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Bivariate Dimension Quasi-polynomials of Difference–Differential Field Extensions with Weighted Basic Operators

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Abstract We prove the existence and determine some invariants of a Hilbert-type bivariate quasi-polynomial associated with a difference–differential field extension with weighted basic derivations and translations. We also show that such a quasi-polynomial can be expressed in terms of univariate Ehrhart quasi-polynomials of rational conic polytopes.

Keywords Difference–differential ring · Dimension quasi-polynomial · reduction · Autoreduced set · Characteristic set

Mathematics Subject Classification Primary 12H05; Secondary 12H10 · 39A05

1 Introduction

Hilbert-type functions in differential and difference algebra have been extensively studied since 1960s when Kolchin [8] (see also [9, Ch. II, Sect. 12]) introduced a concept of a differential dimension polynomial associated with a finitely generated differential field extension. The corresponding characteristics of difference and difference–differential field extensions were introduced in [11,18]. The important role of differential, difference and difference–differential dimension polynomials is determined by at least three factors. First, a dimension polynomial associated with a system of algebraic differential (respectively, difference or difference–differential) equations expresses the strength of such a system in the sense of Einstein. In the case of a system of partial differential equations, this concept, introduced in [5] as a qualitative characteristic of a system, was expressed as a certain differential dimension polynomial in [20]; the corresponding algebraic descriptions of the concepts of strength of systems of difference and difference–differential equations were obtained in [10, Sect. 6.4], [15, Sect. 7.7], and [16]. Second, a dimension polynomial associated with a finite system of generators of a differential, difference or difference–differential field extension carries certain birational invariants, numbers that do not change when we switch to another system of generators. These invariants, which characterize the extension, are closely connected with some other of its important characteristics, e. g., with the differential (respectively, difference or difference–differential) transcendence degree of the extension. Finally,

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properties of dimension polynomials associated with prime differential (respectively, difference or difference–differential) ideals provide a powerful tool in the dimension theory of the corresponding rings (see, for example, [7], [10, Ch.7], [12] and [19]).

The existence theorems on differential and difference dimension polynomials were recently generalized to the cases of differential and difference algebraic structures with weighted basic operators (derivations or translations), see [4,17]. These results show that the corresponding dimension functions are univariate quasi-polynomials that can be expressed as linear combinations of Ehrhart quasi-polynomials associated with certain conic polytopes. It was also proven that the degrees and the leading coefficients of such dimension quasi-polynomials are birational invariants of the differential (or difference) field extensions they are associated with.

In this paper, we prove the existence and determine invariants of a bivariate dimension quasi-polynomial associated with a difference–differential field extension with weighted basic derivations and translations. In order to obtain these results, we generalize the classical method of characteristic sets to the case of two natural rankings in an algebra of difference–differential polynomials where the basic operators are assigned positive integer weights. Then we use properties of characteristic sets with respect to two rankings to describe bivariate difference–differential dimension quasi-polynomials.

2 Preliminaries

In what follows $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of all integers, nonnegative integers, rational numbers, and real numbers, respectively. If $m \in \mathbb{Z}$, $m \geq 1$, then by the product order on $\mathbb{N}^m$ we mean a partial order $\leq_P$ such that $(a_1, \dots, a_m) \leq_P (a'_1, \dots, a'_m)$ if and only if $a_i \leq a'_i$ for $i = 1, \dots, m$.

By a ring we always mean an associative ring with unity. Every ring homomorphism is unitary (maps unity onto unity), every subring of a ring contains the unity of the ring.

By a *difference–differential ring* we mean a commutative ring R considered together with finite sets $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\sigma = \{\alpha_1, \dots, \alpha_n\}$ of derivations and automorphisms of R, respectively, such that any two mappings of the set $\Delta \bigcup \sigma$ commute. The set $\Delta \bigcup \sigma$ is called the *basic set* of the difference–differential ring R, which is also called a Δ-σ-ring. If R is a field, it is called a *difference–differential field* or a Δ-σ-field. In what follows, we will often use prefix Δ-σ- instead of the adjective "difference–differential". Furthermore, we assume that every δ_i, $1 \leq i \leq m$, (respectively, every α_j, $1 \leq j \leq n$) is assigned a positive integer weight v_i (respectively, w_j). We set $V = (v_1, \dots, v_m)$ and $W = (w_1, \dots, w_n)$.

Let T be the free commutative semigroup generated by the set $\Delta \bigcup \sigma$, that is, the semigroup of all power products

$$\tau = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n} \qquad (k_i, l_j \in \mathbb{N}).$$

The numbers

$$\operatorname{ord}_\Delta \tau = \sum_{i=1}^{m} v_i k_i \quad \text{and} \quad \operatorname{ord}_\sigma \tau = \sum_{j=1}^{n} w_j l_j$$

are called the *orders* of τ with respect to Δ and σ (and with respect to the given weights). Furthermore, for every $r, s \in \mathbb{N}$, we set

$$T(r, s) = \{\tau \in T \mid \operatorname{ord}_\Delta \tau \leq r,\ \operatorname{ord}_\sigma \tau \leq s\}.$$

A subring (ideal) R_0 of a Δ-σ-ring R is said to be a difference–differential (or Δ-σ-) subring of R (respectively, a difference–differential (or Δ-σ-) ideal of R) if R_0 is closed with respect to the action of any operator of $\Delta \bigcup \sigma$. In this case the restriction of a mapping from $\Delta \bigcup \sigma$ to R_0 is denoted by the same symbol. If a prime ideal P of R is closed with respect to the action of $\Delta \bigcup \sigma$, it is called a *prime difference–differential* (or Δ-σ-) *ideal* of R.

If R is a Δ-σ-field and R_0 a subfield of R which is also a Δ-σ-subring of R, then R_0 is said to be a Δ-σ-subfield of R; R, in turn, is called a difference–differential (or Δ-σ-) field extension or a Δ-σ-overfield of R_0. In this case we also say that we have a Δ-σ-field extension R/R_0.

If R is a Δ-σ-ring and $S \subseteq R$, then the intersection of all Δ-σ-ideals of R containing the set S is, obviously, the smallest Δ-σ-ideal of R containing S. This ideal is denoted by $[S]$; as an ideal, it is generated by all elements $\tau\eta$ where $\tau \in T$, $\eta \in S$. (Here and below we frequently write $\tau\eta$ for $\tau(\eta)$ for elements $\tau \in T$, $\eta \in R$.) If the set S is finite, $S = \{\eta_1, \dots, \eta_p\}$, we say that the Δ-σ-ideal $I = [S]$ is finitely generated (in this case we write $I = [\eta_1, \dots, \eta_p]$) and call $\eta_1, \dots, \eta_p$ difference–differential (or Δ-σ-) generators of I.

If K_0 is a Δ-σ-subfield of a Δ-σ-field K and $S \subseteq K$, then the intersection of all Δ-σ-subfields of K containing K_0 and S is the unique Δ-σ-subfield of K containing K_0 and S and contained in every Δ-σ-subfield of K containing K_0 and S. It is denoted by $K_0\langle S\rangle$. If S is finite, $S = \{\eta_1, \dots, \eta_p\}$, then K is said to be a finitely generated Δ-σ-extension of K_0 with the set of Δ-σ-generators $\{\eta_1, \dots, \eta_p\}$. In this case we write $K = K_0\langle\eta_1, \dots, \eta_p\rangle$. It is easy to see that the field $K_0\langle\eta_1, \dots, \eta_p\rangle$ coincides with the field $K_0(\{\tau\eta_i | \tau \in T, 1 \le i \le p\})$.

Let R and S be two difference–differential rings with the same basic set $\Delta \bigcup \sigma$, so that elements of the sets Δ and σ act on each of the rings as derivations and automorphisms, respectively, and every two mappings of the set $\Delta \bigcup \sigma$ commute. (More rigorously, we assume that there exist injective mappings of the sets Δ and σ into the sets of derivations and automorphisms of the rings R and S, respectively, such that the images of any two elements of $\Delta \bigcup \sigma$ commute. For convenience we will denote the images of elements of $\Delta \bigcup \sigma$ under these mappings by the same symbols $\delta_1, \dots, \delta_m, \alpha_1, \dots, \alpha_n$). A ring homomorphism $\phi : R \longrightarrow S$ is called a *difference–differential* (or Δ-σ-) *homomorphism* if $\phi(\tau a) = \tau\phi(a)$ for any $\tau \in \Delta \bigcup \sigma$, $a \in R$.

If K is a difference–differential (Δ-σ-) field and $Y = \{y_1, \dots, y_p\}$ is a finite set of symbols, then one can consider a countable set of symbols $TY = \{\tau y_j | \tau \in T, 1 \le j \le p\}$ and the polynomial ring $R = K[\{\tau y_j | \tau \in T, 1 \le j \le p\}]$ in the set of indeterminates TY over the field K. This polynomial ring is naturally viewed as a Δ-σ-ring where $\theta(\tau y_j) = (\theta\tau)y_j$ for any $\theta \in \Delta \bigcup \sigma$, $\tau \in T$, $1 \le j \le p$, and the elements of $\Delta \bigcup \sigma$ act on the coefficients of the polynomials of R as they act in the field K. The ring R is called the *ring of difference–differential* (or Δ-σ-) *polynomials* in the set of difference–differential (Δ-σ-) indeterminates $y_1, \dots, y_p$ over K. This ring is denoted by $K\{y_1, \dots, y_p\}$ and its elements are called difference–differential (or Δ-σ-) polynomials. If $f \in K\{y_1, \dots, y_p\}$ and $\eta = (\eta_1, \dots, \eta_p)$ is a p-dimensional vector with coordinates in some Δ-σ-overfield of K, then $f(\eta)$ (or $f(\eta_1, \dots, \eta_p)$) denotes the result of the replacement of every entry τy_i in f by $\tau\eta_i$ ($\tau \in T$, $1 \le i \le p$).

Let R be a Δ-σ-ring and $\mathcal{U}$ a family of elements of some Δ-σ-overring of R. We say that the family $\mathcal{U}$ is Δ-σ-*algebraically dependent* over R, if the family $T\mathcal{U} = \{\tau u \,|\, \tau \in T,\ u \in \mathcal{U}\}$ is algebraically dependent over R (that is, there exist elements $u_1, \dots, u_k \in T\mathcal{U}$ and a nonzero polynomial f in k variables with coefficients in R such that $f(u_1, \dots, u_k) = 0$). Otherwise, the family $\mathcal{U}$ is said to be Δ-σ-*algebraically independent* over R.

If K is a Δ-σ-field and L a Δ-σ-field extension of K, then a set $B \subseteq L$ is said to be a Δ-σ-*transcendence basis* of L over K if B is Δ-σ-algebraically independent over K and every element $a \in L$ is Δ-σ-algebraic over $K\langle B\rangle$ (it means that the set $\{\tau a \,|\, \tau \in T\}$ is algebraically dependent over the field $K\langle B\rangle$). If L is a finitely generated Δ-σ-field extension of K, then all Δ-σ-transcendence bases of L over K are finite and have the same number of elements (one can easily obtain this result by mimicking the proof of Proposition 4.1.6 of [15]). This number is called the Δ-σ-*transcendence degree* of L over K (or the Δ-σ-transcendence degree of the extension L/K); it is denoted by Δ-σ-$\mathrm{trdeg}_K L$.

3 Dimension Quasi-polynomials of Subsets of $\mathbb{N}^p$

A function $f : \mathbb{Z} \to \mathbb{Q}$ is called a (univariate) *quasi-polynomial* of period q if there exist q polynomials $g_i(x) \in \mathbb{Q}[x]$ $(0 \le i \le q-1)$ such that $f(n) = g_i(n)$ whenever $n \in \mathbb{Z}$ and $n \equiv i\ (mod\ q)$.

An equivalent way of introducing quasi-polynomials is as follows.

A *rational periodic number* $U(n)$ is a function $U : \mathbb{Z} \to \mathbb{Q}$ with the property that there exists (a period) $q \in \mathbb{N}$ such that

$$U(n) = U(n') \ \text{ whenever } \ n \equiv n'\ (mod\ q).$$

A rational periodic number is usually represented by a list of the q its possible values enclosed in square brackets:

$$U(n) = [a_0, \dots, a_{q-1}]_n$$

where $U(n) = a_i$ $(0 \le i \le q-1)$ whenever $n \equiv i\ (mod\ q)$.

For example, $U(n) = \left[\frac{1}{2}, \frac{3}{4}, 1\right]_n$ is a periodic number with period 3 such that $U(n) = \frac{1}{2}$ if $n \equiv 0\ (mod\ 3)$, $U(n) = \frac{3}{4}$ if $n \equiv 1\ (mod\ 3)$, and $U(n) = 1$ if $n \equiv 2\ (mod\ 3)$.

A *quasi-polynomial of degree d* is defined as a function $f : \mathbb{Z} \to \mathbb{Q}$ such that

$$f(n) = c_d(n)n^d + \dots + c_1(n)n + c_0(n) \quad (n \in \mathbb{Z})$$

where $c_i(n)$'s are rational periodic numbers and $c_d(n) \neq 0$ for at least one $n \in \mathbb{Z}$.

One of the main applications of the theory of quasi-polynomials is its application to the problem of counting integer points in polytopes. Recall that a *rational polytope* in $\mathbb{R}^d$ is the convex hull of finitely many points (vertices) in $\mathbb{Q}^d$. If all vertices of a rational polytope P have integer coordinates, P is said to be a *lattice polytope*.

Equivalently, a rational polytope $P \subseteq \mathbb{R}^d$ is the set of solutions of a finite system of linear inequalities

$$A\mathbf{x} \le \mathbf{b},$$

where A is an $m \times d$-matrix with integer entries (m is a positive integer) and $\mathbf{b} \in \mathbb{Z}^m$, provided that the solution set is bounded.

Let $P \subseteq \mathbb{R}^d$ be a rational polytope. (We assume that P has dimension d, that is, P is not contained in a proper affine subspace of $\mathbb{R}^d$.) Then a polytope

$$rP = \{r\mathbf{x} \mid \mathbf{x} \in P\}$$

($r \in \mathbb{N}$) is called the rth *dilate* of P. (Clearly, if $\mathbf{v}_1, \dots, \mathbf{v}_k$ are all vertices of P, then rP is the convex hull of $r\mathbf{v}_1, \dots, r\mathbf{v}_k$.)

Given a rational polytope P, let $L(P, r) = \mathrm{Card}(rP \cap \mathbb{Z}^d)$, the number of integer points in rP. The following theorem of Ehrhart (see [6]) shows that $L(P, r)$ can be expressed as a certain quasi-polynomial.

Theorem 3.1 *Let $P \subseteq \mathbb{R}^d$ be a rational polytope of dimension d. Then there exists a quasi-polynomial $\phi_P(r)$ of degree d such that*

(i) *$\phi_P(r) = L(P, r)$ for all $r \in \mathbb{N}$.*
(ii) *The leading coefficient of $\phi_P(r)$ is a constant that is equal to the Euclidean volume of the polytope P.*
(iii) *The minimum period of $\phi_P(r)$ (that is, the least common multiple of the minimum periods of its coefficients) is a divisor of the number $\mathcal{D}(P) = \min\{n \in \mathbb{N} \mid nP$ is a lattice polytope$\}$.*
(iv) *If P is a lattice polytope, then $\phi_P(r)$ is a polynomial of r with rational coefficients.*

The main tools for computation of Ehrhart quasi-polynomials are Alexander Barvinok's polynomial time algorithm and its modifications, see [1–3]. In some cases, however, the Ehrhart quasi-polynomial can be found directly from the Ehrhart's theorem by evaluating the periodic coefficients of the quasi-polynomial (by computing $L(P, r)$ for the first several values of $r \in \mathbb{N}$ and then solving the corresponding system of linear equations, see [17, Example 1]).

In what follows, $\lambda_W^{(m)}(t)$ will denote the Ehrhart quasi-polynomial that describes the number of integer points in a rational *conic polytope* in $\mathbb{R}^m$ defined as

$$\{(x_1, \dots, x_m) \in \mathbb{R}^m \mid \sum_{i=1}^{m} w_i x_i \leq t,\ x_j \geq 0\ \ (1 \leq j \leq m)\}$$

where $W = (w_1, \dots, w_m)$ is a fixed m-tuple of positive integers. It follows from the Ehrhart's Theorem that $\lambda_W^{(m)}(t)$ is a quasi-polynomial of degree m whose leading coefficient is $\dfrac{1}{m! w_1 \dots w_m}$. A polynomial time algorithm for computing $\lambda_W^{(m)}(t)$ can be found, for example, in [3].

A *k-dimensional periodic number* is a function $U : \mathbb{Z}^k \to \mathbb{Q}$ with the following property: there exists a k-tuple $\mathbf{q} = (q_1, \dots, q_k) \in \mathbb{N}^k$, $q_i > 0$, such that for any $(p_1, \dots, p_k), (p'_1, \dots, p'_k) \in \mathbb{Z}^k$,

$$U(p_1, \dots, p_k) = U(p'_1, \dots, p'_k) \text{ if } p_i \equiv p'_i\ (mod\, q_i),\ \ 1 \leq i \leq k.$$

Such a vector $\mathbf{q} = (q_1, \dots, q_k)$ is called a *k-period* of U.

For example, $[[1, \frac{1}{2}]_{p_2}, [0, \frac{3}{2}]_{p_2}, [-1, \frac{1}{4}]_{p_2}]_{p_1}$ is a 2-dimensional periodic number with a 2-period $\mathbf{q} = (3, 2)$.

A polynomial in k variables $p_1, \dots, p_k$, where each coefficient is a multidimensional periodic number on a subset of $\{p_1, \dots, p_k\}$, is called a *k-variate quasi-polynomial* (in $p_1, \dots, p_k$).

In what follows, for any $m, n \in \mathbb{N}$ and for any $A \subseteq \mathbb{N}^{m+n}$ we set

$X_A = \{\mathbf{x} = (x_1, \dots, x_{m+n}) \mid \mathbf{x}$ is not greater than or equal to any $\mathbf{a} \in A$ with respect to the product order $\leq_P$ on $\mathbb{N}^{m+n}\}$.

Let us fix two vectors $V = (v_1, \dots, v_m) \in \mathbb{N}^m$ and $W = (w_1, \dots, w_n) \in \mathbb{N}^n$ ("weight vectors") and define the orders of an $(m+n)$-tuple $\mathbf{a} = (a_1, \dots, a_{m+n}) \in \mathbb{N}^{m+n}$ with respect to these vectors as

$$\mathrm{ord}_V\, \mathbf{a} = \sum_{i=1}^{m} v_i a_i \quad \text{and} \quad \mathrm{ord}_W\, \mathbf{a} = \sum_{i=1}^{n} w_i a_{m+i},\ \text{ respectively.}$$

Furthermore, for any set $A \subseteq \mathbb{N}^{m+n}$ and any $r, s \in \mathbb{N}$, let

$$A(r, s) = \{\mathbf{a} \in A \mid \mathrm{ord}_V\, \mathbf{a} \leq r,\ \mathrm{ord}_W\, \mathbf{a} \leq s\}.$$

Theorem 3.2 *With the above notation, for any set $A \subseteq \mathbb{N}^{m+n}$, there exists a bivariate quasi-polynomial $\phi_A^{V,W}(t_1, t_2)$ such that*

(i) $\phi_A^{V,W}(r, s) = \mathrm{Card}\, X_A(r, s)$ *for all sufficiently large $(r, s) \in \mathbb{N}^2$. (It means that there is $(r_0, s_0) \in \mathbb{N}^2$ such that the equality holds for all integers $r \geq r_0,\ s \geq s_0$.)*
(ii) $\deg_{t_1} \phi_A^{V,W} \leq m$ *and* $\deg_{t_2} \phi_A^{V,W} \leq n$.
(iii) $\deg \phi_A^{V,W} = m + n$ *if and only if* $A = \emptyset$.
(iv) $\phi_A^{V,W}(t_1, t_2) = 0$ *if and only if* $(0, \dots, 0) \in A$.

Proof Let $A \subseteq \mathbb{N}^{m+n}$ and $X_A = \{\mathbf{x} = (x_1, \dots, x_{m+n}) \in \mathbb{N}^{m+n} \mid a \not\leq_P \mathbf{x}$ for any $a \in A\}$.

Clearly, if one replaces A with the finite set of all its minimal points with respect to $\leq_P$, this replacement does not change X_A. Therefore, we can assume that A is finite:

$$A = \{a^{(1)}, \dots, a^{(d)}\} \text{ where } a^{(i)} = (a_{i,1}, \dots, a_{i,m+n}),\ \ 1 \leq i \leq d.$$

We are going to prove the theorem by induction on n and $\gamma(A) = \displaystyle\sum_{i=1}^{d} \sum_{j=1}^{m+n} a_{i,j}$.

If $n = 0$, the statement is true by [17, Theorem 2].

Let $n \geq 1$. If $\mathbf{x} = (x_1, \dots, x_m, y_1, \dots, y_n) \in X_A(r, s)$, then either $y_n = 0$ and $\displaystyle\sum_{i=1}^{n} w_i y_i \leq s$, or $y_n = y'_n + 1$ with $y'_n \in \mathbb{N}$ and $\displaystyle\sum_{i=1}^{n-1} w_i y_i + w_n y'_n \leq s - w_n$.

Let $A_0 = \{(a_1, \dots, a_m, b_1, \dots, b_{n-1}) \mid (a_1, \dots, a_m, b_1, \dots, b_{n-1}, 0) \in A\}$ and

$A_1 = \{a = (a_1, \dots, a_m, b_1, \dots, b_{n-1}, b'_n) \in \mathbb{N}^{m+n} \mid (a_1, \dots, a_m, b_1, \dots, b_{n-1}, b'_n + 1) \in A$, or $b'_n = 0$ and $(a_1, \dots, a_m, b_1, \dots, b_{n-1}, 0) \in A\}$.

If for any $B \subset \mathbb{N}^{m+n}$ and $r, s \in \mathbb{N}$, we set

$$N_B(r, s) = \text{Card}\{\mathbf{x} = (x_1, \dots, x_m, y_1, \dots, y_n) \in X_B \mid \sum_{i=1}^{m} v_i x_i \le r, \ \sum_{j=1}^{n} w_j y_j \le s\},$$

then it is easy to see that

$$N_A(r, s) = N_{A_0}(r, s) + N_{A_1}(r, s - w_n).$$

Since $|A_1| < |A|$ and $A_0 \subseteq \mathbb{N}^{m+n-1}$, $N_{A_0}(r, s)$ and $N_{A_1}(r, s - w_m)$ are expressed by bivariate quasi-polynomials $\phi_0(t_1, t_2)$ and $\phi_1(t_1, t_2)$ with $\deg_{t_1} \phi_0 \le m$, $\deg_{t_2} \phi_0 \le n-1$, $\deg_{t_1} \phi_1 \le m$, and $\deg_{t_2} \phi_1 \le n$. It follows that $N_A(r, s)$ is expressed by a bivariate quasi-polynomial satisfying the conditions of Theorem 3.2. □

Definition 3.3 The bivariate quasi-polynomial $\phi_{(A)}^{V,W}(t_1, t_2)$ whose existence is established by Theorem 3.2 is called the (m, n)-dimension quasi-polynomial of the set $A \subseteq \mathbb{N}^{m+n}$ associated with the weight vectors $V = (v_1, \dots, v_m)$ and $W = (w_1, \dots, w_n)$.

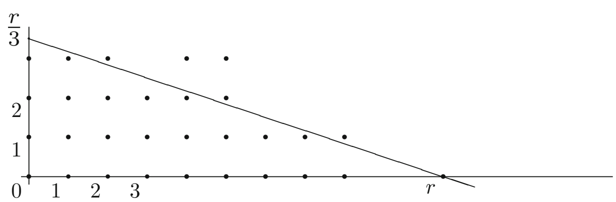
Example Let $m = 2$, $n = 1$, $A = \{(2, 1, 1), (0, 2, 1)\} \subseteq \mathbb{N}^3$ and $v_1 = 1$, $v_2 = 3$, $w_1 = 2$.

Then $A_0 = \emptyset \subseteq \mathbb{N}^2$ and $A_1 = \{(2, 1, 0), (0, 2, 0)\} \subseteq \mathbb{N}^3$. In this case $N_A(r, s) = N_{A_0}(r, s) + N_{A_1}(r, s - 2)$ where

$N_{A_0}(r, s) = \text{Card}\{(x, y) \in \mathbb{N}^2 \mid x + 3y \le r\}$ and

$N_{A_1}(r, s-2) = \text{Card}\{(x, y, z) \in \mathbb{N}^3 \mid (2, 1, 0) \not\le_P (x, y, z),\ (0, 2, 0) \not\le_P (x, y, z)$ and $x+3y \le r,\ 2z \le s-2\}$.

Clearly, $N_{A_0}(r, s) = N_{A_0}(r)$ is the number of integer points in the closed triangle $\triangle_r$ with vertices $(0, 0)$, $(r, 0)$, and $(0, r/3)$:



By the Ehrhart's theorem, $N_{A_0}(r)$ is a quadratic polynomial of r whose leading coefficient is equal to the area of the triangle $\triangle_1$, that is, to $1/6$. Therefore,

$$N_{A_0}(r) = \frac{1}{6}r^2 + [a, b, c]_r r + [d, e, f]_r$$

for some $a, b, c, d, e, f \in \mathbb{Z}$. In order to find these integers, we can count the number of integer points in the triangles $\triangle_r$ with $r = 0, 1, 2, 3, 4$, and 5 and get $N_{A_0}(0) = d = 1$,

$N_{A_0}(1) = \dfrac{1}{6} + b + e = 2$; $N_{A_0}(2) = \dfrac{2}{3} + 2c + f = 3$;

$N_{A_0}(3) = \dfrac{3}{2} + 3a + d = 5$; $N_{A_0}(4) = \dfrac{8}{3} + 4b + e = 7$;

$N_{A_0}(5) = \dfrac{25}{6} + 5c + f = 9$.

It follows that $a = b = c = \dfrac{5}{6}$, $d = e = 1$, and $f = \dfrac{2}{3}$, so

$$N_{A_0}(r, s) = N_{A_0}(r) = \frac{1}{6}r^2 + \frac{5}{6}r + \left[1, 1, \frac{2}{3}\right]_r$$

for all $r \in \mathbb{N}$.

The direct computation of the number of integer points in $X_{A_1}(r, s-2)$ gives

$$N_{A_1}(r, s-2) = \frac{1}{2}rs + \left[0, -\frac{1}{2}\right]_s r + \frac{3}{2}s + \left[0, -\frac{3}{2}\right]_s$$

whence

$$N_A(r, s) = \frac{1}{6}r^2 + \frac{1}{2}rs + \left[\frac{5}{6}, \frac{1}{3}\right]_s r + \frac{3}{2}s + \left[0, 0, -\frac{1}{3}\right]_r + \left[1, -\frac{1}{2}\right]_s.$$

Thus,

$$\phi_A^{V,W}(t_1, t_2) = \frac{1}{6}t_1^2 + \frac{1}{2}t_1t_2 + \left[\frac{5}{6}, \frac{1}{3}\right]_{t_2} t_1 + \frac{3}{2}t_2 + \left[0, 0, -\frac{1}{3}\right]_{t_1} + \left[1, -\frac{1}{2}\right]_{t_2}.$$

Note that the free term in the expression for $\phi_A^{V,W}(t_1, t_2)$ is a 2-dimensional periodic number $\left[\left[1, -\frac{1}{2}\right]_{t_2}, \left[1, -\frac{1}{2}\right]_{t_2}, \left[\frac{2}{3}, -\frac{5}{6}\right]_{t_2}\right]_{t_1}$ with period $(3, 2)$.

As it was mentioned at the beginning of the proof of Theorem 3.2, the (m, n)-dimension quasi-polynomial of the set $A \subseteq \mathbb{N}^{m+n}$ coincides with the (m, n)-dimension quasi-polynomial of the finite set of all minimal points of A with respect to the product order on $\mathbb{N}^{m+n}$. The following theorem gives a formula for computing the (m, n)-dimension quasi-polynomial of a finite subset of $\mathbb{N}^{m+n}$. It can be obtained by mimicking the proof of the corresponding result for (m, n)-dimension polynomials of finite subsets of $\mathbb{N}^{m+n}$ (see [13, Theorem 3.3]) with the replacement of the formula

$$\mathrm{Card}\{(a_1, \dots, a_{m+n}) \in \mathbb{N}^{m+n} \mid \sum_{i=1}^{m} a_i \le r, \sum_{j=m+1}^{m+n} a_j \le s\} = \binom{r+m}{m}\binom{s+n}{n}$$

with the formula

$$\mathrm{Card}\{(a_1, \dots, a_{m+n}) \in \mathbb{N}^{m+n} \mid \sum_{i=1}^{m} v_i a_i \le r, \sum_{j=m+1}^{m+n} w_j a_j \le s\} = \lambda_V^{(m)}(r)\lambda_W^{(n)}(s).$$

(The univariate quasi-polynomials of the form $\lambda_V^{(m)}$ are defined after Theorem 3.1.)

Theorem 3.4 *Let $A = \{a^{(1)}, \dots, a^{(p)}\}$ be a finite subset of $\mathbb{N}^{m+n}$ and let $a^{(i)} = (a_{i1}, \dots, a_{i,m+n})$ $(1 \le i \le p)$. Furthermore, for any $l \in \mathbb{N}$, $0 \le l \le p$, let $\Gamma(l, p)$ denote the set of all l-element subsets of the set $\{1, \dots, p\}$, and for any $\gamma \in \Gamma(l, p)$ let $\bar{a}_{\gamma j} = \max\{a_{ij} | i \in \gamma\}$ $(1 \le j \le m+n)$, $b_\gamma = \sum_{j=1}^{m} \bar{a}_{\gamma j}$, and $c_\sigma = \sum_{j=m+1}^{m+n} \bar{a}_{\gamma j}$. Then, with the notation of Theorem 3.2, one has*

$$\phi_A^{V,W}(t_1, t_2) = \sum_{l=0}^{p}(-1)^l \sum_{\gamma \in \Gamma(l,p)} \lambda_V^{(m)}(t_1 - b_\gamma)\lambda_W^{(n)}(t_2 - c_\gamma). \tag{3.1}$$

4 The Main Theorem

Let K be a Δ-σ-field and let $L = K\langle \eta_1, \dots, \eta_p \rangle$ be a Δ-σ-field extension of K generated by a finite set $\eta = \{\eta_1, \dots, \eta_p\}$. For any $r, s \in \mathbb{N}$, let $L_{r,s} = K(\{\tau(\eta_i) \mid \tau \in T(r, s),\ 1 \le i \le p\})$.

Theorem 4.1 *With the above notation, there exists a bivariate quasi-polynomial $\Phi_{\eta|K}^{(V,W)}(t_1, t_2)$ such that*

(i) $\Phi_{\eta|K}^{(V,W)}(r, s) = \mathrm{trdeg}_K\, L_{r,s}$ *for all sufficiently large* $(r, s) \in \mathbb{N}^2$.

(ii) $\deg_{t_1} \Phi_{\eta|K}^{(V,W)} \leq m = \text{Card}\,\Delta$ *and* $\deg_{t_2} \Phi_{\eta|K}^{(V,W)} \leq n = \text{Card}\,\sigma$.

(iii) $\Phi_{\eta|K}^{(V,W)}$ *is an alternating sum of bivariate quasi-polynomials of the form* $g(t_1)h(t_2)$ *where* $g(t_1)$ *and* $h(t_2)$ *are (univariate) Ehrhart quasi-polynomials associated with rational conic polytopes.*

Definition 4.2 The bivariate quasi-polynomial $\Phi_{\eta|K}^{(V,W)}(t_1, t_2)$ whose existence is established by Theorem 4.1 is called a Δ-σ-dimension quasi-polynomial of the Δ-σ-field extension L/K associated with the system of Δ-σ-generators η (and the weight vectors V and W).

Theorem 4.1 generalizes the theorem on a bivariate dimension polynomial of a finitely generated difference–differential field extension (see [14, Theorem 5.4]) and also allows one to assign a bivariate quasi-polynomial to a system of algebraic difference–differential (Δ-σ-) equations with weighted basic derivations and translations

$$f_i(y_1, \dots, y_p) = 0 \qquad (i = 1, \dots, q) \tag{4.1}$$

($f_i \in R = K\{y_1, \dots, y_p\}$ $(1 \leq i \leq q)$, where $K\{y_1, \dots, y_p\}$ is the ring of Δ-σ-polynomials in p variables over K) such that the Δ-σ-ideal P of R generated by $f_1, \dots, f_q$ is prime (e. g., to a system of linear difference–differential equations). Systems of this form arise in connection with systems of PDEs with weighted derivatives (see [21,22]) and their finite difference approximations.

In this case, the reflexive closure P^* of the Δ-σ-ideal P is also prime, so one can consider the quotient field of R/P^* as a finitely generated Δ-σ-field extension of K: $L = K\langle\eta_1, \dots, \eta_p\rangle$ where η_i is the canonical image of y_i in R/P^*.

The corresponding bivariate dimension quasi-polynomial $\Phi_{\eta|K}^{(V,W)}(t_1, t_2)$ can be viewed as the Einstein's strength of the system (4.1) in the sense of the corresponding concepts for systems of partial differential and difference equations (see [20] and [15, Sect. 7.7] for detailed descriptions of these concepts and their expressions as dimension polynomials).

Proof of the Main Theorem (Theorem 4.1)

Let K be a difference–differential (Δ-σ-) field, with $\Delta = \{\delta_1, \dots, \delta_m\}$ and $\sigma = \{\alpha_1, \dots, \alpha_n\}$. Let T be the free commutative semigroup generated by $\Delta \bigcup \sigma$, and $R = K\{y_1, \dots, y_p\}$ the algebra of Δ-σ-polynomials over K. (As we have seen, R can be viewed as a polynomial ring in the set of indeterminates $TY = \{\tau y_i | \tau \in T, 1 \leq i \leq p\}$ over K with the natural extension of the actions of δ_i and α_j; elements of the set TY will be called *terms*.)

As before, if $\tau = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n} \in T$ $(k_i, l_j \in \mathbb{N})$, then the numbers

$$\text{ord}_\Delta\, \tau = \sum_{i=1}^{m} v_i k_i \qquad \text{and} \qquad \text{ord}_\sigma\, \tau = \sum_{j=1}^{n} w_j l_j$$

are said to be the *orders* of τ with respect to Δ and σ, respectively. We also set $\tau_\Delta = \delta_1^{k_1} \dots \delta_m^{k_m}$ and $\tau_\sigma = \alpha_1^{l_1} \dots \alpha_n^{l_n}$.

Consider two well-orderings $<_\Delta$ and $<_\sigma$ of T such that

$\tau = \delta_1^{k_1} \dots \delta_m^{k_m} \alpha_1^{l_1} \dots \alpha_n^{l_n} <_\Delta \tau' = \delta_1^{k'_1} \dots \delta_m^{k'_m} \alpha_1^{l'_1} \dots \alpha_n^{l'_n}$ if and only if the $(m + n + 2)$-tuple

$(\text{ord}_\Delta\, \tau, \text{ord}_\sigma\, \tau, k_1, \dots, k_m, l_1, \dots, l_n)$ is less than the corresponding $(m + n + 2)$-tuple for τ' with respect to the lexicographic order on $\mathbb{N}^{m+n+2}$. The order $<_\sigma$ is defined similarly: $\tau <_\sigma \tau'$ if and only if the $(m + n + 2)$-tuple $(\text{ord}_\sigma\, \tau, \text{ord}_\Delta\, \tau, l_1, \dots, l_n, k_1, \dots, k_m)$ is less than the corresponding $(m + n + 2)$-tuple for τ' with respect to the lexicographic order on $\mathbb{N}^{m+n+2}$.

The orders $<_\Delta$ and $<_\sigma$ on the set T induce similar well-orderings of the set of terms TY (denoted by the same symbols): if $\tau y_i, \tau' y_j \in TY$, then $\tau y_i <_\Delta$(respectively, $<_\sigma$) $\tau' y_j$ if and only if $\tau <_\Delta$(respectively, $<_\sigma$) τ' or $\tau = \tau'$ and $i < j$.

If $A \in K\{y_1, \dots, y_p\} \setminus K$, then the highest terms of A with respect to $<_\Delta$ and $<_\sigma$ are called the Δ-*leader* and σ-*leader* of A, respectively; they are denoted by u_A and v_A, respectively.

If A is written as a polynomial in u_A, $A = I_d(u_A)^d + I_{d-1}(u_A)^{d-1} + \cdots + I_0$, where all terms of $I_0, \ldots, I_d$ are less than u_A with respect to $<_\Delta$, then I_d and $\partial A/\partial u_A$ are called, respectively, the *initial* and *separant* of A; they are denoted by I_A and S_A, respectively.

If $A, B \in K\{y_1, \ldots, y_p\}$, then A is said to have *lower rank* than B (we write $rk\ A < rk\ B$) if either $A \in K$, $B \notin K$, or the vector $(u_A, \deg_{u_A} A, \mathrm{ord}_\sigma v_A)$ is less than $(u_B, \deg_{u_B} B, \mathrm{ord}_\sigma v_B)$ with respect to the lexicographic order (u_A and u_B are compared with respect to $<_\Delta$).

If the vectors are equal (or $A, B \in K$) we say that A and B are of the same rank and write $rk\ A = rk\ B$.

If $A, B \in K\{y_1, \ldots, y_p\}$, then B is said to be *reduced with respect to* A if

(i) B does not contain terms τu_A such that $\tau_\Delta \neq 1$, and $\mathrm{ord}_\sigma(\tau v_A) \leq \mathrm{ord}_\sigma v_B$.

(ii) If B contains a term τu_A with $\tau_\Delta = 1$, then $\mathrm{ord}_\sigma v_B < \mathrm{ord}_\sigma(\tau v_A)$, or $\mathrm{ord}_\sigma(\tau v_A) \leq \mathrm{ord}_\sigma v_B$ and $\deg_{\tau u_A} B < \deg_{u_A} A$.

If $B \in K\{y_1, \ldots, y_p\}$, then B is said to be *reduced with respect to a set* $\Sigma \subseteq K\{y_1, \ldots, y_p\}$ if B is reduced with respect to every element of Σ.

A set $\mathcal{A} \subseteq K\{y_1, \ldots, y_p\}$ is called *autoreduced* if $\mathcal{A} \bigcap K = \emptyset$ and every element of $\mathcal{A}$ is reduced with respect to any other element of this set.

The proofs of the following four statements (Propositions 4.3 - 4.6 below) can be obtained by repeating the arguments in the proofs of the corresponding results (Propositions 5.4, 5.5, 5.7, and 5.8) of [16]. (Despite the fact that the term orderings introduced above involve weights of the elements of the basic set $\Delta \bigcup \sigma$ the arguments of the proofs of the statements of [16] are still valid.)

Proposition 4.3 *Every autoreduced set is finite.*

Proposition 4.4 *Let $\mathcal{A} = \{A_1, \ldots, A_d\}$ be an autoreduced set in the ring $R = K\{y_1, \ldots, y_p\}$ and let I_k and S_k denote the initial and separant of A_k, respectively. Furthermore, let $I(\mathcal{A}) = \{X \in K\{y_1, \ldots, y_p\} \mid X = 1$ or X is a product of finitely many elements of the form $\tau(I_k)$ and $\tau(S_k)$ where $\tau, \tau' \in T\}$. Then for any Δ-σ-polynomial B, there exist $B_0 \in K\{y_1, \ldots, y_p\}$ and $J \in I(\mathcal{A})$ such that B_0 is reduced with respect to $\mathcal{A}$ and $JB \equiv B_0\ (mod[\mathcal{A}])$ (that is, $JB - B_0 \in [\mathcal{A}]$).*

With the notation of the last proposition, we say that the Δ-σ-polynomial B *reduces to* B_0 modulo $\mathcal{A}$.

In what follows, elements of an autoreduced set will be always arranged in order of increasing rank. If $\mathcal{A} = \{A_1, \ldots, A_d\}$ and $\mathcal{A}' = \{B_1, \ldots, B_e\}$ are two autoreduced sets in $K\{y_1, \ldots, y_p\}$, we say that $\mathcal{A}$ has *lower rank* than $\mathcal{A}'$ if one of the following two cases holds:

(1) There exists $k \in \mathbb{N}$ such that $k \leq \min\{d, e\}$, $rk\ A_i = rk\ B_i$ for $i = 1, \ldots, k-1$ and $rk\ A_k < rk\ B_k$.

(2) $d > e$ and $rk\ A_i = rk\ B_i$ for $i = 1, \ldots, e$.

If $d = e$ and $rk\ A_i = rk\ B_i$ for $i = 1, \ldots, d$, then $\mathcal{A}$ is said to have the same rank as $\mathcal{A}'$.

Proposition 4.5 *In every nonempty family of autoreduced sets of Δ-σ-polynomials there exists an autoreduced set of lowest rank.*

Let J be any ideal of the ring $K\{y_1, \ldots, y_p\}$. Since the set of all autoreduced subsets of J is not empty (if $A \in J$, then $\{A\}$ is an autoreduced subset of J), the last statement shows that the ideal J contains an autoreduced subset of lowest rank. Such an autoreduced set is called a **characteristic set** of the ideal J.

Proposition 4.6 *Let $\mathcal{A} = \{A_1, \ldots, A_d\}$ be a characteristic set of a Δ-σ-ideal J of the ring $R = K\{y_1, \ldots, y_p\}$. Then an element $B \in J$ is reduced with respect to the set $\mathcal{A}$ if and only if $B = 0$.*

Given a characteristic set $\mathcal{A}$ of a Δ-σ-ideal J in $K\{y_1, \ldots, y_p\}$, the last proposition, together with Proposition 4.4, allows us to decide whether a Δ-σ-polynomial B belongs to J. First, we reduce B to a Δ-σ-polynomial B_0 modulo $\mathcal{A}$ (we use the notation of Proposition 4.4; the process of reduction is similar to the corresponding process of reduction for standard difference–differential polynomials, see [10, Theorem 3.5.27]). Then we use the result of Proposition 4.6: $B \in J$ if and only if $B_0 = 0$.

The proof of following proposition can be obtained using the same arguments as in the proof of Theorem 3.1 of [16].

Proposition 4.7 *Let $L = K\langle \eta_1, \dots, \eta_p\rangle$ and P the defining Δ-σ-ideal of the Δ-σ-field extension L/K (i. e., $P = \mathrm{Ker}(K\{y_1, \dots, y_p\} \to L)$, $y_i \mapsto \eta_i$). Let $\mathcal{A} = \{A_1, \dots, A_d\}$ be a characteristic set of P and for any $r, s \in \mathbf{N}$, let*
$U^{(1)}_{r,s} = \{u \in TY \mid \mathrm{ord}_\Delta u \le r,\ \mathrm{ord}_\sigma u \le s$ and $u \ne \tau u_{A_i}$ for any $\tau \in T$, $1 \le i \le d\}$,
$U^{(2)}_{r,s} = \{u \in TY \mid \mathrm{ord}_\Delta u \le r,\ \mathrm{ord}_\sigma u \le s$ and for every $\tau \in T$, $A \in \mathcal{A}$ such that $u = \tau u_A$, one has $\mathrm{ord}_\sigma(\tau v_A) > s\}$, and

$$U_{r,s} = U^{(1)}_{r,s} \bigcup U^{(2)}_{r,s}.$$

Then $\bar{U}_{r,s} = \{u(\eta) | u \in U_{r,s}\}$ is a transcendence basis of the field $K(\bigcup_{j=1}^{n} T(r,s)\eta_j)$ over K.

Now, in order to prove the main theorem, one has to evaluate $\mathrm{Card}\, U_{r,s} = U^{(1)}_{r,s} \bigcup U^{(2)}_{r,s}$. By Theorem 3.2, $\mathrm{Card}\ U^{(1)}_{r,s}$ is expressed by a bivariate quasi-polynomial whose degrees with respect to r and s do not exceed m and n, respectively. Furthermore, using the principle of inclusion and exclusion, one can express $\mathrm{Card}\ U^{(2)}_{r,s}$ as an alternating sum of bivariate quasi-polynomials that describe the numbers of points $(k_1, \dots, k_m, l_1, \dots, l_n) \in \mathbb{N}^{m+n}$ satisfying the inequalities

$$\sum_{i=1}^{m} v_i k_i \le r - a, \qquad s - b < \sum_{j=1}^{n} w_j l_j \le s$$

where $a, b \in \mathbb{N}$. (a and b appear as the orders of the Δ- and σ- leaders of elements of the characteristic set $\mathcal{A}$ with respect to Δ and σ.) It follows that for all sufficiently large $(r, s) \in \mathbb{N}^2$, $\mathrm{Card}\ U_{r,s} = \mathrm{trdeg}_K\, K(\bigcup_{j=1}^{n} T(r,s)\eta_j)$ is expressed by a bivariate quasi-polynomial $\Phi^{(V,W)}_{\eta|K}(t_1, t_2)$ whose degrees with respect to t_1 and t_2 do not exceed m and n, respectively. The last statement of the main theorem follows from the fact that the bivariate quasi-polynomial that expresses $\mathrm{Card}\ U^{(1)}_{r,s}$ is the value at $(t_1, t_2) = (r, s)$ of an alternating sum of the (m, n)-dimension quasi-polynomials of the form $\lambda^{(m)}_V(t_1 - a)\lambda^{(n)}_W(t_2 - b)$ with $a, b \in \mathbb{Z}$ (see Theorem 3.4) and $\mathrm{Card}\ U^{(2)}_{r,s}$ is expressed as a similar alternative sum. □

We conclude with a theorem that gives invariants of the Δ-σ-dimension quasi-polynomial of a Δ-σ-field extension $L = K\langle \eta_1, \dots, \eta_p\rangle$ (we use the notation of Theorem 4.1), that is, numbers carried by the quasi-polynomial $\Phi^{(V,W)}_{\eta|K}(t_1, t_2)$ that do not depend on the set of Δ-σ-generators $\eta = \{\eta_1, \dots, \eta_p\}$ of L/K.

Theorem 4.8 *With the notation of Theorem 4.1, let $E(\Phi^{(V,W)}_{\eta|K})$ be the set of all pairs $(i, j) \in \mathbb{N}^2$ such that the monomial $t_1^i t_2^j$ appears in $\Phi^{(V,W)}_{\eta|K}(t_1, t_2)$ with a nonzero coefficient. Let $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$ be the maximal elements of $E(\Phi^{(V,W)}_{\eta|K})$ with respect to the lexicographic and reverse lexicographic orders on $\mathbb{N}^2$, respectively. Then μ, ν, the coefficients of the monomials $t_1^{\mu_1} t_2^{\mu_2}$ and $t_1^{\nu_1} t_2^{\nu_2}$ in $\Phi^{(V,W)}_{\eta|K}(t_1, t_2)$, as well as the total degree d of the quasi-polynomial $\Phi^{(V,W)}_{\eta|K}$ and the coefficient of $t_1^m t_2^n$ in this quasi-polynomial do not depend on the system of Δ-σ-generators $\eta = \{\eta_1, \dots, \eta_p\}$ of L/K.*

Proof Let $L = K\langle \eta_1, \dots, \eta_p\rangle = K\langle \zeta_1, \dots, \zeta_q\rangle$ where $\eta = \{\eta_1, \dots, \eta_p\}$ and $\zeta = \{\zeta_1, \dots, \zeta_q\}$ are two finite sets of Δ-σ-generators of L/K. Let $N = (N_1, N_2)$ be a minimal (with respect to the product order) element of $\mathbb{N}^2$ such that if (e_1, e_2) is a 2-period of a nonzero coefficient of $\Phi_{\eta|K}^{(V,W)}(t_1, t_2)$ or $\Phi_{\zeta|K}^{(V,W)}(t_1, t_2)$ (such a coefficient is a 2-dimensional periodic number), then e_1 divides N_1 and e_2 divides N_2. Then there exist $2N_1N_2$ bivariate polynomials $g_{ij}(t_1, t_2),\ h_{ij}(t_1, t_2) \in \mathbb{Q}[t_1, t_2]$ $(0 \leq i \leq N_1 - 1, 0 \leq j \leq N_2 - 1)$ such that

$$\Phi_{\eta|K}^{(V,W)}(r, s) = g_{ij}(r, s) \ \text{ and } \ \Phi_{\zeta|K}^{(V,W)}(r, s) = h_{ij}(r, s)$$

if $r \equiv i\ (mod\ N_1)$, $s \equiv j\ (mod\ N_2)$.

Since $\eta \subseteq K\langle\zeta_1, \dots, \zeta_q\rangle$ and $\zeta \subseteq K\langle\eta_1, \dots, \eta_p\rangle$, there exists $d \in \mathbb{N}$ such that

$$\eta \subseteq K\left(\bigcup_{l=1}^{q} T(dN_1, dN_2)\zeta_l\right) \ \text{ and } \ \zeta \subseteq K\left(\bigcup_{k=1}^{p} T(dN_1, dN_2)\eta_k\right).$$

Then

$$K\left(\bigcup_{k=1}^{p} T(r, s)\eta_k\right) \subseteq K\left(\bigcup_{l=1}^{q} T(r + dN_1, s + dN_2)\zeta_l\right) \ \text{ and}$$
$$K\left(\bigcup_{l=1}^{q} T(r, s)\zeta_l\right) \subseteq K\left(\bigcup_{k=1}^{p} T(r + dN_1, s + dN_2)\eta_k\right)$$

for every $(r, s) \in \mathbb{N}^2$.

Therefore, for all $(i, j) \in \{0, 1, \dots, N_1 - 1\} \times \{0, 1, \dots, N_2 - 1\}$ and for all sufficiently large $(r, s) \in \mathbb{N}^2$, one has

$$g_{ij}(r, s) \leq h_{ij}(r + dN_1, s + dN_2) \tag{4.2}$$

and

$$h_{ij}(r, s) \leq g_{ij}(r + dN_1, s + dN_2) \tag{4.3}$$

for all $(i, j) \in \{0, 1, \dots, N_1 - 1\} \times \{0, 1, \dots, N_2 - 1\}$ and all sufficiently large $(r, s) \in \mathbb{N}^2$.

Since g_{ij} and h_{ij} are numerical polynomials (that is, polynomials with rational coefficients that take integer values for all sufficiently large integer values of their arguments), they can be written as

$$g_{ij}(t_1, t_2) = \sum_{k=0}^{m}\sum_{l=0}^{n} a_{kl}^{(ij)}\binom{t_1 + k}{k}\binom{t_2 + l}{l} \ \text{ and } \ h_{ij}(t_1, t_2) = \sum_{k=0}^{m}\sum_{l=0}^{n} b_{kl}^{(ij)}\binom{t_1 + k}{k}\binom{t_2 + l}{l},$$

respectively, where $a_{kl}^{(ij)}, b_{kl}^{(ij)} \in \mathbb{Z}$. Let

$$E(g_{ij}) = \{(k, l) \in \mathbb{N}^2 \,|\, 0 \leq k \leq m,\ 0 \leq l \leq n \ \text{ and } \ a_{kl}^{(ij)} \neq 0\},$$
$$E(h_{ij}) = \{(k, l) \in \mathbb{N}^2 \,|\, 0 \leq k \leq m,\ 0 \leq l \leq n \ \text{ and } \ b_{kl}^{(ij)} \neq 0\}.$$

Furthermore, let (μ_{1ij}, μ_{2ij}) and (ν_{1ij}, ν_{2ij}) (respectively, (μ'_{1ij}, μ'_{2ij}) and (ν'_{1ij}, ν'_{2ij})) be the maximal elements of $E(g_{ij})$ (respectively, of $E(h_{ij})$) relative to the lexicographic and reverse lexicographic orders on $\mathbb{N}^2$, respectively. Then, as it is shown in [13, Theorem 5.4], the polynomials g_{ij} and h_{ij} have the same total degree, $a_{mn}^{(ij)} = b_{mn}^{(ij)}$ and also $(\mu_{1ij}, \mu_{2ij}) = (\mu'_{1ij}, \mu'_{2ij})$, $(\nu_{1ij}, \nu_{2ij}) = (\nu'_{1ij}, \nu'_{2ij})$, $a_{\mu_{1ij}\mu_{2ij}}^{(ij)} = b_{\mu_{1ij}\mu_{2ij}}^{(ij)}$, and $a_{\nu_{1ij}\nu_{2ij}}^{(ij)} = b_{\nu_{1ij}\nu_{2ij}}^{(ij)}$.

Since these equalities hold for all pairs of polynomials g_{ij} and h_{ij} $(0 \leq i \leq N_1 - 1,\ 0 \leq j \leq N_2 - 1)$ that represent the bivariate quasi-polynomials $\Phi_{\eta|K}^{(V,W)}(t_1, t_2)$ and $\Phi_{\zeta|K}^{(V,W)}(t_1, t_2)$, respectively, we obtain all statements of our theorem. □

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