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Positive curvature and symmetry in small dimensions

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Extending existing work in small dimensions, Dessai computed the Euler characteristic, signature, and elliptic genus for 8-manifolds of positive sectional curvature in the presence of torus symmetry. He also computes the diffeomorphism type by restricting his results to classes of manifolds known to admit non-negative curvature, such as biquotients. The first part of this paper extends Dessai's calculations to even dimensions up to 16. In particular, we obtain a first characterization of the Cayley plane in such a setting. The second part studies a closely related family of manifolds called positively elliptic manifolds, and we prove a conjecture of Halperin in this context for dimensions up to 16 or Euler characteristics up to 16.

Keywords: Positive sectional curvature; torus symmetry; Euler characteristic; elliptic genus; biquotient; Halperin conjecture.

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0. Introduction

In this paper, we prove topological obstructions to the existence of Riemannian metrics with positive sectional curvature and large symmetry. This is part of a well-established symmetry program, which was initiated by Grove in the 1990s and has led to new obstructions in the presence of symmetry, the construction of many interesting examples, and the development of new tools for studying Riemannian manifolds with curvature bounds and symmetry. For surveys, we refer the reader to [29, 60, 62, 63].

In this paper, we focus on torus symmetry, as it is perhaps the most understood. Three fundamental results in this area are due to Grove and Searle [30], Fang and Rong [19], and Wilking [59]. They prove, respectively, equivariant diffeomorphism, homeomorphism, and homotopy and cohomology classifications for positively curved manifolds with torus symmetry, where the rank of the torus action is bounded from below by a constant that only depends on the dimension of the manifold.

While these results provide strong classifications that hold in arbitrary dimensions, they do not always reduce to the best known results in small dimensions. For example, in dimensions 2 and 3, positive sectional curvature by itself already implies that the manifold is diffeormorphic to a sphere. This follows from the Gauss–Bonnet theorem, the classification of surfaces, and Hamilton's work on the Ricci flow. Furthermore, in dimension 4, the best result is due to Hsiang–Kleiner [36] and Grove–Wilking [32] (cf. [23, 25, 47]), while in dimension 5, the best result is due to Rong [50] (cf. [24, 26, 53]). In dimensions 6 and 7, the above results are the best known, however there remains a large gap in our understanding due to the vast number of known positively curved examples (see [9, 31, 49, 62]).

This paper focuses on even dimensions 8 through 16. Our philosophy is motivated by Dessai [12], which considers positively curved manifolds with torus symmetry in dimension 8. The result of Fang and Rong implies that a closed, simply connected, positively curved 8-manifold with T^3 symmetry is homeomorphic to $\mathbb{S}^8$, $\mathbb{C}\mathrm{P}^4$, or $\mathbb{H}\mathrm{P}^2$, i.e. to one of the manifolds known to admit a positively curved metric. Dessai studies precisely this problem, except that he only assumes T^2 symmetry, and he computes a number of topological invariants, including the Euler characteristic $\chi(M)$ and signature $\sigma(M)$. These computations provide obstructions to the existence of positively curved Riemannian metrics on 8-manifolds with T^2 symmetry.

Theorem ([12]). *If M^8 is a closed, simply connected Riemannian manifold with positive sectional curvature and T^2 symmetry, then one of the following occurs*:

- $\chi(M) = 2$, $\sigma(M) = 0$, *and, if M is spin, the elliptic genus vanishes.*
- $\chi(M) = 3$, $\sigma(M) = \pm 1$, *and, if M is spin, the elliptic genus is constant.*
- $\chi(M) = 5$, $\sigma(M) = \pm 1$, *and M is not spin.*

Dessai also restricts his results to well-studied classes of non-negatively curved manifolds (e.g. biquotients and certain cohomogeneity one manifolds), and for such manifolds his results imply stronger classification (e.g. up to diffeomorphism).

The main results of this paper extend this work of Dessai into dimensions 10, 12, 14, and 16. For example, we prove the following (see Theorem 5.1 for a more detailed statement).

Theorem (Theorem 5.1). *If M^{10} is a closed, simply connected Riemannian manifold with positive sectional curvature and T^3 symmetry, then one of the following*

occurs:

- M *is homeomorphic to* $\mathbb{S}^{10}$.
- $\chi(M) = \chi(\mathbb{C}\mathrm{P}^5)$ *and* $H_i(M;\mathbb{Z}) \cong H_i(\mathbb{C}\mathrm{P}^5;\mathbb{Z})$ *for* $i \leq 3$.

We remark that the sphere and complex projective space are the only manifolds known to admit positive curvature in dimension 10. There is a similar story in dimension 14.

Theorem (Theorem 7.1). *If* M^{14} *is a closed, simply connected, positively curved Riemannian manifold with* T^4 *symmetry, then one of the following occurs*:

- $\chi(M) = 2$ *and* M *is 3-connected.*
- $H_*(M;\mathbb{Z}) \cong H_*(\mathbb{C}\mathrm{P}^7;\mathbb{Z})$, *and the cohomology is generated by some* $z \in H^2(M;\mathbb{Z})$ *and* $x \in H^4(M;\mathbb{Z})$ *subject to the relation* $z^2 = mx$ *for some integer* m.

In dimensions divisible by four, the quaternionic projective spaces arise as examples. In addition, dimension 16 is home to the Cayley plane. The sphere, the Cayley plane, $\mathbb{H}\mathrm{P}^4$, and $\mathbb{C}\mathrm{P}^8$ admit positively curved metrics with T^4 symmetry. The next result shows that any closed, simply connected 16-manifold with positive curvature and T^4 symmetry has the same Euler characteristic as one of these spaces (2, 3, 5, or 9, respectively). It also provides a sharp calculation of the signature and elliptic genus under these assumptions.

Theorem (Theorem 8.1). *If* M^{16} *is a closed, simply connected Riemannian manifold with positive sectional curvature and* T^4 *symmetry, then one of the following occurs*:

- $\chi(M) = 2$ *and* $\sigma(M) = 0$.
- $\chi(M) = 3$, $\sigma(M) = \pm 1$, *and* M *is 2-connected.*
- $\chi(M) = 5$, $\sigma(M) = \pm 1$, *and* $H_{2+4i}(M;\mathbb{Z}) = 0$ *for all* i.
- $\chi(M) = 9$, $\sigma(M) = \pm 1$, $H_i(M;\mathbb{Z}) \cong H_i(\mathbb{C}\mathrm{P}^8;\mathbb{Z})$ *for* $i \leq 4$, *and* M *is not spin.*

In any case, if M *is spin, then the elliptic genus is constant.*

We now discuss dimension 12. The rank one symmetric spaces $\mathbb{S}^{12}$, $\mathbb{C}\mathrm{P}^6$, and $\mathbb{H}\mathrm{P}^3$ admit positively curved metrics with T^4 symmetry, and they are the only spaces up to tangential homotopy that admit such metrics by Wilking [59, Theorem 2] (cf. [13]). If one requires only T^3 symmetry, there is no classification, and there is one known additional example, the Wallach manifold $W^{12} = \mathrm{Sp}(3)/\mathrm{Sp}(1)^3$.

The Euler characteristics of these manifolds are 2, 7, 4, and 6, respectively, and their signatures are 0 or ± 1, according to the parity of $\chi(M)$. Moreover, the elliptic genus is constant for each of these manifolds, and $\mathbb{C}\mathrm{P}^6$ is not spin while the other three are spin. The next theorem provides a partial recovery of these properties. To state it, we denote by $C(6)$ the maximum Euler characteristic achieved by a closed, simply connected, positively curved 6-manifold with T^2 symmetry. Note

that $C(6) < \infty$ by Gromov's Betti number estimate (see [27]). In fact, it is not difficult to see that $C(6) \in \{6, 8, \ldots, 14\}$ (see Lemma 6.1).

Theorem (Theorem 6.2). *Let M^{12} be a closed, simply connected, positively curved Riemannian manifold with T^3 symmetry. Either $\chi(M) \in \{2, 4, 6, \ldots, C(6)\}$ or M is not spin and $7 \leq \chi(M) \leq \frac{7}{4}C(6)$. Moreover,*

(1) *if $C(6) = 6$, then $\chi(M) \in \{2, 4, 6\}$ or M is not spin and $\chi(M) \in \{7, 8, 9\}$.*
(2) *if M is rationally elliptic, then $\chi(M) \in \{2, 4, 6, 7, 8, 9, 10, 12\}$.*

Regarding the signature and elliptic genus, the following hold:

(1) *If $\chi(M) \leq 13$, then $|\sigma(M)| \in \{0, 1\}$ according to the parity of $\chi(M)$.*
(2) *If M is spin, then the elliptic genus is constant.*

In light of the known examples in dimension 12, this result is sharp in the spin case under the additional assumption that $C(6) = 6$, which is maximal with regard to the known examples due to Wallach [56] and Eschenburg [18].

We give some context here for the symmetry assumptions. In each theorem above, if the torus T^r is replaced by T^{r+1}, there are stronger classifications due to Fang–Rong [19] and Wilking [59]. In particular, these classifications exclude the Cayley plane and the Wallach 12-manifold whereas Theorems 8.1 and 6.2 include them. If instead the T^r is replaced by T^{r-1}, then little is known beyond the positivity of the Euler characteristic and the vanishing of some higher $\hat{A}$ genera (see [10, 11, 51]).

For the proofs of these results, we build upon on a great deal of previous work that draws on a diverse set of analytic, topological, and combinatorial tools. In numerous cases, we refine in some fixed, small dimension the conclusion of a result that holds in all dimensions. We outline below some of the tools we use, refinements we prove, and new lemmas that play a role in the proofs.

One major source of ideas is [59], where the following are developed:

(1) The connectedness lemma and the resulting periodicity in cohomology,
(2) techniques for studying the fixed-point sets of involutions, including the use of the error correcting codes to analyze isotropy representations,
(3) spherical recognition theorems, and
(4) the classification of maximal rank, smooth torus actions on $\mathbb{H}\mathrm{P}^n$.

We also rely on the refinements of (1) and (2) from our previous work (see [1, 3, 38]). In this paper, we add to this a simple but useful result related to (2) which we call the *containment lemma* (see Lemma 2.1). Also useful for our purposes is Lemma 3.3, which we prove using (3).

A second important source of ideas and motivation is [30]. First, we apply in a crucial way the *equivariant* diffeomorphism classification of positively curved manifolds with maximal symmetry rank. We apply this to prove Lemma 6.7, which is crucial to the proof of our result in dimension 12. In addition, we prove a partial

generalization of their classification of fixed-point homogeneous circle actions, which we call the *codimension two lemma* (see Lemma 4.1).

A third important source of ideas, and the main motivation for this paper, is [12]. There is a fair amount of work required to understand the *global* picture of the fixed-point set data. Our basic strategy is the same: When there are not enough fixed-point sets of small codimension to classify the homotopy type using the connectedness lemma, one obtains *isotropy rigidity* at fixed-point sets of the torus action. For the reader hoping to get just a passing idea of how this strategy works, we recommend the proof of the dimension 14 case, as it involves the smallest number of special cases and other hiccups. For the more interested reader, the proof of the dimension 12 case is by far the most involved but also, we believe, the most interesting. In particular, there is a significant amount of combinatorial analysis required in this dimension that comes out of the isotropy rigidity.

Fourth, regarding the elliptic genus calculations, see Sec. 9. These calculations are motivated by a question of Dessai [10, 11] and work of Weisskopf [57].

All manifolds studied in this paper are shown to have positive Euler characteristic. In addition, they are positively curved, so the ellipticity conjecture of Bott–Grove–Halperin suggests that they are rationally elliptic (see [28]). Putting these properties together, it is suspected that the class of manifolds studied in this paper is closely related to the class of *positively rationally elliptic*, or F_0, spaces. Motivated by this, we compute the homotopy groups of F_0 spaces of formal dimension at most 16 (see Sec. 10). The resulting tables are used to study the four questions we discuss next.

First, we specialize the results above to rationally elliptic spaces and derive rational homotopy classifications in dimensions 10, 14, and 16. We also provide a partial classification of this kind in dimension 12. For this, the Euler characteristic calculations are helpful but not sufficient. One must also apply the conclusions above regarding the product structure in cohomology. See Sec. 12.

Second, we specialize further to biquotients, a large class of manifolds that contains all homogeneous spaces and that provides a source of numerous examples of manifolds admitting positive sectional curvature, as well as weaker notions such as positive curvature almost everywhere (see, for example, [14, 15, 17, 39–41, 58, 62]). In our context, our results in dimensions 10, 14, and 16 imply diffeomorphism classifications when restricted to the case of biquotients.

Corollary (Theorem 13.1). *Let M^n be a closed, simply connected biquotient that independently admits a positively curved Riemannian metric with T^r symmetry.*

- *If $n = 10$ and $r \geq 3$, then M is diffeomorphic to $\mathbb{S}^{10}$, $\mathbb{C}\mathrm{P}^5$, $G_2/U(2)$, $\mathbb{S}^2 \tilde{\times} \mathbb{H}\mathrm{P}^2$, $\mathrm{SO}(7)/(\mathrm{SO}(5) \times \mathrm{SO}(2))$, or $\Delta\mathrm{SO}(2)\backslash\mathrm{SO}(7)/\mathrm{SO}(5)$.*
- *If $n = 14$ and $r \geq 4$, then M is diffeomorphic to $\mathbb{S}^{14}$, $\mathbb{C}\mathrm{P}^7$, $\mathbb{S}^2 \tilde{\times} \mathbb{H}\mathrm{P}^3$, $\mathrm{SO}(9)/(\mathrm{SO}(7) \times \mathrm{SO}(2))$, or $\Delta\mathrm{SO}(2)\backslash\mathrm{SO}(9)/\mathrm{SO}(7)$.*
- *If $n = 16$ and $r \geq 4$, then M is diffeomorphic to $\mathbb{S}^{16}$, $\mathbb{C}\mathrm{P}^8$, $\mathbb{H}\mathrm{P}^4$, or CaP^2.*

Here $\mathbb{S}^2 \tilde{\times} \mathbb{H}\mathrm{P}^m$ *denotes one of the two diffeomorphism types of total spaces of* $\mathbb{H}\mathrm{P}^m$*-bundles over* $\mathbb{S}^2$ *whose structure group reduces to circle acting linearly.*

This corollary is proved in two steps. First, we apply the rational ellipticity of biquotients to derive the rational homotopy type. We then apply a diffeomorphism classification of Kapovitch and Ziller [37], together with a recent generalization due to DeVito [16], to conclude the result, up to a small number of other possibilities. To exclude these, we return to what our calculations imply about the integral cohomology, and this is sufficient to complete the classification.

We remark that the 16-dimensional case of this corollary provides a diffeomorphism characterization of the Cayley plane among biquotients admitting positively curved metrics with T^4 symmetry. To our knowledge, previous results along these lines have either had too strong a symmetry assumption to allow the Cayley plane or too weak an assumption to recognize it.

Third, we specialize even further in dimension 12 to the class of symmetric spaces, and here we obtain a diffeomorphism classification (see Theorem 13.3).

Finally, we study the conjecture of Halperin that any fibration $E \to B$ of simply connected spaces where the fiber F is an F_0 space has the property that $H^*(E;\mathbb{Q}) \cong H^*(B;\mathbb{Q}) \otimes H^*(F;\mathbb{Q})$ as $H^*(B;\mathbb{Q})$-modules.

Theorem (Theorem 11.6). *The Halperin conjecture holds for* F_0 *spaces of formal dimension at most* 16 *or Euler characteristic at most* 16.

For further discussion of F_0 spaces and Halperin's conjecture, cases in which this conjecture is known to hold, and the proof of this theorem, see Secs. 10 and 11.

1. Preliminaries

There are a large number of relatively old results from the theory of transformation groups and newer tools developed over the past two decades that have grown out of work on the Grove program. We attempt to efficiently summarize those which we use.

Let M be an even-dimensional, closed Riemannian manifold with positive sectional curvature, and let T denote a torus that acts isometrically on M. A theorem of Berger states that the fixed-point set $M^T = \{x \in M \mid g(x) = x \text{ for all } g \in T\}$ of the torus is non-empty. In general, for $g \in T$, each component N of the fixed-point set $M^g = \{x \in M \mid g(x) = x\}$ is a closed, even-dimensional, totally geodesic (hence positively curved) embedded submanifold on which T acts, possibly ineffectively. Applying Berger's theorem to the induced T-action on N, we see that every fixed-point component of every isometry in T contains a fixed point of T. We use this fact frequently.

Another issue is proving that components $N \subseteq M^g$ are orientable. If N is a fixed-point component of some subgroup $H \subseteq T$ not equal to $\mathbb{Z}_2$, then N inherits orientability from M. Also, if N has dimension greater than $\frac{1}{2}\dim M$, then Wilking's connectedness lemma (see below) implies that N is simply connected. In some

cases, other arguments are used. For example, if $N \subseteq M^{\mathbb{Z}_2}$ is a component and $\mathbb{Z}_2 \subseteq \mathrm{S}^1 \subseteq T$, and if $Q \subseteq N$ is a fixed-point component of S^1 inside N such that $\dim(Q) \geq \frac{1}{2}\dim(N)$, then Q is orientable, and hence simply connected, and so N is simply connected, and hence orientable, by the connectedness lemma.

Next, we recall some results from Smith theory. The first is due to Conner and Kobayashi (see [8, 42]): The Euler characteristic of M and its fixed-point set M^T coincide. The same holds for the signature, i.e. $\sigma(M) = \sigma(M^T)$ (see [12, Theorem 2.4]). For this latter result, one has to take care how one assigns orientations to the components of M^T, however for our purposes this will not matter since we will always show $|\sigma(M)| \leq 1$ by showing that $|\sigma(M^T)| \leq \sum |\sigma(F)| \leq 1$, where the sum runs over components $F \subseteq M^T$. Another result due to Conner is that sum of the odd Betti numbers of M^T is at most that of M, and likewise for the even Betti numbers.

Regarding the Euler characteristic, we use frequently the following inclusion–exclusion property. If M^T is contained in the union $N_1 \cup N_2$ where N_1 and N_2 admit induced T-actions, then $M^T = (N_1 \cup N_2)^T$. Applying the property above together with the Mayer–Vietoris sequence, we conclude

$$\chi(M) = \chi(N_1) + \chi(N_2) - \chi(N_1 \cap N_2).$$

There are obvious extensions of this formula in the case of three or more submanifolds.

To close this section on preliminaries, we discuss a collection of results related to Wilking's connectedness lemma. We recall these at many points in the paper, so for ease of reference within this paper, we give them names.

Theorem 1.1 ([21]). *Let M^n be a closed Riemannian manifold with positive sectional curvature. If $N_1, N_2 \subseteq M$ are closed, totally geodesic, embedded submanifolds such that $\mathrm{cod}(N_1) + \mathrm{cod}(N_2) \leq n$, then N_1 and N_2 non-trivially intersect.*

Throughout this paper, $\mathrm{cod}(N)$ denotes the codimension of a submanifold $N \subseteq M$.

Theorem 1.2 (Connectedness lemma [59]). *Let M^n be a closed Riemannian manifold with positive sectional curvature.*

(1) *If $N^{n-k} \to M$ is a closed, totally geodesic, embedded submanifold, then the inclusion is $(n-2k+1)$-connected.*
(2) *If $N^{n-k} \to M$ is as above, and if N is a fixed-point component of an isometric action by a Lie group G, then the inclusion is $(n-2k+1+\delta)$-connected, where δ is the dimension of the principal orbits of G.*
(3) *If $N_i^{n-k_i} \to M$ are closed, totally geodesic, embedded submanifolds with $k_1 \leq k_2$, then the inclusion $N_1 \cap N_2 \to N_2$ is $(n-k_1-k_2)$-connected.*

Note that, for our purposes, if M is a closed, positively curved manifold and if H acts isometrically on M, then every component of M^H is a closed, embedded, totally geodesic submanifold.

When the codimensions in the connectedness lemma are sufficiently small, there are strong cohomological consequences. The following is a corollary of the connectedness lemma together with [59, Lemma 2.2].

Lemma 1.3 (Periodicity lemma). *Let M^n be a closed, simply connected, positively curved Riemannian manifold.*

(1) *If $N^{n-k} \to M$ is a closed, totally geodesic, embedded submanifold and $n - 3k + 2 > 0$, it follows that $H^{k-1 \leq * \leq n-k+1}(M; \mathbb{Z})$ is k-periodic.*
(2) *If $N^{n-k} \to M$ is a fixed-point component of an isometric action by a Lie group G and $n - 3k + 2 + 2\delta > 0$, then $H^{k-1-\delta \leq * \leq n-k+1+\delta}(M; \mathbb{Z})$ is k-periodic, where δ is the dimension of the principal orbits of G.*
(3) *If $N_i^{n-k_i} \to M$ are closed, totally geodesic, embedded submanifolds with $k_1 \leq k_2$ and $n - k_1 - k_2 > 0$, and if the intersection $N_1 \cap N_2$ is transverse, then $H^*(N_2; \mathbb{Z})$ is k_1-periodic.*

Following [59] and throughout this paper, we say that $H^{l \leq * \leq n-l}(M; \mathbb{Z})$ is k-periodic if $n - k - 2l > 0$ and there exists $x \in H^k(M; \mathbb{Z})$ such that the multiplication map $H^i(M; \mathbb{Z}) \to H^{i+k}(M; \mathbb{Z})$ induced by x is a surjection for $l \leq i < n - l - k$ and an injection for $l < i \leq n - l - k$. When $l = 0$, we say simply that $H^*(M; \mathbb{Z})$ is k-periodic. Note that part (a) does not apply in dimension 10 when $k = 4$, for example, so in such cases we will either apply part (b) or other arguments.

In [38], refinements to this corollary are proved by applying Steenrod powers to k-periodic cohomology rings. For this paper, we only require the following result.

Lemma 1.4 (Classification of 4-periodic cohomology). *Let M^n be a simply connected closed manifold of dimension $n \geq 8$ such that $H^*(M; \mathbb{Z})$ is 4-periodic.*

(1) *If $n \equiv 0 \bmod 4$, then M is a cohomology $\mathbb{S}^n$, $\mathbb{C}\mathrm{P}^{\frac{n}{2}}$, or $\mathbb{H}\mathrm{P}^{\frac{n}{4}}$.*
(2) *If $n \equiv 2 \bmod 4$, then either M is a mod 2 cohomology sphere or M is a homology $\mathbb{C}\mathrm{P}^{\frac{n}{2}}$ with cohomology generated by some $z \in H^2(M; \mathbb{Z})$ and $x \in H^4(M; \mathbb{Z})$ such that $z^2 = mx$ for some $m \in \mathbb{Z}$.*

Proof. First suppose $n \equiv 0 \bmod 4$. Since M is simply connected, $H^1(M; \mathbb{Z}) = 0$ and $H^{n-1}(M; \mathbb{Z})$. By the definition of periodicity and Poincaré duality, it follows that all cohomology groups are zero in odd degrees and that $H^*(M; \mathbb{Z})$ is torsion-free. Next, periodicity implies that $H^0(M; \mathbb{Z}) \cong \mathbb{Z}$ surjects onto $H^4(M; \mathbb{Z})$, so $H^4(M; \mathbb{Z})$ is zero or $\mathbb{Z}$. If it is zero, then periodicity implies that M is a cohomology sphere, so assume $H^4(M; \mathbb{Z}) \cong \mathbb{Z}$. If $H^2(M; \mathbb{Z}) = 0$, then periodicity implies that M is a cohomology $\mathbb{H}\mathrm{P}^{\frac{n}{4}}$. Finally, if $H^2(M; \mathbb{Z})$ is non-zero, then it contains an element y of infinite order that is not a multiple. By Poincaré duality, there is an element $z \in H^{n-2}(M; \mathbb{Z})$ such that yz equals the top class. By periodicity, the top class is $\pm x^{\frac{n}{4}}$ and $z = x^{\frac{n}{4}-1} w$ for some $w \in H^2(M; \mathbb{Z})$. By periodicity again, it follows that x factors as $\pm yw$, and moreover that y induces periodicity in $H^*(M; \mathbb{Z})$. Note for example, that the isomorphism $H^2(M; \mathbb{Z}) \to H^6(M; \mathbb{Z})$ given by multiplication by x now factors through $H^4(M; \mathbb{Z}) \cong \mathbb{Z}$, so $H^2(M; \mathbb{Z}) \cong \mathbb{Z}$.

For $n \equiv 2 \bmod 4$, the proof requires an additional argument that shows $H^3(M;\mathbb{Z}_2) = 0$ (see [38, Sec. 6]). In particular, $H^4(M;\mathbb{Z}) \neq \mathbb{Z}_{2k}$ for some $k \geq 1$ by the universal coefficients theorem. By periodicity, we conclude that $H^4(M;\mathbb{Z})$ is either $\mathbb{Z}$ or $\mathbb{Z}_{2k+1}$ for some $k \geq 0$. In the latter case, the element inducing periodicity is two-torsion, and by periodicity M is a mod two homology sphere. In the former, we get that $H_2(M;\mathbb{Z}) \cong \mathbb{Z}$ by periodicity and Poincaré duality. Using $H^3(M;\mathbb{Z}_2) = 0$ again, we have $H^3(M;\mathbb{Z}) = 0$. It now follows by periodicity and Poincaré duality again that $H^*(M;\mathbb{Z})$ is as described in the conclusion of the lemma. □

2. Containment Lemma

The first result applies Frankel's theorem to provide a sufficient condition for the fixed-point set M^T of the torus action to be contained in a small number of fixed-point components of involutions. We will use this lemma many times, so we will call it the *containment lemma*.

Lemma 2.1 (Containment lemma). *Suppose M^n is a closed, positively curved Riemannian manifold, and assume T is a torus acting isometrically on M. Fix $x \in M^T$. Let $H \subseteq T$ denote a subgroup isomorphic to $\mathbb{Z}_2^r$. Let $\delta = 4$ if $n \equiv 0 \bmod 4$ and M is spin, and otherwise let $\delta = 2$ if n is even and $\delta = 1$ if n is odd. If*

$$\operatorname{cod}(M_x^H) + \operatorname{cod}(M_y^H) < \left(\frac{2^r - 1}{2^{r-1}}\right)(n + \delta),$$

then $y \in \bigcup_{\iota \in H \setminus \{1\}} M_x^\iota$.

Note that every component of M^H has codimension at most n. Hence, the containment lemma implies that, if

$$\operatorname{cod}(M_x^H) < n + 2\delta - \frac{n+\delta}{2^{r-1}},$$

then $M^T \subseteq \cup M_x^\iota$, where the union runs over non-trivial elements $\iota \in H$. For example, when M has even dimension, this assumption is satisfied in each of the following two cases:

- $H \cong \mathbb{Z}_2^2$ and $\operatorname{cod}(M_x^H) \leq \frac{n}{2} + 2$.
- $H \cong \mathbb{Z}_2^3$ and $\operatorname{cod}(M_x^H) \leq \frac{3n}{4} + 3$.

We will use this consequence frequently. We remark that, if we take $r = 1$ in this statement, this is just a restatement of Frankel's theorem for the case of components of the fixed-point set of some copy of $\mathbb{Z}_2$ in T.

Proof. Let $y \in M^T$ and $\iota \in H$. If $y \notin M_x^\iota$ then the components M_x^ι and M_y^ι are disjoint, so Frankel's theorem implies

$$\operatorname{cod}(M_y^\iota) + \operatorname{cod}(M_x^\iota) > n.$$

Since both codimensions are even, the left-hand side is at least $n+1$ or $n+2$ if n is odd or even, respectively. Moreover, if $n \equiv 0 \bmod 4$ and M is spin, then $\operatorname{cod}(M_x^\iota) + \operatorname{cod}(M_y^\iota) \equiv n \equiv 0 \bmod 4$ and hence the left-hand side is at least $n+4$ (see [35, p. 317]). Hence the left-hand side is at least $n+\delta$, where δ is defined as in the theorem. The lemma follows by summing these inequalities over $\iota \in H$ and using the fact that $\sum_{\iota \in H} \operatorname{cod} M_x^\iota = 2^{r-1} \operatorname{cod}(M_x^H)$ since H is some copy of $\mathbb{Z}_2^r$ inside T (see [4]). □

3. Recognition Theorems for Spheres

It is a basic result from Smith theory that a smooth action by $\mathbb{Z}_p$ on a mod p homology sphere has fixed-point set a mod p homology sphere (see [5, Theorem 5.1, Chap. III]). Wilking proved two results which can be viewed as partial converses to this theorem. The first provides a sufficient condition for recognizing when a manifold is homeomorphic to a sphere (see [59, Theorem 4.1]).

Theorem 3.1 (Recognition theorem, even-dimensional case). *Let M^{2m} be a compact $(m-k)$-connected manifold. Suppose that T^{2k-1} acts smoothly and effectively on M with non-empty fixed-point set. Assume that every $\sigma \in T^{2k-1}$ of prime order p has the property that its fixed-point set is either empty or a mod p homology sphere. Then M is homeomorphic to a sphere.*

The second provides a partial recognition result for mod p homology spheres (see [59, Theorem 5.1]).

Theorem 3.2. *Let M^n be a closed, simply connected, positively curved Riemannian manifold, and let p be a prime. If $\mathbb{Z}_p$ acts isometrically on M with connected fixed-point set of codimension k, then $H^i(M;\mathbb{Z}_p) = 0$ for $k \le i \le n-2k+1$. If, moreover, $M^{\mathbb{Z}_p}$ is fixed by a circle acting isometrically on M, then $H^{k-1}(M;\mathbb{Z}_p) = 0$ and $H^{n-2k+2}(M;\mathbb{Z}_p) = 0$ as well.*

Proof. The first statement is just [59, Theorem 5.1], which is based on the fact that $M^{\mathbb{Z}_p} \to M$ is $(n-2k+1)$-connected according to the connectedness lemma. When $M^{\mathbb{Z}_p}$ is fixed by a circle as in the second assertion, the connectedness lemma implies that the inclusion $M^{\mathbb{Z}_p} \to M$ is $(n-2k+2)$-connected, and the proof of [59, Theorem 5.1] implies that $H^{k-1}(M;\mathbb{Z}_p) = 0$ and $H^{n-2k+2}(M;\mathbb{Z}_p) = 0$ as well. □

Using Theorem 3.2, we prove the following, which can be viewed as another recognition theorem for the sphere.

Lemma 3.3. *Let M^n be a closed, simply connected, positively curved Riemannian manifold with $n = 2m \ge 10$. Assume there exists a circle S^1 acting isometrically on M such that M^{S^1} has a component N of codimension four. If $b_2(M) = 0$ and $\chi(M) = \chi(N)$ then M is homeomorphic to a sphere.*

Note, in particular, that $b_2(M) = 0$ and $\chi(M) = \chi(N) = 2$ if M is a rational homology sphere.

Proof. Let $\mathbb{Z}_p \subset \mathrm{S}^1$. If $M^{\mathbb{Z}_p}$ has codimension two, then the result follows from the periodicity lemma. Suppose that $M^{\mathbb{Z}_p}$ has codimension four. By Frankel's theorem, N is the only component of codimension four and all others are points or positively curved 2-manifolds. Since $\chi(M) = \chi(M^{\mathbb{Z}_p})$, it follows that $M^{\mathbb{Z}_p}$ is connected. By Theorem 3.2 it follows that $H^i(M; \mathbb{Z}_p) = 0$ for all $3 \leq i \leq n-6$. Since $n \geq 10$, we have $H^3(M; \mathbb{Z}_p) = 0$. By the universal coefficients theorem and the assumption that $b_2(M) = 0$, it follows that $H^2(M; \mathbb{Z}_p) = 0$. For $n \geq 12$, it follows immediately from Poincaré duality that M is a mod p homology sphere. For $n = 10$, the same holds by a similar argument using the additional observation that $b_5(M) = \chi(M) - 2 = \chi(N) - 2 = 0$, which holds since $b_2(N) = b_2(M) = 0$, $b_3(N) = b_3(M) = 0$, and $b_4(M) = 0$. Note also that $\dim H^5(M; \mathbb{Z}_p) = b_5(M)$ by the universal coefficients theorem since $H^6(M; \mathbb{Z}_p) \cong H^4(M; \mathbb{Z}_p) = 0$. Since this argument holds for all primes, the lemma follows by the resolution of the Poincaré conjecture. □

4. Low Codimension Lemmas

It follows immediately from the periodicity lemma and the resolution of the Poincaré conjecture that a positively curved, simply connected, closed manifold of odd dimension containing a codimension two, totally geodesic submanifold is homeomorphic to $\mathbb{S}^n$. It is an open question whether an analogous result holds in the even-dimensional case. By Grove and Searle's result, if the codimension two submanifold is fixed by an isometric circle action, then M is in fact diffeomorphic to $\mathbb{S}^n$ or $\mathbb{C}\mathrm{P}^{n/2}$. For our purposes, we will use the following, different partial result in this direction.

Lemma 4.1 (Codimension two lemma). *Let M^n be a closed, simply connected, positively curved Riemannian manifold with T^2 symmetry. If some involution $\iota \in T^2$ has fixed-point set of codimension two, then M is homeomorphic to $\mathbb{S}^n$ or $\mathbb{C}\mathrm{P}^{n/2}$.*

Proof. We may assume that $n \geq 6$, since otherwise this holds by the theorem of Hsiang and Kleiner. Let $N \subseteq M^\iota$ be a component of codimension two. By the periodicity lemma, it follows that M is an cohomology $\mathbb{S}^n$ or $\mathbb{C}\mathrm{P}^{n/2}$ if $b_2(M) \leq 1$. In the first of these cases, the homeomorphism classification follows by Perelman's resolution of the Poincaré conjecture. In the second case, the homeomorphism classification follows from [19, Lemma 3.6]. Hence it suffices to show that $b_2(M) \leq 1$.

Suppose first that an involution $\iota' \in T^2 \backslash \langle \iota \rangle$ exists whose fixed-point set contains a component N' has codimension at most $\frac{n-1}{2}$. The intersection is transverse, so the periodicity lemma implies that N' is 2-periodic. By the connectedness lemma, $b_2(M) \leq b_2(N') \leq 1$, as required.

Suppose now that $\mathrm{cod}(M^{\iota'}) \geq \frac{n}{2}$ for all involutions $\iota' \in T^2 \backslash \langle \iota \rangle$. Consider the isotropy representation at any point $x \in M^T \backslash N$. The codimensions of $M_x^{\iota'}$ and $M_x^{\iota\iota'}$ sum to n and hence equal $\frac{n}{2}$. Fixing any such involution ι' and denoting

its component of maximal dimension by N', it follows by Frankel's theorem that $M^{T^2} \subseteq N \cup N'$. Applying the inclusion-exclusion property to this containment, we have

$$\chi(M) - \chi(N) = \chi(N') - \chi(N \cap N').$$

By the periodicity lemma, Parts (1) and (3), the left-hand side equals $b_2(M)$, the right-hand side equals $b_2(N')$, and $b_2(N') \leq 1$. This completes the proof. □

Using the codimension two lemma, we prove a similar lemma for the case of codimension four fixed-point sets.

Lemma 4.2 (Codimension four lemma, part 1). *Let M^n be a closed, simply connected, positively curved Riemannian manifold with even dimension $n \geq 12$ and T^r symmetry with $r \geq 3$. Assume there exists $x \in M$ and involutions $\iota_1, \iota_2 \in T^r$ such that $\operatorname{cod}(M_x^{\iota_1}) = 4$ and $\operatorname{cod}(M_x^{\iota_2}) < \frac{n}{2}$. One of the following occurs*:

(1) *M is a cohomology $\mathbb{S}^n$, $\mathbb{C}\mathrm{P}^{\frac{n}{2}}$, or $\mathbb{H}\mathrm{P}^{\frac{n}{4}}$, or M is a homology $\mathbb{C}\mathrm{P}^{2m+1}$ with cohomology generated by some $z \in H^2(M;\mathbb{Z})$ and $x \in H^4(M;\mathbb{Z})$ such that $z^2 = mx$ for some $m \in \mathbb{Z}$.*
(2) *$n = 4m + 2$, $M \sim_{\mathbb{Z}_2} \mathbb{S}^n$, and T^r acts almost effectively on $M_x^{\iota_1}$.*
(3) *$n = 4m + 2$, $\operatorname{cod}(M_x^{\iota_2}) = \frac{n}{2} - 1$ and T^r acts almost effectively on $M_x^{\iota_1}$ or $M_x^{\iota_2}$.*

Proof. Let $N_1 \subseteq M^{\iota_1}$ denote the component of codimension four, and let $N_2 \subseteq M^{\iota_2}$ denote the component of codimension $k_2 \leq \frac{n}{2} - 1$. Also let $N_{12} \subseteq M^{\iota_1\iota_2}$ be the component containing $N_1 \cap N_2$, which is connected by the connectedness lemma. We may assume that N_2 has maximal dimension among all such choices, so it admits T^2 symmetry. As a result, N_2 has 4-periodic cohomology by the periodicity lemma (if $N_1 \cap N_2$ is transverse) or the codimension two lemma (if not).

First assume $k_2 \leq \frac{n}{2} - 2$. We claim that $b_2(M) = b_6(M)$. Since N_2 has 4-periodic cohomology, $H^3(N_2;\mathbb{Z}_2) = 0$ and $H^5(N_2;\mathbb{Z}) = 0$ by Lemma 1.4. Since the inclusion $N_2 \to M$ is 5-connected, $H^3(M;\mathbb{Z}_2) = 0$ and $H^5(M;\mathbb{Z}) = 0$ as well. By the periodicity lemma, $H^{2i+1}(M;\mathbb{Z}_2) = 0$ for all i and

$$\chi(M) - \chi(N_1) = b_4(M) + b_6(M). \tag{4.1}$$

By the connectedness lemma, N_2, N_{12}, and the common 2-fold intersection $M_x^{\langle \iota_1, \iota_2 \rangle}$ of any three of submanifolds N_1, N_2, and N_{12} are 4-periodic with common values of b_2 and b_4. Moreover, the M^{T^r} is contained in $N_1 \cup N_2 \cup N_{12}$ by the containment lemma, so the inclusion–exclusion property of Euler characteristics implies

$$\chi(M) - \chi(N_1) = b_2(M) + b_4(M). \tag{4.2}$$

Indeed, if $N_1 \cap N_2$ is transverse, then $N_{12} = N_1 \cap N_2$ by the connectedness lemma and hence

$$\chi(M) - \chi(N_1) = \chi(N_2) - \chi(N_1 \cap N_2) = b_2(N_2) + b_4(N_2)$$

by 4-periodicity. Equation (4.2) follows from the connectedness lemma. Similarly, if $N_1 \cap N_2$ is not transverse, then N_2, N_{12}, and $N_1 \cap N_2$ are actually 2-periodic and

$$\begin{aligned}\chi(M) - \chi(N_1) &= \chi(N_2) - \chi(N_1 \cap N_2) + \chi(N_{12}) - \chi(N_1 \cap N_2) \\ &= b_2(N_2) + b_2(N_{12}).\end{aligned}$$

By two periodicity, $b_2(N_{12}) = b_4(N_{12})$, so again Eq. (4.2) follows by the connectedness lemma. Comparing Eqs. (4.1) and (4.2), we conclude the claim that $b_2(M) = b_6(M)$.

Given this claim, we complete the proof of the lemma assuming $k_2 \leq \frac{n}{2} - 2$ by considering two cases. Suppose first that N_2 is not a mod two homology sphere. By Lemma 1.4 and the connectedness lemma, it follows that $H^i(M;\mathbb{Z}) = 0$ for $i \in \{3,5\}$ and that $H^4(M;\mathbb{Z}) \cong \mathbb{Z}$. In particular, $H^{2i+1}(M;\mathbb{Z}) = 0$ for all i by the periodicity lemma and Poincaré duality. Hence $H^2(M;\mathbb{Z})$ and $H^6(M;\mathbb{Z})$ are isomorphic and either 0 or $\mathbb{Z}$ since they are torsion-free and of the same rank as $H^2(N_2;\mathbb{Z})$. If $b_2(M) = 0$, then the periodicity lemma and Poincaré duality imply that M is a cohomology $\mathbb{H}\mathrm{P}^{\frac{n}{4}}$. If $b_2(M) \neq 0$, then $H^2(M;\mathbb{Z}) \cong H^6(M;\mathbb{Z}) \cong \mathbb{Z}$. Let $x \in H^4(M;\mathbb{Z})$ and $z \in H^2(M;\mathbb{Z})$ be generators. By the naturality of cup products and the four-periodicity of $H^*(N_2;\mathbb{Z})$, the product $xz \in H^6(M;\mathbb{Z}) \cong \mathbb{Z}$ is not a multiple (or zero). In other words, multiplication by x induces an isomorphism $H^2(M;\mathbb{Z}) \to H^6(M;\mathbb{Z})$. Combining this with the periodicity lemma and Poincaré duality, it follows that $H^*(M;\mathbb{Z})$ is four-periodic and hence as in conclusion (1).

Now suppose instead that N_2 is a mod two homology sphere. By the connectedness lemma, $H^i(M;\mathbb{Z}_2) = 0$ for $1 \leq i \leq 5$ and hence $H^{2i+1}(M;\mathbb{Z}_2) = 0$ for all i by Poincaré duality and the periodicity lemma. Together with the fact that $b_2(M) = b_6(M)$ and the universal coefficients theorem, this implies that $H^6(M;\mathbb{Z}_2) = 0$. Applying Poincaré duality and the periodicity lemma again, it follows that M is a mod two homology sphere. We can refine this conclusion as follows. First, if some circle in T^r fixes N_1, then M is homeomorphic to $\mathbb{S}^n$ by Lemma 3.3 and conclusion (1) holds. Second, if $n \equiv 0 \bmod 4$, then Poincaré duality and the periodicity lemma imply that $H_3(M;\mathbb{Z}) \cong H^5(M;\mathbb{Z})$, which vanishes since it injects into $H^5(N_2;\mathbb{Z}) = 0$. Therefore, $H^4(M;\mathbb{Z})$ is torsion-free and hence zero, which implies that N_2 and M are integral homology spheres, and again conclusion (1) holds. Finally, if $n \equiv 2 \bmod 4$ and N_1 is not fixed by some circle in T^r, then conclusion (2) holds.

Assume from now on that $k_2 = \frac{n}{2} - 1$. Note that $n \equiv 2 \bmod 4$ since k_2 is even. Moreover, assume N_1 is fixed by some circle in T^r and likewise for N_2, since otherwise conclusion (3) holds. In particular, N_2 has 4-periodic cohomology and the inclusion $N_2 \to M$ is 4-connected. We may assume further that $b_2(M) \geq 1$. Indeed, if $b_2(M) = 0$, then the periodicity lemma implies $b_4(M) = 0$ and hence that M is a rational homology sphere. By Lemma 3.3, M is homeomorphic to $\mathbb{S}^n$ and conclusion (1) holds.

Lemma 1.4 implies that N_2 is a homology $\mathbb{C}\mathrm{P}^{\frac{n+2}{4}}$ and that $H^*(N_2;\mathbb{Z})$ is generated by some $z \in H^2(N_2;\mathbb{Z})$ and $x \in H^4(N_2;\mathbb{Z})$ such that $z^2 = mx$ for some $m \in \mathbb{Z}$. By the connectedness lemma, $H^2(M;\mathbb{Z}) = \mathbb{Z}$ and $H^3(M;\mathbb{Z}) = 0$. Moreover, the map $H^4(M;\mathbb{Z}) \to H^4(N_2;\mathbb{Z}) \cong \mathbb{Z}$ is an injection and hence is either an isomorphism or the zero map. If the map is an isomorphism, then the naturality of cup products and the periodicity lemma, part (b), imply that $H^*(M;\mathbb{Z})$ is generated by elements in degree two and four as is the case for N_2 and conclusion (1) holds.

To complete the proof, we assume that $H^4(M;\mathbb{Z}) \to H^4(N_2;\mathbb{Z})$ is the zero map and derive a contradiction. By the periodicity lemma and the connectedness lemma, $M \sim_{\mathbb{Z}} \mathbb{S}^2 \times \mathbb{S}^{n-2}$ and $N_1 \sim_{\mathbb{Z}} \mathbb{S}^2 \times \mathbb{S}^{n-6}$. In particular, $\chi(M) = \chi(N_1)$, so M^{ι_1} is connected by Frankel's theorem. By the connectedness lemma again, the action of ι_1 on N_2 has connected fixed-point set $(N_2)^{\iota_1} = M^{\iota_1} \cap N_2 = N_1 \cap N_2$. If $\dim(N_2) \geq 12$, this implies $H^4(N_2;\mathbb{Z}_2) = 0$ by Lemma 3.2, a contradiction, so we may assume $\dim(N_2) \leq 10$, which is equivalent to $n \leq 18$. Note that $\chi(M) = 4$ and $\chi(N_2) = \frac{n+2}{4} + 1 > 4$, so there exists a component $M_y^{\iota_2} \subseteq M^{\iota_2}$ with negative Euler characteristic. By Frankel's theorem, $\dim(M_y^{\iota_2}) \leq 6$. But $\chi(M_y^{\iota_2}) > 0$ if $M_y^{\iota_2}$ has dimension 0, 2, or 4, or if $M_y^{\iota_2}$ admits an isometric circle action, so we may assume $\dim(M_y^{\iota_2}) = 6$ and that T^r fixes $M_y^{\iota_2}$. Since $r \geq 3$, there exists an involution $\iota \in T^r \backslash \langle \iota_1, \iota_2 \rangle$ such that M_y^{ι} strictly contains $M_y^{\iota_2}$. By replacing M_y^{ι} by $M_y^{\iota_2 \iota}$, if necessary, we may assume $\operatorname{cod}(M_y^{\iota}) \leq \frac{1}{2}\operatorname{cod}(M_y^{\iota_2}) = 6$. But now $M_x^{\iota_1}$ and M_y^{ι} intersect by Frankel, and we may proceed as in the case where $\operatorname{cod}(M^{\iota_2}) \leq \frac{n}{2} - 2$. Altogether one concludes that this case does not actually occur. □

In the case where there is no involution ι_2 as in Lemma 4.2, we need a companion lemma.

Lemma 4.3 (Codimension four lemma, part 2). *Let M^n be a closed, simply connected, positively curved Riemannian manifold with even dimension and T^r symmetry such that $r \geq 3$. Assume there exist an involution $\iota_1 \in T^r$ and a component $N_1 \subseteq M^{\iota_1}$ of codimension four, and assume that every other non-trivial involution ι satisfies $\operatorname{cod}(M^{\iota}) \geq \frac{n}{2}$. Either*

(1) *M^{ι_1} is connected, or*
(2) *2^r divides n, and $\operatorname{cod}(M_y^{\iota}) = \frac{n}{2}$ for every involution $\iota \in T^r \backslash \langle \iota_1 \rangle$ and $y \in M^{T^r} \backslash N_1$.*

Proof. Assume M^{ι_1} is not connected, and let $y \in (M^{\iota_1})^T \backslash N_1$. Frankel's theorem implies $\operatorname{cod}(M_y^{\iota_1}) \geq n - 2$. Under the map $\mathbb{Z}_2^r \to \mathbb{Z}_2^{n/2}$ induced by the isotropy representation of T^r at y,

$$\iota_1 \mapsto (1, 1, 1, \ldots, 1, *),$$

where the asterisk is 0 or 1 according to whether $\operatorname{cod}(M_y^{\iota})$ is $n-2$ or n. In the first case, we may choose $\iota_2 \in \mathbb{Z}_2^r \backslash \langle \iota_1 \rangle$ whose image has a 0 in the last entry. It

then follows that $M_y^{\iota_2}$ or $M_y^{\iota_1\iota_2}$ has codimension less than $\frac{n}{2}$, a contradiction. In the second case, we have that

$$\operatorname{cod}(M_y^{\iota}) + \operatorname{cod}(M_y^{\iota\iota_1}) = n$$

for all $\iota \in \mathbb{Z}_2^r \backslash \langle \iota_1 \rangle$. Under the assumption that $\operatorname{cod}(M_y^{\iota}) \geq \frac{n}{2}$ for all $\iota \in \mathbb{Z}_2^r \backslash \langle \iota_1 \rangle$, it follows that every such codimension equals $\frac{n}{2}$. It is not hard to see that there exists a decomposition $\mathbb{Z}_2^{\frac{n}{2}} = V_1 \oplus \cdots \oplus V_{2^{r-1}}$ such that

- for all $\iota \in \mathbb{Z}_2^r$, the projection onto V_i of image of ι has all 0s or all 1s, and
- the V_i have equal dimensions.

In particular, $\frac{n}{2}$ is divisible by 2^{r-1}, which proves that 2^r divides n. □

5. Dimension 10

The only two compact, simply connected, smooth 10-manifolds known to admit positive sectional curvature are $\mathbb{S}^{10}$ and $\mathbb{C}\mathrm{P}^5$. Equipped with the standard metrics, each of these spaces admits T^5 symmetry. We partially recover a classification of these two spaces under the assumption of T^3 symmetry.

Theorem 5.1. *Let M^{10} be a closed, simply connected Riemannian manifold with positive sectional curvature and T^3 symmetry. One of the following occurs*:

- *M is homeomorphic to $\mathbb{S}^{10}$.*
- *$\chi(M) = \chi(\mathbb{C}\mathrm{P}^5)$, $H_i(M;\mathbb{Z}) = H_i(\mathbb{C}\mathrm{P}^5;\mathbb{Z})$ for $i \leq 3$, and $H^{10}(M;\mathbb{Z})$ is generated by an element of the form x^3y with $x \in H^2(M;\mathbb{Z})$ and $y \in H^4(M;\mathbb{Z})$.*
- *$H_*(M;\mathbb{Z}) \cong H_*(\mathbb{C}\mathrm{P}^5;\mathbb{Z})$, and the cohomology is generated by some $z \in H^2(M;\mathbb{Z})$ and $x \in H^4(M;\mathbb{Z})$ subject to the relation $z^2 = mx$ for some $m \in \mathbb{Z}$.*

In any case, $H_2(M;\mathbb{Z})$ is 0 or $\mathbb{Z}$, $H_3(M) = 0$, and $\chi(M) = 2 + 4b_2(M) \in \{2, 6\}$.

For context, we remark that, if M is as in the theorem but has T^4 symmetry, work of Fang and Rong implies that M is homeomorphic to $\mathbb{S}^{10}$ or $\mathbb{C}\mathrm{P}^5$ (see [19]). On the other hand, it follows from [51] that a T^1 action on M^{10} as in the theorem is sufficient to show $\chi(M) \geq 2$ (cf. [12]). We also remark that this classification can be strengthened in the special case where M is rationally elliptic or admits a biquotient structure (see Theorems 12.1 and 13.1).

We spend the rest of this section on the proof. Denote the torus by T. If some involution $\iota \in T$ has fixed-point set M^{ι} of codimension two, then M is homeomorphic to $\mathbb{S}^{10}$ or $\mathbb{C}\mathrm{P}^5$ by Lemma 4.1. We assume therefore that $\operatorname{cod}(M^{\iota}) \geq 4$ for all non-trivial involutions $\iota \in T$. In this situation, there exist two involutions with fixed-point sets of codimension four, so we are in the setting of the codimension four lemma, part 1. However, the arguments there do not suffice in dimension 10. Fortunately, four is large enough relative to the dimension of M in this case to force isotropy rigidity. The main consequence of this rigidity is the following.

Lemma 5.2 (Containment lemma for dimension 10). *Assume no involution $\iota \in T^3$ has fixed-point set of codimension two. There exist $x \in M^T$ and independent involutions $\iota_1, \iota_2 \in T^3$ such that both $\operatorname{cod}(M_x^{\iota_i}) = 4$ and $M^T \subseteq M_x^{\iota_1} \cup M_x^{\iota_2} \cup M_x^{\iota_1\iota_2}$.*

Proof. Consider the map $\mathbb{Z}_2^3 \to \mathbb{Z}_2^5$ induced by the isotropy representation $T \to \mathrm{SO}(T_xM)$ at some fixed point $x \in M^T$. We claim that one of the following possibilities occurs.

Case 1. There exist $x \in M^T$ and independent involutions $\iota_1, \iota_2 \in T$ such that $\operatorname{cod}(M_x^{\iota_1}) = \operatorname{cod}(M_x^{\iota_2}) = 4$ and $M_x^{\iota_1} \cap M_x^{\iota_2}$ is not transverse.

Case 2. At every $x \in M^T$, there exist involutions $\iota_1, \iota_2 \in \mathbb{Z}_2^3$ such that

$$\operatorname{cod}(M_x^{\iota}) = \begin{cases} 4 & \text{if } \iota \in \{\iota_1, \iota_2\}, \\ 6 & \text{if } \iota \notin \langle \iota_1, \iota_2 \rangle, \\ 8 & \text{if } \iota = \iota_1\iota_2. \end{cases}$$

Indeed, suppose Case 1 does not occur, and let $x \in M^T$. It is not possible for the image of every $\iota \in \mathbb{Z}_2^3$ to have weight at least three, so we may choose $\iota_1 \in \mathbb{Z}_2^3$ with $\operatorname{cod}(M_x^{\iota_1}) = 4$. Next, we may choose $\iota_2 \in \mathbb{Z}_2^3 \backslash \langle \iota_1 \rangle$ such that $M_x^{\iota_1} \cap M_x^{\iota_2}$ is transverse. Supposing for a moment that $\operatorname{cod}(M_x^{\iota_2}) = 6$, it follows that some $\iota \in \mathbb{Z}_2^3 \backslash \langle \iota_1, \iota_2 \rangle$ exists such that M_x^{ι} has codimension four and intersects $M_x^{\iota_1}$ non-transversely. This is a contradiction since we assumed that Case 1 does not occur. It follows that $\operatorname{cod}(M_x^{\iota_2}) = 4$. Finally, using again the fact that Case 1 does not occur, it follows that every $\iota \in \mathbb{Z}_2^3 \backslash \langle \iota_1, \iota_2 \rangle$ has $\operatorname{cod}(M_x^{\iota}) = 6$. This concludes the proof of the claim.

If Case 1 occurs, then the lemma follows immediately from the containment lemma. We suppose now that Case 2 occurs and proceed by contradiction. If $z \in M^T$ does not lie in $M_x^{\iota_1} \cup M_x^{\iota_2}$, then $\operatorname{cod}(M_z^{\iota_1}) > 6$ and $\operatorname{cod}(M_z^{\iota_2}) > 6$ by Frankel's theorem. But this contradicts the fact that, at each point, there exists a unique involution whose fixed-point set component containing that point has codimension greater than six, so the proof is complete. □

The proof of Theorem 5.1 is finished below in the following two lemmas. In both, we fix involutions ι_1 and ι_2 as in Lemma 5.2, and we keep the notation $N_i = M_x^{\iota_i}$ for $i \in \{1, 2\}$.

Lemma 5.3. *If $N_1 \cap N_2$ is not transverse, or if N_1 or N_2 admits T^3 symmetry, then*

- *M is homeomorphic to $\mathbb{S}^{10}$, or*
- *$\chi(M) = \chi(\mathbb{C}\mathrm{P}^5)$, $H_i(M;\mathbb{Z}) = H_i(\mathbb{C}\mathrm{P}^5;\mathbb{Z})$ for $i \leq 3$, and $H^{10}(M;\mathbb{Z})$ is generated by an element of the form x^3y with $x \in H^2(M;\mathbb{Z})$ and $y \in H^4(M;\mathbb{Z})$.*

Proof. We first calculate $H_2(M;\mathbb{Z})$, $H_3(M;\mathbb{Z})$, and $\chi(M)$. Suppose first that N_1 and N_2 intersect transversely, and assume without loss of generality that N_1 is

admits T^3 symmetry. It follows from Grove and Searle's diffeomorphism classification that N_1 is diffeomorphic to $\mathbb{S}^6$ or $\mathbb{C}\mathrm{P}^3$. By either applying the same reasoning to N_2 or by using the connectedness lemma, it follows that $H_3(M;\mathbb{Z}) \cong H_3(N_2;\mathbb{Z}) = 0$ and $H_2(N_2;\mathbb{Z}) \cong H_2(M;\mathbb{Z}) \cong H_2(N_1;\mathbb{Z})$, which is isomorphic to 0 or $\mathbb{Z}$. Note that $N_1 \cap N_2 \to N_i \to M$ is 2-connected by the connectedness lemma, so $M_x^{\iota_1\iota_2} = N_1 \cap N_2 = \mathbb{S}^2$. In particular, $M^T \subseteq N_1 \cup N_2$, so $\chi(M) = 2 + 4b_2(M) \in \{2, 6\}$ by the inclusion–exclusion formula for Euler characteristics.

Suppose now that N_1 and N_2 do not intersect transversely. By the connectedness lemma, all two-fold intersections of N_1, N_2, and $N_{12} = M_x^{\iota_1\iota_2}$ coincide and equal $M_x^{\langle \iota_1, \iota_2 \rangle}$. By the codimension two lemma, all four of these submanifolds have 2-periodic integral cohomology, which means their third homology groups vanish, while their second homology groups coincide and equal 0 or $\mathbb{Z}$. By the connectedness lemma, $H_3(M;\mathbb{Z}) = 0$ and $H_2(M;\mathbb{Z}) \in \{0, \mathbb{Z}\}$. Since $N_1 \cup N_2 \cup N_{12}$ contains M^T, the inclusion–exclusion formula for Euler characteristics implies $\chi(M) = 2 + 4b_2(M) \in \{2, 6\}$.

This completes the calculation of $H_2(M;\mathbb{Z})$, $H_3(M;\mathbb{Z})$, and $\chi(M)$. We now complete the proof. Suppose first that $\chi(M) = 6$. Let N be a six-dimensional submanifold of M homotopy equivalent to $\mathbb{C}\mathrm{P}^3$ such that the inclusion $N \to M$ is 3-connected. By the naturality of cup products, it follows that the third power of a generator $x \in H^2(M;\mathbb{Z})$ is non-zero and not a non-trivial multiple. By Poincaré duality, there exists $y \in H^4(M;\mathbb{Z})$ such that x^3y generates $H^{10}(M;\mathbb{Z})$.

Suppose now that $\chi(M) = 2$. By the calculations above, it follows that M is 3-connected. Moreover, we claim that the T^3-action on M has the property that every subgroup $H \subseteq T^3$ has fixed-point set homeomorphic to a sphere. Since $\chi(M^H) = \chi(M) = 2$, it suffices to show that each component $P \subseteq M^H$ is homeomorphic to a sphere or a point. This clearly holds if $\dim P = 8$ by the periodicity lemma and the fact that M is 3-connected. If $\dim P = 6$, then P is 2-connected by the connectedness lemma, and again it follows that P is homeomorphic to the sphere either by the connectedness lemma (if some circle in T^3 fixes P) or the classification of Grove and Searle (if the T^3 acts almost effectively on P). If $\dim P = 4$, then some circle $T^1 \subseteq T^3$ fixes P, and there exists a fixed-point component Q of T^1 containing P. If $P \subseteq Q$ is an equality or has codimension two, then P is orientable by the connectedness lemma and Synge's theorem. If not, then Q has dimension eight, which implies that M and hence P are homeomorphic to spheres. Finally, if $\dim P \leq 2$, then P is either a point or a two-dimensional fixed-point component inside a fixed-point component of some circle in T^3, and we may argue as in the previous case that P is orientable and hence $\mathbb{S}^2$. The homeomorphism classification now follows from Wilking's spherical recognition theorem (Theorem 3.1). □

Lemma 5.4. *If N_1 and N_2 are fixed by circles in T^3 and intersect transversely, then*

- *M is homeomorphic to $\mathbb{S}^{10}$, or*

- $H_*(M;\mathbb{Z}) \cong H_*(\mathbb{C}\mathrm{P}^5;\mathbb{Z})$, *and the cohomology is generated by* $H^2(M;\mathbb{Z})$ *and* $H^4(M;\mathbb{Z})$.

Proof. Since N_1 and N_2 intersect transversely, the connectedness lemma implies that the inclusions $N_1 \cap N_2 \to N_i \to M$ are 2-connected. In particular, $N_1 \cap N_2$ is diffeomorphic to $\mathbb{S}^2$ and equals the fixed-point component $M_x^{\iota_1\iota_2}$. In particular, $M^T \subseteq N_1 \cup N_2$ by Lemma 5.2. Note also that $H_2(M;\mathbb{Z})$ is either finite or isomorphic to $\mathbb{Z}$ since the natural map $H_2(N_1 \cap N_2;\mathbb{Z}) \to H_2(M;\mathbb{Z})$ is surjective.

First suppose $H_2(M;\mathbb{Z})$ is finite. By the periodicity lemma, the Betti numbers of M satisfy $b_2(M) = b_6(M) = 0$ and $b_4(M) = b_8(M) = 0$. By Poincaré duality, the Euler characteristic of M is even and equals $2 - 2b_3(M) - b_5(M)$. But $\chi(M) > 0$, so $b_3(M) = b_5(M) = 0$. Hence M is a rational sphere, and N_1 is as well by the connectedness lemma. Hence M is homeomorphic to $\mathbb{S}^{10}$ by Lemma 3.3.

Now suppose that $H_2(M;\mathbb{Z}) \cong \mathbb{Z}$. Since each N_j is fixed by a circle in T, the connectedness lemma implies that $N_j \to M$ is 4-connected. Applying the inclusion–exclusion formula for the Euler characteristic, we conclude that

$$\chi(M) = \chi(N_1 \cup N_2) = \chi(N_1) + \chi(N_2) - \chi(N_1 \cap N_2) = 6 - 2b_3(M).$$

Comparing with the alternating sum of Betti numbers formula for $\chi(M)$, we conclude that $2 = 2b_4(M) - b_5(M)$. The periodicity lemma and Poincaré duality imply that $b_4(M) \leq 1$, so this equality implies $b_4(M) = 1$ and $b_5(M) = 0$. Applying Poincaré duality and the periodicity lemma again, we conclude that $H_5(M;\mathbb{Z}) = 0$ and that $x \in H^4(M;\mathbb{Z})$ and $z \in H^2(M;\mathbb{Z})$ exist such that xz generates $H^6(M;\mathbb{Z})$, that $x^2 z$ generates $H^{10}(M;\mathbb{Z})$, and hence that x^2 generates $H^8(M;\mathbb{Z})$. Finally, it follows as in the proof of Lemma 1.4 that $H^3(M;\mathbb{Z}_2) = 0$, so $H^3(M;\mathbb{Z}) = 0$ and M is as in the second conclusion of the lemma. □

6. Dimension 12

Let $C(6)$ denote the maximum Euler characteristic achieved by a closed, simply connected 6-manifold that admits a Riemannian metric with positive sectional curvature and T^2 symmetry. Note that $C(6) < \infty$ by Gromov's Betti number estimate. In fact, we have the following.

Lemma 6.1. *The maximum Euler characteristic* $C(6)$ *of a closed, simply connected, positively curved manifold with* T^2 *symmetry satisfies* $6 \leq C(6) \leq 14$.

Proof. The Wallach flag $M^6 = \mathrm{SU}(3)/T^2$ admits a metric with both positive curvature and an isometric T^2 action, so $C(6) \geq 6$. For the upper bound, let M^6 be a manifold as in the theorem. Suppose for a moment that some $g \in T^2$ has $\mathrm{cod}(M^g) = 2$. Let $N^4 \subseteq M^g$ denote a four-dimensional component. By the result of Hsiang and Kleiner, N has $b_2(N) \leq 1$. By the connectedness lemma and the periodicity lemma, it follows that M is homotopy equivalent to $\mathbb{S}^6$ or $\mathbb{C}\mathrm{P}^3$.

Suppose therefore that every $g \in T^2$ has fixed-point set of codimension at least four. In this case, the fixed-point set of T^2 is made up of isolated fixed points. Moreover, this property implies that the isotropy representations of T^2 at fixed points are such that each fixed point projects to an extremal point in M/T^2. Since this space is a four-dimensional Alexandrov space, the number of extremal points is strictly less than 2^4 (see [43]). Since, on the other hand, the number equals $\chi(M)$, which is even, the proof is complete. □

The main result in dimension 12 is presented in terms of the constant $C(6)$.

Theorem 6.2. *Let M^{12} be a closed, simply connected, positively curved manifold with T^3 symmetry. Either $\chi(M) \in \{2, 4, \ldots, C(6)\}$ or M is not spin and $7 \leq \chi(M) \leq \frac{7}{4}C(6)$. Moreover, the following hold.*

(1) *If $C(6) = 6$, then $\chi(M) \in \{2, 4, 6, 7, 8, 9\}$.*
(2) *If M is rationally elliptic, then $\chi(M) \in \{2, 4, 6, 7, 8, 9, 10, 12\}$.*

In any case, the following hold for the signature and elliptic genus.

(1) *If $\chi(M) \leq 13$, then $|\sigma(M)| \in \{0, 1\}$ according to the parity of $\chi(M)$.*
(2) *If M is spin, then the elliptic genus is constant.*

Recall that 2, 4, 6, and 7 are realized as Euler characteristics of positively curved manifolds with T^3 symmetry, namely, $\mathbb{S}^{12}$, $\mathbb{H}\mathrm{P}^3$, the Wallach flag W^{12}, and $\mathbb{C}\mathrm{P}^6$. As for the possibilities of $\chi(M) > 7$, we note that there are many examples of non-negatively curved manifolds M with isometric T^3-actions such that $\chi(M) \in \{8, 9, 10, 12\}$ and $\sigma(M) \in \{0, 1\}$. Indeed, one finds such examples among compact symmetric spaces of rank two (e.g. the Grassmannian $\mathrm{SO}(8)/\mathrm{SO}(2) \times \mathrm{SO}(6)$ or products of rank one spaces such as $\mathbb{S}^{12-2m} \times \mathbb{C}\mathrm{P}^m$ or $\mathbb{C}\mathrm{P}^2 \times \mathbb{H}\mathrm{P}^2$) or among certain connected sums of rank one symmetric spaces (e.g. $\mathbb{C}\mathrm{P}^6 \# \mathbb{H}\mathrm{P}^3$) endowed with Cheeger metrics (see [7]).

The proof of Theorem 6.2 takes the rest of the section. The bulk of the proof is contained in a sequence of lemmas that together prove the Euler characteristic calculation claimed in Theorem 6.2. The signature calculation is then proved at the end of the section, and the elliptic genus calculation is proved in Sec. 9.

We assume for the rest of the section that M is a 12-dimensional, compact, simply connected Riemannian manifold with positive curvature and an effective, isometric T^3 action.

Lemma 6.3. *If there exists an involution with fixed-point set of codimension two, or if there exist two involutions whose fixed-point sets have codimension four, then M has the integral cohomology of $\mathbb{S}^{12}$, $\mathbb{C}\mathrm{P}^6$, and $\mathbb{H}\mathrm{P}^3$.*

This follows immediately from the codimension two and part 1 of the codimension four lemma (Lemmas 4.1 and 4.2). The next case we consider is the following.

Lemma 6.4. *If the codimension of the fixed-point set is four for one involution but at least six for every other, then M is homeomorphic to $\mathbb{S}^{12}$.*

Proof. Note that $N_1 = M^{\iota_1}$ is connected by the codimension four lemma, part 2 (see Lemma 4.3), so $H^i(M;\mathbb{Z}_2) = 0$ for $4 \leq i \leq 5$ by Theorem 3.2. Moreover, if N_1 is fixed by a circle in T^3, then Theorem 3.2 implies $H^3(M;\mathbb{Z}_2) = 0$ and $H^6(M;\mathbb{Z}_2) = 0$ as well. Since M^{ι_1} is connected, we have $\chi(N_1) = \chi(M) = 2 + 2b_2(M)$. But N_1 admits an isometric T^2 action, so Dessai's Euler characteristic calculation in dimension 8 implies that $\chi(N_1) \in \{2, 3, 5\}$. Hence $b_2(M) = 0$ and M is a rational sphere. It now follows that M is homeomorphic to $\mathbb{S}^{12}$ by Lemma 3.3.

We may assume therefore that the induced T^3-action on N_1 has finite kernel. By Fang and Rong's homeomorphism classification, N_1 is homeomorphic to $\mathbb{S}^8$, $\mathbb{C}\mathrm{P}^4$, or $\mathbb{H}\mathrm{P}^2$. By the connectedness lemma, $N_1 \to M$ is 5-connected, so N_1 is homeomorphic to $\mathbb{S}^8$ and M is 5-connected. It follows that $H^6(M;\mathbb{Z})$ is torsion-free with rank $b_6(M) = \chi(M) - 2 = \chi(N_1) - 2 = 0$, and hence that M is again homeomorphic to $\mathbb{S}^{12}$. □

Before continuing with the proof, we remark that it might be surprising that the Euler characteristic of $\mathbb{C}\mathrm{P}^6$ does not appear in Lemma 6.4. The example below shows that, while there exist T^3-actions on $\mathbb{C}\mathrm{P}^6$ that realize the above isotropy data at one fixed point, they need not globally realize the isotropy data at all fixed points.

Example 6.5. Denote points in $\mathbb{C}\mathrm{P}^6$ as equivalence classes $[z_0, z_1, \ldots, z_6]$ where $z_j \in \mathbb{C}$ such that $\sum |z_j|^2 = 1$. Define the actions of three circles on $\mathbb{C}\mathrm{P}^6$ by the following three maps $\mathrm{S}^1 \to \mathrm{PU}(7)$:

$$w \mapsto \mathrm{diag}(1, w, w, 1, 1, 1, 1),$$

$$w \mapsto \mathrm{diag}(1, w, 1, w, w, 1, 1),$$

$$w \mapsto \mathrm{diag}(1, 1, 1, w, 1, w, w).$$

Note that, at the point $x = [1, 0, 0, 0, 0, 0, 0]$, the isotropy representation implies that the codimension of the fixed-point component at x is four for one involution but is greater than four for all of the others. Also note, however, that the product of the involutions in the first and third circle factors of T^3 acts in the same way as $\mathrm{diag}(-1, 1, 1, 1, -1, 1, 1)$ and hence has a codimension-four fixed-point set. Hence this action does not satisfy the hypotheses of Lemma 6.4.

To complete the proof of the Euler characteristic calculation claimed in Theorem 6.2, we need to consider the case where every involution in the torus acts with fixed-point set of codimension at least six.

The key aspect of this case is the rigidity of the maps $\mathbb{Z}_2^3 \to \mathbb{Z}_2^6$ induced by the isotropy representations at fixed points of the torus action. More specifically, at each fixed point x, there exists a choice of basis for the tangent space at x and a choice

of $\rho, \sigma, \tau \in \mathbb{Z}_2^3$ such that the map $\mathbb{Z}_2^3 \to \mathbb{Z}_2^6$ induced by the isotropy representation at x takes the form

$$\rho \mapsto (1,0,0,1,1,0),$$
$$\sigma \mapsto (0,1,0,1,0,1),$$
$$\tau \mapsto (0,0,1,0,1,1).$$

Note that

$$\operatorname{cod}(M_x^\iota) = \begin{cases} 6 & \text{if } \iota \in \{\rho, \sigma, \tau, \rho\sigma\tau\}, \\ 8 & \text{if } \iota \in \{\rho\sigma, \rho\tau, \sigma\tau\}. \end{cases}$$

In particular, every $x \in M^T$ is an isolated fixed point, and we can associate to it a subgroup isomorphic to $\mathbb{Z}_2^2$ inside $\mathbb{Z}_2^3$, the complement of which has the property that every member ι has $\operatorname{cod}(M_x^\iota) = 6$. We call these complements "clubs". In particular, the club $C(x)$ at x consists of the four members ρ, σ, τ, and their three-fold product $\rho\sigma\tau$. In fact, it is an important property that the product of any three elements of a club is in that club. We call this the "triple product property" of clubs.

We analyze how these clubs overlap at distinct fixed points $x, y \in M^T$. One possibility is that the clubs at x and y coincide. By Frankel's theorem, this implies that $M_x^\iota = M_y^\iota$ for all $\iota \in C(x) = C(y)$. As it turns out, there is only one other possibility, namely, that the clubs $C(x)$ and $C(y)$ intersect in exactly two members. Indeed,

- $C(x) \cap C(y)$ contains at least one member, since each club consists of four of the seven non-trivial elements of $\mathbb{Z}_2^3$,
- if $C(x) \cap C(y)$ contains exactly one member, ι, then the product of the three elements of $C(x) \backslash C(y)$ is both equal to ι (by the triple product property) but not in $C(y)$ (since clubs are complements of subgroups), a contradiction, and
- if $C(x) \cap C(y)$ contains at least three members, then the three-fold product equals both the fourth member of $C(x)$ and that of $C(y)$ by the triple product property applied to both clubs, hence these clubs coincide in this case.

Next, we analyze how clubs at three distinct fixed points $x, y, z \in M^T$ might over lap. There are two possibilities (up to relabeling the involutions in $\mathbb{Z}_2^3$). The first is

$$\text{(Type I)} \quad \begin{cases} C(x) = \{\rho, \sigma\tau, \sigma, \rho\tau\}, \\ C(y) = \{\rho, \sigma\tau, \tau, \rho\sigma\}, \\ C(z) = \{\sigma, \rho\tau, \tau, \rho\sigma\}. \end{cases}$$

The second is (Type II), in which $C(x)$ and $C(y)$ are exactly the same as above, and $C(z)$ contains $\rho\sigma\tau$ as well as exactly one involution from each of the sets $C(x) \cap C(y)$, $C(x) \backslash C(y)$, and $C(y) \backslash C(x)$. Note however these choices are neither

unique nor arbitrary since $C(z)$ satisfies the triple product property. We omit the proof, as it follows simply from further analysis using the triple product property of clubs.

Using this club analysis, we claim the following.

Lemma 6.6 (Club analysis). *One of the following occurs*:

(1) *There exists* $\iota \in \mathbb{Z}_2^3$ *whose fixed-point set is connected, has dimension six, and admits an effective, isometric* T^2*-action.*
(2) *There exist three clubs with Type I intersection data, and* M *is not spin.*

Proof. It might happen that some involution ι is in every club. In this case, Frankel's theorem implies that M^ι has dimension six and is connected. Moreover, by the rigidity of the isotropy representation, M^ι is fixed by at most a circle, so in this case the first possibility of the conclusion occurs.

We claim that such an involution ι exists in each of the following three cases:

(1) There exists exactly one club.
(2) There exist exactly two clubs.
(3) There exist at least three clubs, and every subset of three has intersection data of Type II.

Indeed, in each of the first two cases, the club analysis above implies that some involution is in all clubs. In the last case, we verify this as follows. Suppose $C(x) = \{\iota_1, \iota_2, \iota_3, \iota_4\}$ and $C(y) = \{\iota_1, \iota_2, \iota_5, \iota_6\}$ are two of the clubs. We assume that no involution is in every club and proceed by contradiction. Since the intersection data of every club with $C(x)$ and $C(y)$ is of Type II, there exist clubs $C(z)$ and $C(w)$ such that one contains ι_1 but not ι_2 and vice versa for the other club. Next our assumption implies that the clubs $C(y)$, $C(z)$, and $C(w)$ have intersection data of Type II. A general property of such triples of clubs is that their union contains all seven of the non-trivial involutions in $\mathbb{Z}_2^3$. In particular, $C(z) \cup C(w)$ contains both ι_3 and ι_4. By applying the same line of reasoning to the triple of clubs $C(x)$, $C(z)$, and $C(w)$, we conclude that $C(z) \cup C(w)$ contains ι_5 and ι_6. Altogether we have that $C(z) \cup C(w)$ contains all seven non-trivial involutions in $\mathbb{Z}_2^3$. This contradicts the fact that any two clubs either coincide or intersect in exactly two elements. This completes the proof of the claim.

Assuming the claim now, there exist at least three (distinct) clubs, and among these there exist three with intersection data of Type I. We see immediately in this case that M is not spin. Indeed, fixed-point sets of involutions on spin manifolds have all components of codimension congruent to c modulo four, for some $c \in \{0, 2\}$. In our setting, no involution is in every club, so any member $\iota \in C(x)$ has fixed-point components of codimension six and eight. □

We require one more lemma whose proof relies on further club analysis, together with the equivariant diffeomorphism classification of Grove and Searle.

Lemma 6.7. *Assume M is as in Theorem* 6.2, *and that every involution in the torus T has fixed-point set of codimension at least six. Suppose $\iota \in T$ is an involution and $P \subseteq M^\iota$ is a component of dimension four.*

- *If P is $\mathbb{S}^4$, then the two fixed-points in $P^T = P \cap M^T$ have the same club.*
- *If P is $\mathbb{C}\mathrm{P}^2$, then the three fixed points in P^T represent three distinct clubs.*

Proof. Fix $\iota \in \mathbb{Z}_2^3$ and a four-dimensional component $P \subseteq M^\iota$. By the rigidity of the isotropy maps, the induced T^3-action on P has one-dimensional kernel that does not contain any other involution. Let $\overline{T}$ denote the two-dimensional torus equal to the quotient of T^3 by the kernel of this induced action. The induced action of $\overline{T}$ on P is equivariant to a linear action on $\mathbb{S}^4$ or $\mathbb{C}\mathrm{P}^2$. Applying isotropy rigidity again, it follows that, for all $p \in P^{\overline{T}} = P \cap M^T$, there exist ι_1 and ι_2 such that the isotropy map at p takes the form

$$\iota \mapsto (1,1,1,1,0,0),$$
$$\iota_1 \mapsto (*,*,*,*,1,0),$$
$$\iota_2 \mapsto (*,*,*,*,0,1).$$

By the rigidity of the isotropy at p, it follows that the club at p is given by

$$C(p) = \{\iota_1, \iota\iota_1, \iota_2, \iota\iota_2\}.$$

To complete the proof, first suppose P is a sphere, and denote the fixed points in $P \cap M^T$ by p_1 and p_2. Since the $\overline{T}$-action on P is equivariant to a linear action on $\mathbb{S}^4$, it follows that $P^\tau_{p_1}$ and $P^\tau_{p_2}$ have the same dimension for all involutions τ in $\overline{T}$. In particular, the ι_1 and ι_2 from the previous paragraph are the same for p_1 and p_2, and hence the clubs coincide at these points.

Now suppose P is $\mathbb{C}\mathrm{P}^2$. We use the fact that the $\overline{T}$-action on P is equivariant to a linear action. In particular, each involution $\iota \in \overline{T}$ has the property that P^ι consists of a copy of $\mathbb{S}^2$ together with an isolated point. In particular, if p_1 and p_2 are two points in $P^{\overline{T}}$ with the same club, and if ι_1 and ι_2 are the involutions as above such that $C(p_1) = C(p_2) = \{\iota_1, \iota\iota_1, \iota_2, \iota\iota_2\}$, then the third point, $p_3 \in P^{\overline{T}}$, is an isolated fixed point of the actions of ι_1 and ι_2 on P. But then the product $\iota_1\iota_2$ acts on $T_{p_3}P$ as the identity and hence fixes P. This contradicts the rigidity of the isotropy maps. □

We now complete the proof of the Euler characteristic calculation claimed in Theorem 6.2. We do this is three steps (see Lemmas 6.8–6.10).

Lemma 6.8. *If every involution has fixed-point set of codimension at least six, then one of the following occurs*:

(1) $\chi(M) \in \{2, 4, 6, \ldots, C(6)\}$, *or*
(2) $7 \leq \chi(M) \leq \frac{7}{4}C(6)$ *and M is not spin.*

Proof. If there exists $\iota \in \mathbb{Z}_2^3$ with connected fixed-point set of dimension six as in case (1) of Lemma 6.6, then we have $\chi(M) = \chi(M^\iota) \in \{2, 4, 6, \ldots, C(6)\}$.

Assume instead that we are in case (2) of Lemma 6.6. There exist $x, y, z \in M^T$ whose clubs $C(x)$, $C(y)$, and $C(z)$ are distinct and have Type I intersection data. In other words, we may label the non-trivial involutions by $\iota_0, \ldots, \iota_6$ so that $C(x) = \{\iota_1, \iota_2, \iota_3, \iota_4\}$, $C(y) = \{\iota_1, \iota_2, \iota_5, \iota_6\}$ and $C(z) = \{\iota_3, \iota_4, \iota_5, \iota_6\}$. In particular, ι_i is in some club for all $1 \leq i \leq 6$, so the fixed-point set of M^{ι_i} has a component N_i of codimension six. Moreover, there is a unique such fixed-point component by Frankel's theorem. If ι_0 is in some club (different from one of the three above), let N_0 denote the six-dimensional component of its fixed-point set; otherwise, let N_0 denote the empty set.

Let $X \subseteq M^T$ denote the set of fixed points whose club equals $C(x)$. Define Y and Z similarly, and let W denote the fixed points of T whose club is distinct from $C(x)$, $C(y)$, and $C(z)$. Note the following facts:

- X, Y, Z, and W partition the fixed-point set M^T.
- $N_{2i-1} \cap W$ and $N_{2i} \cap W$ partition W for all $i \in \{1, 2, 3\}$. To see this, first suppose $w \in W \backslash (N_{2i} \cup N_{2i-1})$. Then ι_{2i-1} and ι_{2i} are not in $C(w)$. Since the complements of clubs are rank two subgroups, we have $\iota_0 = \iota_{2i-1}\iota_{2i} \notin C(w)$ and hence that $C(w)$ equals $C(x)$, $C(y)$, or $C(z)$, a contradiction. Next suppose $w \in W \cap N_{2i-1} \cap N_{2i}$. Then $\iota_{2i-1}, \iota_{2i} \in C(w)$. Since $C(w)$ is distinct from both $C(x)$ and $C(y)$, we have $\iota_3, \ldots, \iota_6 \notin C(w)$ by the triple product property. But now $C(w)$ has at most three involutions, a contradiction.
- Each $w \in W$ lies in exactly one or all three of N_1, N_3 or N_5. Indeed, w is in at least one of these sets by the previous fact and the triple product property. In addition, if for example $w \in N_1 \cap N_3$, then $\iota_1, \iota_3 \in C(w)$. By the triple product property and the definition of W, we have $\iota_2 \notin C(w)$ and $\iota_4 \notin C(w)$. Hence $\iota_6 = \iota_2\iota_4 \notin C(w)$ and we have $C(w) = \{\iota_1, \iota_3, \iota_5, \iota_0\}$. In particular, $w \in N_1 \cap N_3 \cap N_5$.
- $w \in W$ if and only if $\iota_0 \in C(w)$. If $\iota_0 \in C(w)$, then $w \in W$ by definition of W. Conversely, if $w \in W$ and $w \in N_1 \cap N_3 \cap N_5$, then $\iota_0 \in C(w)$ by the proof of the previous point. If instead, say, $w \in W \cap N_1$ but $w \notin N_3 \cup N_5$, then the complement of $C(w)$ contains ι_3, ι_5, and $\iota_3\iota_5 = \iota_2$. Hence $\iota_0 \in C(w)$, as claimed.

We denote $\chi(X)$ by $|X|$ and similarly for the orders of Y, Z, and W. The last point above implies that $W = (N_0)^T$. Hence

$$|W| = \chi(N_0). \tag{6.1}$$

Next, note that $X \cup Y = (N_1 \cap N_2)^T$ by the second point above. Similarly, $X \cup Z = (N_3 \cap N_4)^T$ and $Y \cup Z = (N_5 \cap N_6)^T$. Hence

$$|X| + |Y| = \chi(N_1 \cap N_2), \tag{6.2}$$

$$|X| + |Z| = \chi(N_3 \cap N_4), \tag{6.3}$$

$$|Y| + |Z| = \chi(N_5 \cap N_6). \tag{6.4}$$

Next, note that $X \cup Y \cup W = (N_1 \cup N_2)^T$ and that we have similar statements for $N_3 \cup N_4$ and $N_5 \cup N_6$. Hence

$$|X| + |Y| + |W| = \chi(N_1) + \chi(N_2) - \chi(N_1 \cap N_2), \tag{6.5}$$

$$|X| + |Z| + |W| = \chi(N_3) + \chi(N_4) - \chi(N_3 \cap N_4), \tag{6.6}$$

$$|Y| + |Z| + |W| = \chi(N_5) + \chi(N_6) - \chi(N_5 \cap N_6). \tag{6.7}$$

Adding together Eqs. (6.1)–(6.7), we conclude

$$4\chi(M) = \sum_{j=0}^{6} \chi(N_i). \tag{6.8}$$

We first use this equation to bound $\chi(M)$ from below. Since no ι_j is in every club, every M^{ι_j} is composed of the six-manifold N_j together with a disjoint union of 4-manifolds, each of which has Euler characteristic at least two. Hence $\chi(M) \geq \chi(N_j) + 2$ for all $1 \leq j \leq 6$. In addition, M^{ι_0} contains N_0 as well as X, Y, and Z, so $\chi(M) \geq \chi(N_0) + 3$. Hence Eq. (6.8) implies

$$4\chi(M) \leq (\chi(M) - 3) + \sum_{j=1}^{6} (\chi(M) - 2),$$

which implies that $\chi(M) \geq 5$. In fact, $\chi(M) = 5$ would imply that $\chi(N_j) \leq 3$ for all $1 \leq j \leq 6$. But $\chi(N_j)$ is both positive and even, so $\chi(N_j) = 2$ for all $1 \leq j \leq 6$. Since $\chi(N_0) \leq \chi(M) - 3 = 2$, Equation (6.8) implies $4\chi(M) \leq 7(2) = 14$, a contradiction. In fact, $\chi(M) = 6$ also cannot occur. Indeed, suppose $\chi(M) = 6$. Then $\chi(N_0) \leq 3$, so $\chi(N_0) = 2$. But now $\chi(M^{\iota_0} \backslash N_0) = 4$, a contradiction to Lemma 6.7, which implies that these four points must come in two pairs such that, in each pair, the two clubs are the same. But these four points together represent three clubs (their union is $X \cup Y \cup Z$), a contradiction.

To complete the proof of the lemma, it suffices to prove $\chi(M) \leq \frac{7}{4}C(6)$. By the rigidity of the isotropy maps, N_i admits an effective, isometric T^2-action and hence has $\chi(N_i) \leq C(6)$ for all $1 \leq i \leq 6$. The same estimate on $\chi(N_0)$ holds, even if it is empty. The upper bound now follows from Eq. (6.8). □

Lemma 6.9. *Let M be as in Lemma* 6.8. *If $C(6) = 6$, then $\chi(M) \leq 9$.*

Proof. We keep the notation from above, but now we assume $\chi(N_i) \leq 6$ for all $0 \leq i \leq 6$. Equation (6.8) implies $4\chi(M) \leq 42$, and hence $\chi(M) \leq 10$.

Suppose that $\chi(M) = 10$. Equation (6.8) implies $\chi(N_0) \geq 4$. Moreover, if $\chi(N_0) = 6$, then we have $|X| = |Y| = 1$ and $|Z| = 2$ without loss of generality, and one can show that x and y can be replaced by suitable choices of w_1 and w_2 in W so that the three clubs $C(z)$, $C(w_1)$, and $C(w_2)$ are distinct, have intersection data of Type I, and have the property that the involution not in $C(z) \cup C(w_1) \cup C(w_2)$ has maximal component of Euler characteristic four. In

other words, we may assume without loss of generality that $\chi(N_0) = 4$. By Eq. (6.8), $\chi(N_j) = 6$ for all $1 \leq j \leq 6$. By Eqs. (6.3)–(6.7), it follows that $|X| = |Y| = |Z| = 2$. Write $W = \{w_1, w_2, w_3, w_4\}$. Up to relabeling the w_i, we have that

$$N_1^T = X \cup Y \cup \{w_1, w_2\}.$$

Now every $w \in W$ lies in exactly one or three of N_1, N_3, and N_5. Without loss of generality, this implies that

$$N_3^T = X \cup Z \cup \{w_1, w_3\}, \quad N_5^T = Y \cup Z \cup \{w_1, w_4\}.$$

In particular, ι_3 is in the club $C(w_3)$ but not $C(w_4)$, so $C(w_3) \neq C(w_4)$.

On the other hand, consider the fixed-point set M^{ι_1}. One component is N_1, which has Euler characteristic six. The others are closed, oriented, positively curved 4-manifolds. By Lemma 6.7, we have that $M^{\iota_1} = N_1 \cup P \cup Q$ where P and Q are diffeomorphic to $\mathbb{S}^4$, and where the two fixed-points in P have equal clubs and likewise for the fixed points of Q. In particular, the four points in $(P \cup Q)^T$ represent only two clubs. But $(P \cup Q)^T$ contains the two points in Z, as well as w_3 and w_4. These points represent three distinct clubs, so we have the desired contradiction. □

To complete the proof of the Euler characteristic calculation claimed in Theorem 6.2, it suffices to prove the following.

Lemma 6.10. *Let M be as in Lemma 6.8. If M is rationally elliptic, then $\chi(M) \leq 10$ or $\chi(M) = 12$.*

Proof. We keep the notation from the proof of Lemma 6.8. Consider one of the submanifolds N_i in Eq. (6.8). If N_i admits an effective, isometric T^3 action, then $\chi(N_i) \in \{2, 4\}$ by Grove and Searle's classification. Otherwise, N_i is a fixed-point component of a circle action on M. Since M is rationally elliptic, it follows that N_i is rationally elliptic. Since N_i is a simply connected, closed manifold, it follows that $\chi(N_i) \leq \chi(\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2) = 8$. Since this estimate holds for all $0 \leq i \leq 6$, Eq. (6.8) implies that $\chi(M) \leq 14$.

To complete the proof, it suffices to observe that no rationally elliptic, simply connected, closed manifold of dimension 12 can have Euler characteristic 11, 13, or 14. For a proof, one can use the existence of a pure minimal model for such a space and apply the properties listed in the beginning of the proof of Theorem 10.2. The calculation also immediately follows from the results in Table 4. □

This completes the proof of the Euler characteristic calculation. For the elliptic genus, see Sec. 9. It suffices to compute the signature.

Proof of Theorem 6.2 (Signature calculation). First, if M is as in Lemmas 6.3 or 6.4, then M is an integral cohomology $\mathbb{S}^{12}$, $\mathbb{C}\mathrm{P}^6$, or $\mathbb{H}\mathrm{P}^3$. In each of these cases, it follows that $|\sigma(M)|$ is 0 or 1, according to the parity of $\chi(M)$.

Second, if M is as in the first possibility of Lemma 6.6, then there exists an involution ι such that M^ι is a closed, connected, simply connected 6-manifold. Hence $\sigma(M) = \sigma(M^\iota) = 0$. Note that this is consistent with the fact that $\chi(M)$ is even in this case by the proof of Lemma 6.8.

We may therefore assume M is as in the second possibility of Lemma 6.6. In particular, M is not spin, no fixed-point set M^{ι_i} is connected, and $7 \leq \chi(M) \leq \frac{7}{4}C(6)$. Since $C(6) \leq 14$, it follows that $7 \leq \chi(M) \leq 24$.

We cannot calculate the signature in all cases, so we further assume $\chi(M) \leq 13$. The proof of Lemma 6.8 (in particular, Eq. (6.8)) implies $\chi(M) - \chi(N_i) \in \{2, 3, 4, 5\}$ for some $0 \leq i \leq 6$, where the N_i are the 6-dimensional fixed-point components of the ι_i as in the proof of Lemma 6.8. In particular, we have the following possibilities:

- $M^{\iota_i} = N_i \cup \mathbb{S}^4$, and so $\sigma(M) = \sigma(\mathbb{S}^4) = 0$,
- $M^{\iota_i} = N_i \cup \mathbb{CP}^2$, and so $|\sigma(M)| = |\sigma(\mathbb{CP}^2)| = 1$,
- $M^{\iota_i} = N_i \cup \mathbb{S}^4 \cup \mathbb{S}^4$, and so $\sigma(M) = 0$, or
- $M^{\iota_i} = N_i \cup \mathbb{S}^4 \cup \mathbb{CP}^2$, and so $|\sigma(M)| = 1$.

Hence the signature is 0 or ± 1 according to the parity of $\chi(M)$. □

We remark that an extension of this argument shows that $|\sigma(M)| = 1$ if there exists some N_i with $\chi(M) - \chi(N_i) = 7$. Using this together with the fact that $|\sigma(M)| \equiv \chi(M) \bmod 2$, one can further compute that $|\sigma(M)| = 1$ if $\chi(M) \in \{15, 17, 19\}$.

7. Dimension 14

The only simply connected, smooth, closed manifolds of dimension 14 known to admit positive sectional curvature are $\mathbb{S}^{14}$ and $\mathbb{CP}^7$. The following result provides a sharp calculation of the Euler characteristic and the second and third homology groups of a positively curved 14-manifold in the presence of T^4 symmetry.

Theorem 7.1. *If M^{14} is a closed, simply connected, positively curved Riemannian manifold with T^4 symmetry, then one of the following occurs.*

- *M is 3-connected and $\chi(M) = 2$.*
- *$H_*(M;\mathbb{Z}) \cong H_*(\mathbb{CP}^7;\mathbb{Z})$, and $H^*(M;\mathbb{Z})$ is generated by some $z \in H^2(M;\mathbb{Z})$ and $x \in H^4(M;\mathbb{Z})$ subject to the relation $z^2 = mx$ for some $m \in \mathbb{Z}$.*

Note that, if T^4 is replaced by T^5 in this statement, then M is tangentially homotopy equivalent to $\mathbb{S}^{14}$ or $\mathbb{CP}^7$ (see [13, 59]). We proceed to the proof.

Lemma 7.2. *If a non-trivial involution $\iota_1 \in T^4$ exists such that* $\operatorname{cod}(M^{\iota_1}) \leq 4$, *then one of the following occurs*:

- *M is homeomorphic to $\mathbb{S}^{14}$.*
- *$H_*(M;\mathbb{Z}) \cong H_*(\mathbb{CP}^7;\mathbb{Z})$, and $H^*(M;\mathbb{Z})$ is generated by some $z \in H^2(M;\mathbb{Z})$ and $x \in H^4(M;\mathbb{Z})$ subject to the relation $z^2 = mx$ for some $m \in \mathbb{Z}$.*

As the proof indicates, a weak version of this lemma is an easy consequence of the codimension four lemma. For the stronger conclusion, we require our calculation in dimension 10 (see Theorem 5.1).

Proof. Let $M_x^{\iota_1}$ be a fixed-point component of an involution of maximum dimension. By the codimension two lemma, we may assume $\operatorname{cod}(M_x^{\iota_1}) = 4$. By maximality, $M_x^{\iota_1}$ has T^3 (or T^4) symmetry. Theorem 5.1 implies that $b_2(M_x^{\iota_1}) = 1$ or $M_x^{\iota_1}$ is homeomorphic to $\mathbb{S}^{10}$. In the latter case, the connectedness lemma implies M is homeomorphic to $\mathbb{S}^{14}$. In the former case, if $M_x^{\iota_1}$ has T^4 symmetry, then it is a cohomology $\mathbb{C}\mathrm{P}^5$ by Wilking's homotopy classification, and M is a cohomology $\mathbb{C}\mathrm{P}^7$ by the connectedness lemma. We assume now that $H_2(M;\mathbb{Z}) \cong \mathbb{Z}$ and that $M_x^{\iota_1}$ is fixed by a circle in T^4.

Consider the map $\mathbb{Z}_2^4 \to \mathbb{Z}_2^7$ induced by the isotropy representation of T^4 at x. There exists $\iota_2 \in \mathbb{Z}_2^4 \backslash \langle \iota_1 \rangle$ such that $\operatorname{cod}(M_x^{\iota_2}) \leq 6$, and the codimension four lemma, part 1, applies. The lemma follows if $\operatorname{cod}(M_x^{\iota_2}) \leq 4$, so we may assume $\operatorname{cod}(M_x^{\iota_2}) = 6$. Returning to the isotropy map $\mathbb{Z}_2^4 \to \mathbb{Z}_2^7$, we see that we may choose ι_2 so that, in addition, $M_x^{\iota_1}$ and $M_x^{\iota_2}$ intersect non-transversely. From the previous paragraph, we have without loss of generality that $M_x^{\iota_2}$ has T^4 symmetry and hence is diffeomorphic to $\mathbb{C}\mathrm{P}^4$ by the diffeomorphism classification of Grove and Searle. Since the intersection of $M_x^{\iota_1}$ and $M_x^{\iota_2}$ is not transverse, the intersection of any two of $M_x^{\iota_1}$, $M_x^{\iota_2}$, and $M_x^{\iota_1\iota_2}$ equals the 6-dimensional manifold $M_x^{\langle \iota_1, \iota_2 \rangle}$. By the connectedness lemma, $M_x^{\iota_2}$, $M_x^{\iota_1\iota_2}$, and $M_x^{\langle \iota_1, \iota_2 \rangle}$ are cohomology complex projective spaces. By the containment lemma,

$$M^{T^4} \subseteq M_x^{\iota_1} \cup M_x^{\iota_2} \cup M_x^{\iota_1\iota_2},$$

so the inclusion–exclusion property of Euler characteristics implies

$$\chi(M) - \chi(M_x^{\iota_1}) = 5 + 5 - 3(4) + (4) = 2.$$

On the other hand, the periodicity lemma together with the fact that $M_x^{\iota_1}$ is fixed by a circle in T^4 imply that

$$\chi(M) - \chi(M_x^{\iota_1}) = -b_5(M) + 2b_6(M) - b_7(M) \leq 2b_2(M) = 2.$$

Combining the previous two computations, we conclude that M has the Betti numbers of $\mathbb{C}\mathrm{P}^7$. Applying the periodicity lemma again, we conclude that $\mathbb{Z} \cong H^2(M;\mathbb{Z}) \cong H^6(M;\mathbb{Z})$ and hence that $H_5(M;\mathbb{Z}) = 0$. Recalling that $H_3(M;\mathbb{Z}) = 0$, it follows that $H_*(M;\mathbb{Z}) \cong H_*(\mathbb{C}\mathrm{P}^7;\mathbb{Z})$. Finally, the connectedness lemma applied to the inclusion $M_x^{\iota_2} \to M$ and Poincaré duality imply that $H^*(M;\mathbb{Z}) \cong H^*(\mathbb{C}\mathrm{P}^7;\mathbb{Z})$. □

To complete the proof of the theorem, it suffices to prove the following.

Lemma 7.3. *If the fixed-point set of every involution has codimension at least* 6, *then* M *is* 3*-connected and has Euler characteristic two.*

The proof in this case is similar to, but not as hard as, the corresponding lemma in dimension 12. As there, the key aspect is the rigidity of the maps $\mathbb{Z}_2^4 \to \mathbb{Z}_2^7$ induced by the isotropy representations at fixed points $x \in M^{T^4}$.

Proof. Let T denote a torus of rank four acting effectively and isometrically on M. Since every non-trivial involution $\iota \in T$ has $\operatorname{cod}(M^\iota) \geq 6$, we have rigidity in the isotropy representation $\mathbb{Z}_2^4 \to \mathbb{Z}_2^7 \subseteq \mathrm{SO}(T_xM)$. Indeed, for every $x \in M^T$, there exist a basis of T_xM and a generating set of involutions $\iota_1, \ldots, \iota_4 \in \mathbb{Z}_2^4$ such that the map $\mathbb{Z}_2^4 \to \mathbb{Z}_2^7$ can be represented as follows:

$$\iota_1 \mapsto (1,1,0,1,0,0,0),$$

$$\iota_2 \mapsto (1,0,1,0,1,0,0),$$

$$\iota_3 \mapsto (0,1,1,0,0,1,0),$$

$$\iota_4 \mapsto (1,1,1,0,0,0,1).$$

In particular, at every $x \in M^T$, there exist seven distinct involutions whose fixed-point component containing x has codimension six, seven with codimension eight, and exactly one with codimension 14.

As a consequence of this rigidity and Frankel's theorem, if ι_1 and ι_2 are distinct involutions such that $N_1 = M_x^{\iota_1}$ and $N_2 = M_x^{\iota_2}$ have codimension six, then $N_1 \cup N_2$ contains M^T. By the connectedness lemma, $N_1 \cap N_2$ is one-connected and $b_2(N_1 \cap N_2) \geq b_2(N_i) = b_2(M)$ for $i \in \{1,2\}$. By the rigidity of the isotropy representation, N_1 and N_2 have T^3 symmetry and $N_1 \cap N_2$ has T^2 symmetry. By the classifications of Grove–Searle and Fang–Rong, all three submanifolds are homotopy spheres or complex or quaternionic projective spaces. By the inclusion–exclusion property of Euler characteristics, we obtain the estimate

$$\chi(M) = \chi(N_1) + \chi(N_2) - \chi(N_1 \cap N_2) \leq 7.$$

In fact, $\chi(M) \equiv b_7(M) \equiv 0 \bmod 2$ by Poincaré duality, so $\chi(M) \in \{2,4,6\}$.

In particular, the number of fixed points of T is less than the number of involutions $\iota \in T$ with $\operatorname{cod}(M_x^\iota) = 6$. At most one of these involutions can have $\operatorname{cod}(M_y^\iota) = 14$ at any one fixed point $y \in M^T$, so there exists some involution ι with $\operatorname{cod}(M_y^\iota) < 14$ for all $y \in M^T$. By the rigidity of the isotropy representation and Frankel's theorem, M^ι is connected. Applying Fang and Rong's classification again, we conclude that M^ι either is homeomorphic to $\mathbb{S}^8$ or has odd Euler characteristic. Since $\chi(M) = \chi(M^\iota)$, we conclude that M^ι is homeomorphic to $\mathbb{S}^8$, $\chi(M) = 2$, and M is 3-connected. □

8. Dimension 16

The four known, simply connected, compact, positively curved examples in dimension 16 are the rank one symmetric spaces, $\mathbb{S}^{16}$, CaP^2, $\mathbb{H}\mathrm{P}^4$, and $\mathbb{C}\mathrm{P}^8$, and each of these admits a positively curved metric with T^4 symmetry.

Theorem 8.1. *If M^{16} is a closed, simply connected Riemannian manifold with positive sectional curvature and T^4 symmetry, then one of the following occurs*:

- $\chi(M) = 2$ *and* $\sigma(M) = 0$.
- $\chi(M) = 3$, $\sigma(M) = \pm 1$, *and M is 2-connected.*
- $\chi(M) = 5$, $\sigma(M) = \pm 1$, $H_{2+4i}(M;\mathbb{Z}) = 0$ *for all i, and* $b_4(M) = 1 + 2b_3(M)$.
- $\chi(M) = 9$, $\sigma(M) = \pm 1$, $H_2(M;\mathbb{Z}) = \mathbb{Z}$, $H_3(M;\mathbb{Z}) = 0$, $H_4(M;\mathbb{Z}) = \mathbb{Z}$, *and* $H^{16}(M;\mathbb{Z})$ *is generated by an element of the form x^5y with $x \in H^2(M;\mathbb{Z})$ and $y \in H^6(M;\mathbb{Z})$. Moreover, M is not spin.*

If moreover M is spin, then the elliptic genus is constant.

The proof of the last claim is in Sec. 9. The proof of the first claim takes the rest of the section and is contained in Lemmas 8.2, 8.4, and 8.5.

Let M be as Theorem 8.1, let T denote a torus of rank four acting effectively and isometrically on M, and let $\mathbb{Z}_2^4 \subset T$ denote the subgroup of involutions.

Lemma 8.2. *If there exists $\iota_1 \in \mathbb{Z}_2^4$ with $\operatorname{cod}(M^{\iota_1}) \in \{2, 4\}$, then*

(1) *M is homeomorphic to $\mathbb{S}^{16}$,*
(2) *M is homotopy equivalent to $\mathbb{C}\mathrm{P}^8$, or*
(3) *$\chi(M) = 5$, $\sigma(M) = \pm 1$, $H_{2+4i}(M;\mathbb{Z}) = 0$ for all i, and $b_4(M) = 1 + 2b_3(M)$.*

The proof is a bit involved, and we would like to illustrate by example some of the structure we recover in the most difficult case of the proof.

Example 8.3. Denote points in $\mathbb{H}\mathrm{P}^4$ as equivalence classes $[q_0, q_1, \ldots, q_4]$ where $q_s \in \mathbb{H}$ such that $\sum |q_s|^2 = 1$. Define the actions of four circles on $\mathbb{H}\mathrm{P}^4$ by the following three maps $\mathrm{S}^1 \to \mathrm{Sp}(5)$:

$$w \mapsto \operatorname{diag}(w, 1, 1, 1, 1),$$
$$w \mapsto \operatorname{diag}(w, w, w, w, w),$$
$$w \mapsto \operatorname{diag}(1, 1, w, 1, w),$$
$$w \mapsto \operatorname{diag}(1, 1, 1, w, w).$$

The involution ι_1 in the first circle fixes a component $\mathbb{H}\mathrm{P}^3$ of codimension four. The involution ι_2 in the second circle acts trivially, so really we consider the action by the quotient of the second circle by $\{\pm 1\}$. The involution in this circle is then represented by (i, i, i, i, i), and its fixed-point set is a $\mathbb{C}\mathrm{P}^4$. Notice that all fixed points of the T^4 action are isolated, and that they come in two types. If $\iota_1, \ldots, \iota_4$ denote the involutions in the four circles above, then at the point $z = [1, 0, 0, 0, 0]$ one has

$$\operatorname{cod}(M_z^{\iota}) = \begin{cases} 8 & \text{if } \iota \notin \langle \iota_1 \rangle, \\ 16 & \text{if } \iota = \iota_1 \end{cases}$$

while at any of the other fixed points y, one has

$$\operatorname{cod}(M_y^\iota) = \begin{cases} 4 & \text{if } \iota = \iota_1, \\ 8 & \text{if } \iota \notin \iota_1 C(y), \\ 12 & \text{if } \iota \in \iota_1 C(y) \backslash \{\iota_1\} \end{cases}$$

for some subgroup $C(y)$ isomorphic to $\mathbb{Z}_2^2$ depending on y. For example, $C(y) = \langle \iota_3, \iota_4 \rangle$ at $y = [0,1,0,0,0]$. Notice also the fixed-point component M_y^ι is $\mathbb{C}\mathrm{P}^4$ for $\iota \in \iota_2 C(y) \cup \iota_1\iota_2 C(y)$ and is $\mathbb{H}\mathrm{P}^2$ for non-trivial $\iota \in C(y)$. Finally, note that the fixed-point set of the T^4 action is contained in the two-fold union of a large number of choices of eight-dimensional fixed-point components of involutions. We recover this combinatorial and topological fixed point data in one case of the proof of Lemma 8.2, however we are unable to fully recover the topology of M.

Proof of Lemma 8.2. By the codimension two lemma, we may assume that $\operatorname{cod}(M^{\iota_1}) = 4$. Choose $x \in M^T$ such that $\operatorname{cod}(M_x^{\iota_1}) = 4$. By the codimension four lemmas, we may assume that every other non-trivial involution $\iota \in T$ has $\operatorname{cod}(M^\iota) \geq 8$ and that $\operatorname{cod}(M_z^\iota) = 8$ for any $z \in M^T \backslash M_x^{\iota_1}$ and $\iota \in \mathbb{Z}_2^4 \backslash \langle \iota_1 \rangle$.

Let $y \in M^T \cap M_x^{\iota_1}$. The isotropy map at y is also rigid in the sense that there exist a rank-two subgroup $C(y) \subseteq \mathbb{Z}_2^4$ and an involution $\iota_2(y) \notin C(y)$ such that $\operatorname{cod}(M_y^\iota) = 8$ for each of the 11 non-trivial involutions ι not in the coset $\iota_1 C(y)$. Moreover, if ι is one of these 11 involutions, the intersection $M_x^{\iota_1} \cap M_y^\iota$ is transverse if and only if $\iota \in C(y)$.

Fix ι_2 to be any choice of $\iota_2(x)$, and fix any choice of distinct $\iota_3, \iota_4 \in C(x)$. From this isotropy rigidity and Frankel's theorem, all of the following hold:

(1) $\operatorname{cod}(M_x^\iota) = 8$ and $M^T \subset M_x^{\iota_1} \cup M_x^\iota$ for all $\iota \in \{\iota_2, \iota_3, \iota_4, \iota_3\iota_4\}$.
(2) $M_x^{\iota_1} \cap M_x^{\iota_2} \subset M_x^{\iota_2}$ is 4-connected and has codimension two.
(3) $M_x^{\iota_1} \cap M_x^\iota \subset M_x^\iota$ is 4-connected and has codimension four for $\iota \in \{\iota_3, \iota_4, \iota_3\iota_4\}$.
(4) $M^T \subseteq M_x^{\iota_3} \cup M_x^{\iota_4} \cup M_x^{\iota_3\iota_4}$.

Set $\epsilon = \chi(M) - \chi(M_x^{\iota_1})$. By the first claim above, $\epsilon = \chi(M_x^{\iota_2}) - \chi(M_x^{\iota_1} \cap M_x^{\iota_2})$. By the second claim and the codimension two lemma, $M_x^{\iota_2}$ and $M_x^{\iota_1} \cap M_x^{\iota_2}$ are cohomology spheres or complex projective spaces. In either case, the difference in their Euler characteristics is $\epsilon \in \{0,1\}$, and $\chi(M_x^{\iota_2}) = 2 + 3\epsilon$. Note that the fixed points of T are isolated in this case, so

$$\chi(M) \geq \chi(M_x^{\iota_2}) \geq 2 + 3\epsilon.$$

We prove now that the opposite inequality holds. Fix for a moment $\iota \in \{\iota_3, \iota_4, \iota_3\iota_4\}$. By the first claim, $\epsilon = \chi(M_x^\iota) - \chi(M_x^{\iota_1} \cap M_x^\iota)$. By the third claim, M_x^ι has 4-periodic integral cohomology and second Betti number equal to that of $M_x^{\iota_1} \cap M_x^\iota$. In particular, since $\epsilon \in \{0,1\}$, we have $\chi(M_x^\iota) = 2 + \epsilon$ for all $\iota \in \{\iota_3, \iota_4, \iota_3\iota_4\}$. Now consider any two-fold intersection $M_x^\iota \cap M_x^{\iota'}$ of these three

submanifolds. Since the fixed points of T are isolated, the Euler characteristic of $M_x^\iota \cap M_x^{\iota'}$ is at least that of its component $M_x^{\langle \iota, \iota' \rangle}$. This component is a closed, oriented, positively curved 4-manifold, so its Euler characteristic is at least two. In addition, this Euler characteristic is at least that of the three-fold intersection $M_x^{\iota_3} \cap M_x^{\iota_4} \cap M_x^{\iota_3\iota_4}$. Putting these estimates together, the fourth claim above and the inclusion–exclusion property of Euler characteristics imply

$$\chi(M) \leq 3(2+\epsilon) - 2 - 2 - 0 = 2 + 3\epsilon.$$

Since the opposite inequality also holds, we conclude $\chi(M) = 2 + 3\epsilon \in \{2, 5\}$.

For the signature, recall that $M_x^{\iota_2}$ has 2-periodic cohomology and is either $\mathbb{S}^8$ or $\mathbb{C}\mathrm{P}^4$, according to whether ϵ is 0 or 1. In either case $\chi(M) = \chi(M_x^{\iota_2})$, so M^{ι_2} is connected since the fixed points of T are isolated. In particular, $\sigma(M) = \sigma(M_x^{\iota_2})$, which is either 0 or ± 1, according to whether $\chi(M)$ is 2 or 5, respectively.

We conclude the rest of the lemma by considering cases. Let $N_1 = M_x^{\iota_1}$, and let $\dim \ker(T|_{N_1})$ denote the dimension of the kernel of the induced T-action on N_1.

- If $\dim \ker(T|_{N_1}) = 0$, Wilking's homotopy classification implies that N_1 is a cohomology sphere or quaternionic projective space. By the connectedness lemma, M is as well.
- If $\dim \ker(T|_{N_1}) = 1$ and $\chi(M) = 2$, then $\chi(N_1) = 2$ as well. Since the fixed points of T are isolated, M^{ι_1} is connected and hence $H^i(M; \mathbb{Z}_2) = 0$ for $3 \leq i \leq n-3$. Since $\chi(M) = 2 + 2b_2(M)$, we have $b_2(M) = 0$ and hence that M is homeomorphic to $\mathbb{S}^{16}$ by Lemma 3.3.
- If $\dim \ker(T|_{N_1}) = 1$ and $\chi(M) = 5$, the proof shows that $M_x^{\iota_3}$ is homeomorphic to $\mathbb{H}\mathrm{P}^2$. By Wilking's maximal smooth symmetry rank classification for an integral quaternionic projective spaces (see [59, Theorem 3]), $M_x^{\iota_3}$ is fixed by a circle in T. In particular, $M_x^{\iota_3} \to M$ is 2-connected, so $H_2(M; \mathbb{Z}) = 0$. The periodicity lemma now implies that $H^{2+4i}(M; \mathbb{Z}) = 0$ and $H^4(M; \mathbb{Z}) \supseteq \mathbb{Z}$. □

We now consider the possibility that the minimal codimension of a fixed-point set of an involution in T is six.

Lemma 8.4. *Suppose* $\min \operatorname{cod}(M^\iota) = 6$, *where the minimum runs over involutions in* T. *One of the following occurs*:

- $\chi(M) = 2$, $\sigma(M) = 0$, *and* M *is* 5*-connected.*
- $\chi(M) = 9$, $\sigma(M) = \pm 1$, $H_2(M; \mathbb{Z}) = \mathbb{Z}$, $H_3(M; \mathbb{Z}) = 0$, $H_4(M; \mathbb{Z}) = \mathbb{Z}$, $H_5(M; \mathbb{Z}) = 0$, *and* $H^{16}(M; \mathbb{Z})$ *is generated by an element of the form* $x^5 y$ *with* $x \in H^2(M; \mathbb{Z})$ *and* $y \in H^6(M; \mathbb{Z})$, *and* M *is not spin.*

Proof. Choose $x \in M^T$ such that $N_1 = M_x^{\iota_1}$ has codimension six. The isotropy at x implies that some $\iota_2 \in \mathbb{Z}_2^4 \backslash \langle \iota_1 \rangle$ exists such that $M_x^{\iota_2}$ has codimension six. Moreover, if N_1 and N_2 intersect transversely, there exists $\iota_3 \in \mathbb{Z}_2^4 \backslash \langle \iota_1, \iota_2 \rangle$ such that $M_x^{\iota_3}$ has codimension six and intersects N_1 non-transversely. We

may assume therefore that $N_2 = M_x^{\iota_2}$ has codimension six and intersects N_1 non-transversely.

By the containment lemma,

$$M^T \subseteq N_1 \cup N_2 \cup N_{12},$$

where $N_{12} = M_x^{\iota_1\iota_2}$. By the connectedness lemma, all two-fold intersections of N_1, N_2, and N_{12} are connected and hence equal $N_1 \cap N_2 = M_x^{\langle \iota_1, \iota_2 \rangle}$. Since the intersection $N_1 \cap N_2$ is not transverse, $N_1 \cap N_2 \subseteq N_{12}$ has codimension two. Since no involution ι has $\operatorname{cod}(M^\iota) < 6$, N_{12} has T^3 symmetry, so N_{12} and $N_1 \cap N_2$ are both cohomology spheres or both cohomology complex projective spaces. We consider two cases:

(1) Suppose $N_1 \cap N_2$ is homeomorphic to $\mathbb{S}^6$. Since $N_1 \cap N_2 \to N_i$ is 4-connected for $i \in \{1, 2\}$, each N_i is a 4-connected 10-manifold. Since N_i is positively curved and has T^3 symmetry, $\chi(N_i) > 0$, which implies that N_i is homeomorphic to $\mathbb{S}^{10}$. By the inclusion–exclusion property of the Euler characteristic, $\chi(M) = 2$. Moreover, since $N_1 \to M$ is 5-connected, M is 5-connected as well. For the signature, note that $\sigma(M) = \sigma(M^{\iota_1}) = \sigma(N_1) = 0$.

(2) Suppose $N_1 \cap N_2$ is a cohomology $\mathbb{C}\mathrm{P}^3$. Since $N_1 \cap N_2 \to N_i$ is 4-connected for $i \in \{1, 2\}$, it follows by arguments similar to those in the dimension 10 result that each N_i is a cohomology $\mathbb{C}\mathrm{P}^5$. It follows that $\chi(M) = 9$, $H_2(M; \mathbb{Z}) = \mathbb{Z}$, $H_3(M; \mathbb{Z}) = 0$, $H_4(M; \mathbb{Z}) = \mathbb{Z}$, $H_5(M; \mathbb{Z}) = 0$, and some $x \in H^2(M; \mathbb{Z})$ satisfies the property that x^5 is not a multiple. In particular, $H^{16}(M; \mathbb{Z})$ is generated by an element of the form $x^5 y$ with $y \in H^6(M; \mathbb{Z})$.

For the signature, note that $\sigma(M) = \pm 1$ follows immediately from the formula $\sigma(M) = \sigma(M^{\iota_1})$ unless M^{ι_1} consists of N_1 together with three isolated points y_j. In this latter case, no $y_j \in N_{12}$, as that would imply that $\operatorname{cod}(M_{y_j}^{\iota_2}) = 8$, a contradiction to Frankel's theorem since $\operatorname{cod}(M^{\iota_2}) = 6$. It follows that each $y_j \in N_2$ and hence that each y_j lies in a six-dimensional component of $M^{\iota_1\iota_2}$. Since $\chi(N_{12}) = 6$, the fixed-point set of $\iota_1\iota_2$ is comprised of N_{12} together with some number of six-dimensional components whose Euler characteristic is at least, and hence sums to, three. This contradicts the fact that closed, oriented six-manifolds have even Euler characteristics.

Finally, if M is spin, then M^{ι_1} has, in addition to N_1^{10}, components of dimension two. These two-dimensional components are oriented, so they are diffeomorphic to spheres. This is a contradiction since $\chi(M) - \chi(N_1) = 3$. □

To complete the proof of Theorem 8.1, we prove the following.

Lemma 8.5 (Rigid isotropy lemma for $n = 16$). *If $\operatorname{cod}(M^\iota) \geq 8$ for every non-trivial involution $\iota \in T$, then one of the following occurs*:

- $\chi(M) = 2$ *and* $\sigma(M) = 0$.
- $\chi(M) = 3$, $\sigma(M) = \pm 1$, *and M is 2-connected.*

Proof. The isotropy representation is rigid in this case. Indeed, for each $x \in M^T$, there exists an involution ι_0 such that $\operatorname{cod}(M_x^{\iota_0}) = 16$ and $\operatorname{cod}(M_x^{\iota}) = 8$ for every other non-trivial involutions ι.

Fix $x \in M^T$ and choose distinct, non-trivial involutions $\iota_1, \iota_2 \in T$ such that each $M_x^{\iota_i}$ has codimension eight. By the rigidity of the isotropy, each N_i has T^3 symmetry, so $\chi(N_i) \leq \chi(\mathbb{C}\mathrm{P}^4) = 5$ by Fang and Rong's homeomorphism classification in dimension eight. By Frankel's theorem, $M^T \subseteq M_x^{\iota_1} \cup M_x^{\iota_2}$, so we have $\chi(M) \leq 5 + 5 < 14$.

Since there are fourteen involutions ι with $\operatorname{cod}(M_x^{\iota}) = 8$, and since at most one of these has $\operatorname{cod}(M_y^{\iota}) \neq 8$ at any other fixed point $y \in M^T$, not every involution gets a turn at having a 0-dimensional fixed-point component. By Frankel's theorem, there exists an involution $\iota \in T^4$ such that M^{ι} is connected and has dimension eight. By the rigidity of the isotropy representation, M^{ι} has T^3 symmetry and hence is homeomorphic to $\mathbb{S}^8$, $\mathbb{H}\mathrm{P}^2$, or $\mathbb{C}\mathrm{P}^4$ by the Fang–Rong classification. We consider these three cases separately.

(1) $M^{\iota} = \mathbb{S}^8$. The Euler characteristic and signature of M and M^{ι} are the same, so the calculation $\chi(M) = 2$ and $\sigma(M) = 0$ follows immediately. This is the first possible conclusion of Lemma 8.5.
(2) $M^{\iota} = \mathbb{H}\mathrm{P}^2$. As in the previous case, we immediately conclude $\chi(M) = 3$ and $\sigma(M) = \pm 1$. Moreover, we conclude by Wilking's maximal smooth symmetry rank bound for $\mathbb{H}\mathrm{P}^2$ that there is a circle in T^4 fixing M^{ι}. The connectedness lemma implies that $M^{\iota} \to M$ is 2-connected, so we have the additional conclusion in this case that $\pi_2(M) = 0$.
(3) $M^{\iota} = \mathbb{C}\mathrm{P}^4$. We claim this case cannot occur. Indeed, choose another non-trivial involution τ, and consider the induced action of τ on M^{ι}. Since M^{ι} is homeomorphic to $\mathbb{C}\mathrm{P}^4$, the fixed-point set $(M^{\iota})^{\tau}$ is comprised of exactly two components, and these two components are integral complex projective spaces whose dimensions add to $\dim(\mathbb{C}\mathrm{P}^4) - 2 = 6$ in accordance with the condition that $\chi(M^{\iota}) = \chi((M^{\iota})^{\tau})$. In particular, one of the components of $(M^{\iota})^{\tau}$ has dimension two or six. This contradicts the rigidity of the isotropy representation, which implies that every component of $M^{\langle \iota, \tau \rangle}$ has dimension zero or four.

This concludes the proof of Lemma 8.5 and hence of Theorem 8.1. □

9. Elliptic Genus Calculation

We present a unified proof of the elliptic genus claims in dimensions 12 and 16. In fact, the proof only requires T^2 symmetry. Recall that the *elliptic genus*

$$\Phi(M^{4m}) = q^{-2m} \cdot \hat{A}\left(M, \prod_{k \geq 1, \text{odd}}^{\infty} \bigwedge_{-q^k} T_{\mathbb{C}} \cdot \prod_{k \geq 2, \text{even}}^{\infty} S_{q^k} T_{\mathbb{C}}\right)$$

of an oriented, compact manifold M^{4m} is a Laurent series in the variable q with coefficients $\hat{A}(X)$, $\hat{A}(X, T_{\mathbb{C}}X)$, $\hat{A}(X, \bigwedge^2 T_{\mathbb{C}}X)$, etc. We refer the reader to [34] for a description of the notation and an introduction to index theory.

Theorem 9.1. *Let M^{4m} be a closed, simply connected, spin manifold admitting positive sectional curvature and T^2 symmetry. If $m \in \{3, 4\}$, the elliptic genus is constant.*

Note that if T^1 or T^2 acts on M^4 or M^8, respectively, as in this theorem, then the elliptic genus is constant by the work of Hsiang–Kleiner in dimension four and by Dessai's work in dimension eight [12, 36].

Proof. Since M is spin, every involution $\iota \in T^2$ has the property that there exists $c \in \{0, 2\}$ such that $\operatorname{cod}(N) \equiv c \bmod 4$ for all components $N \subseteq M^\iota$. If $c = 2$ for some involution, the action of that involution is of odd type and it follows that the elliptic genus vanishes by a result of Hirzebruch–Slodowy (see Corollary in [35, p. 317]). We assume therefore that every component of the fixed-point set of every involution has codimension divisible by four.

If there exists an involution $\iota \in T^2$ for which $\operatorname{cod}(M^\iota) \geq \frac{1}{2}\dim(M)$, then the elliptic genus is constant by another result of Hirzebruch–Slodowy (see the above-cited corollary). In particular, we are done if $\operatorname{cod}(M^\iota) \geq 8$ for some involution $\iota \in T^2$. We may assume therefore that every non-trivial involution $\iota \in T^2$ has $\operatorname{cod}(M^\iota) = 4$.

Let $\iota_1, \iota_2 \in T^2$ be distinct, non-trivial involutions and choose components $N_i \subseteq M^{\iota_i}$ with $\operatorname{cod}(N_i) = 4$. Suppose first that N_1 and N_2 intersect transversely. Fix $x \in N_1 \cap N_2 \cap M^{T^2}$. Such a point exists by Frankel's theorem and Berger's theorem. From the isotropy representation at x, we see that the fixed point component $N_{12} \subseteq M^{\iota_1\iota_2}$ containing x has codimension eight. But then Frankel's theorem implies that every component of $M^{\iota_1\iota_2}$ has codimension at least eight, and hence $\operatorname{cod}(M^{\iota_1\iota_2}) = 8$, a contradiction.

Suppose now that N_1 and N_2 do not intersect transversely. Choose $x \in N_1 \cap N_2 \cap M^{T^2}$ as above. The intersection $N_1 \cap N_2$ has codimension six in M and hence codimension two in N_i for both $i \in \{1, 2\}$. We claim that N_1 and N_2 are rational spheres. Indeed, if N_1 and N_2 are fixed by circles in T^2, then $N_1 \cap N_2$ is fixed by T^2. In particular, the induced action of T^2 on N_1 gives rise to a fixed point homogenous action on N_1 with codimension two fixed point component $N_1 \cap N_2$. Hence N_1 is diffeomorphic to an $\mathbb{S}^{4m-4}$ or $\mathbb{C}\mathrm{P}^{2m-2}$ by Grove and Searle's classification. In the latter case, the connectedness lemma and Poincaré duality imply that M is an integral $\mathbb{C}\mathrm{P}^{2m}$, and this contradicts the spin assumption. Hence the claim holds in this case. If instead the restriction of T^2 to, say, N_1 has finite kernel, then the codimension two lemma applies and we conclude that N_1 is an integral $\mathbb{S}^{4m-4}$ or $\mathbb{C}\mathrm{P}^{2m-2}$. Once again the latter case cannot occur, so N_1 is an integral $\mathbb{S}^{4m-4}$. By the connectedness lemma, N_2 is $(4m-8)$-connected and hence a sphere by Poincaré duality.

Finally, we assume the claim and complete the proof. If $\dim M = 16$, the connectedness lemma immediately implies that M is an integral sphere. Since M is simply connected, it follows that M is homeomorphic to a sphere and hence that the elliptic genus vanishes by Novikov's theorem. Similarly, if $\dim M = 12$, it suffices to prove that M is an integral $\mathbb{S}^{12}$. Note by the connectedness lemma that M is 5-connected and that $\chi(M) = 2 + b_6(M)$. On the other hand, the containment lemma implies that M^{T^2} is contained in $N_1 \cup N_2 \cup N_{12}$, where $N_{12} \subseteq M^{\iota_1 \iota_2}$ is the component containing x, which is a point in the intersection and in M^{T^2}. But N_{12} and $N_1 \cap N_2$ are also spheres by the connectedness lemma, so $\chi(M) = \chi(N_1 \cup N_2 \cup N_{12}) = 2$ by the inclusion–exclusion principle. Hence $b_6(M) = 0$. The universal coefficients theorem now implies that M is an integral sphere, as claimed. □

10. Low Dimensional Positively Elliptic Spaces

A simply connected, rationally elliptic topological space with positive Euler characteristic is called an F_0, or *positively elliptic*, space. These spaces have a (formal) dimension and satisfy Poincaré duality, and they admit pure minimal models (see [20]). Specifically, they admit minimal models $(\Lambda V, d)$ with generators $x_1, \ldots, x_k \in V$ of even degree and generators $y_1, \ldots, y_k \in V$ of odd degree such that each $\mathrm{d}x_i = 0$ and each $\mathrm{d}y_i$ is a homogeneous polynomial in the $x_1, \ldots, x_k$. We classify all possible tuples

$$(\deg x_1, \ldots, \deg x_k, \deg y_1, \ldots, \deg y_k)$$

of degrees for F_0 spaces of dimension up to 16.

In dimensions up to eight, this follows from previous work (see [33, 47, 48]).

Theorem 10.1 (Paternain–Petean, Pavlov, Herrmann). *If M is an F_0 space of formal dimension* 2, 4, 6, *or* 8, *then M admits a pure model whose tuples of homotopy generator degrees satisfy one of the following*:

- $\dim M = 2$ *and the tuple of degrees is* $(2, 3)$, *and* $M \simeq_{\mathbb{Q}} \mathbb{S}^2$.
- $\dim M = 4$ *and the tuple of degrees is* $(2, 5)$, $(4, 7)$, *or* $(2, 2, 3, 3)$, *and M is rationally homotopy equivalent to* $\mathbb{S}^4$, $\mathbb{C}\mathrm{P}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$, *or* $\mathbb{C}\mathrm{P}^2 \# -\mathbb{C}\mathrm{P}^2$.
- $\dim M = 6$ *and the tuple of degrees appears in Table* 1.
- $\dim M = 8$ *and the tuple of degrees appears in Table* 2.

Table 1. Dimension 6.

$\deg \pi_*(M) \otimes \mathbb{Q}$	$\chi(M)$
$(6, 11)$	2
$(2, 7)$, $(2, 4, 3, 7)$	4
$(2, 2, 3, 5)$	6
$(2, 2, 2, 3, 3, 3)$	8

Table 2. Dimension 8.

$\deg \pi_*(M) \otimes \mathbb{Q}$	$\chi(M)$
$(8, 15)$	2
$(4, 11)$	3
$(2, 6, 3, 11)$, $(4, 4, 7, 7)$	4
$(2, 9)$	5
$(2, 4, 5, 7)$	6
$(2, 2, 3, 7)$, $(2, 2, 4, 3, 3, 7)$	8
$(2, 2, 5, 5)$	9
$(2, 2, 2, 3, 3, 5)$	12
$(2, 2, 2, 2, 3, 3, 3, 3)$	16

In dimensions six and eight, Herrmann [33] proves a partial classification of the rational homotopy types in dimensions, but we will not need this here. We proceed to the computation in dimensions 10, 12, 14, and 16. Let M be an F_0-space.

Theorem 10.2. *If M is an F_0 space of formal dimension* 10, 12, 14, *or* 16, *then M admits a pure model whose tuples of homotopy generator degrees appear in one of Tables* 3, 4, 5, *or* 6. *Moreover, each tuple that appears in this table is realized by such a space.*

Proof. Let $(\Lambda V, d)$ be a pure model of M with k generators x_i of even degree $2a_i$ and k generators y_i of odd degree $2b_i - 1$. Following [20, Sec. 32], we may choose such x_i and y_i such that the following hold:

- $1 \leq a_1 \leq \cdots \leq a_k$ and $2 \leq b_1 \leq \cdots \leq b_k$.
- $b_i \geq 2a_i$ for all $1 \leq i \leq k$.
- $2\sum_{i=1}^{k}(b_i - a_i) = \dim M$.

Note in particular that $\dim(M) \geq 2\sum a_i \geq k$, so in each dimension there are finitely many possible values for k and for the a_i. It follows that there are only finitely many possible values for the b_i as well. We lead a computer-based

Table 3. Dimension 10.

$\deg \pi_*(M) \otimes \mathbb{Q}$	$\chi(M)$
$(10, 19)$	2
$(2, 8, 3, 15)$, $(4, 6, 7, 11)$	4
$(2, 11)$, $(2, 4, 3, 11)$, $(2, 6, 5, 11)$	6
$(2, 4, 7, 7)$, $(2, 2, 6, 3, 3, 11)$, $(2, 4, 4, 3, 7, 7)$	8
$(2, 2, 3, 9)$	10
$(2, 2, 5, 7)$, $(2, 2, 4, 3, 5, 7)$	12
$(2, 2, 2, 3, 3, 7)$, $(2, 2, 2, 4, 3, 3, 3, 7)$	16
$(2, 2, 2, 3, 5, 5)$	18
$(2, 2, 2, 2, 3, 3, 3, 5)$	24
$(2, 2, 2, 2, 2, 3, 3, 3, 3, 3)$	32

search using Mathematica to enumerate all possible tuples $(2a_1, \ldots, 2a_k, 2b_1-1, \ldots, 2b_k-1)$ where k ranges from one up to half the dimension of M. Moreover, we compute in each case the Euler characteristic, which satisfies the following formula:

$$\chi(M) = \prod_{i=1}^{k} \frac{\deg y_i + 1}{\deg x_i} = \prod_{i=1}^{k} \frac{b_i}{a_i}.$$

Of course, we can rule out any tuples for which the Euler characteristic is non-integral. However, there is a single criterion called the "arithmetic condition" due to Friedlander and Halperin that characterizes precisely which tuples of degrees are realized by F_0 spaces (see [20, Proposition 32.9]). In practice, it is not too difficult to check whether any given tuple of degrees can be realized using the integrality of the Euler characteristic together with direct arguments.

Let us demonstrate such a direct argument in one exemplary case, in which we exclude the existence of such a model. Suppose that homotopy groups are given degrees $(2, 4, 4, 5, 7, 9)$. That is, $V = \langle x_1, x_2, x_3, y_1, y_2, y_3\rangle$ with $\deg x_1 = 2, \deg x_2 = 4, \deg x_3 = 4$, $\deg y_1 = 5, \deg y_2 = 7, \deg y_3 = 9$. Let us see that there is no differential d on ΛV making $H(\Lambda V, \mathrm{d})$ finite-dimensional. In fact, for degree reasons we see that $\mathrm{d}y_1 \in x_1 \cdot \langle x_1^2, x_2, x_3\rangle$ and $\mathrm{d}y_3 \in x_1 \cdot \langle x_1^4, x_2x_3, x_2^2, x_3^2\rangle$. Thus these two relations taken together cannot reduce the Krull dimension of $\mathbb{Q}[x_1, x_2, x_3]$ by two.

For all the configurations in the tables it is easy to construct minimal Sullivan models. (Again, one can alternatively check that the arithmetic condition of Friedlander and Halperin holds.) Indeed, nearly all of them can be realized by products of spaces with singly generated cohomology algebra. Let us also provide the arguments in a few cases where we need slightly more complicated realizing models:

The first configuration which is not realizable as a non-trivial rational product is given by $(4, 6, 9, 11)$. Here we may consider

$$(\Lambda\langle x_1, x_2, y_1, y_2\rangle, \mathrm{d})$$

with $\deg x_1 = 4$, $\deg x_2 = 6$, $\deg y_1 = 9$, $\deg y_2 = 11$ and $\mathrm{d}x_1 = \mathrm{d}x_2 = 0$, $\mathrm{d}y_1 = x_1x_2$, $\mathrm{d}y_2 = x_1^3 + x_2^2$.

The next case is $(2, 4, 6, 3, 9, 11)$, which can be realized by the product of the model above and the one of the 2-sphere.

The case $(4, 6, 11, 13)$ is realized by

$$(\Lambda\langle x_1, x_2, y_1, y_2\rangle, \mathrm{d})$$

with $\deg x_1 = 4$, $\deg x_2 = 6$, $\deg y_1 = 11$, $\deg y_2 = 13$ and $\mathrm{d}x_1 = \mathrm{d}x_2 = 0$, $\mathrm{d}y_1 = x_1^3 + x_2^2$, $\mathrm{d}y_2 = x_1^2x_2$.

The next case is $(2, 4, 6, 5, 9, 11)$, which can be realized by the product of the model above and the one of $\mathbb{C}\mathrm{P}^2$. In the case $(2, 2, 4, 6, 3, 3, 9, 11)$, we use two sphere factors instead. For $(2, 2, 4, 6, 3, 3, 9, 11)$, we use two sphere factors.

The configuration $(4, 4, 6, 7, 9, 11)$ can be realized by

$$(\Lambda\langle x_1, x_2, x_3, y_1, y_2, y_3\rangle, \mathrm{d})$$

with $\deg x_1 = \deg x_2 = 4$, $\deg x_3 = 6$, $\deg y_1 = 7$, $\deg y_2 = 9$, $\deg y_3 = 11$ and $\mathrm{d}x_1 = \mathrm{d}x_2 = \mathrm{d}x_3 = 0$, $\mathrm{d}y_1 = x_1^2$, $\mathrm{d}y_2 = x_2x_3$, $\mathrm{d}y_3 = x_2^3 + x_2^2$. □

11. The Halperin Conjecture

In this section, we will show that F_0-spaces of dimension at most 16 satisfy the Halperin conjecture; and so do F_0-spaces with Euler characteristic at most 16. Recall that the Halperin conjecture states that any fibration with an F_0-space as a fiber should yield a Leray–Serre spectral sequence degenerating at the E_2-term.

Keeping the notation from the previous section, let M be an F_0 space of dimension $\dim M \leq 16$, and let $(\Lambda(x_1, \ldots, x_k, y_1, \ldots, y_k), \mathrm{d})$ be a pure model satisfying the following properties:

- The degrees $\deg(x_i)$ are increasing, and likewise for the degrees $\deg(y_i)$.
- $\deg(\mathrm{d}y_i) \geq 2\deg(x_i)$ for all i.
- each $\mathrm{d}x_i = 0$ and each $\mathrm{d}y_i$ is a homogeneous polynomial in $x_1, \ldots, x_k$.

Recall that the Halperin conjecture is known in the following cases:

- $k \leq 3$, i.e. if the cohomology algebra of M is generated by at most three elements (see [44]).
- $\deg(x_1) = \cdots = \deg(x_k)$, i.e. if all cohomology generators have the same degree (see Lemma 11.1).
- (Markl's Theorem) If the model splits as the total space of a rational fibration whose base and fiber satisfy the Halperin conjecture, then it too satisfies the Halperin conjecture (see Markl [45]).

We use the following characterization of the Halperin conjecture due to Meier [46]. If the rational cohomology algebra $H^*(M; \mathbb{Q})$ does not admit a derivation of degree $d < 0$, then the Halperin conjecture holds for M. Using this, the following is a well known and easy consequence.

Lemma 11.1. *Let δ be a derivation of negative degree on $H^*(M; \mathbb{Q})$, where M is an F_0 space. If $x \in H^i(M; \mathbb{Q})$ for some $i > 0$ such that $\delta(x) \in H^0(M; \mathbb{Q})$, then $\delta(x) = 0$. In particular, if $H^*(M; \mathbb{Q})$ is generated by elements of the same degree, then $\delta = 0$.*

In other words, any image of δ landing in $H^0(M; \mathbb{Q})$ is zero.

Proof. Suppose x^a is the maximal non-zero power of x. Since $\delta(x)$ is a non-zero element of $H^0(M; \mathbb{Z})$, we have $\delta(x)x^a \neq 0$. However, δ is a derivation, so it follows that $\delta(x^{a+1}) \neq 0$. This is a contradiction.

To prove the last statement, note that $\delta = 0$ if δ is zero on all generators. By the assumption on the degrees of the generators, these images either land in $H^0(M;\mathbb{Q})$ or a zero group. Either way, these images are zero, so the proof is complete. □

The proof of the Halperin conjecture for Euler characteristics up to 16 now follows easily, so we prove it first.

Theorem 11.2. *The Halperin conjecture holds for F_0-spaces M with $\chi(M) \leq 16$.*

Proof. Recall from the previous section that the Euler characteristic of M is given by the formula $\chi(M) = \prod \deg(\mathrm{d}y_i)/\deg(x_i)$. Since each factor in the product is at least 2, we have $\chi(M) \geq 2^k$ where k is the number of cohomology generators, as above. By assumption, either $k \leq 3$ or $k = 4$ and $\deg(\mathrm{d}y_i) = 2\deg(x_i)$ for all i. In the first case, the conjecture holds by Lupton's result above. Suppose therefore that $k = 4$ and $\deg(\mathrm{d}y_i) = 2\deg(x_i)$ for all i.

Let δ be a degree d derivation on $H^*(M;\mathbb{Q})$ for some $d < 0$. If all of the x_i have the same degree, then Lemma 11.1 implies that $\delta = 0$. If this is not the case, then there exists some integer l such that

$$\deg(x_1) = \cdots = \deg(x_l) < \deg(x_{l+1}) \leq \cdots \leq \deg(x_4).$$

Since $\deg(\mathrm{d}y_i) = 2\deg(x_i)$ for all i, it follows for degree reasons that $\mathrm{d}y_i \in \Lambda(x_1, \ldots, x_l)$ for all $i \leq l$. In particular, the model splits as a rational fibration over the subalgebra $\Lambda(x_1, \ldots, x_l, y_1, \ldots, y_l)$. Since both the base and fiber have fewer than four generators, we conclude from Markl's result that the Halperin conjecture holds for M as well. □

We proceed to the proof of the Halperin conjecture for F_0 spaces of dimension up to 16. We require two more observations that will facilitate the arguments.

Lemma 11.3. *Suppose M is an F_0 space and that its pure model $(\Lambda V, \mathrm{d})$ is chosen as above. If δ is a derivation of negative degree on $H^*(M;\mathbb{Q})$ such that $\delta(x_i) = 0$ for $k-1$ of the k generators x_i, then $\delta = 0$.*

Proof. For this proof only, we reorder the x_i so that $\delta(x_i) = 0$ for all $i \geq 2$. We proceed by contradiction, i.e. we assume that $\delta(x_1) \neq 0$. Then, by Poincaré duality, there is an element $x \in H^*(M;\mathbb{Q})$ such that $\delta(x_1)x$ generates the top cohomology group. Moreover, we may choose x to be a monomial in the x_i. Write $x = x_1^l x'$ where x' is a monomial in $x_2, \ldots, x_k$. It follows that $\delta(x_1^{l+1})x'$ generates top cohomology. But $x_1^{l+1}x' = 0$ for degree reasons and $\delta(x') = 0$ by assumption, so we compute that

$$0 = \delta(x_1^{l+1}x') - x_1^{l+1}\delta(x') = \delta(x_1^{l+1})x' \neq 0,$$

a contradiction. □

Lemma 11.4. *Suppose M is an F_0 space and that its pure model $(\Lambda V, \mathrm{d})$ is chosen as above. If there exists $l < k$ such that $\deg(\mathrm{d}y_l) < \deg(x_1) + \deg(x_{l+1})$, then $(\Lambda V, \mathrm{d})$ splits as a rational fibration with base algebra generated by $x_1, \ldots, x_l, y_1, \ldots, y_l$.*

Proof. This lemma is proved by the observation that under the above assumptions, the regular sequence $\mathrm{d}y_1, \ldots, \mathrm{d}y_l$ lies in $\Lambda\langle x_1, \ldots, x_l\rangle$, so splitting follows. □

With these preparations, we proceed to the proof of the Halperin conjecture for F_0 spaces with dimension at most 16. As a warm-up, we provide here a short proof in the case where there are at most three cohomology generators. Of course, this already follows by Lupton's theorem, but we include it here for the convenience of the reader.

Lemma 11.5. *The Halperin conjecture holds for F_0 spaces M such that M has dimension at most* 16 *and $H^*(M; \mathbb{Q})$ has at most three generators.*

Proof. If there are at most two generators of the cohomology algebra, the proof follows immediately from Lemmas 11.1 and 11.3. Suppose there are exactly three cohomology generators.

We denote by $(\Lambda\langle x_1, x_2, x_3, y_1, y_2, y_3\rangle, \mathrm{d})$ a corresponding minimal model with $\mathrm{d}x_1 = \mathrm{d}x_2 = \mathrm{d}x_3 = 0$. Denote by δ a non-trivial derivation of negative degree on its cohomology algebra. Let the x_i and the y_i be ordered by degree. We want to show that it has to be trivial and assume the contrary. Denote the cohomology classes represented by x_i by $[x_i]$. We make the following observations:

- $\delta([x_1]) = 0$ by Lemma 11.1.
- $\delta([x_2]) \neq 0$ and $\delta([x_3]) \neq 0$ without loss of generality by Lemma 11.3.
- $\deg(x_1) < \deg(x_2)$ by degree reasons and Lemma 11.1.
- $\deg(x_2) < \deg(x_3)$. Indeed, since δ maps $H^{\deg(x_2)}(M; \mathbb{Q})$ linearly into a zero- or one-dimensional cohomology group, the property $\deg(x_2) = \deg(x_3)$ would imply that, up to a change of basis, $\delta([x_2]) = 0$, a contradiction.
- $\deg(\mathrm{d}y_1) \geq \deg(x_1) + \deg(x_2)$ without loss of generality by Lemma 11.4. In fact, $\deg(\mathrm{d}y_1) \geq \deg(x_1) + \deg(x_3)$ since otherwise $\mathrm{d}y_1 \in \Lambda(x_1, x_2)$. If $\mathrm{d}y_1 \in \Lambda(x_1)$, the model splits as a rational fibration. If, on the other hand, $\mathrm{d}y_1 \notin \Lambda(x_1)$, then applying δ to $[\mathrm{d}y_1] = 0$ in cohomology yields a relation among $[x_1]$ and $[x_2]$, and this is a contradiction since $\mathrm{d}y_1$ induces the relation of smallest degree.
- $\deg(\mathrm{d}y_2) \geq \deg(x_1) + \deg(x_3)$ without loss of generality by Lemma 11.4.

Recall also that $\deg(\mathrm{d}y_i) \geq 2\deg(x_i)$ for all $1 \leq i \leq 3$. With these estimates in hand, the dimension formula,

$$16 \geq n = \sum_{i=1}^{3} (\deg(\mathrm{d}y_i) - \deg(x_i)),$$

implies that the tuple of degrees is $(2, 4, 6, 7, 7, 11)$.

By the proof of the estimate $\deg(\mathrm{d}y_1) \geq \deg(x_1)+\deg(x_3)$, we may assume that $\mathrm{d}y_1$ contains a non-zero term involving x_3. By degree reasons, up to a scaling of y_1, we may assume $\mathrm{d}y_1 = x_1x_3 + p(x_1, x_2)$. By a similar argument, we may assume that $\mathrm{d}y_2$ also has a non-zero x_1x_3 term, and hence that $\mathrm{d}y_2 = x_1x_3+q(x_1, x_2)$ after scaling y_2. Finally, by replacing y_1 by $y_1 - y_2$, we have that $\mathrm{d}y_1 \in \Lambda(x_1, x_2)$. Again, we obtain a contradiction if $\mathrm{d}y_1$ has a non-zero term involving x_2, so we actually have $\mathrm{d}y_1 \in \Lambda(x_1)$ and hence that the model of M splits as a rational fibration. □

Theorem 11.6. *The Halperin conjecture holds for F_0 spaces of dimension up to* 16.

Proof. The proof proceeds by stepping through Tables 1–6. Let k denote the number of cohomology generators.

First, any model with $k \leq 3$ is covered by Lemma 11.5. Second, any case where the first $k-1$ cohomology generators have the same degree is covered by Lemmas 11.1 and 11.3. Third, any case where the last $k-1$ cohomology generators have the same degree also follows from these two lemmas. Indeed, if $\deg(x_1) < \deg(x_2) = \cdots = \deg(x_k)$, then δ maps $H^{\deg(x_2)}(M;\mathbb{Q})$ linearly into a zero- or one-dimensional space. By a change of basis, we can arrange that $\delta([x_1]) = \cdots = \delta([x_{k-1}]) = 0$ and hence that $\delta = 0$. Finally, if some $\deg(\mathrm{d}y_i) < \deg(x_1) + \deg(x_{i+1})$ as in Lemma 11.4, then the model splits as a rational fibration where the base and fiber are F_0 spaces of smaller dimension. By induction over the dimension, this implies by Markl's theorem that M satisfies the Halperin conjecture in this case.

Taking into account these observations, one can either scan the tables or apply the dimension formula to show that we only need to provide special arguments for the following five cases:

- $(2, 2, 4, 4, 3, 5, 7, 7)$ and $\dim(M) = 14$.
- $(2, 2, 4, 4, 3, 7, 7, 7)$ and $\dim(M) = 16$.
- $(2, 2, 4, 4, 5, 5, 7, 7)$ and $\dim(M) = 16$.
- $(2, 2, 4, 6, 3, 5, 7, 11)$ and $\dim(M) = 16$.
- $(2, 2, 2, 4, 4, 3, 3, 5, 7, 7)$ and $\dim(M) = 16$.

We consider these cases one at a time. In some cases, we prove directly that there is no non-trivial derivation of negative degree. In the remaining cases, we show that model for M splits as a rational fibration and apply Markl's theorem.

$(2, 2, 4, 4, 3, 5, 7, 7)$: For degree reasons, we may write

$$\mathrm{d}y_3 = p(x_3, x_4) + q,$$
$$\mathrm{d}y_4 = p'(x_3, x_4) + q',$$

where $p, p' \in \Lambda^2\langle x_3, x_4\rangle$ and where q and q' lie in the ideal $(x_1, x_2) \subseteq \Lambda\langle x_1, x_2, x_3, x_4\rangle$ generated by x_1 and x_2. Suppose for a moment that p is a rational multiple of p'. Up to a change of basis, we may assume that $\mathrm{d}y_3 = q \in (x_1, x_2)$. We derive a contradiction as follows. By the finite-dimensionality of $H^*(M;\mathbb{Q})$,

some power $[x_3]^m$ is zero. At the level of the model, we have that some $\alpha_i \in \Lambda\langle x_1, x_2, x_3, x_4\rangle$ exist such that

$$x_3^m = \mathrm{d}\left(\sum_{i=1}^{4} \alpha_i y_i\right) = \sum_{i=1}^{4} \alpha_i \mathrm{d}y_i = \sum_{i=1}^{3} \alpha_i \mathrm{d}y_i + \alpha_4(p' + q').$$

Since $\mathrm{d}y_i \in (x_1, x_2)$ for $i \leq 3$ and since $q' \in (x_1, x_2)$, it follows that p' is a rational multiple of x_3^2. On the other hand, we can apply the same argument using a sufficiently large power of x_4 to conclude that p' is a rational multiple of x_4^2. This is a contradiction, hence we may assume that p and p' are not multiples of each other.

Let δ be a non-trivial derivation on $H^*(M;\mathbb{Q})$ of negative degree. By Lemma 11.1, we may assume that $\delta([x_1]) = \delta([x_2]) = 0$ and that δ has degree -2. Moreover, in view of Lemma 11.4, we may assume that $\delta([x_3])$ and $\delta([x_4])$ are linearly independent, since otherwise a change of basis would yield $\delta([x_3]) = 0$ and hence that $\delta = 0$. By a further change of basis, we may assume that $\delta([x_3]) = [x_1]$ and $\delta([x_4]) = [x_2]$.

Note, in particular, that δ maps $[q], [q'] \in H^*(M;\mathbb{Q})$ into the subalgebra generated by $[x_1]$ and $[x_2]$. Hence δ^2 maps $[q]$ and $[q']$ to zero in $H^4(M;\mathbb{Q})$. Moreover, if we write

$$p = ax_3^2 + bx_3x_4 + cx_4^2,$$

for $a, b, c \in \mathbb{Q}$, then we have that

$$\delta^2([p]) = 2a[x_1]^2 + 2b[x_1][x_2] + 2c[x_2]^2,$$

and similarly for $[p']$. Since the triple (a, b, c) and the corresponding triple for p' are not multiples of each other, we have constructed two independent relations in degree four. This contradicts the fact that there is only one homotopy generator in degree three, so the proof is complete.

$(2, 2, 4, 4, 3, 7, 7, 7)$: As in the previous case, we may assume that $\delta([x_3]) = [x_1]$, $\delta([x_4]) = [x_2]$, and $\delta([x_i]) = 0$ for $i \in \{1, 2\}$. Moreover, it follows as in the last case that there is some pair y_i and y_j such that $\mathrm{d}y_i$ and $\mathrm{d}y_j$ are linearly independent modulo the ideal (x_1, x_2). As in the previous case, this implies the existence of two linearly independent relations in degree four and hence provides a contradiction to degree considerations.

$(2, 2, 4, 4, 5, 5, 7, 7)$: As in the previous case, we may assume that $\delta([x_3]) = [x_1]$, $\delta([x_4]) = [x_2]$, and $\delta([x_i]) = 0$ for $i \in \{1, 2\}$. The proof can be handled similarly to the previous two cases, but there is a shortcut here. Without restriction, we may assume that $\mathrm{d}y_3$ has x_3^2 and possibly x_4^2 as non-trivial summands, up to multiples. We compute that $\delta^2(\mathrm{d}y_3)$ has summands x_1^2 and possibly x_2^2, up to non-trivial multiples. Since there is no relation in degree 3, we have a contradiction.

$(2, 2, 4, 6, 3, 5, 7, 11)$: As in the previous cases, we may assume $\delta([x_1]) = \delta([x_2]) = 0$ and that δ has degree two. In view of Lemma 11.3, we may assume further that

$\delta([x_3]) \in \langle [x_1], [x_2] \rangle$ and $\delta([x_4]) \in \langle [x_3], [x_1]^2, [x_1][x_2], [x_2]^2 \rangle$ are non-zero. Up to an isomorphism of the minimal model, we may assume that either $\delta([x_4]) = [x_3]$ or $\delta([x_4]) \in \langle [x_1]^2, [x_1][x_2], [x_2]^2 \rangle$. Further, we may assume $\delta([x_3]) = [x_1]$.

Using the finite-dimensionality of $H^*(M; \mathbb{Q})$, it follows that x_4^2 appears as a non-trivial summand of $\mathrm{d}y_4$. Up to scaling x_4, we may express

$$\mathrm{d}y_4 = x_4^2 + k_1 x_3^3 + l(x_1, x_2) x_3 x_4 + p,$$

where $k_1 \in \mathbb{Q}$, $l(x_1, x_2)$ is a linear function of x_1 and x_2, and $p \in \ker(\delta^3)$. We break the proof into two cases.

First, we suppose that $\delta([x_4]) = [x_3]$. For this we apply δ^4 to the relation in cohomology induced by y_4. Since $\delta^4([x_3]^3) = 0$, we conclude $0 = \delta^4([x_4]^2) = 6[x_1]^2$. This implies that $\mathrm{d}y_1$ is a multiple of x_1^2 and hence that the model splits as a rational fibration over $\Lambda(x_1, y_1)$. By Markl's result, the Halperin conjecture follows.

Second, we suppose that $\delta([x_4]) \in \langle [x_1]^2, [x_1][x_2], [x_2]^2 \rangle$. For this case, we apply δ^3 to the relation induced by $\mathrm{d}y_4$. This time, $\delta^3([x_4]^2) = 0$, so we conclude that $0 = \delta^3(k_1[x_3]^3) = 6k_1[x_1]^3$. We consider now two subcases, according to whether $k_1 = 0$ or $[x_1]^3 = 0$.

First suppose $k_1 = 0$. Using the finite-dimensionality of $H^*(M; \mathbb{Q})$, the fact that x_3^3 does not appear in $\mathrm{d}y_4$ implies that x_3^2 must appear as a summand in $\mathrm{d}y_3$. Up to scaling y_3, we may write

$$\mathrm{d}y_3 = x_3^2 + l'(x_1, x_2) x_4 + p',$$

where $l'(x_1, x_2)$ is a linear function of x_1 and x_2 and $p' \in \ker(\delta^2)$. In fact, δ^2 also sends $[x_4]$ to zero since we are assuming $\delta([x_4])$ is a polynomial in $[x_1]$ and $[x_2]$. Hence applying δ^2 to the induced relation in cohomology yields $0 = 2[x_1]^2$. As before, this implies a rational fibration splitting, and hence that the Halperin conjecture holds.

Finally, suppose that $[x_1]^3 = 0$. This implies that

$$x_1^3 = m(x_1, x_2)\mathrm{d}y_1 + c\mathrm{d}y_2$$

for some linear function $m(x_1, x_2)$ of x_1 and x_2 and some $c \in \mathbb{Q}$. Now, for degree considerations, we may express

$$\mathrm{d}y_2 = l''(x_1, x_2) x_3 + p'',$$

where $l''(x_1, x_2)$ is a linear function of x_1 and x_2 and $p'' \in \Lambda\langle x_1, x_2 \rangle \subseteq \ker(\delta)$. Moreover, unless the model of M splits as a rational fibration over the subalgebra $(\Lambda\langle x_1, x_2, y_1, y_2 \rangle, \mathrm{d})$, we may assume that $l''(x_1, x_2) \neq 0$. Applying δ to the induced relation in cohomology yields $0 = l''([x_1], [x_2])[x_1]$. Since the only relation in this degree is that induced by y_1, we conclude from this that $\mathrm{d}y_1 \in \Lambda\langle x_1, x_2 \rangle$ is divisible by x_1. But now the relation above for x_1^3 implies that $\mathrm{d}y_2$ is also divisible by x_1. Looking again at the expression for $\mathrm{d}y_2$, we conclude that $l''(x_1, x_2)$ is a (non-trivial) multiple of x_1. Since $l''([x_1], [x_2])[x_1] = 0$, we conclude that $[x_1]^2 = 0$ and

hence that, again, the model for M splits as a rational fibration over $\Lambda\langle x_1, y_1\rangle$. This proves the Halperin conjecture for this tuple of homotopy generator degrees.

$(2,2,2,4,4,3,3,5,7,7)$: Here we may assume $\delta([x_i]) = 0$ for $i \in \{1,2,3\}$ and that $\delta([x_4]) = [x_1]$ and $\delta([x_5]) = [x_2]$. The proof here is similar to the first few exceptional cases, where the ideal (x_1, x_2) is replaced by the ideal (x_1, x_2, x_3).

This concludes the proof of Halperin's conjecture in the five special cases above. Altogether, this completes the proof of the conjecture in dimensions up to 16. □

12. Positive Curvature and Rational Ellipticity

We now combine the information of the previous sections in order to classify rationally elliptic Riemannian manifolds of dimension at most 16 with positive sectional curvature and torus symmetry. The additional assumption of rational ellipticity does not add to our understanding in dimensions two, four, and six since the existing theorems do not see improvement upon adding the assumption of rational ellipticity. Indeed, each of these results shows that the manifold is rational elliptic. Starting in dimension eight, however, Dessai's result is one where the conclusion (an Euler characteristic, signature, and elliptic genus calculation) is improved to a rational homotopy classification by adding the assumption of rational ellipticity (see [12, Theorem 1.2]). We have corresponding rational homotopy classifications in dimensions 10, 14, and 16, and a partial result along these lines in dimension 12.

Theorem 12.1. *If M^{10} is a closed, simply connected, rationally elliptic Riemannian manifold with positive curvature and T^3 symmetry, then M is rationally homotopy equivalent to $\mathbb{S}^{10}$, $\mathbb{C}\mathrm{P}^5$, or $\mathbb{S}^2 \times \mathbb{H}\mathrm{P}^2$.*

Proof. We may assume that M is not homeomorphic to $\mathbb{S}^{10}$. By our classification in dimension 10 (Theorem 5.1), $\chi(M) = 6$, $H_2(M;\mathbb{Q}) = \mathbb{Q}$, and $H^{10}(M;\mathbb{Q})$ is generated by a product of elements of degree two and four. Rational ellipticity implies $H^{2i+1}(M;\mathbb{Q}) = 0$ for all i, and it follows that $H_*(M;\mathbb{Q}) \cong H_*(\mathbb{C}\mathrm{P}^5;\mathbb{Q})$. Fix generators $z \in H^2(M;\mathbb{Q})$ and $x \in H^4(M;\mathbb{Q})$.

By Theorem 5.1, in the case when cohomology is generated by x and z, the manifold M has the rational type of a complex projective space (corresponding to $m \neq 0$) or $\mathbb{S}^2 \times \mathbb{H}\mathrm{P}^2$ (when $m = 0$) using intrinsic formality of these spaces.

In the remaining case, the element x^3 does not vanish and generates $H^6(M;\mathbb{Q})$. Hence, by Poincaré duality, $M \simeq_{\mathbb{Q}} \mathbb{C}\mathrm{P}^5$. □

Theorem 12.2. *Let M^{12} be a closed, simply connected, rationally elliptic Riemannian manifold with positive curvature and T^3 symmetry. If $\chi(M) \in \{2,4,7\}$, then one of the following occurs*:

- *If $\chi(M) = 2$, then $M \simeq_{\mathbb{Q}} \mathbb{S}^{12}$.*

- *If* $\chi(M) = 4$, *then* $M \simeq_{\mathbb{Q}} \mathbb{H}\mathrm{P}^3$, $M \simeq_{\mathbb{Q}} \mathbb{S}^2 \times \mathbb{S}^{10}$, $M \simeq_{\mathbb{Q}} \mathbb{S}^4 \times \mathbb{S}^8$, *or the rational type of* M *comes out of a* 1*-parameter family tensoring to the real homotopy type of* $\mathbb{S}^6 \times \mathbb{S}^6$.
- *If* $\chi(M) = 7$, *then* $M \simeq_{\mathbb{Q}} \mathbb{C}\mathrm{P}^6$.

If $\chi(M) \notin \{2, 4, 7\}$, *then* $\chi(M) \in \{6, 8, 9, 10, 12\}$ *and correspondingly* M *has the Betti numbers and homotopy Betti numbers of one of the following spaces*:

- $\mathbb{S}^8 \times \mathbb{C}\mathrm{P}^2$ *or* $\mathbb{S}^4 \times \mathbb{H}\mathrm{P}^2$.
- $\mathbb{S}^6 \times \mathbb{C}\mathrm{P}^3$, $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^8$, $\mathbb{S}^2 \times \mathbb{S}^4 \times \mathbb{S}^6$, *or* $\mathbb{S}^4 \times \mathbb{S}^4 \times \mathbb{S}^4$.
- $\mathbb{C}\mathrm{P}^2 \times \mathbb{H}\mathrm{P}^2$.
- $\mathbb{S}^4 \times \mathbb{C}\mathrm{P}^4$.
- $\mathbb{S}^2 \times \mathbb{C}\mathrm{P}^5$, $\mathbb{S}^2 \times \mathbb{S}^6 \times \mathbb{C}\mathrm{P}^2$, $\mathbb{C}\mathrm{P}^2 \times \mathbb{S}^4 \times \mathbb{S}^4$, *or* $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{H}\mathrm{P}^2$.

Remark 12.3. Theorem 6.2 further computes the signature in this setting, but this additional information does not exclude any of the manifolds listed in the conclusion of this theorem. For example, in the case of rational homotopy generators in degrees 2, 6, 7, and 11, we construct one class of possible minimal models by

$$(\Lambda\langle x, y, x', y'\rangle, \mathrm{d})$$

with $\deg x = 2$, $\deg y = 6$, $\deg x' = 7$, $\deg y' = 11$, $\mathrm{d}x = \mathrm{d}y = 0$, $\mathrm{d}x' = xy$, $\mathrm{d}y' = x^6 + y^2$. Its intersection form is represented by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For the configuration $(2, 4, 6, 3, 7, 11)$ we may use $\mathbb{S}^2 \times \mathbb{S}^4 \times \mathbb{S}^6$ with the intersection matrix being the same with respect to the basis $\frac{1}{2}([\mathbb{S}^6] + [\mathbb{S}^2][\mathbb{S}^4])$, $\frac{1}{2}([\mathbb{S}^6] - [\mathbb{S}^2][\mathbb{S}^4])$.

Similarly, when $\chi(M) \in \{10, 12\}$, we obtain spaces which can be realized as manifolds by a product containing a sphere factor, i.e. by boundaries. Since the signature is a bordism invariant, these manifolds have vanishing signatures.

Proof. If $\chi(M) = 2$ or $\chi(M) = 7$, Table 4 implies that M has the rational homotopy groups of $\mathbb{S}^{12}$ or $\mathbb{C}\mathrm{P}^6$. By formality, the rational homotopy classification follows in this case.

Suppose next that $\chi(M) = 4$. Table 4 implies that M has the rational homotopy groups of one of four types. We consider these cases one at a time. Listed first in each case is the sequence of degrees of homotopy generators.

- $(4, 15)$. In this case, M has the rational homotopy groups of $\mathbb{H}\mathrm{P}^3$ and hence the same rational homotopy type.
- $(2, 10, 3, 19)$. In this case, the generator in degree three must kill the square of the generator in degree two, so it is straightforward to see in this case that M has the rational homotopy type of $\mathbb{S}^2 \times \mathbb{S}^{10}$.

Table 4. Dimension 12.

$\deg \pi_*(M) \otimes \mathbb{Q}$	$\chi(M)$
$(12, 23)$	2
$(6, 17)$	3
$(4, 15)$, $(2, 10, 3, 19)$, $(4, 8, 7, 15)$, $(6, 6, 11, 11)$	4
$(4, 6, 9, 11)$	5
$(2, 8, 5, 15)$, $(4, 4, 7, 11)$	6
$(2, 13)$	7
$(2, 6, 7, 11)$, $(2, 2, 8, 3, 3, 15)$, $(2, 4, 6, 3, 7, 11)$, $(4, 4, 4, 7, 7, 7)$	8
$(2, 4, 5, 11)$	9
$(2, 4, 7, 9)$	10
$(2, 2, 3, 11)$, $(2, 2, 4, 3, 3, 11)$, $(2, 2, 6, 3, 5, 11)$, $(2, 4, 4, 5, 7, 7)$	12
$(2, 2, 5, 9)$	15
$(2, 2, 7, 7)$, $(2, 2, 4, 3, 7, 7)$, $(2, 2, 2, 6, 3, 3, 3, 11)$, $(2, 2, 4, 4, 3, 3, 7, 7)$	16
$(2, 2, 4, 5, 5, 7)$	18
$(2, 2, 2, 3, 3, 9)$	20
$(2, 2, 2, 3, 5, 7)$, $(2, 2, 2, 4, 3, 3, 5, 7)$	24
$(2, 2, 2, 5, 5, 5)$	27
$(2, 2, 2, 2, 3, 3, 3, 7)$, $(2, 2, 2, 2, 4, 3, 3, 3, 3, 7)$	32
$(2, 2, 2, 2, 3, 3, 5, 5)$	36
$(2, 2, 2, 2, 2, 3, 3, 3, 3, 5)$	48
$(2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3)$	64

Table 5. Dimension 14.

$\deg \pi_*(M) \otimes \mathbb{Q}$	$\chi(M)$
$(14, 27)$	2
$(2, 12, 3, 23)$, $(4, 10, 7, 19)$, $(6, 8, 11, 15)$	4
$(2, 6, 3, 17)$, $(2, 10, 5, 19)$, $(4, 6, 11, 11)$	6
$(2, 15)$, $(2, 4, 3, 15)$, $(2, 8, 7, 15)$, $(2, 2, 10, 3, 3, 19)$, $(2, 4, 8, 3, 7, 15)$, $(2, 6, 6, 3, 11, 11)$, $(4, 4, 6, 7, 7, 11)$	8
$(2, 6, 9, 11)$, $(2, 4, 6, 3, 9, 11)$	10
$(2, 4, 7, 11)$, $(2, 2, 8, 3, 5, 15)$, $(2, 4, 4, 3, 7, 11)$, $(2, 4, 6, 5, 7, 11)$	12
$(2, 2, 3, 13)$	14
$(2, 2, 6, 3, 7, 11)$, $(2, 4, 4, 7, 7, 7)$, $(2, 2, 2, 8, 3, 3, 3, 15)$, $(2, 2, 4, 6, 3, 3, 7, 11)$, $(2, 4, 4, 4, 3, 7, 7, 7)$	16
$(2, 2, 5, 11)$, $(2, 2, 4, 3, 5, 11)$, $(2, 2, 6, 5, 5, 11)$	18
$(2, 2, 7, 9)$, $(2, 2, 4, 3, 7, 9)$	20
$(2, 2, 2, 3, 3, 11)$, $(2, 2, 4, 5, 7, 7)$, $(2, 2, 2, 4, 3, 3, 3, 11)$, $(2, 2, 2, 6, 3, 3, 5, 11)$, $(2, 2, 4, 4, 3, 5, 7, 7)$	24
$(2, 2, 2, 3, 5, 9)$	30
$(2, 2, 2, 3, 7, 7)$, $(2, 2, 2, 4, 3, 3, 7, 7)$, $(2, 2, 2, 2, 6, 3, 3, 3, 3, 11)$, $(2, 2, 2, 4, 4, 3, 3, 3, 7, 7)$	32
$(2, 2, 2, 5, 5, 7)$, $(2, 2, 2, 4, 3, 5, 5, 7)$	36
$(2, 2, 2, 2, 3, 3, 3, 9)$	40
$(2, 2, 2, 2, 3, 3, 5, 7)$, $(2, 2, 2, 2, 4, 3, 3, 3, 5, 7)$	48
$(2, 2, 2, 2, 3, 5, 5, 5)$	54
$(2, 2, 2, 2, 2, 3, 3, 3, 3, 7)$, $(2, 2, 2, 2, 2, 4, 3, 3, 3, 3, 3, 7)$	64
$(2, 2, 2, 2, 2, 3, 3, 3, 5, 5)$	72
$(2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 5)$	96
$(2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3)$	128

Table 6. Dimension 16.

$\deg \pi_*(M) \otimes \mathbb{Q}$	$\chi(M)$
$(16, 31)$	2
$(8, 23)$	3
$(2, 14, 3, 27)$, $(4, 12, 7, 23)$, $(6, 10, 11, 19)$, $(8, 8, 15, 15)$	4
$(4, 19)$	5
$(2, 12, 5, 23)$, $(4, 6, 7, 17)$, $(4, 8, 11, 15)$	6
$(4, 6, 11, 13)$	7
$(2, 10, 7, 19)$, $(4, 4, 7, 15)$, $(2, 2, 12, 3, 3, 23)$, $(2, 4, 10, 3, 7, 19)$, $(2, 6, 8, 3, 11, 15)$, $(4, 4, 8, 7, 7, 15)$, $(4, 6, 6, 7, 11, 11)$	8
$(2, 17)$, $(2, 6, 5, 17)$, $(4, 4, 11, 11)$	9
$(2, 8, 9, 15)$, $(4, 4, 6, 7, 9, 11)$	10
$(2, 4, 5, 15)$, $(2, 6, 11, 11)$, $(2, 2, 6, 3, 3, 17)$, $(2, 2, 10, 3, 5, 19)$, $(2, 4, 6, 3, 11, 11)$, $(2, 4, 8, 5, 7, 15)$, $(2, 6, 6, 5, 11, 11)$, $(4, 4, 4, 7, 7, 11)$	12
$(2, 4, 7, 13)$	14
$(2, 4, 9, 11)$, $(2, 4, 6, 5, 9, 11)$	15
$(2, 2, 3, 15)$, $(2, 2, 4, 3, 3, 15)$, $(2, 2, 8, 3, 7, 15)$, $(2, 4, 6, 7, 7, 11)$, $(2, 2, 2, 10, 3, 3, 3, 19)$, $(2, 2, 4, 8, 3, 3, 7, 15)$, $(2, 2, 6, 6, 3, 3, 11, 11)$, $(2, 4, 4, 6, 3, 7, 7, 11)$, $(4, 4, 4, 4, 7, 7, 7, 7)$	16
$(2, 2, 8, 5, 5, 15)$, $(2, 4, 4, 5, 7, 11)$	18
$(2, 2, 6, 3, 9, 11)$, $(2, 4, 4, 7, 7, 9)$, $(2, 2, 4, 6, 3, 3, 9, 11)$	20
$(2, 2, 5, 13)$	21
$(2, 2, 7, 11)$, $(2, 2, 4, 3, 7, 11)$, $(2, 2, 6, 5, 7, 11)$, $(2, 2, 2, 8, 3, 3, 5, 15)$, $(2, 2, 4, 4, 3, 3, 7, 11)$, $(2, 2, 4, 6, 3, 5, 7, 11)$, $(2, 4, 4, 4, 5, 7, 7, 7)$	24
$(2, 2, 9, 9)$	25
$(2, 2, 4, 5, 5, 11)$	27
$(2, 2, 2, 3, 3, 13)$	28
$(2, 2, 4, 5, 7, 9)$	30
$(2, 2, 4, 7, 7, 7)$, $(2, 2, 2, 6, 3, 3, 7, 11)$, $(2, 2, 4, 4, 3, 7, 7, 7)$, $(2, 2, 2, 2, 8, 3, 3, 3, 3, 15)$, $(2, 2, 2, 4, 6, 3, 3, 3, 7, 11)$, $(2, 2, 4, 4, 4, 3, 3, 7, 7, 7)$	32
$(2, 2, 2, 3, 5, 11)$, $(2, 2, 2, 4, 3, 3, 5, 11)$, $(2, 2, 2, 6, 3, 5, 5, 11)$, $(2, 2, 4, 4, 5, 5, 7, 7)$	36
$(2, 2, 2, 3, 7, 9)$, $(2, 2, 2, 4, 3, 3, 7, 9)$	40
$(2, 2, 2, 5, 5, 9)$	45
$(2, 2, 2, 5, 7, 7)$, $(2, 2, 2, 2, 3, 3, 3, 11)$, $(2, 2, 2, 4, 3, 5, 7, 7)$, $(2, 2, 2, 2, 4, 3, 3, 3, 3, 11)$, $(2, 2, 2, 2, 6, 3, 3, 3, 5, 11)$, $(2, 2, 2, 4, 4, 3, 3, 5, 7, 7)$	48
$(2, 2, 2, 4, 5, 5, 5, 7)$	54
$(2, 2, 2, 2, 3, 3, 5, 9)$	60
$(2, 2, 2, 2, 3, 3, 7, 7)$, $(2, 2, 2, 2, 4, 3, 3, 3, 7, 7)$, $(2, 2, 2, 2, 2, 6, 3, 3, 3, 3, 3, 11)$, $(2, 2, 2, 2, 4, 4, 3, 3, 3, 3, 7, 7)$	64
$(2, 2, 2, 2, 3, 5, 5, 7)$, $(2, 2, 2, 2, 4, 3, 3, 5, 5, 7)$	72
$(2, 2, 2, 2, 2, 3, 3, 3, 3, 9)$	80
$(2, 2, 2, 2, 5, 5, 5, 5)$	81
$(2, 2, 2, 2, 2, 3, 3, 3, 5, 7)$, $(2, 2, 2, 2, 2, 4, 3, 3, 3, 3, 5, 7)$	96
$(2, 2, 2, 2, 2, 3, 3, 5, 5, 5)$	108
$(2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 7)$, $(2, 2, 2, 2, 2, 2, 4, 3, 3, 3, 3, 3, 3, 7)$	128
$(2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 5, 5)$	144
$(2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 5)$	192
$(2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3)$	256

- $(4, 8, 7, 15)$. This case is similar to the previous case and yields the rational homotopy type of $\mathbb{S}^4 \times \mathbb{S}^8$.
- $(6, 6, 11, 11)$. In this case, M has a minimal model of the form $(\Lambda\langle u, v, u', v'\rangle, \mathrm{d})$ with $\deg u = \deg v = 6$, $\deg u' = \deg v' = 11$, $\mathrm{d}u = \mathrm{d}v = 0$. Since the signature vanishes, we obtain that the intersection form over the reals is represented by the matrix
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
This relates to the minimal model over $\mathbb{R}$ by determining the differentials $\mathrm{d}u' = u^2 + v^2$ and $\mathrm{d}v' = uv$ up to isomorphism. Let us now investigate how many rational homotopy types tensor to this real homotopy type. Since the product of the two elements u, v is exact over $\mathbb{R}$ if and only if it is exact over $\mathbb{Q}$, we see that all the rational types of this space are given by
$$\begin{aligned} &(\Lambda\langle u, v, u', v'\rangle, \mathrm{d}),\\ &\deg u = \deg v = 6, \quad \deg u' = \deg v' = 11, \quad \mathrm{d}u = \mathrm{d}v = 0,\\ &\mathrm{d}u' = u^2 + kv^2, \quad \mathrm{d}v' = uv, \quad k \in \mathbb{Q} \end{aligned}$$
and $k > 0$ may be chosen not to be a square number over $\mathbb{Q}$. It is easy to see that for $k = 1$, this is the model for $\mathbb{S}^6 \times \mathbb{S}^6$.

We proceed to the proof in the second case, where $\chi(M) \notin \{2, 4, 7\}$. Here our Euler characteristic calculation (Theorem 6.2) implies that $\chi(M) \in \{6, 8, 9, 10, 12\}$. In particular, M is an F_0 space and hence admits a pure model. Recall from [20, p. 446] that the Betti numbers of a pure space only depend on the rational homotopy groups, i.e. pure spaces with the same rational homotopy groups have the same Betti numbers. As a result, we only need to compute the dimensions of the rational homotopy groups. This information can be immediately deduced from Table 4. □

Next we come to the rational homotopy classifications in dimensions 14 and 16.

Theorem 12.4. *If M^{14} is a closed, simply connected, rationally elliptic Riemannian manifold with positive curvature and T^4 symmetry, then M is rationally homotopy equivalent to $\mathbb{S}^{14}$, $\mathbb{C}\mathrm{P}^7$, or $\mathbb{S}^2 \times \mathbb{H}\mathrm{P}^3$.*

Proof. This follows directly from Theorem 7.1, since $\chi(M) = 2$ for a rationally elliptic space M^n implies $M^n \simeq_{\mathbb{Q}} \mathbb{S}^n$ and since the rational homotopy types of $\mathbb{C}\mathrm{P}^7$ and $\mathbb{S}^2 \times \mathbb{H}\mathrm{P}^3$ (corresponding to $m \neq 0$ and $m = 0$ in the conclusion of Theorem 7.1, respectively) are intrinsically formal. □

Theorem 12.5. *If M^{16} is a closed, simply connected, rationally elliptic Riemannian manifold with positive curvature and T^4 symmetry, then M is rationally homotopy equivalent to a compact, rank one symmetric space, i.e. to $\mathbb{S}^{10}$, $\mathbb{C}\mathrm{P}^5$, $\mathbb{H}\mathrm{P}^4$, or $\mathrm{Ca}\mathrm{P}^2$.*

Proof. From Theorem 8.1 we derive that $\chi(M) \in \{2, 3, 5, 9\}$. In the first three cases, Table 6 implies that M has singly generated rational cohomology with generator in degree 16, 8, or 4, and so the result follows by formality. Suppose $\chi(M) = 9$. By Theorem 8.1 again, the fifth power of a generator $x \in H^2(M; \mathbb{Q})$ is non-zero. Since M is an F_0 space, the odd Betti numbers are zero, and by this fact the even Betti numbers are equal to one. By Poincaré duality, M is a rational cohomology $\mathbb{C}\mathrm{P}^8$ and hence a rational homotopy $\mathbb{C}\mathrm{P}^8$ by formality. □

In [2], the authors proved the Wilhelm conjecture for spaces with rational cohomology algebra generated by one element. We have the following consequence.

Corollary 12.6. *The Wilhelm conjecture holds for closed, simply connected, rationally elliptic manifolds of dimension* 16 *that admit a Riemannian metric with positive sectional curvature and* T^4 *symmetry.*

13. Biquotients

In this section, we combine our Euler characteristic and other calculations with work of Kapovitch–Ziller, Totaro, and DeVito to provide diffeomorphism classifications for biquotients in dimensions 10, 14, and 16 that admit positively curved metrics with torus symmetry. In particular, we obtain a characterization of the Cayley plane in this context. We also consider dimension 12 and restrict further the case of symmetric spaces to obtain a partial diffeomorphism classification.

The main additional ingredient is the classification of biquotients whose rational cohomology is four-periodic in the sense of [38, Definition 1.1]. This class includes those with singly generated cohomology, and this case was classified in Kapovitch and Ziller [37]. In even dimensions, there is one more possible rational cohomology ring, namely, that of $\mathbb{S}^2 \times \mathbb{H}\mathrm{P}^m$. Here we require recent work of DeVito [16].

To be clear, we state explicitly that in the subsequent theorem the biquotient structure does not have to be related to the positively curved metric.

Theorem 13.1. *Let* M^n *be a closed, simply connected biquotient that independently admits a positively curved Riemannian metric with* T^r *symmetry.*

- *If* $n = 10$ *and* $r \geq 3$, *then* M *is diffeomorphic to* $\mathbb{S}^{10}$, $\mathbb{C}\mathrm{P}^5$, $G_2/U(2)$, $\mathbb{S}^2 \tilde{\times} \mathbb{H}\mathrm{P}^2$, N_1^{10}, *or* N_2^{10}.
- *If* $n = 14$ *and* $r \geq 4$, *then* M *is diffeomorphic to* $\mathbb{S}^{14}$, $\mathbb{C}\mathrm{P}^7$, $\mathbb{S}^2 \tilde{\times} \mathbb{H}\mathrm{P}^3$, N_1^{14}, *or* N_2^{14}.
- *If* $n = 16$ *and* $r \geq 4$, *then* M *is diffeomorphic to* $\mathbb{S}^{16}$, $\mathbb{C}\mathrm{P}^8$, $\mathbb{H}\mathrm{P}^4$, *or* $\mathrm{Ca}\mathrm{P}^2$.

Here $\mathbb{S}^2 \tilde{\times} \mathbb{H}\mathrm{P}^m$ *denotes one of the two diffeomorphism types of total spaces of* $\mathbb{H}\mathrm{P}^m$*-bundles over* $\mathbb{S}^2$ *whose structure group reduces to a circle acting linearly, and* N_1^{4k-2} *and* N_2^{4k-2} *denote the free circle quotients* $\mathrm{SO}(2k+1)/(\mathrm{SO}(2k-1) \times \mathrm{SO}(2))$ *and* $\Delta\mathrm{SO}(2)\backslash\mathrm{SO}(2k+1)/\mathrm{SO}(2k-1)$ *of the unit tangent bundle of* $\mathbb{S}^{2k}$ *described in* [37].

Proof. First suppose M has the rational homotopy type of $\mathbb{S}^2 \times \mathbb{H}\mathrm{P}^2$ or $\mathbb{S}^2 \times \mathbb{H}\mathrm{P}^3$. From DeVito's classification we deduce that M^{10} is diffeomorphic to one of two bundles over $\mathbb{S}^2$ or with fiber $\mathbb{H}\mathrm{P}^2$ or $[G_2/\mathrm{SO}(4)] \times \mathbb{S}^2$. For M^{14} we obtain that M is one of the two respective $\mathbb{H}\mathrm{P}^3$-bundles over $\mathbb{S}^2$. Now, in dimension 10, we know that $H_2(M;\mathbb{Z}) = \mathbb{Z}$, whereas $H_2([G_2/\mathrm{SO}(4)] \times \mathbb{S}^2) = \mathbb{Z}_2 \oplus \mathbb{Z}$.

Next suppose that the rational homotopy type is not as above. The calculations above for rationally elliptic spaces imply that M has the rational homotopy type of a compact, rank one symmetric space (see Theorems 12.1, 12.4, and 12.5). In particular, the rational cohomology of M is generated by one element, so we can apply the classification of Kapovitch and Ziller (see [37, Theorem 0.1]). In these dimensions, either the conclusion of Theorem 13.1 holds, or M is diffeomorphic to one of the two rational complex projective spaces in dimensions 10 or 14. These spaces are N_1 and N_2 from the conclusion of the theorem. □

A similarly strong result in dimension 12 seems out of reach. We make a few general remarks on the structure of a 12-dimensional biquotient admitting positive sectional curvature and T^3 symmetry. We then restrict further to the case of symmetric spaces and provide a diffeomorphism classification in this case.

Suppose that M^{12} is a closed, simply connected Riemannian manifold with positive sectional curvature and T^3 symmetry. Assume moreover that M admits a possibly independent biquotient structure, i.e. suppose that M is diffeomorphic to a quotient $G/\!\!/H$ of G by a two-sided action by a subgroup $H \subseteq G$. By the results of Kapovitch–Ziller [37] or Totaro [55], we may replace G and H, if necessary, so that G is a connected, simply connected Lie group and H is a product of a torus T^r with a connected, simply connected Lie group. In other words, $G = \prod_{i=1}^{k} G_i$ and $H = T^r \times \prod_{i=1}^{l} H_i$, where G_i and H_i are compact, one-connected, simple Lie groups. Since $\chi(M) > 0$ under our assumptions, the ranks of G and H agree, so $l = k - r$. Moreover, an extension of the arguments in [37, Sec. 1] or [55, Lemma 3.3 and Corollary 4.6] shows that G has at most three simple factors, i.e. $k \leq 3$. Next, we require Totaro's work on the "contribution" of each simple group G_i to the odd rational homotopy groups of M. This work implies that, if G_i has the rational homotopy type of $\prod_j \mathbb{S}^{2n_j - 1}$, then the largest value of n_j appearing is at most $2n = 24$. In particular, there exist finitely many such G_i. Finally, we recall that the quotient map $G \to M$ is a fibration with fiber H, so the associated long exact homotopy sequence relates the rational homotopy groups of G and H (which are both concentrated in odd degrees) with those of M. In particular, this implies that $r = b_2(M) = \dim \pi_2(M) \otimes \mathbb{Q} \leq 2$ by Table 4. Putting all of this together, we have the following structure theorem.

Theorem 13.2. *Let M^{12} be a closed, simply connected Riemannian manifold with positive sectional curvature and T^3 symmetry. Assume M admits a possibly independent biquotient structure. Then there exist $k \in \{1, 2, 3\}$ and $r \in \{1, 2\}$ and compact, one-connected, simple Lie groups G_i and H_i such that*

$\pi_j(G_i) \otimes \mathbb{Q} = \pi_j(H_i) \otimes \mathbb{Q} = 0$ for $j \geq 48$ and such that M is diffeomorphic to a biquotient of the form $G/\!\!/H$, where $G = \prod_{i=1}^{k} G_i$ and $H = T^r \times \prod_{i=1}^{k-r} H_i$.

Using the last statement together with the classification in Table 4 of the rational homotopy groups of M (with dimension 12 and Euler characteristic 2, 4, 6, 7, 8, 9, 10, or 12), one could analyze the possible groups G and H appearing in the conclusion of this theorem. Indeed, the theorem together with the classification of compact, simply connected, simple Lie groups implies that there are finitely many choices of G. Given G and M, the rational homotopy groups of H follow from the long exact homotopy sequence associated to the fibration $G \to M$, so there are only finitely many possibilities for H. We could therefore enumerate the finite number of possibilities for G and H, however we do not pursue this here.

Instead, we restrict ourselves further to the class of symmetric spaces. These are homogeneous spaces and therefore biquotients, so everything above carries over. However, we proceed by a different route to get a classification in this case.

Theorem 13.3. *Let M^{12} be a compact, simply connected symmetric space. Assume M admits a possibly independent Riemannian metric with positive sectional curvature and T^3 symmetry. One of the following occurs*:

- $\chi(M) = 2$, *M is spin, and M is* $\mathbb{S}^{12}$.
- $\chi(M) = 4$, *M is spin, and M is* $\mathbb{H}\mathrm{P}^3$ *or a product of the form* $\mathbb{S}^{2k} \times \mathbb{S}^{12-2k}$.
- $\chi(M) = 6$, *M is spin, and M is* $\mathbb{S}^4 \times \mathbb{H}\mathrm{P}^2$ *or* $\mathbb{S}^4 \times (\mathrm{G}_2/\mathrm{SO}(4))$.
- $\chi(M) = 6$, *M is not spin, and M is* $\mathrm{SO}(7)/\mathrm{SO}(3) \times \mathrm{SO}(4)$ *or* $\mathbb{S}^8 \times \mathbb{C}\mathrm{P}^2$.
- $\chi(M) = 7$, *M is not spin, and M is* $\mathbb{C}\mathrm{P}^6$.
- $\chi(M) = 8$, *M is spin, and M is* $\mathrm{SO}(8)/\mathrm{SO}(2) \times \mathrm{SO}(6)$, $\mathrm{Sp}(3)/\mathrm{U}(3)$, $\mathbb{S}^6 \times \mathbb{C}\mathrm{P}^3$, *or a product of the form* $\mathbb{S}^{2k} \times \mathbb{S}^{2l} \times \mathbb{S}^{12-2k-2l}$.
- $\chi(M) = 8$, *M is not spin, and M is* $\mathbb{S}^6 \times (\mathrm{SO}(5)/\mathrm{SO}(2) \times \mathrm{SO}(3))$.
- $\chi(M) = 9$, *M is not spin, and M is* $\mathbb{C}\mathrm{P}^2 \times \mathbb{H}\mathrm{P}^2$ *or* $\mathbb{C}\mathrm{P}^2 \times (\mathrm{G}_2/\mathrm{SO}(4))$.
- $\chi(M) = 10$, *M is not spin, and M is* $\mathbb{S}^4 \times \mathbb{C}\mathrm{P}^4$.
- $\chi(M) = 12$, *M is not spin, and M is* $\mathbb{S}^2 \times (\mathrm{SO}(7)/\mathrm{SO}(2) \times \mathrm{SO}(5))$ *or a product of the form* $\mathbb{S}^{2k} \times \mathbb{S}^{8-2k} \times \mathbb{C}\mathrm{P}^2$.

We note that the signature in each case is 0 or ± 1, according to the parity of M. In addition, the elliptic genus is constant for all of these manifolds since they are homogeneous (see [35]). These remarks are consistent with the calculation in Theorem 6.2 of the Euler characteristic, signature, and elliptic genus.

We also remark that Theorem 13.3 could be improved if the constant $C(6)$ defined in the introduction is shown to be six. Indeed, if $C(6) = 6$, our Euler characteristic calculation in dimension 12 implies that $\chi(M) \neq 12$, $\chi(M) \neq 10$, and that $\chi(M) = 8$ only if M is not spin.

Proof. Let M^{12} be a closed, simply connected symmetric space that admits a (possibly independent) Riemannian metric with positive curvature and T^3 symmetry. By Theorem 6.2, either $\chi(M) \in \{2, 4, 6, 8\}$ or M is not spin and $\chi(M) \in$

$\{7, 9, 10, 12\}$. Note in the first case, the theorem does not claim that M is spin, however we will see for symmetric spaces, $\chi(M) \in \{2, 4\}$ only if M is spin.

By the classification of symmetric spaces, $M = \prod_{i=1}^{k} M_i$ where each M_i is a compact, simply connected, irreducible symmetric space. Since $\dim(M) = 12$, there are a finite number of possibilities to check. In fact, the classification also implies that each M_i has Euler characteristic zero or Euler characteristic at least two. Since $\chi(M) > 0$, we conclude that each $\chi(M_i) \geq 2$ (in particular, no Lie group factors appear) and that the number of factors is at most three.

Consider first the case where $M = M_1$, i.e. where M is irreducible. For this case, we simply enumerate all compact, simply connected, irreducible symmetric spaces of dimension 12. We first eliminate those spaces with Euler characteristic zero (i.e. any space G/H where G and H have unequal rank). Second, we eliminate the rank two complex Grassmannian $\mathrm{U}(5)/\mathrm{U}(2) \times \mathrm{U}(3)$, which has Euler characteristic 10 and is not spin, because it has signature ± 2 (see [52, Sec. 5]). Third, we record each space's signature and whether the spaces are spin, and we check that this data is consistent with our obstructions mentioned above (Theorem 6.2). For determining which irreducible symmetric spaces are spin, we rely on [6] (cf. [22, Table 4]). For the signature, we note that the signature vanishes for any simply connected, spin homogeneous space of dimension not divisible by eight (see [35, Sec. 2.3]). For the signature of the oriented Grassmannians, see [54]. Finally, we exclude $\mathrm{SO}(8)/\mathrm{U}(4)$ from the table, since it is diffeomorphic to $\mathrm{SO}(8)/\mathrm{SO}(2) \times \mathrm{SO}(6)$ by way of low-dimensional, accidental isomorphisms (see [61, Sec. 6.3]). The result is Table 7.

Next consider the case where $M = M_1 \times M_2$ or $M = M_1 \times M_2 \times M_3$ where each M_i is irreducible. In these cases, $2 \leq \dim(M_i) \leq 10$ and $2 \leq \chi(M_i) \leq 6$ by the multiplicativity of the Euler characteristic and the fact that $\chi(M) \leq 12$. Using [61, Sec. 6.3] again, we eliminate redundancy in our list by observing the following accidental isomorphisms:

(1) $\mathrm{SO}(4)/\mathrm{SO}(2) \times \mathrm{SO}(2)$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$, and hence it is not irreducible.
(2) $\mathrm{Sp}(2)/\mathrm{U}(2)$ is diffeomorphic to $\mathbb{C}\mathrm{P}^3$.
(3) $\mathrm{SO}(6)/\mathrm{U}(3)$ is diffeomorphic to $\mathbb{C}\mathrm{P}^3$.

We also record which spaces are spin using the classification of Cahen and Gutt mentioned above. The list of possible irreducible factors is presented in Table 8.

Table 7. Irreducible symmetric spaces M^{12} with $2 \leq \chi(M) \leq 12$ and $|\sigma(M)| \leq 1$.

M	$\chi(M)$	spin?	$\lvert\sigma(M)\rvert$
$\mathbb{S}^{12}$	2	yes	0
$\mathbb{H}\mathrm{P}^3$	4	yes	0
$\mathrm{SO}(7)/\mathrm{SO}(3) \times \mathrm{SO}(4)$	6	no	0
$\mathbb{C}\mathrm{P}^6$	7	no	1
$\mathrm{SO}(8)/\mathrm{SO}(2) \times \mathrm{SO}(6)$, $\mathrm{Sp}(3)/\mathrm{U}(3)$	8	yes	0

Table 8. Irreducible symmetric spaces M^n with $2 \leq n \leq 10$ and $2 \leq \chi(M) \leq 6$.

M	$\dim(M)$	$\chi(M)$	spin?
$\mathbb{S}^n$	n	2	yes
$\mathbb{C}\mathrm{P}^2$	4	3	no
$\mathbb{H}\mathrm{P}^2$, $\mathrm{G}_2/\mathrm{SO}(4)$	8	3	yes
$\mathbb{C}\mathrm{P}^3$	6	4	yes
$\mathrm{SO}(5)/\mathrm{SO}(2) \times \mathrm{SO}(3)$	6	4	no
$\mathbb{C}\mathrm{P}^4$	8	5	no
$\mathrm{SO}(6)/\mathrm{SO}(2) \times \mathrm{SO}(4)$, $\mathrm{U}(4)/\mathrm{U}(2) \times \mathrm{U}(2)$	8	6	yes
$\mathrm{SO}(7)/\mathrm{SO}(2) \times \mathrm{SO}(5)$	10	6	no
$\mathbb{C}\mathrm{P}^5$	10	6	yes

With Table 8 complete, it is not difficult to enumerate the possible products $M_1 \times M_2$ and $M_1 \times M_2 \times M_3$ that have dimension 12 and Euler characteristic satisfying $\chi(M) \in \{2, 4, 6, 8\}$ or $\chi(M) \in \{7, 9, 10, 12\}$, where the latter case only occurs if M is not spin. Here it is useful to recall that the product of manifolds is spin if and only if each factor is spin. For example, we can exclude the products of the form $\mathbb{S}^2 \times N$ where N is one of the three spin manifolds in Table 8 with Euler characteristic six. □

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