$\left| \sum_{(i,j)\in C_1} (2^{-i}u_i^T)A_s(2^{-j}v_j) \right| \leq 3\sqrt{d}\left(\sum_{i\in[r]} s_i 2^{-2i} + \sum_{j\in[\ell]} t_j 2^{-2j} \right).$$

Proof. The sum is conditioned over the set of tuples (i,j) in C_1, where

$$C_1 = \left\{ (i \in [r], j \in [\ell]) \mid \left(j \geq i + \frac{1}{2}\log d \right) \text{ or } (\max(s_i,t_j) \geq d\min(s_i,t_j)) \right\}.$$

Let

$$C_1' := \left\{ (i,j) \in [r]\times[\ell] : j \geq i + \frac{1}{2}\log d \right\} \text{ and}$$

$$C_1'' := \left\{ (i,j) \in [r]\times[\ell] : i \leq j < i + \frac{1}{2}\log d, \max(s_i,t_j) \geq d\min(s_i,t_j) \right\}.$$

By the triangle inequality,

$$\left| \sum_{(i,j)\in C_1} 2^{-i-j}u_i^T A_s v_j \right| \leq \left| \sum_{(i,j)\in C_1'} 2^{-i-j}u_i^T A_s v_j \right| + \left| \sum_{(i,j)\in C_1''} 2^{-i-j}u_i^T A_s v_j \right|.$$

We note that the number of edges out of any set S is bounded by $d|S|$. So, $|u_i^T A_s v_j| \leq d\min(s_i,t_j)$ for any $u_i, v_j \in \{-1,0,+1\}^n$. We now bound the two terms above. For the first term,

$$\begin{aligned}
\left| \sum_{(i,j)\in[r]\times[\ell]:j\geq i+\frac{1}{2}\log d} 2^{-i-j}u_i^T A_s v_j \right| &\leq \sum_{i\in[r]} \sum_{j=i+\frac{1}{2}\log d}^{\ell} 2^{-i-j}|u_i^T A_s v_j| \\
&\leq \sum_{i\in[r]} \sum_{j=i+\frac{1}{2}\log d}^{\ell} 2^{-i-j} d\cdot\min(s_i,t_j) \\
&\leq \sum_{i\in[r]} \sum_{j=i+\frac{1}{2}\log d}^{\ell} 2^{-i-j} d\cdot s_i \\
&\leq 2\sqrt{d}\sum_{i\in[r]} 2^{-2i}s_i.
\end{aligned}$$

For the second term,

$$\begin{aligned}\left|\sum_{\substack{(i,j)\in[r]\times[\ell]: i\le j< i+\frac{1}{2}\log d,\\ \max(s_i,t_j)\ge d\min(s_i,t_j)}} 2^{-i-j}u_i^T A_s v_j\right| &\le \sum_{\substack{i\in[r], j\in[\ell]: i\le j< i+\frac{1}{2}\log d,\\ \max(s_i,t_j)\ge d\min(s_i,t_j)}} 2^{-i-j}|u_i^T A_s v_j| \\ &\le \sum_{\substack{i\in[r], j\in[\ell]: i\le j< i+\frac{1}{2}\log d,\\ \max(s_i,t_j)\ge d\min(s_i,t_j)}} 2^{-i-j} d\min(s_i,t_j) \\ &\le \sum_{\substack{i\in[r], j\in[\ell]: i\le j< i+\frac{1}{2}\log d,\\ \max(s_i,t_j)\ge d\min(s_i,t_j)}} 2^{-i-j}\max(s_i,t_j) \\ &\le \sum_{\substack{i\in[r], j\in[\ell]: i\le j< i+\frac{1}{2}\log d,\\ \max(s_i,t_j)\ge d\min(s_i,t_j)}} 2^{-i-j}(s_i+t_j) \\ &= \sum_{i\in[r]} 2^{-i}s_i \sum_{j=i}^{i+\frac{1}{2}\log d} 2^{-j} + \sum_{j\in[\ell]} 2^{-j}t_j \sum_{i=j-\frac{1}{2}\log d}^{j} 2^{-i} \\ &\le 2\sum_{i\in[r]} s_i 2^{-2i} + 2\sqrt{d}\sum_{j\in[\ell]} t_j 2^{-2j}. \qquad \square\end{aligned}$$

Claim 5.5.

$$\left|\sum_{(i,j)\in C_2} (2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \le 28\max\left(\sqrt{d}, \sqrt{\lambda\log d}\right)\sum_{i\in[r]} s_i 2^{-2i}.$$

Proof. The sum is conditioned over the set of tuples (i,j) in C_2, where

$$C_2 = \left\{(i,j)\in[r]\times[\ell] | \left(i\le j< i+\frac{1}{2}\log d\right) \text{ and } (t_j\le s_i< d\cdot t_j)\right\}.$$

By the triangle inequality the required sum is at most $\sum_{(i,j)\in C_2} 2^{-i-j}|u_i^T A_s v_j|$. We note that $u_i, v_j \neq \overline{0}$ since $t_j \le s_i < dt_j$. Consider the term $|u_i^T A_s v_j|$, where (i,j) is in C_2. We have two cases:

Case 1: If $(d/\lambda)\sqrt{s_i t_j} \ge n$, then we use Lemma 5.1 for the choice $a \leftarrow u_i, b_0 \leftarrow v_j$. This choice satisfies the conditions of Lemma 5.1. Hence,

$$|u_i^T A_s v_j| \le 14\sqrt{d\cdot s_i^2\cdot\frac{t_j}{n}\log\left(\frac{2n}{t_j}\right)} \le 14\sqrt{d}s_i.$$

Here, the last inequality follows by using $x\log(\frac{2}{x}) \le 1$ for $x<1$.

Case 2: If $(d/\lambda)\sqrt{s_i t_j} < n$, then we use Corollary 5.3 for the choice $a \leftarrow v_j, b \leftarrow u_i$. This choice satisfies the conditions of Corollary 5.3 since $t_j \le s_i < dt_j$, condition (II) of the lemma implies $s_i > n/d^2$, and $(d/\lambda)\sqrt{s_i t_j} < n$. Hence,

$$|u_i^T A_s v_j| \le 14\sqrt{\lambda\sqrt{t_j s_i}s_i\log\left(\frac{2\cdot d\cdot t_j}{s_i}\right)} \le 14\sqrt{\lambda\log d}s_i.$$

The last inequality follows since $t_j \le s_i$.

Thus, for $(i,j) \in C_2$, we have $|u_i^T A_s v_j| \le 14 \max(\sqrt{d}, \sqrt{\lambda \log d}) s_i$. Therefore,

$$\begin{aligned}\left|\sum_{(i,j)\in C_2} (2^{-i}u_i^T) A_s (2^{-j}v_j)\right| &\le \sum_{(i,j)\in C_2} 2^{-i-j}|u_i^T A_s v_j| \\ &\le 14 \sum_{i\in[r]}\sum_{j=i}^{\infty} 2^{-i-j}\max(\sqrt{d}, \sqrt{\lambda\log d}) s_i \\ &\le 28 \max\left(\sqrt{d}, \sqrt{\lambda \log d}\right) \sum_{i\in[r]} s_i 2^{-2i}. \qquad \square\end{aligned}$$

CLAIM 5.6.

$$\left|\sum_{(i,j)\in C_3} (2^{-i}u_i^T) A_s (2^{-j}v_j)\right| = \left(\frac{\lambda}{5} + 0.95 \cdot 10^{12}\sqrt{d}\right) \sum_{j\in[\ell]} t_j 2^{-2j}.$$

Proof. The sum is conditioned over the set of tuples (i,j) in C_3, where

$$\begin{aligned}C_3 = \Bigg\{(i,j) | &\left(i \le j \le i + \frac{1}{2}\log d\right) \wedge (s_i \le t_j < d s_i) \\ &\wedge \left(\frac{d}{\lambda}\sqrt{s_i t_j} < n\right) \wedge \left(s_i 2^{-2i} < \frac{\lambda}{\sqrt{d}} t_j 2^{-2j}\right)\Bigg\}.\end{aligned}$$

By the triangle inequality,

$$\left|\sum_{(i,j)\in C_3} (2^{-i}u_i^T) A_s (2^{-j}v_j)\right| \le \sum_{(i,j)\in C_3} 2^{-i-j}|u_i^T A_s v_j|.$$

We note that $u_i, v_j \ne \overline{0}$ since $s_i \le t_j < d s_i$. We use Corollary 5.3 to bound each term $|u_i^T A_s v_j|$. We use Corollary 5.3 with the choice $a \leftarrow u_i$ and $b \leftarrow v_j$. This choice satisfies the conditions of Corollary 5.3 since $s_i \le t_j \le d s_i$, condition (II) of the lemma implies $t_j > n/d^2$, and $(d/\lambda)\sqrt{s_i t_j} < n$. Hence,

$$\begin{aligned}&\sum_{(i,j)\in C_3} 2^{-i-j}|u_i^T A_s v_j| \\ &\le 10 \sum_{(i,j)\in C_3} 2^{-i-j}\sqrt{\lambda\sqrt{s_i t_j}\, t_j \log\left(\frac{2ds_i}{t_j}\right)} \\ &< 10 \sum_{(i,j)\in C_3} \frac{\lambda^{3/4}}{d^{1/8}} t_j 2^{-i-j}\sqrt{2^{-(j-i)}\log\left(\frac{2\lambda\sqrt{d}}{2^{2j-2i}}\right)} \qquad \left(\text{since } s_i 2^{-2i} < \frac{\lambda}{\sqrt{d}} t_j 2^{-2j}\right) \\ &\le 10 \frac{\lambda^{3/4}}{d^{1/8}} \sum_{j\in[\ell]} t_j 2^{-2j} \sum_{i=j-\frac{1}{2}\log d+1}^{i=j} \sqrt{2^{j-i}\log\left(\frac{2\lambda\sqrt{d}}{2^{2j-2i}}\right)}.\end{aligned}$$

By assumption, $n \geq 9d^2$. Hence, $\lambda \geq \sqrt{d \cdot \frac{n-d}{n-1}} > \sqrt{d} - 1$. Applying Lemma 2.2, we get

$$\sum_{(i,j)\in C_3} 2^{-i-j}|u_i^T A_s v_j| \leq 90\frac{\lambda^{3/4}}{d^{1/8}}\sqrt{\sqrt{d}\log\left(\frac{2(\lambda+1)}{\sqrt{d}}\right)}\sum_{j\in[\ell]} t_j 2^{-2j}$$
$$= 90\lambda\sqrt{\sqrt{\frac{\sqrt{d}}{\lambda}}\log\left(\frac{2(\lambda+1)}{\sqrt{d}}\right)}\sum_{j\in[\ell]} t_j 2^{-2j}.$$

By Lemma 2.3, we can chose an appropriate constant c_1 such that the above quantity is bounded by

$$\left(\frac{\lambda}{5} + 0.95 \cdot 10^{12}\sqrt{d}\right)\sum_{j\in[\ell]} t_j 2^{-2j}. \qquad \square$$

Claim 5.7.

$$\left|\sum_{(i,j)\in C_4} (2^{-i}u_i^T)A_s(2^{-j}v_j)\right| = 136\sqrt{d}\sum_{i\in[r]} s_i 2^{-2i}.$$

Proof. The sum is conditioned over the set of tuples (i,j) in C_4, where

$$C_4 = \left\{(i,j) | \left(i \leq j < i + \frac{1}{2}\log d\right) \wedge (s_i \leq t_j < ds_i) \right.$$
$$\left. \wedge \left(\frac{d}{\lambda}\sqrt{s_i t_j} < n\right) \wedge \left(s_i 2^{-2i} \geq \frac{\lambda}{\sqrt{d}} t_j 2^{-2j}\right)\right\}.$$

By the triangle inequality,

$$\left|\sum_{(i,j)\in C_4} (2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \leq \sum_{(i,j)\in C_4} 2^{-i-j}|u_i^T A_s v_j|.$$

We note that $u_i, v_j \neq \overline{0}$ since $s_i \leq t_j < ds_i$. We use Corollary 5.3 to bound each term $|u_i^T A_s v_j|$. We use Corollary 5.3 with the choice $a \leftarrow u_i$ and $b \leftarrow v_j$. This choice satisfies the conditions of Corollary 5.3 since $s_i \leq t_j \leq ds_i$, condition (I) of the lemma implies $t_j > n/d^2$, and $(d/\lambda)\sqrt{s_i t_j} < n$. Hence,

$$\left|\sum_{(i,j)\in C_4} (2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \leq \sum_{(i,j)\in C_4} 2^{-i-j}|u_i^T A_s v_j|$$
$$\leq 10\sum_{(i,j)\in C_4} 2^{-i-j}\sqrt{\lambda\sqrt{s_i t_j}t_j\log\left(\frac{2ds_i}{t_j}\right)}$$
$$= 10\sum_{(i,j)\in C_4} 2^{-i-j}\sqrt{\lambda}s_i\sqrt{\left(\frac{t_j}{s_i}\right)^{\frac{3}{2}}\log\left(\frac{2ds_i}{t_j}\right)}$$
$$\leq 10\sum_{(i,j)\in C_4} 2^{-i-j}\frac{d^{3/8}}{\lambda^{1/4}}s_i\sqrt{2^{3j-3i}\log\left(\frac{2\lambda\sqrt{d}}{2^{2j-2i}}\right)}.$$

Above we use the fact that $x^{\frac{3}{2}}\log(\frac{c}{x})$ is an increasing function if $x \le \frac{c}{2}$ and $s_i 2^{-2j} \ge \frac{\lambda}{\sqrt{d}} t_j 2^{-2j}$. Therefore,

$$\begin{aligned}
&\left|\sum_{(i,j)\in C_4} (2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \\
&\quad\le 10\sum_{i\in[r]} \frac{d^{3/8}}{\lambda^{1/4}} s_i 2^{-2i} \sum_{j=i}^{i+\frac{1}{2}\log d-1} \sqrt{2^{j-i}2\log\left(\frac{2\lambda\sqrt{d}}{2^{2j-2i}}\right)} \\
&\quad= 90\sum_{i\in[r]} \frac{d^{3/8}}{\lambda^{1/4}} s_i 2^{-2i} \sqrt{\sqrt{d}\log\left(\frac{2\lambda}{\sqrt{d}}\right)} \qquad \text{(by Lemma 2.2)} \\
&\quad= 90\sum_{i\in[r]} d^{\frac{1}{2}} s_i 2^{-2i} \sqrt{\sqrt{\frac{\sqrt{d}}{\lambda}}\log\left(\frac{2\lambda}{\sqrt{d}}\right)}.
\end{aligned}$$

By assumption, $n \ge 9d^2$. Hence, $\lambda \ge \sqrt{d\cdot\frac{n-d}{n-1}} > \frac{\sqrt{d}}{2}$. It can be verified that for $x > \frac{1}{2}$, $\frac{1}{x^{1/4}}(\log 2x)^{1/2} \le 1.502$. Substituting this bound in the above equation, we get

$$\left|\sum_{(i,j)\in C_4} (2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \le 136\sqrt{d}\sum_{i\in[r]} s_i 2^{-2i}. \qquad \square$$

CLAIM 5.8.

$$\left|\sum_{(i,j)\in C_5} (2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \le 56\sqrt{d}\left(\sum_{j\in[l]} t_j 2^{-2j} + \sum_{i\in[r]} s_i 2^{-2i}\right).$$

Proof. The sum is conditioned over the set of tuples (i,j) in C_5, where

$$\begin{aligned}
C_5 = \Bigg\{(i,j)|\left(i \le j < i+\frac{1}{2}\log d\right) \wedge (s_i \le t_j < ds_i) \\
\wedge \left(\frac{d}{\lambda}\sqrt{s_i t_j} \ge n\right) \wedge \left(s_i 2^{-2i} < t_j 2^{-2j}\right)\Bigg\}.
\end{aligned}$$

By the triangle inequality,

$$\left|\sum_{(i,j)\in C_5} (2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \le \sum_{j\in[\ell]:\exists i\in[r] \text{ with } (i,j)\in C_5} 2^{-2j}\left|\sum_{i:(i,j)\in C_5} 2^{-i+j}u_i^T A_s v_j\right|.$$

We note that $u_i, v_j \ne \overline{0}$ since $s_i \le t_j < ds_i$ for every $(i,j)\in C_5$. Let us fix j such that there exists $(i,j)\in C_5$. We bound

$$\left|\sum_{\substack{i\in\{j-(1/2)\log d,\dots,j\}:\\(i,j)\in C_5}} 2^{-i+j}u_i^T A_s v_j\right|$$

using Lemma 5.1. We will use Lemma 5.1 for the choice $a \leftarrow v_j$ and for every $k = 0, 1, \ldots, (1/2)\log d$, we take $b_k \leftarrow u_{j-k}$ if $(j-k, j) \in C_5$ and $b_k \leftarrow \overline{0}$ if $(j-k, j) \notin C_5$. This choice satisfies the conditions of Lemma 5.1 since (i) condition (I) of the lemma implies the $S(b_k)$ are mutually nonintersecting, (ii) $|S(v_j)| = t_j \geq 2^{2j-2i}s_i = 2^{2j-2i}|S(u_i)|$ for every $(i,j) \in C_5$ implies $|S(a)| \geq 2^{2k}|S(b_k)|$ for every $k = 0, 1, \ldots, (1/2)\log d$, and (iii) b_k is nonzero iff $(j-k, j) \in C_5$ implies $(d/\lambda)\sqrt{|S(b_k)||S(a)|} \geq n$ for every nonzero b_k. Hence, by Lemma 5.1, we have

$$\sum_{j\in[\ell]} 2^{-2j} \left| \sum_{i:(i,j)\in C_5} 2^{-i+j} u_i^T A_s v_j \right| \leq 14 \sum_{j\in[\ell]} 2^{-2j} \sqrt{\frac{dt_j^2}{n} \sum_{i=j-\frac{1}{2}\log d}^{j} s_i 2^{-2i+2j} \log\left(\frac{2n}{t_j}\right)}$$

$$= 14\sqrt{d} \sum_{j\in[\ell]} \sqrt{2^{-2j}\frac{t_j^2}{n} \log\left(\frac{2n}{t_j}\right) \sum_{i=j-\frac{1}{2}\log d}^{j} s_i 2^{-2i}}.$$

Next, we group the v_j according to their support sizes and then sum them together. For $c = 0, 1, 2, \ldots, \log(n)$, let J_c be the set of indices $j \in [\ell]$ s.t. $n/2^c \leq t_j < 2n/2^c$ and for nonempty sets J_c, define $j_c := \min(j \in J_c)$. With this notation, the above sum is

$$\leq 14\sqrt{d} \sum_{c=0}^{\log n} \sum_{j\in J_c} \sqrt{4n2^{-2j-2c}\log(2\cdot 2^c) \sum_{i=j-1/2\log d+1}^{i=j} s_i 2^{-2i}} \quad \left(\text{since } \frac{n}{2^c} \leq t_j < \frac{2n}{2^c}\right)$$

$$\leq 14\sqrt{d} \sum_{c=0}^{\log n} \sum_{j\in J_c} \frac{1}{2}\left(4n2^{-j-j_c-c} + 2^{-j+j_c-c}\log(2\cdot 2^c) \sum_{i=j-1/2\log d+1}^{i=j} s_i 2^{-2i}\right)$$

(since geometric mean is at most arithmetic mean)

$$= 28\sqrt{d} \sum_{c=0}^{\log n} \sum_{j\in J_c} n2^{-j-j_c-c} + 7\sqrt{d} \sum_{c=0}^{\log n} \sum_{j\in J_c} \sum_{i=j-1/2\log d+1}^{i=j} 2^{-j+j_c-c}\log(2\cdot 2^c) s_i 2^{-2i}$$

$$\leq 28\sqrt{d} \sum_{c=0}^{\log n} \sum_{j\in J_c} \frac{n}{2^c} 2^{-j-j_c} + 7\sqrt{d} \sum_{i\in[r]} s_i 2^{-2i} \sum_{c=0}^{\log n} \frac{\log(2\cdot 2^c)}{2^c} \sum_{j\in J_c} 2^{-j+j_c}.$$

We observe that

$$\sum_{c=0}^{\log n} \sum_{j\in J_c} \frac{n}{2^c} 2^{-j-j_c} \leq \sum_{c=0}^{\log n} \sum_{j\in J_c} t_j 2^{-j-j_c} \leq \sum_{c=0}^{\log n} \sum_{j\in J_c} t_j 2^{-2j} = \sum_{j\in[\ell]} t_j 2^{-2j}.$$

Moreover, $\sum_{j\in J_c} 2^{-j+j_c} \leq 2$ and $\sum_{c=0}^{\log n} \frac{\log(2.2^c)}{2^c} \leq 4$. Substituting these we have the claim. ∎

Claim 5.9.

$$\left| \sum_{(i,j)\in C_6} (2^{-i}u_i^T) A_s (2^{-j}v_j) \right| = 154\sqrt{d} \sum_{i\in[r]} s_i 2^{-2i}.$$

Proof. The sum is conditioned over the set of tuples (i,j) in C_6, where

$$C_6 = \left\{(i,j) | \left(i \le j \le i + \frac{1}{2}\log d\right) \wedge (s_i \le t_j < ds_i) \right.$$
$$\left. \wedge \left(\frac{d}{\lambda}\sqrt{s_i t_j} \ge n\right) \wedge \left(s_i 2^{-2i} \ge t_j 2^{-2j}\right) \right\}.$$

By the triangle inequality,

$$\left|\sum_{(i,j)\in C_6} (2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \le \sum_{(i,j)\in C_6} 2^{-i-j}|u_i^T A_s v_j|.$$

We will use Lemma 5.1 to bound each term $|u_i^T A_s v_j|$. We use Lemma 5.1 with the choice $a \leftarrow v_j, b_0 \leftarrow u_i$. This choice satisfies the conditions of Lemma 5.1 since $s_i \le t_j < ds_i$ and $(d/\lambda)\sqrt{s_i t_j} \ge n$. Hence,

$$\sum_{(i,j)\in C_6} 2^{-i-j}|u_i^T A_s v_j| \le 14 \sum_{(i,j)\in C_6} 2^{-i-j}\sqrt{\frac{ds_i t_j^2}{n}\log\left(\frac{2n}{t_j}\right)}.$$

Next, we divide the tuples in C_6 into two parts depending on the values of i and j:

$$C_6' := \left\{(i,j)|(i,j)\in C_6, \left(i \le j < i + \frac{1}{2}\log(n/s_i)\right)\right\} \text{ and}$$
$$C_6'' := \left\{(i,j)|(i,j)\in C_6, \left(j \ge i + \frac{1}{2}\log(n/s_i)\right)\right\}.$$

Let us consider the above RHS sum over tuples (i,j) in C_6':

$$\begin{aligned}
&14 \sum_{(i,j)\in C_6'} 2^{-i-j}\sqrt{\frac{ds_i t_j^2}{n}\log\left(\frac{2n}{t_j}\right)} \\
&= 14\sqrt{d} \sum_{(i,j)\in C_6'} 2^{-2i} s_i \sqrt{2^{-2j+2i}\frac{1}{ns_i}t_j^2 \log\left(\frac{2n}{t_j}\right)} \\
&\le 14\sqrt{d} \sum_{(i,j)\in C_6'} s_i 2^{-2i}\sqrt{\frac{s_i 2^{2j-2i}}{n}\log\left(\frac{2n}{s_i 2^{2j-2i}}\right)} \qquad (t_j 2^{-2j} \le s_i 2^{-2i}) \\
&\le 14\sqrt{d}\sum_{i\in[r]} s_i 2^{-2i} \sum_{j=i}^{j=i+\frac{1}{2}\log(\frac{n}{s_i})} \sqrt{\frac{s_i 2^{2j-2i}}{n}\log\left(\frac{2n}{s_i 2^{2j-2i}}\right)} \\
&\le 126\sqrt{d}\sum_{i\in[r]} s_i 2^{-2i}.
\end{aligned}$$

In the above, the last inequality is by using Lemma 2.2 for $\sum_{j=i}^{j=i+\frac{1}{2}\log(\frac{n}{s_i})}$

$\sqrt{\frac{s_i 2^{2j-2i}}{n}\log(\frac{2n}{s_i 2^{2j-2i}})}$. Next, let us consider the RHS sum over tuples (i,j) in C_6'':

$$\begin{aligned}
&14\sum_{(i,j)\in C_6''} 2^{-i-j}\sqrt{\frac{ds_i t_j^2}{n}\log\left(\frac{2n}{t_j}\right)}\\
&= 14\sqrt{d}\sum_{(i,j)\in C_6''} s_i 2^{-i-j}\sqrt{\frac{t_j}{s_i}}\sqrt{\frac{t_j}{n}\log\left(\frac{2n}{t_j}\right)}\\
&\le 14\sqrt{d}\sum_{(i,j)\in C_6''} s_i 2^{-i-j}\sqrt{\frac{n}{s_i}} && \left(t_j\le n, x\log\frac{2}{x}\le 1\right)\\
&\le 14\sqrt{d}\sum_{i\in[r]} s_i 2^{-2i}\sum_{j=i+\frac{1}{2}\log(n/s_i)}^{\infty} 2^{-j+i}\sqrt{\frac{n}{s_i}}\\
&\le 28\sqrt{d}\sum_{i\in[r]} s_i 2^{-2i}.
\end{aligned}$$

The claim follows from the above two bounds. ∎

We now obtain the required bound for conclusion 1 of the lemma from Claims 5.4, 5.5, 5.6, 5.7, 5.8, and 5.9:

$$\left|\sum_{(i,j)\in[r]\times[\ell]}(2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \le 377\max\left(\sqrt{\lambda\log d},\sqrt{d}\right)\sum_{i\in[r]} s_i 2^{-2i} + \left(\frac{\lambda}{5}+10^{12}\sqrt{d}\right)\sum_{j\in[\ell]} t_j 2^{-2j}.$$

For conclusion 2 of the lemma, we observe that if $s_i \ge t_j$ for all $i\in[r], j\in[\ell]$, then C_3, C_4, C_5, C_6 are empty. Thus the bound follows from Claims 5.4 and 5.5:

$$\left|\sum_{(i,j)\in[r]\times[\ell]}(2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \le 31\max\left(\sqrt{\lambda\log d},\sqrt{d}\right)\left(\sum_{i\in[r]} s_i 2^{-2i}+\sum_{j\in[l]} t_j 2^{-2j}\right). \quad ∎$$

6. Proof of Theorem 1.2.

In this section, we prove Theorem 1.2. We need the following modified version of Lemma 4.2.

LEMMA 6.1. *Let G be a d-regular n-vertex graph, where $2\le d\le\sqrt{n/3\ln n}$, with largest (in magnitude) nontrivial eigenvalue λ, and let A be the adjacency matrix of G. Let A' be a random $n\times n$ real matrix whose entries $A'(i,j)$ are random variables with mean 0, $|A'(i,j)|\le A(i,j)$ for all i,j, and the entries $A'(i,j)$ are independent of all other entries except $A'(j,i)$. There exist constants $c_1, c_2\ge 1000, c_3, c_4$ such that the following statements hold with probability at least $1-e^{-(n/d^2)}$ (over the random choice of A').*

1. *For all $u_1,u_2,\ldots,u_r\in\{0,\pm1,\pm2^{-1}\}^n$, $v_1,v_2\ldots,v_\ell\in\{0,\pm1,\pm2^{-1}\}^n$ satisfying*

(I) *$S(u_i) \cap S(u_j) = \phi$ for every $i, j \in [r]$ and $S(v_i) \cap S(v_j) = \phi$ for every $i, j \in [\ell]$, and*

(II) *either $|S(u_i)| > n/d^2$ for every $i \in [r]$ with nonzero u_i, or $|S(v_i)| > n/d^2$ for every $i \in [\ell]$ with nonzero v_i,*

we have

$$\left|\sum_{i\le j}(2^{-i}u_i^T)A'(2^{-j}v_j)\right| \le c_1 \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)\sum_{i=1}^{r}|S(u_i)|2^{-2i} + \left(\frac{\lambda}{c_2} + c_3\sqrt{d}\right)\sum_{j=1}^{\ell}|S(v_j)|2^{-2j}.$$

2. *For all $u_1, u_2, \ldots, u_r \in \{0, \pm 1, \pm 2^{-1}\}^n$, $v_1, v_2 \ldots, v_\ell \in \{0, \pm 1, \pm 2^{-1}\}^n$ satisfying* (I), (II), *and*

(III) *$|S(u_i)| > |S(v_j)|$ for every $i \in [r], j \in [\ell]$ with nonzero u_i,*

we have

$$\left|\sum_{i\le j}(2^{-i}u_i^T)A'(2^{-j}v_j)\right| \le c_4 \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)\left(\sum_{i=1}^{r}|S(u_i)|2^{-2i} + \sum_{j=1}^{\ell}|S(v_j)|2^{-2j}\right).$$

The proof of Lemma 6.1 is identical to that of Lemma 4.2. In the proof of Lemma 4.2, we used the concentration inequalities from Lemma 5.1 and Corollary 5.3. We note that these concentration inequalities were obtained using Hoeffding's inequality. Since Hoeffding's inequality is applicable when the random variables are bounded, we have the version of Lemma 5.1 and Corollary 5.3 applicable to the random matrix A'. As a consequence, we obtain Lemma 6.1 by following the same proof as that of Lemma 4.2. We avoid repeating the proof in the interests of brevity.

Proof of Theorem 1.2. The proof is very similar to the proof of Theorem 4.1. However, in order to avoid a loss of factor 4, we avoid discretizing in the first step, but discretize only for certain cases. Using Lemma 2.10, we know that for a shift k-lift, λ_{new} is the maximum absolute value in the set

$$\bigcup_{\omega:\ \omega \text{ is a } k\text{th primitive root of unity},\ \omega\neq 1} \text{eigenvalues}\left(A_s(\omega)\right).$$

We will bound the probability that the maximum eigenvalue of $A_s(\omega)$ is large for ω being a fixed primitive kth root of unity. A union bound over the $k-1$ primitive kth roots of unity bounds the maximum eigenvalues of all $k-1$ matrices simultaneously.

Let us fix ω to be a primitive kth root of unity and bound the eigenvalues of $A_s(\omega)$. We need to bound $\max_{x\in\mathbb{C}^n}|x^*A_s(\omega)x|/x^*x$, where x^* denotes the complex conjugate of vector x. Let $x = q + iw \in \mathbb{C}^n$, where $q, w \in \mathbb{R}^n$. By rescaling we may assume that the absolute value of every coordinate of x is at most $1/2$. We consider a decomposition of q, w (similar to but not the same as the dyadic decomposition) into a sequence of vectors y_i's and z_i's for $i = 0, 1, \ldots$, respectively as follows:

$$[y_i]_j := \begin{cases} q_j & \text{if } 2^{-i-1} < |q_j| \le 2^{-i}, \\ 0 & \text{otherwise,} \end{cases}$$

$$[z_i]_j := \begin{cases} w_j & \text{if } 2^{-i-1} < |w_j| \le 2^{-i}, \\ 0 & \text{otherwise.} \end{cases}$$

Let us partition the set of indices $\{0, 1, \ldots\}$ into two sets $M_r := \{i : |S(y_i)| < n/d^2\}$ and $L_r := \{i : |S(y_i)| \ge n/d^2\}$ and define $y_{M_r} := \sum_{i \in M_r} y_i$ and $y_{L_r} := \sum_{i \in L_r} y_i$. Similarly, define M_c and L_c based on the support of z_i's and define z_{M_c} and z_{L_c}. We will refer to vectors y_{M_r}, z_{M_c} as "type M" vectors, and y_{L_r} and z_{L_c} as "type L" vectors. We note that

$$x^*x = \|y_{M_r}\|^2 + \|y_{L_r}\|^2 + \|z_{M_c}\|^2 + \|z_{L_c}\|^2.$$

By splitting the terms in $|x^* A_s(\omega)x|$, we get

$$\begin{aligned} |x^* A_s(\omega)x| \le{} & |(y_{M_r} + iz_{M_c})^* A_s(\omega)(y_{M_r} + iz_{M_c})| + |z_{L_c}^T A_s(\omega) y_{L_r}| + |y_{L_r}^T A_s(\omega) z_{L_c}| \\ & + |y_{L_r}^T A_s(\omega) y_{L_r}| + |y_{L_r}^T A_s(\omega) y_{M_r}| + |y_{M_r}^T A_s(\omega) y_{L_r}| \\ & + |z_{L_c}^T A_s(\omega) z_{L_c}| + |z_{L_c}^T A_s(\omega) z_{M_c}| + |z_{M_c}^T A_s(\omega) z_{L_c}| \\ & + |y_{L_r}^T A_s(\omega) z_{M_c}| + |z_{M_c}^T A_s(\omega) y_{L_r}| + |z_{L_c}^T A_s(\omega) y_{M_r}| + |y_{M_r}^T A_s(\omega) z_{L_c}|. \end{aligned} \tag{6.1}$$

To derive an upper bound on $|x^* A_s(\omega)x|$, we will show upper bounds for each of the terms in the RHS using Lemma 6.1. We note that the concentration inequalities given in parts 1 and 2 of Lemma 6.1 hold with probability at least $1 - e^{-n/d^2}$ for some constants $c_1, c_2 \ge 1000, c_3, c_4$. Assuming parts 1 and 2 of Lemma 6.1, we will show the following claims.

Claim 6.2.

$$|(y_{M_r} + iz_{M_c})^* A_s(\omega)(y_{M_r} + iz_{M_c})| \le \left(\lambda + \frac{128}{d}\right) \|y_{M_r} + iz_{M_c}\|^2.$$

Claim 6.3. *For every pair of type L vectors a and b,*

$$|a^T A_s(\omega) b| \le \left(\frac{32\lambda}{c_2} + 32(c_1 + c_3)\left(\max\left(\sqrt{\lambda \log(d)}, \sqrt{d}\right)\right)\right)\left(\|a\|^2 + \|b\|^2\right).$$

Claim 6.4. *For every vector a of type M and every vector b of type L,*

$$|a^T A_s(\omega) b| \le \frac{32\lambda}{c_2}\|b\|^2 + 32(c_1 + c_3 + c_4)\left(\max\left(\sqrt{\lambda \log(d)}, \sqrt{d}\right)\right)\left(\|b\|^2 + \|a\|^2\right).$$

We note that all terms in the RHS of inequality (6.1) fall into one of the three categories given in Claims 6.2, 6.3, and 6.4 above. Using these bounds, the following holds with probability at least $1 - e^{-(n/d^2)}$:

$$\begin{aligned} & |x^* A_s(\omega)x| \\ & \le \left(\lambda + \frac{128}{d}\right)\|y_{M_r} + iz_{M_c}\|^2 + \frac{256\lambda}{c_2}\left(\|y_{L_r}\|^2 + \|z_{L_c}\|^2\right) \\ & \quad + 256(c_1 + c_3 + c_4)\left(\max\left(\sqrt{\lambda \log(d)}, \sqrt{d}\right)\right)\left(\|y_{M_r}\|^2 + \|z_{M_c}\|^2 + \|y_{L_r}\|^2 + \|z_{L_c}\|^2\right) \\ & \le \left(\lambda + 288(c_1 + c_3 + c_4)\left(\max\left(\sqrt{\lambda \log(d)}, \sqrt{d}\right)\right)\right) x^* x. \end{aligned}$$

The last inequality is because $c_2 \geq 1000$ and $d \geq 2$. Taking a union bound over the $k-1$ primitive roots of unity shows that there exists a constant c such that with probability at least $1 - ke^{-(n/d^2)}$, all new eigenvalues of a random shift k-lift have absolute value at most

$$\lambda + c \max\left(\sqrt{\lambda \log(d)}, \sqrt{d}\right). \qquad \square$$

Proof of Claim 6.2. We observe that $|(y_{M_r} + iz_{M_c})^* A_s(\omega)(y_{M_r} + iz_{M_c})| \leq y'^T A y'$, where y' is a real vector whose jth coordinate is equal to the absolute value of the jth coordinate in $y_{M_r} + iz_{M_c}$ and A is the adjacency matrix of the base graph. Let $J = vv^T$ and $J' = v'v'^T$, where v is the all-ones vector and v' is defined as $v'_i = 1$ for $i \in \{1, \ldots, n/2\}$ and $v'_i = -1$ for $i \in \{n/2+1, \ldots, n\}$. For nonbipartite graph G, we have

$$\begin{aligned} y'^T A y' &= y'^T \left(A - \frac{d}{n}J\right) y' + y'^T \left(\frac{d}{n}J\right) y' \leq \lambda \|y'\|^2 + y'^T \left(\frac{d}{n}J\right) y' \\ &= \lambda \|(y_{M_r} + iz_{M_c})\|^2 + y'^T \left(\frac{d}{n}J\right) y'. \end{aligned}$$

Above, we have used the fact that the maximum eigenvalue of $A - (\frac{d}{n}J)$ is λ. Similarly, for bipartite graphs, we have

$$\begin{aligned} y'^T A y' &= y'^T \left(A - \frac{d}{n}J + \frac{d}{n}J'\right) y' + y'^T \left(\frac{d}{n}J\right) y' - y'^T \left(\frac{d}{n}J'\right) y' \\ &\leq \lambda \|y'\|^2 + y'^T \left(\frac{d}{n}J\right) y' - y'^T \left(\frac{d}{n}J'\right) y' \\ &\leq \lambda \|(y_{M_r} + iz_{M_c})\|^2 + y'^T \left(\frac{2d}{n}J\right) y'. \end{aligned}$$

It remains to bound $|y'^T \frac{d}{n} J y'|$. Let y'_{M_r} and z'_{M_c} be vectors obtained by taking the absolute values of the coordinates of y_{M_r} and z_{M_c}, respectively. We have

$$y'^T \left(\frac{d}{n}J\right) y' \leq (y'_{M_r} + z'_{M_c})^T \left(\frac{d}{n}J\right) (y'_{M_r} + z'_{M_c}).$$

We recall that the number of entries between 2^{-i-1} and 2^{-i} in y'_{M_r} and z'_{M_c} are less than $\frac{n}{d^2}$. We will show that $|u^T(\frac{d}{n})Jv| \leq \frac{4}{d}(\|u\|^2 + \|v\|^2)$, where $u, v \in \{y'_{M_r}, z'_{M_c}\}$.

Let $u, v \in \{y'_{M_r}, z'_{M_c}\}$. By Lemma 2.1, there exist u', v' s.t. $|u^T \frac{d}{n} J v| \leq |u'^T \frac{d}{n} J v'|$, where $u', v' \in \{0, \pm 2^{-1}, \pm 2^{-2}, \ldots\}^n$, $\|u'\|^2 \leq 4\|u\|^2$, and $\|v'\|^2 \leq 4\|v\|^2$. Consider the dyadic decomposition of $u' = \sum_{i=0}^{\infty} 2^{-i} u_i$ obtained as follows: a coordinate of u_i is 1 if the corresponding coordinate of u' is 2^{-i}, it is -1 if the corresponding coordinate of u' is -2^{-i}, and is 0 otherwise. Similarly define the dyadic decomposition of $v' = \sum_{j=0}^{\infty} 2^{-j} v_j$. We note that all entries between 2^{-i-1} and 2^{-i} in u and v are rounded to either 2^{-i-1} or 2^{-i} in u' and v' and all entries between -2^{-i-1} and -2^{i} are rounded to either -2^{-i-1} or -2^{-i}. Since the number of entries in u, v with absolute value between 2^{-i-1} and 2^{-i} is at most n/d^2, we get $|S(u_i)|, |S(v_j)| < \frac{2n}{d^2}$ for all i, j.

Thus,

$$\begin{aligned}
\left|u'\left(\frac{d}{n}J\right)v'\right| &= \left|\sum_{i,j=0}^{\infty} 2^{-i-j}u_i^T\left(\frac{d}{n}J\right)v_j\right| \\
&\leq \sum_{i=0}^{\infty}\sum_{j=i}^{\infty} 2^{-i-j}\frac{d}{n}|u_i^TJv_j| + \sum_{j=0}^{\infty}\sum_{i=j+1}^{\infty} 2^{-i-j}\frac{d}{n}|u_i^TJv_j| \\
&\leq \sum_{i=0}^{\infty}\sum_{j=i}^{\infty} 2^{-i-j}\frac{d|S(u_i)||S(v_j)|}{n} + \sum_{j=0}^{\infty}\sum_{i=j+1}^{\infty} 2^{-i-j}\frac{d|S(v_j)||S(u_i)|}{n} \\
&\leq 2\sum_{i=0}^{\infty} 2^{-2i}\frac{|S(u_i)|}{d}\sum_{j=i}^{\infty} 2^{-j+i} + 2\sum_{j=0}^{\infty} 2^{-2j}\frac{|S(v_j)|}{d}\sum_{i=j+1}^{\infty} 2^{-i+j} \\
&\leq \frac{4}{d}\left(\sum_{i=0}^{\infty}|S(u_i)|2^{-2i} + \sum_{j=0}^{\infty}|S(v_j)|2^{-2j}\right) \\
&\leq \frac{4}{d}\left(\|u'\|^2 + \|v'\|^2\right).
\end{aligned}$$

For $u, v \in \{y'_{M_r}, z'_{M_c}\}$, $|u^T(\frac{d}{n}J)v| \leq |u'^T(\frac{d}{n}J)v'| \leq \frac{4}{d}(\|u'\|^2 + \|v'\|^2) \leq \frac{16}{d}(\|u\|^2 + \|v\|^2)$. Therefore,

$$\begin{aligned}
&y'^T\left(\frac{d}{n}J\right)y' \\
&\quad\leq (y'_{M_r} + z'_{M_c})^T\left(\frac{d}{n}J\right)(y'_{M_r} + z'_{M_c}) \\
&\quad\leq y'^T_{M_r}\left(\frac{d}{n}J\right)y'_{M_r} + y'^T_{M_r}\left(\frac{d}{n}J\right)z'_{M_c} + z'^T_{M_c}\left(\frac{d}{n}J\right)y'_{M_r} + z'^T_{M_c}\left(\frac{d}{n}J\right)z'_{M_c} \\
&\quad\leq \frac{16}{d}\left(\|y'_{M_r}\|^2 + \|y'_{M_r}\|^2 + \|y'_{M_r}\|^2 + \|z'_{M_c}\|^2 + \|z'_{M_c}\| + \|y'_{M_r}\|^2 + \|z'_{M_c}\| + \|z'_{M_c}\|\right) \\
&\quad= \frac{64}{d}\left(\|y'_{M_r}\|^2 + \|z'_{M_c}\|^2\right) = \frac{64}{d}\left(\|y_{M_r}\|^2 + \|z_{M_c}\|^2\right) = \frac{64}{d}\|y_{M_r} + iz_{M_c}\|^2. \qquad \square
\end{aligned}$$

Thus, we have

$$|(y_{M_r} + iz_{M_c})^* A_s(\omega)(y_{M_r} + iz_{M_c})| \leq y'Ay' \leq (\lambda + (128/d))\|(y_{M_r} + iz_{M_c})\|^2.$$

In order to show Claims 6.3 and 6.4, we divide the matrix into its real and imaginary parts: $A_s(\omega) = A_s^1(\omega) + iA_s^2(\omega)$, where $A_s^1(\omega)$ and $A_s^2(\omega)$ are real matrices. For any two vectors $a, b \in \mathbb{R}^n$,

$$|a^T A_s(\omega)b| \leq |a^T A_s^1(\omega)b| + |a^T A_s^2(\omega)b|.$$

We will bound $|a^T A'_s(\omega)b|$, where $A'_s(\omega) \in \{A_s^1(\omega), A_s^2(\omega)\}$ for a, b as in Claims 6.3 and 6.4. We start by discretizing a and b. By Lemma 2.1, there exist a', b' such that $|a^T A'_s(\omega)b| \leq |a'^T A'_s(\omega)b'|$, where $a', b' \in \{0, \pm 2^{-1}, \pm 2^{-2}, \dots\}^n$ and $\|a'\|^2 \leq 4\|a\|^2$ and $\|b'\|^2 \leq 4\|b\|^2$. Moreover, every entry of a and b between 2^{-i-1} and 2^{-i} is rounded to either 2^{-i-1} or 2^{-i} in a' and b', respectively (similarly, every entry between -2^{-i-1} and -2^{-i} is rounded to either -2^{-i-1} or -2^{-i}). Consider the following vectors

$\{u_i\}_{i\in\{0,1,\ldots\}}$, $\{v_i\}_{i\in\{0,1,\ldots\}}$ obtained from a', a and b, b', respectively:

$$[u_i]_j := \begin{cases} 2^i a'_j & \text{if } 2^{-i-1} \le |a_j| < 2^{-i}, \\ 0 & \text{otherwise,} \end{cases}$$

$$[v_i]_j := \begin{cases} 2^i b'_j & \text{if } 2^{-i-1} \le |b_j| < 2^{-i}, \\ 0 & \text{otherwise.} \end{cases}$$

We observe that $u_i, v_i \in \{0, \pm 2^{-1}, \pm 1\}^n$, $|a'^T A'_s(\omega) b'| = |\sum_{i,j=0}^{\infty} 2^{-i-j} u_i^T A'_s(\omega) v_j|$, $\|a'\|^2 = \sum_i 2^{-2i} \|u_i\|^2 \ge \frac{1}{4} \sum_i 2^{-2i} |S(u_i)|$, and

$$\left| \sum_{i,j=0}^{\infty} 2^{-i-j} u_i^T A'_s(\omega) v_j \right| \le \left| \sum_{i \le j} 2^{-i-j} u_i^T A'_s(\omega) v_j \right| + \left| \sum_{i<j} 2^{-i-j} v_i^T A'_s(\omega) u_j \right|.$$

Proof of Claim 6.3. Since a and b are type L vectors, we have $|S(u_i)|, |S(v_j)| \ge \frac{n}{d^2}$ for all nonzero u_i, v_j. By part 1 of Lemma 6.1,

$$\begin{aligned} &\left| \sum_{i \le j} 2^{-i-j} u_i^T A'_s(\omega) v_j \right| \\ &\quad \le c_1 \left(\max\left(\sqrt{\lambda \log(d)}, \sqrt{d} \right) \right) \sum_{i=0}^{\infty} |S(u_i)| 2^{-2i} + \left(\frac{\lambda}{c_2} + c_3 \sqrt{d} \right) \sum_{j=0}^{\infty} |S(v_j)| 2^{-2j}, \\ &\left| \sum_{i<j} 2^{-i-j} v_i^T A'_s(\omega) u_j \right| \\ &\quad \le c_1 \left(\max\left(\sqrt{\lambda \log(d)}, \sqrt{d} \right) \right) \sum_{i=0}^{\infty} |S(v_i)| 2^{-2i} + \left(\frac{\lambda}{c_2} + c_3 \sqrt{d} \right) \sum_{j=0}^{\infty} |S(u_j)| 2^{-2j}. \end{aligned}$$

Combining the above two we get

$$\begin{aligned} &\left| a'^T A'_s(\omega) b' \right| \\ &\quad = \left| \sum_{i,j=0}^{\infty} 2^{-i-j} u_i^T A_s(\omega) v_j \right| \\ &\quad \le \left(\frac{\lambda}{c_2} + (c_1 + c_3) \left(\max\left(\sqrt{\lambda \log(d)}, \sqrt{d} \right) \right) \right) \left(\sum_{i=0}^{\infty} |S(u_i)| 2^{-2i} + \sum_{j=0}^{\infty} |S(v_j) 2^{-2j} \right) \\ &\quad \le \left(\frac{4\lambda}{c_2} + 4(c_1 + c_3) \left(\max\left(\sqrt{\lambda \log(d)}, \sqrt{d} \right) \right) \right) (\|a'\|^2 + \|b'\|^2). \end{aligned}$$

Hence,

$$\begin{aligned} \left| a^T A'_s(\omega) b \right| &\le |y^T A'_s(\omega) z| \\ &\le \left(\frac{4\lambda}{c_2} + 4(c_1 + c_3) \left(\max\left(\sqrt{\lambda \log(d)}, \sqrt{d} \right) \right) \right) (\|a'\|^2 + \|b'\|^2) \\ &\le \left(\frac{16\lambda}{c_2} + 16(c_1 + c_3) \left(\max\left(\sqrt{\lambda \log(d)}, \sqrt{d} \right) \right) \right) \left(\|a\|^2 + \|b\|^2 \right). \end{aligned}$$

Therefore,

$$\begin{aligned}\left|a^T A_s(\omega)b\right| &\le |a^T A_s^1(\omega)b| + |a^T A_s^2(\omega)b| \\ &\le \left(\frac{32\lambda}{c_2} + 32(c_1+c_3)\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\right)\left(\|a\|^2 + \|b\|^2\right). \qquad \square\end{aligned}$$

Proof of Claim 6.4. Since, a is a vector of type M, b is a vector of type L, we have $|S(u_i)| < \frac{n}{d^2} \le |S(v_j)|$ for all nonzero v_j. Applying parts 1 and 2 of Lemma 6.1, we get

$$\begin{aligned}&\left|\sum_{i\le j} 2^{-i-j} u_i^T A_s'(\omega) v_j\right| \\ &\quad \le c_1\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\sum_{i=0}^{\infty}|S(u_i)|2^{-2i} + \left(\frac{\lambda}{c_2} + c_3\sqrt{d}\right)\sum_{j=0}^{\infty}|S(v_j)|2^{-2j}, \\ &\left|\sum_{i<j} 2^{-i-j} v_i^T A_s'(\omega) u_j\right| \\ &\quad \le c_4\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\left(\sum_{i=0}^{\infty}|S(v_i)|2^{-2i} + \sum_{j=0}^{\infty}|S(u_j)|2^{-2j}\right).\end{aligned}$$

Combining the above two, we get

$$\begin{aligned}&\left|a'^T A_s'(\omega) b'\right| \\ &\quad = \left|\sum_{i,j} 2^{-i-j} u_i^T A_s(\omega) v_j\right| \\ &\quad \le \frac{\lambda}{c_2}\sum_j |S(v_j)|2^{-2j} \\ &\qquad + (c_1+c_3+c_4)\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\left(\sum_j |S(v_j)|2^{-2j} + \sum_i |S(u_i)|2^{-2i}\right) \\ &\quad \le \frac{4\lambda}{c_2}\|b'\|^2 + 4(c_1+c_3+c_4)\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\left(\|b'\|^2 + \|a'\|^2\right).\end{aligned}$$

Hence,

$$\begin{aligned}\left|a^T A_s'(\omega) b\right| &\le \left|a'^T A_s'(\omega) b'\right| \\ &\le \frac{4\lambda}{c_2}\|b'\|^2 + 4(c_1+c_3+c_4)\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\left(\|b'\|^2 + \|a'\|^2\right) \\ &\le \frac{16\lambda}{c_2}\|b\|^2 + 16(c_1+c_3+c_4)\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\left(\|b\|^2 + \|a\|^2\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\left|a^T A_s(\omega)b\right| &\leq \left|a^T A_s^1(\omega)b\right| + \left|a^T A_s^2(\omega)b\right| \\ &\leq \frac{32\lambda}{c_2}\|b\|^2 + 32(c_1+c_3+c_4)\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\left(\|b\|^2 + \|a\|^2\right). \quad \square\end{aligned}$$

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