A General Accelerated Destructive Degradation Testing Model for Reliability Analysis

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Abstract: In recent years accelerated destructive degradation testing (ADDT) has been applied to obtaining the reliability information of an asset (component) at use conditions when the component is highly reliable. In ADDT, degradation data are measured under stress levels more severe than usual so that more component failures can be observed in a short period. In the literature, most application-specific ADDT models assume a parametric degradation process under different accelerating conditions. Models without strong parametric assumptions are desirable to describe the complex ADDT processes. This research proposes a general ADDT model that consists of a nonparametric part to describe the degradation path and a parametric part to describe the accelerating-variable effect. The proposed model not only provides more model flexibility with few assumptions, but also retains the physical mechanisms of degradation. Due to the complexity of parameter estimation, an efficient method based on self-adaptive differential evolution is developed to estimate model parameters. A simulation study is implemented to verify the developed methods. Two real-world case studies are conducted and the results show the superior performance of the developed model compared with the existing methods.

Index Terms—Accelerated model; ADDT; Degradation model; Long-term property evaluation; B-spline; Akaike Information Criterion

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ACRONYMS

ADDT	Accelerated Destructive Degradation Testing
SD	Standard Deviation
MSE	Mean Squared Error
AIC	Akaike Information Criterion
REML	Restricted Maximum Likelihood
MLE	Maximum Likelihood Estimation

NOTATION

x_i	Transformed temperature level
$y_i(t)$	Degradation measurement under x_i at time t
AF_i	Accelerating value at level i
$(\alpha,\ \beta)$	Unknown parameters associated with the accelerating variable
$x_{\max}$	The transformed value of the maximum of the accelerating variable
$\boldsymbol{\theta}$	Parameters of the proposed model
$G_{ql}(z)$	The l^{th} B-spline basis function of degree q evaluated at z

d_r The r^{th} interior knot

M Number of accelerating conditions

N Number of measurement time points for each sample under an accelerating condition

ε The unit-to-unit variability and the measurement error

$loglik$ The log-likelihood value under a given parameter combination

AIC The AIC value under a given parameter combination and knot selection

I. INTRODUCTION

In modern industries, high-quality products are expected to perform for years to decades. Degradation testing is a method developed in the reliability literature to obtain product reliability by using the degradation of performance characteristics, such as the light intensity [1], crack length [2], and material elongation [3]. With the development of highly-quality products, degradation testing under a normal use condition is not efficient to meet industrial requirements due to the limitation of testing times. In practice, accelerating variables such as temperature, voltage, or pressure are incorporated into the degradation testing so that the failures can be reached in less time. In some cases, the physical property being tested may only not survive the measurement procedure; that is, the measurement can only be done through destructive means. This kind of testing is referred to as the Accelerated Destructive Degradation Testing (ADDT).

In the reliability literature for degradation data analysis, Whitmore and Schenkelberg [4], Lu, et al. [5], Nelson [6], Crk [7] conducted early research in degradation modeling. Chen, et al. [8]

summarized the methods for degradation data including non-destructive and destructive degradation testing. Nelson [9] used accelerated life testing as a tool to estimate performance degradation. Lu and Meeker [10] used the repeated-measures degradation testing (RMDT) data to estimate the failure time. Meeker, et al. [11] developed a nonlinear mixed-effects model for RMDT data. Gorjian, et al. [12] and Meeker, et al. [13] made an introductory level description of some degradation models. In related literature, Lu and Meeker [10] used RMDT data to estimate the failure time and Meeker, et al. [11] proposed a nonlinear mixed-effects model for analyzing RMDT data. Gorjian, et al. [12] and Meeker, et al. [13] further developed statistical degradation models for reliability analysis. Xu, et al. [14] proposed a nonlinear general path model for analyzing degradation data with dynamic covariates.

In the ADDT literature, several parametric models have been developed to analyze ADDT data. Shi, et al. [15] designed an ADDT plan to study combinations of different accelerating variables. Tsai, et al. [16] proposed an optimal design approach for ADDT. , Lim and Yum [17], and Jeng, et al. [18] developed methods for ADDT planning and analysis. In recent years, Xiao and Ye [19] analyzed destructive degradation tests for highly reliable products with random initial degradation values. Hong, et al. [20] proposed models in which the acceleration can be represented by stochastic processes under certain conditions. Tsai, et al. [16] developed parametric models to analyze ADDT data collected from different materials with multiple accelerating levels.

Although the existing parametric ADDT models have been widely used in practice, the common disadvantages of parametric models are their being application specific and the significantly biased conclusions if the model assumption is not satisfied. Instead of a case-by-case parametric modeling approach, Xie, et al. [21] proposed a semiparametric model to analyze ADDT data. In this research, we propose a general ADDT model with a nonparametric part to describe

the degradation path and a parametric part to describe the accelerating-variable effect. The parametric part takes both the scale and shape parameters of the response distribution into consideration. The major contribution of this paper is two-fold. First, the proposed model is more general and robust than the existing semiparametric model in the literature. Through a simulation study and real-world case studies, we later show that the performance of the proposed general ADDT model is much more improved over the existing one. Second, the traditional optimization method cannot be directly applied to estimating the model parameters due to the complexity of the objective function. To overcome this challenge, we develop an efficient model parameter estimation method based on self-adaptive differential evolution.

The remainder of this paper is organized as follows. In Section II, we present the notation and propose the general ADDT model. In Section III, we develop the method for model parameter estimation. In Section IV, we conduct a simulation study to investigate the performance of the developed methods. In Section V, we apply the proposed model to analyzing ADDT data of two real-world examples and compare the performance to existing models. In Section VI, a conclusion and future work are provided.

II. DATA AND MODEL

For a given ADDT, denote the degradation measurement at time *t* under acceleration factor condition x_i by y_i, which is related to x_i by

$$y_i(t) = H(t; x_i) + \varepsilon_i, \tag{1}$$

where ε_i represent both *the unit-to-unit* variability and the measurement error, which is assumed to follow the normal distribution, i.e., $\varepsilon_i \sim N(0, \sigma^2)$. Here σ^2 is the variance of ε_i. When there

are multiple test units in the ADDT experiment under the same condition x_i at the same time t, their corresponding error correlation is ρ. As an example of an ADDT model, consider a temperature-accelerated process, which is occurance of this type of test. A popular model to describe the relationship between the degradation and temperature is the Arrhenius model. In this model, the temperature is transformed into the following form

$$x_i = h(AF_i) = \frac{-11605}{Temp_i + 273.16}, \tag{2}$$

where $Temp_i$ is the temperature in degrees Celsius, and the value 11,605 is the reciprocal of the Boltzmann's constant. The value 273.16 is the difference between the Kelvin temperature scale and the Celsius temperature scale. Without loss of generality, y_i is assumed to be monotonic decreasing over time.

In this research, we propose a general model form to fit ADDT data. Specifically, $H(t; x_i)$ in equation (1) is expressed as

$$H(t; x_i) = f[\eta_{\alpha,\beta}(t; x_i)], \tag{3}$$

where $f(\cdot)$ is a nonparametric construction and $\eta_{\alpha,\beta}(t; x_i)$ is a scale-shape-acceleration model with parameters (α, β). An example of this model for the case of temperature acceleration would be

$$\eta_{\alpha,\beta}(t; x_i) = \frac{t^{\exp[\alpha(x_{\max} - x_i)]}}{\exp[\beta(x_{\max} - x_i)]}, \tag{4}$$

where x_i is the accelerating variable defined in equation (2), and $x_{\max} = -11605 / [\max_i(Temp_i) + 273.16]$.

We choose the monotonic B-spline method to describe the nonparametric portion of the proposed model. Let $d_1 \le \cdots \le d_r$ be the interior knots and let d_0 and d_{r+1} be the two boundary points. The ordered knots are $d_{-q} = \cdots = d_0 \le d_1 \le \cdots \le d_r \le d_{r+1} = \cdots = d_{r+q+1}$. Here the lower and upper boundary points are repeated q times, where q is the polynomial degree. In the following, we define the ordered knot sequence as $d_1, \cdots, d_{r+2q+2}$ for simplicity. The B-spline basis functions are defined recursively. For degree 0, the first spline function is

$$G_{0l}(z) = \mathbf{1}(d_l \le z \le d_{l+1}), \tag{5}$$

and for degree q, the function is

$$G_{q,l}(z) = \frac{z - d_l}{d_{l+q} - d_l} G_{q-1,l}(z) + \frac{d_{l+q+1} - z}{d_{l+q+1} - d_{l+1}} G_{q-1,l+1}(z). \tag{6}$$

Here $l = 1, \ldots, p,$ and $\mathbf{1}(\cdot)$ is an indicator function.

With the combination of the nonparametric and parametric portions, the model in (1) is represented in its entirety as

$$y_i(t) = H(t; x_i) + \varepsilon = \sum_{l=1}^{p} \gamma_l G_{q,l}[\eta_{\alpha,\beta}(t; x_i)] + \varepsilon_i. \tag{7}$$

Here, γ_l is the coefficients of the B-spline basis function $G_{q,l}[\eta_{\alpha,\beta}(t; x_i)]$. The proposed model in (7) falls within the class of shape-scale-acceleration models. For every stress level i, $H(t; x_i)$ is a decreasing function on time t, with the effects of the time-scale and the time shape controlled through shape acceleration factor (SHAF) $\exp[\alpha(x_{\max} - x_i)]$ and the scale acceleration factor (SCAF) $\exp[\beta(x_{\max} - x_i)]$, respectively, as shown in equation (4). When the SHAF=1, the time

acceleration rate $d\eta/dt$ is determined by the SCAF and so does not depend on a specific time point. When the SHAF$\neq 1$, the time acceleration rate also varies with time. Thus, the introduction of the SHAF allows for more flexibility in modeling the time acceleration rate.

When the acceleration level is at its highest, i.e., $x_{\max} - x_{\max} = 0$, $\eta_i(t; x_i \alpha, \beta) = t$, indicating that the degradation path no longer relies on α and β. Under this situation, the corresponding baseline degradation path for the scale-shape-acceleration model is

$$H(t; x_{\max}) = \sum_{l=1}^{p} \gamma_l G_{q,l}(t). \tag{8}$$

The first derivative of $H(t; x_i)$ with respect to $\eta_{\alpha,\beta}(t; x_i)$ can be calculated as follows.

$$\frac{dH(t; x_i)}{d\eta_{\alpha,\beta}(t; x_i)} = \sum_{l=2}^{p} (q-1) \frac{\gamma_l - \gamma_{l-1}}{d_{l+q+1} - d_l} G_{q-1,l}(\eta_{\alpha,\beta}(t; x_i)). \tag{9}$$

According to the assumption of monotone decreasing behavior in the degradation path, Equation (9) should always be negative. As the B-spline basis functions are nonnegative, this indicates that for $2 \le l \le p$, the condition $\gamma_l \le \gamma_{l-1}$ is sufficient for a monotone decreasing degradation path when the degree of the basis function $q = 1, 2$. A low degree of spline ($q \le 4$) and a small number of interior knots ($1 \le \gamma \le 5$) are usually sufficient to provide a good fit to the data. It was proved in the literature [22] that the optimal number of knots is in the order of $n^{1/5}$, where n is the number of observation. Thus, a small number of knots ($1 \le \gamma \le 5$) is sufficient for the modest size of datasets, such as the datasets in the case study of this research.

III. MODEL PARAMETER ESTIMATION AND KNOT SELECTION

To estimate the unknown parameters in the proposed model (7), we develop a likelihood-based optimization method. As shown in Figure 1, the developed method is an iterative process that consists of two parts: (1) knot selection of B-splines, and (2) parameter estimation. The parameter estimation is based on the initial selected knot setting from part 1, and the updated knot selection is based on the profile likelihood value calculated from part 2.

To find the best knot setting, we use Akaike Information Criterion (AIC), which considers both the profile log-likelihood values and the number of model parameters. For a given knot setting, the model parameters can be estimated by maximizing profile log-likelihood values. The objective function of the corresponding optimization problem is nonlinear and complex. As a result, the traditional gradient-based optimization methods cannot be directly applied. To overcome this difficulty, we develop an efficient parameter estimation method by applying the framework of self-adaptive differential evolution (SaDE) (). The SaDE method is a population-based stochastic search strategy which aims to solve global optimization problems like other evolution algorithms. In the classical differential evolution (DE) method (), the control parameters and learning strategies highly dependents on the problems under consideration so that significant time needs to be spent to try through various strategies and adjust the corresponding parameters. The key improvement of the SaDE over the classical DE method is that the learning strategy and parameter settings are gradually self-adapted during the evaluation process based on the learning experience. Besides that, The authors of [23] also give the numerical analysis of the convergence of the SaDE algorithm which shows that the convergence speed of SaDE performs much better than the original DE algorithm especially in the high-dimension situation.

After obtaining the optimum profile log-likelihood by using the developed optimization method, the *AIC* value can be calculated and used as the performance index to select optimal knot setting of the B-splines. Thus, the two parts run iteratively until the best knot setting and the optimal model parameters are obtained. The details of the parameter estimation and knot selection stages are discussed in Sections 3.1 and 3.2, respectively.

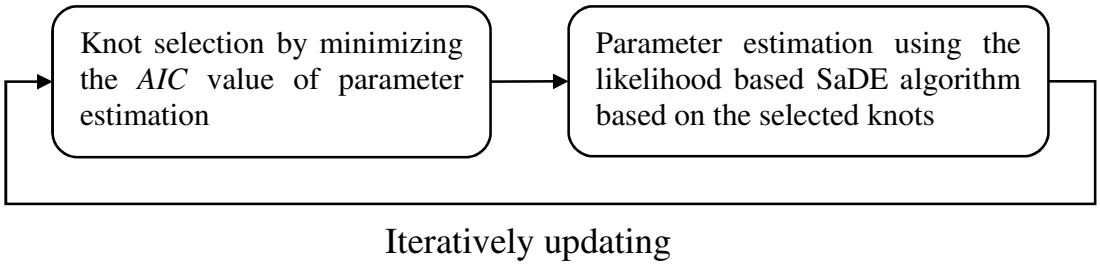


Figure 1. the framework of the knot selection and parameter estimation

3.1 Parameter estimation

For simplicity, we express the proposed model using the matrix form. Suppose there are *M* samples under testing, and for each sample the degradation path consists of *N* measurement time points. The response vector $\mathbf{y}=(y_{11},\cdots,y_{1N},\cdots,y_{M1},\cdots,y_{MN})$ contains all the degradation measurements. The degradation model can then be written as

$$\mathbf{y} = \mathbf{X}_{\alpha,\beta}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}, \tag{10}$$

where

$$\mathbf{X}_{\alpha,\beta} = \begin{bmatrix} G_{q,1}(\eta_1(t_1;x_1,\alpha,\beta)) & \cdots & G_{q,p}(\eta_1(t_1;x_1,\alpha,\beta)) \\ G_{q,1}(\eta_1(t_2;x_2,\alpha,\beta)) & \cdots & G_{q,p}(\eta_1(t_2;x_2,\alpha,\beta)) \\ \vdots & \ddots & \vdots \\ G_{q,1}(\eta_N(t_k;x_M,\alpha,\beta)) & \cdots & G_{q,p}(\eta_N(t_k;x_M,\alpha,\beta)) \end{bmatrix}.$$

In equation (10), $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_p)'$ are the coefficients, $\boldsymbol{\varepsilon} = (\varepsilon_{11}, \cdots, \varepsilon_{1n}, \cdots, \varepsilon_{M1}, \cdots, \varepsilon_{MN})'$, $\boldsymbol{\varepsilon} \sim N(0, \boldsymbol{\Sigma})$. $\boldsymbol{\Sigma} = Diag(\boldsymbol{\Sigma}_{11}, \cdots, \boldsymbol{\Sigma}_{1N}, \cdots, \boldsymbol{\Sigma}_{M1}, \cdots, \boldsymbol{\Sigma}_{MN})$ and $\boldsymbol{\Sigma}_{mn} = \sigma^2[(1-\rho)\mathbf{I}_N + \rho\mathbf{J}_N]$, where $\mathbf{I}_N$ is an $N \times N$ identity matrix and $\mathbf{J}_N$ is an $N \times N$ matrix of $\mathbf{1}'s$. We also rewrite $\boldsymbol{\Sigma} = \sigma^2\mathbf{R}$, where $\mathbf{R} = Diag(\mathbf{R}_{11}, \cdots, \mathbf{R}_{1N}, \cdots, \mathbf{R}_{M1}, \cdots, \mathbf{R}_{MN})$ and $\mathbf{R}_{mn} = (1-\rho)\mathbf{I}_N + \rho\mathbf{J}_N$.

To calculate the objective function of the SaDE algorithm which is the profile log-likelihood function of (α, β), the corresponding estimation of $(\boldsymbol{\gamma}', \sigma, \rho)'$ for a given pair of (α, β) needs to be obtained. We develop an iterative procedure to calculate the estimate of $(\boldsymbol{\gamma}', \sigma, \rho)'$. Specifically, given the estimates $(\sigma^{(m-1)}, \rho^{(m-1)})'$ at the m^{th} iteration, the value of $\hat{\boldsymbol{\gamma}}_i^{(m)}$ is calculated by minimizing

$$Q(\boldsymbol{\gamma}) = (\mathbf{y} - \mathbf{X}_{\alpha,\beta}\boldsymbol{\gamma})'(\hat{\boldsymbol{\Sigma}}^{(m-1)})^{-1}(\mathbf{y} - \mathbf{X}_{\alpha,\beta}\boldsymbol{\gamma}),$$

$$\text{subject to: } \gamma_l \le \gamma_{l-1}, 2 \le l \le p, \tag{11}$$

which is a quadratic objective function with linear constraints and can be solved by traditional optimization methods (e.g., the dual method in Goldfarb and Idnani [24] or the Hinge algorithm in Meyer [25]).

After obtaining $\hat{\boldsymbol{\gamma}}^{(m)}$, the estimates of $(\sigma^{(m)}, \rho^{(m)})'$ can be calculated by using the Restricted Maximum Likelihood (REML) if $\hat{\boldsymbol{\gamma}}^{(m)}$ does not take values on the boundary of the linear

constraints. Otherwise, we use the approximate REML to replace the estimates of σ and ρ. To derive the approximate REML, let $\hat{\boldsymbol{\gamma}}_u^{(m)}$ represent all of the unique values in $\hat{\boldsymbol{\gamma}}^{(m)}$ and let p_u be the length of $\hat{\gamma}_u^{(m)}$. For each unique value $\hat{\gamma}_{i,u}^{(m)}$, let $x_{i,\alpha\beta u}$ be the sum of the corresponding columns in $\mathbf{X}_{\alpha,\beta}$. Then we have $\mathbf{X}_{\alpha,\beta}\hat{\boldsymbol{\gamma}}^{(m)} = \mathbf{X}_{\alpha\beta u}\boldsymbol{\gamma}_u^{(m)}$, where $\mathbf{X}_{\alpha\beta u} = (x_{1,\alpha\beta u}, \cdots, x_{p_u,\alpha\beta u})$. The approximate REML log-likelihood is calculated as

$$L_{REML}(\sigma, \rho \mid \hat{\boldsymbol{\gamma}}^{(m)}) = -\frac{1}{2}\{\log|\boldsymbol{\Sigma}| + \log\left|\mathbf{X}_{u;\alpha,\beta}^{'}\boldsymbol{\Sigma}^{-1}\mathbf{X}_{u;\alpha,\beta}\right| + (\mathbf{y} - \mathbf{X}_{\alpha,\beta}\boldsymbol{\gamma}^{(m)})^{'}\boldsymbol{\Sigma}^{-1}(\mathbf{y} - \mathbf{X}_{\alpha,\beta}\hat{\boldsymbol{\gamma}}^{(m)})\}. \quad (12)$$

The parameter estimates of $(\sigma^{(m)}, \rho^{(m)})'$ are those values that maximize (12), and $\sigma^{(m)}$ has the following closed-form expression

$$\hat{\sigma}^{(m)} = [\frac{(\mathbf{y} - \mathbf{X}_{\alpha,\beta}\hat{\boldsymbol{\gamma}}^{(m)})^{'}(\hat{\mathbf{R}}^{(M-1)})^{-1}(\mathbf{y} - \mathbf{X}_{\alpha,\beta}\boldsymbol{\gamma}^{(m)})}{N - p_u}]^{\frac{1}{2}}. \quad (13)$$

Then $\rho^{(m)}$ can be obtained from a one-dimensional optimization problem. That is,

$$\hat{\rho}^{(m)} = \arg\max_{\rho}\{-\log\left|(\hat{\sigma}^{(m)})^2\mathbf{R}\right| - \log\left|(\sigma^{(m)})^{-2}\mathbf{X}_{u;\alpha,\beta}^{'}\mathbf{R}^{-1}\mathbf{X}_{u;\alpha,\beta}\right| - (\hat{\sigma}^{(m)})^{-2}(\mathbf{y} - \mathbf{X}_{\alpha,\beta}\boldsymbol{\gamma}^{(m)})^{'}\mathbf{R}^{-1}(\mathbf{y} - \mathbf{X}_{\alpha,\beta}\hat{\boldsymbol{\gamma}}^{(m)})\}. \quad (14)$$

Upon convergence, the estimates of $(\hat{\boldsymbol{\gamma}}, \hat{\sigma}, \rho)^{'}$, denoted by $(\hat{\boldsymbol{\gamma}}_{\alpha\beta}^{'}, \hat{\sigma}_{\alpha\beta}, \rho_{\alpha\beta})^{'}$, are obtained for given parameters (α, β). The initial values $(\sigma^{(0)}, \rho^{(0)})'$ can be obtained by fitting an unconstrained model.

The goal of the SaDE method is to maximize the profile log-likelihood of (α, β), which is calculated as

$$l(\alpha,\beta,\hat{\boldsymbol{\gamma}},\hat{\sigma},\rho)=\log\{\frac{1}{(2\pi)^{MN/2}\left|\hat{\boldsymbol{\Sigma}}_{\alpha,\beta}\right|^{1/2}}\exp[-\frac{(\mathbf{y}-\mathbf{X}_{\alpha,\beta}\hat{\boldsymbol{\gamma}}_{\alpha,\beta})'\hat{\boldsymbol{\Sigma}}_{\alpha,\beta}^{-1}(\mathbf{y}-\mathbf{X}_{\alpha,\beta}\boldsymbol{\gamma}_{\alpha,\beta})}{2}]\}. \quad (15)$$

Here, the objective function (15) has a nonlinear structure and the corresponding surface profile over the parameters (α,β) is complex. As a result, the classical gradient-based optimization methods cannot be directly applied. To overcome this challenge, we develop the parameter estimation method based on the SaDE framework.

In the SaDE framework, there are four steps: initialization, mutation, crossover and selection. Here, the mutation strategy is given as

$$(\alpha',\beta')_{i,G}=(\alpha,\beta)_{i,G}+K\cdot[(\alpha,\beta)_{r_1,G}-(\alpha,\beta)_{i,G}]+F\cdot[(\alpha,\beta)_{r_2,G}-(\alpha,\beta)_{r_3,G}], \quad (16)$$

where the control parameter F is generated by a normal distribution with mean value 0.5 and standard deviation 0.3 and $(\alpha,\beta)_{i,G}$ is a target vector at generation G . The indices $r_1,r_2,r_3\in\{1,2,3,\cdots,N_p\}$ are mutually exclusive and randomly generated integers, and all of them distinct from index i . Finally, $(\alpha',\beta')_{i,G}$ is the vector after mutation and K is a randomly generated number on the unit interval $[0,1]$.

3.2 Knot selection

Given Equation (15), the *AIC* can be calculated as

$$AIC=-2\log\{\frac{1}{(2\pi)^{MN/2}\left|\hat{\boldsymbol{\Sigma}}_{\alpha,\beta}\right|^{1/2}}\exp[-\frac{(\mathbf{y}-\mathbf{X}_{\alpha,\beta}\hat{\boldsymbol{\gamma}}_{\alpha,\beta})'\hat{\boldsymbol{\Sigma}}_{\alpha,\beta}^{-1}(\mathbf{y}-\mathbf{X}_{\alpha,\beta}\boldsymbol{\gamma}_{\alpha,\beta})}{2}]\}+2\times edf. \quad (17)$$

The initial condition of the parameter estimation is a given group of B-spline knots equally distributed in the interval. Here, we use the *AIC* value to evaluate the performance of the knot

setting and choose the best group of knots with the smallest *AIC* value. In Equation (17), *edf* is the effective degrees of freedom, which is calculated as the degrees of freedom in γ plus four degrees of freedom for the parameters $(\alpha, \beta, \sigma, \rho)^{'}$. With $p-1$ linear constraints, the effective degrees of freedom in γ has a value from 1 to *p*, where *p* corresponds to the release of the constraint of (11). Letting *q* denote the degree of the B-spline functions, the knot selection procedure can be summarized as the following two steps.

Step 1. Determine the optimal number of interior knots $r_{opt,q}$ by minimizing (17). The knots are to be evenly distributed in the given interval.

Step 2. Calculate the change of the *AIC* value that is caused by the deletion of each interior knots in sequence. Remove the knot that leads to the greatest increase of the *AIC* value. Repeat until no more existing knots can be removed.

Stepwise knot selection methods have been used in the literature including [26-29]. We start from a set of uniform knots (in percentile ranks) to keep the amount of computation as small as possible. The convergence of the iterative framework is proved by the research work [22], which is based on the asymptotic analysis and the existing work [30, 31].

3.3 Summary of parameters estimation

The detailed steps of the estimation method are listed in the following algorithm 1.

Algorithm 1.

Initialize the interior knots sequence

Do

Update the constraint of (11) based on the result of knot selection;

For *k* in 1,2, …

Initialize $P_0 = \{(\hat{\alpha}_{1,0}, \hat{\beta}_{1,0}), (\alpha_{2,0}, \beta_{2,0}), \cdots\}$;

For *n* in 1,2, …

Mutation with strategy (16) and Crossover to decide whether to accept the mutation result under a given probability;

Initialize $(\hat{\gamma}_i^{(0)}, \hat{\sigma}_i^{(0)}, \rho_i^{(0)})$ by solving Equations (11), (13), and (14) with $(\hat{\alpha}_{i,n-1}, \hat{\beta}_{i,n-1})$

and no constraint;

For *m* in 0,1,2, …

Minimize (11) to get $\hat{\gamma}_i^{(m)}$;

Maximize (12) to get $(\hat{\sigma}_i^{(m)}, \hat{\rho}_i^{(m)})$;

End For

Calculate (15) with $(\hat{\gamma}_i^{(m)}, \hat{\sigma}_i^{(m)}, \rho_i^{(m)})$ and P_0 ;

Selection with objective function (15);

Calculate (17) based on the result of parameter estimation;

End For

While (the deletion of any one of the knots would lead to the reduction of the *AIC* value)

Remove the interior knot whose deletion leads to the greatest reduction of the *AIC*;

End For

3.4 Reliability Measures

At first, the failure threshold D_f can be derived by the following equation.

$$D_f(\hat{\mu}_f, x_f; \hat{\boldsymbol{\theta}}) = \sum_{l=1}^{p} \hat{\gamma}_l G_{ql}\left(\frac{\hat{\mu}_f^{\exp[\hat{\alpha}(x_{\max}-x_f)]}}{\exp[\hat{\beta}(x_{\max}-x_f)]}\right) = D_f, \tag{18}$$

where $\hat{\mu}_f$ is the mean time to failure (MTTF) at use condition x_f. Thus, the failure time distribution can be derived based on the general ADDT model as follows.

$$F_T(t) = P(T \le t) = P(y_t \le D_f) = \Phi\left(\frac{D_f - \eta[\frac{t^{\exp(\alpha s)}}{\exp(\beta s)}; \gamma]}{\sigma} \right), \ t \ge 0, \qquad (19)$$

where the failure time T is equivalent to the degradation measurement at time t. The function $\Phi(\cdot)$ is the cdf of the standard normal distribution. The quantile function can then be calculated as the inverse of the cdf which the α quantile is $t_\alpha = F_T^{-1}(\alpha)$.

IV. SIMULATION STUDY

We conduct a simulation study to investigate the performance of the developed parameter estimation method. In the simulation study, temperature is used as the acceleration factor. We use two types of temperature settings: (1) $M = 3$ temperature levels at 50°C, 65°C, and 80°C and (2) $M = 6$ temperature levels at 30°C, 40°C, 50°C, 60°C, 70°C, and 80°C. There are $N = 5$ measuring times set as 8, 25, 75, 130, and 170 days.

The data are simulated from the following parametric model:

$$Y(t) = \xi_0 + \xi_1 \exp(\xi_2 x_i) t + \varepsilon, \qquad (20)$$

where $x_i = -11,605 / (Temp_i + 273.15)$ and $Temp_i$ denotes the i^{th} temperature level. The parameters of the parametric model are set as $(\xi_0, \xi_1, \xi_2) = (1, -3.5, 0.3)$, and $(\sigma, \rho) = (0.02, 0)$.

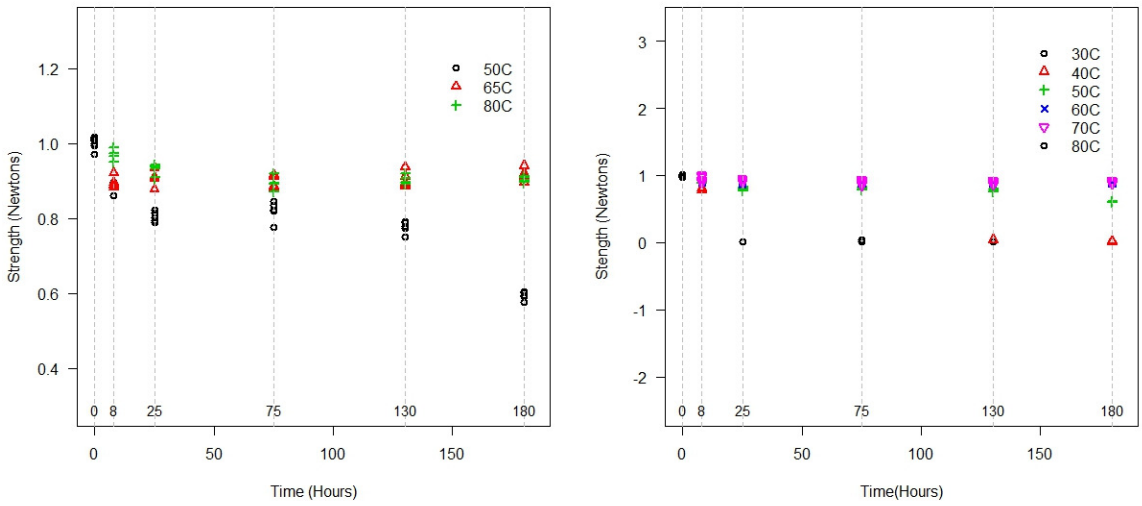


Figure 2. The simulated response under the two settings

Simulated responses based on these settings are shown in Figure 2. Using algorithm 1, we get the estimated parameters listed in Table 1 and the fitted degradation results as shown Figure 3.

Table 1: The estimation results under specific settings

	$\hat{\alpha}$	$\hat{\beta}$
M=3, N=5	0.9052	2.3571
M=6, N=5	-0.0157	-1.5122

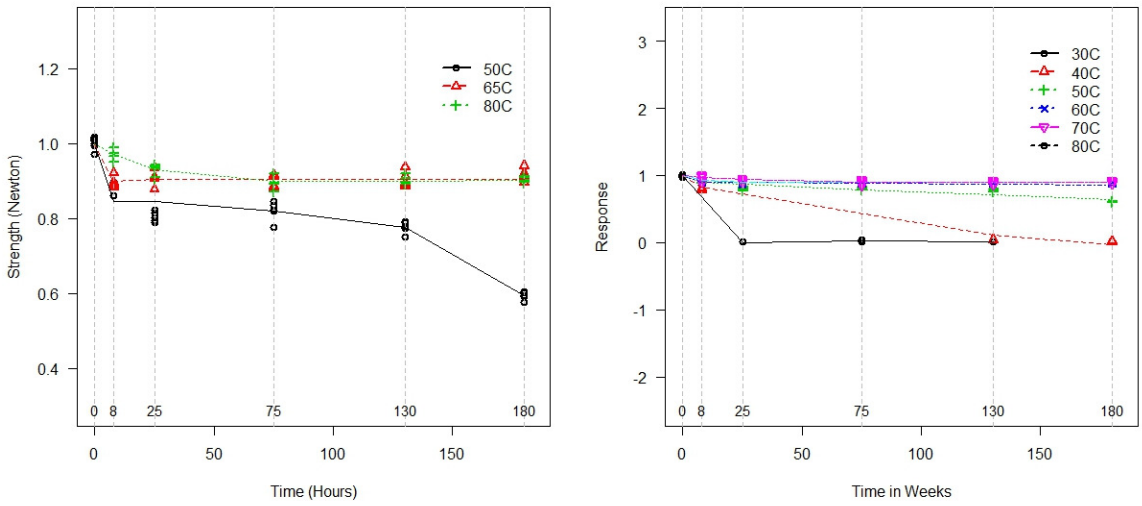


Figure 3. The fitted degradation paths of simulated data

We calculate the Mean Squared Error (MSE) and AIC values of the proposed general ADDT model, and compared with the semiparametric model in Xie, et al. [21]. Specifically, the MSE values are calculated as

$$MSE(y) = E((Y(t) - \hat{Y}(t))^2). \tag{21}$$

Table 2 shows the simulation results of the two semiparametric models when the data are simulated using parametric model.

Table 2. Comparison of results with different two settings

	MSE	*AIC*
General ADDT (*M*=3, *N*=5)	4.3869×10^{-4}	-418.5742
Semiparametric model (*M*=3, *N*=5)	2.1314×10^{-3}	-385.3262
General ADDT (*M*=6, *N*=5)	8.8380×10^{-4}	-586.2078
Semiparametric model (*M*=6, *N*=5)	9.2530×10^{-3}	-584.6612

It can be seen from Table 2 that the general ADDT model we proposed has smaller *MSE* and *AIC* values compared to the semiparametric model, indicating that the new model outperforms the existing semiparametric method under different settings.

We further the simulation study to analyze the performance of the General ADDT model under the use condition. The temperature of use condition is set to be 20°C. The simulated data is based on the first setting which is $M = 3$ temperature levels at 50°C, 65°C, 80°C and $N = 5$ measuring times set as 8, 25, 75, 130, and 170 days. Simulated responses based on these settings are shown in Figure 4 and the fitted degradation results as shown Figure 5.

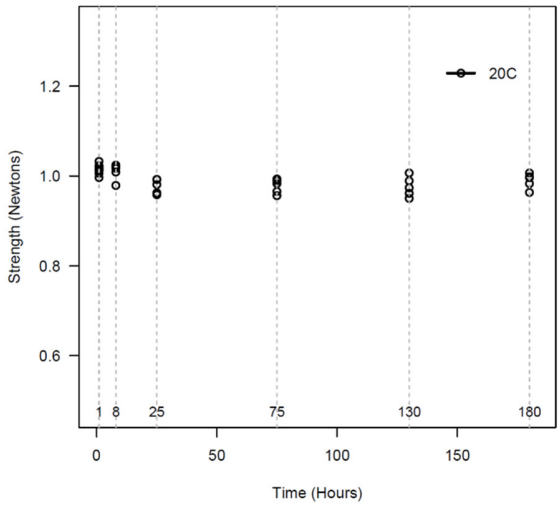
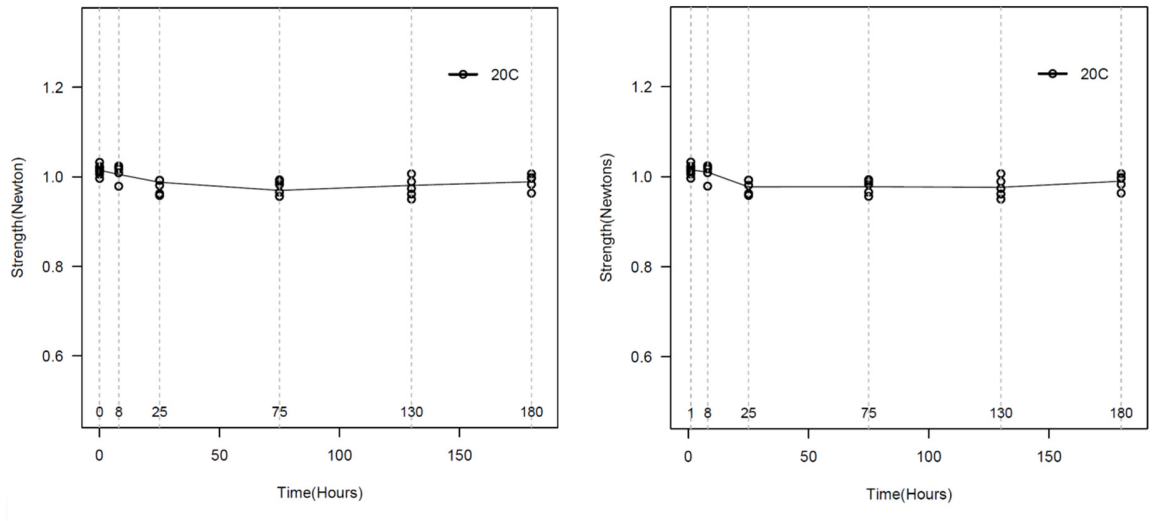


Figure 4. The simulated response under use condition 20°C



(a) Semiparametric model (b) General ADDT model

Figure 5. The fitted degradation paths by using two models under use condition

We calculate the MSE values of the fitted degradation paths under use condition and the result are showed in Table 3. It can be seen in Table 3 that the general ADDT model proposed in this paper outperforms the existing semiparametric method.

Table 3. Comparison of results at use condition

	General ADDT (M=3, N=5)	Semiparametric model (M=3, N=5)
MSE	2.1042×10^{-4}	2.4131×10^{-4}

Based on the existing simulated results, we further our experiments. For each simulated dataset, the MTTF at 30°C (considered as the normal use condition) for each simulated dataset by using the true parametric model, the semiparametric model [21], and the proposed general ADDT model. The mean, bias, standard derivation (SD), and root MSE of the MTTF for all the three different models are summarized in the following Table 1. The results in Table 1 show that the

estimated mean of MTTF using the general ADDT model is closer to the true value, and general ADDT method has a smaller bias, SD and RMSE of the MTTF estimation than the semiparametric model [21]. So the proposed general ADDT model outperforms the existing semiparametric model [21].

Table 4. Comparison of mean, bias, SD, and root MSE of MTTF estimation using the true model, the semiparametric model in [21], and the general ADDT model

Model	Mean	Bias	SD	RMSE
True Model	82.60	0.01	2.99	2.99
Semiparametric model	82.77	0.16	4.22	4.22
General ADDT model	82.70	0.11	3.56	3.56

V. CASE STUDY

Two real-life case studies are conducted to demonstrate the proposed ADDT model. In each case study, we also demonstrate the advantages of the proposed model by comparing its performance with the parametric model and an existing semi-parametric model in the literature.

1. The adhesive formulation K data

An ADDT of the adhesive formulation K is conducted to estimate the lifetime of a certain kind of adhesive that has a new formulation with a new additive compound to help enhance the performance. The adhesive's viscidity would decrease over time till failure. During the ADDT process, the temperature is the acceleration factor and is set at three levels: 40°C, 50°C and 60°C. First, ten samples were tested at nominal temperature as baseline units. Next, for each temperature, 30 samples are placed and measured at different times to obtain the degradation paths of the material. The strength of the adhesive is measured in Newtons. Figure 6 shows the raw data of the Adhesive Formulation K.

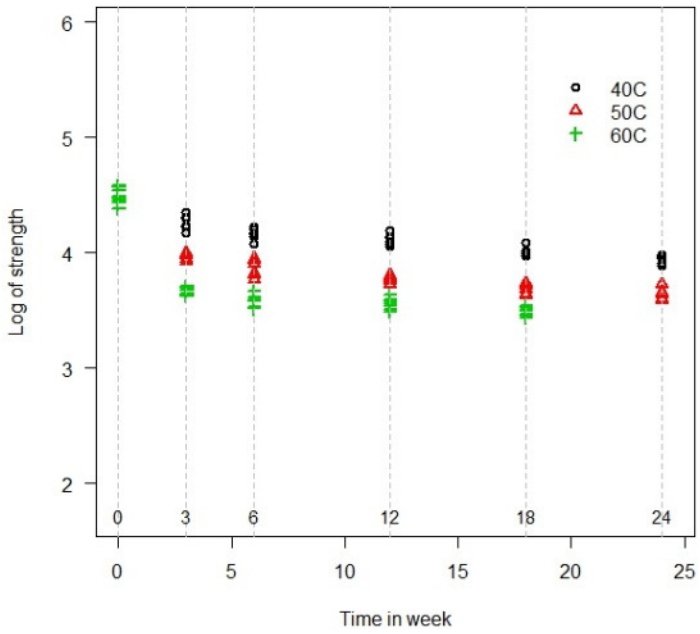


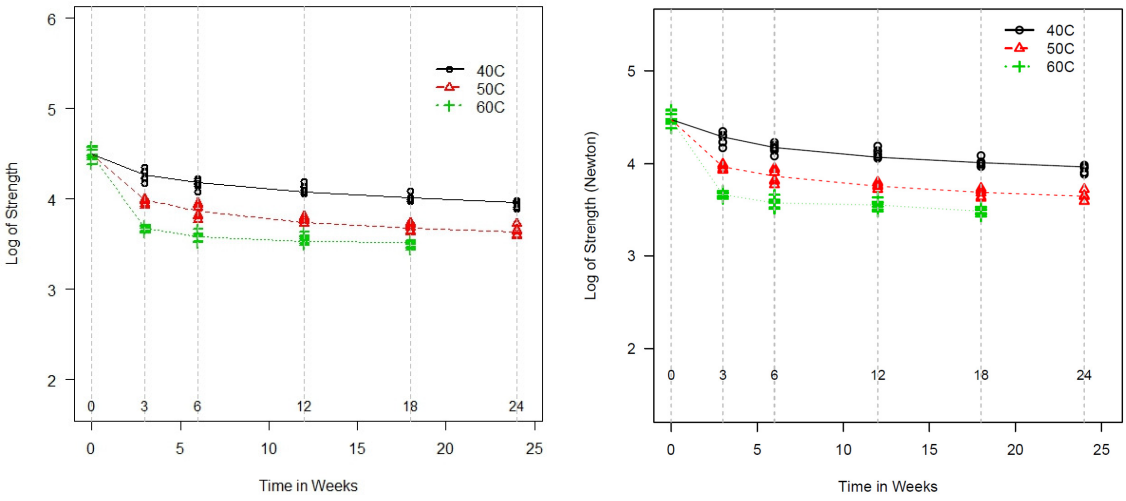
Figure 6. Measurement of adhesive formulation K data at different temperature settings

The parametric model used to describe the adhesive formulation K data is

$$y_i(t) = \log(90) + \xi_0(1 - \exp\{-\xi_1 \exp[\xi_2(x_i - x_2)]t\}) + \varepsilon_i, \tag{22}$$

where $Y(t)$ is the strength of adhesive formulation K in log-Newtons at time *t*, $x_i = -11,605/(Temp_i + 273.15)$, $x_2 = -11,\ 605/(50 + 273.15)$, and $\varepsilon \sim N(0, \sigma^2)$. The estimates of the model parameters are $\xi_0 = -0.9978$, $\xi_1 = 0.4091$, $\xi_2 = 0.8371$, and $\hat{\sigma} = 0.0501$. For the general ADDT model, the parameter estimates are $\alpha = 0.0631$, $\beta = 1.5510$.

Figure 7(a) shows the scatterplot of the parametric model and the Figure 7(b) is the fitted degradation path using the proposed general ADDT model.



(b) Parametric model (b) General ADDT model

Figure 7. Fitted degradation paths of the Adhesive Formulation K data

Table 4 shows the log-likelihood and AIC values of the proposed general ADDT model, compared with the parametric model and the existing semiparametric model in Xie, et al. [21]. As the general ADDT model has a smaller *AIC* value, the new model outperforms the existing models.

Table 5. Comparison of the general ADDT model and the existing semiparametric model

	Loglik	*df*	*AIC*
Parametric model	158.950	4	-309.901
Semiparametric model	162.607	8	-309.213
General ADDT	171.932	9	-325.864

2. The polymer Y data

Tsai, et al. [16] studied the Polymer Y data measured from a type of new polymer material exposed to an alkaline environment at elevated temperatures. Three temperatures were chosen: $Temp_1 = 50\ ^\circ\text{C}$, $Temp_2 = 65\ ^\circ\text{C}$, and $Temp_3 = 80\ ^\circ\text{C}$. At each of the selected temperatures, 25 coupons were destructively tested to assess their initial tensile strength. Then, 5 coupons were tested at selected measurement times to assess degradation. Figure 8 shows the data of the Polymer Y.

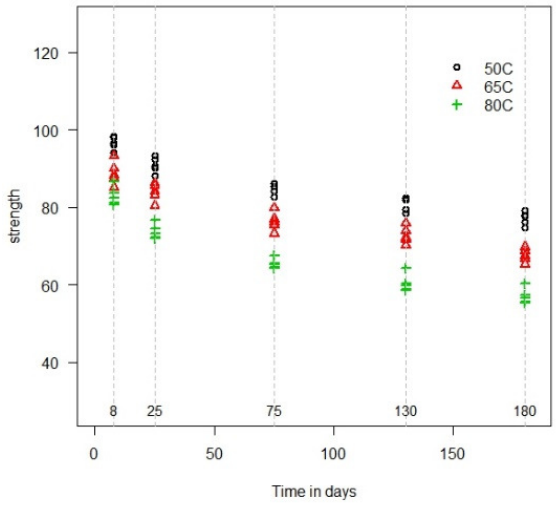


Figure 8. The observed points of polymer Y

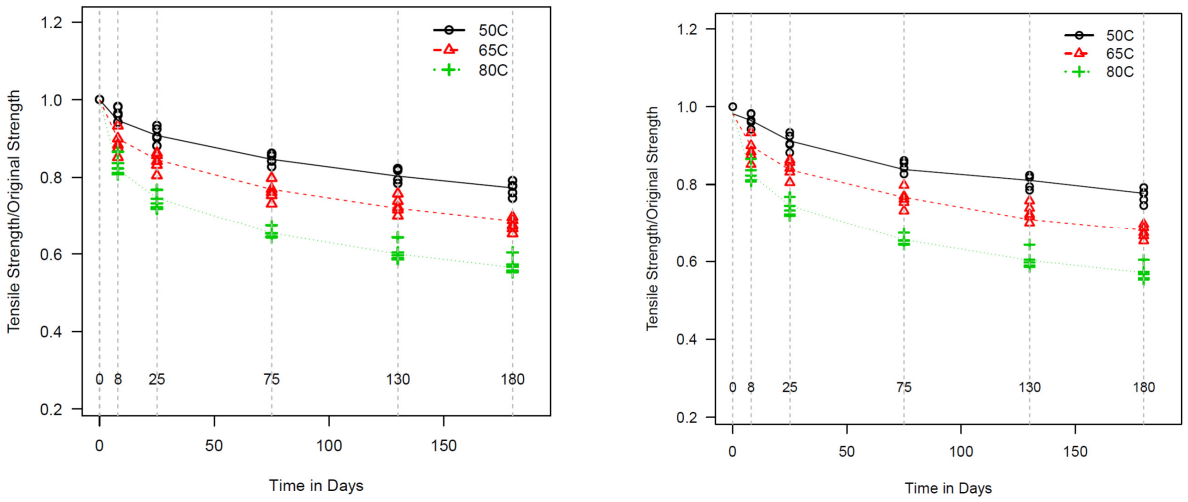
In Tsai, et al. [16], a parametric model is used to describe the mechanical process of the polymer material in ADDT:

$$y_i(t) = e^{-\beta_i t^{\alpha_i}} + \varepsilon, t > 0;\ \alpha_i = \exp(\gamma_1 + \frac{\gamma_2}{273.15 + Temp_i}),\ \beta_i = \exp(\gamma_3 + \frac{\gamma_4}{273.15 + Temp_i}). \quad (23)$$

Here, $y_i(t)$, the strength of polymer Y, is the ratio of the tensile strength to the original strength under stress $Temp_i$ at time *t*. When the ratio is less than a pre-defined failure level, the material is

considered to have failed. The parameters α_i and β_i are fixed and unknown and ε is the error term. The estimates for the parametric model are $\gamma_1 = -5.22$, $\gamma_2 = 1462.42$, $\gamma_3 = 15.12$, $\gamma_4 = -6160.68$, $\sigma = 0.02$. For the general ADDT model, the parameters are estimated as $\alpha = 0.1057$, $\beta = 1.4218$.

Figure 9(a) shows the scatterplot of the parametric model and Figure 9(b) represents the fitted degradation path using the general ADDT model.



(a) Parametric model (b) General ADDT model

Figure 9. Fitted degradation paths of the Polymer Y data

Table 5 shows the log-likelihood and the AIC values of the proposed general ADDT model compared with the parametric model and the existing semiparametric model in the literature. As the general ADDT model has a smaller *AIC* value, the new model outperforms the existing models.

Table 6. Comparison of the general ADDT model and the existing semiparametric model

	Loglik	*df*	*AIC*
Parametric model	189.1715	5	-368.343
Semiparametric model	195.1065	5	-380.213
General ADDT	201.560	6	-391.120

VI. CONCLUSION AND FUTURE WORK

In this article, we developed a new general model for analyzing ADDT data. The degradation path is modeled using the B-spline and includes both location and scale parameters to describe the acceleration factor effect. We developed methods to select the optimal number and location of interior knots of B-splines based on AIC values. We also developed an MLE method based on self-adaptive differential evolution to estimate the model parameters.

We implement a simulation study and a case study to assess the performance of the developed methods. The results show that the proposed model performs well in fitting degradation paths. We also compare the proposed model to the existing parametric and semiparametric ADDT model. Both the simulation results and the real-world case studies show that the proposed model outperforms the existing models in terms of model-fitting. Compared to the existing models, the newly proposed model's is more general and flexible to describe complex relationships between the response and the acceleration factors.

The newly proposed model can be applied to a wide range of applications. An interesting future research topic would be using the model as a basis for optimal test planning. Also, we can assess

the performance of the model using other types of nonparametric methods, such as cubic splines or wavelets, which are also applied widely in the engineering and science research fields. In addition, this research only focuses on ADDT data; a similar model could be developed for RMDT data or other similar data types.

Acknowledgment

The authors thank the Associate Editor and the referees for their valuable comments that helped to improve this article. The work by Hong was partially supported by the National Science Foundation under Grants CNS-1565314 to Virginia Tech.

References

[1] Z.-S. Ye, Y. Wang, K.-L. Tsui, and M. Pecht, "Degradation data analysis using Wiener processes with measurement errors," *IEEE Transactions on Reliability,* vol. 62, pp. 772-780, 2013.

[2] W. Si, Q. Yang, and X. Wu, "A physical–statistical model of overload retardation for crack propagation and application in reliability estimation," *IIE Transactions,* vol. 48, pp. 347-358, 2016.

[3] W. Si, Q. Yang, X. Wu, and Y. Chen, "Reliability analysis considering dynamic material local deformation," *Journal of Quality Technology,* vol. 50, pp. 1-14, 2018.

[4] G. Whitmore and F. Schenkelberg, "Modelling accelerated degradation data using Wiener diffusion with a time scale transformation," *Lifetime data analysis,* vol. 3, pp. 27-45, 1997.

[5] J.-C. Lu, J. Park, and Q. Yang, "Statistical inference of a time-to-failure distribution derived from linear degradation data," *Technometrics,* vol. 39, pp. 391-400, 1997.

[6] W. Nelson, "Analysis of performance-degradation data from accelerated tests," *IEEE Transactions on Reliability,* vol. 30, pp. 149-155, 1981.

[7] V. Crk, "Reliability assessment from degradation data," in *Reliability and Maintainability Symposium, 2000. Proceedings. Annual*, 2000, pp. 155-161.

[8] D.-G. D. Chen, Y. Lio, H. K. T. Ng, and T.-R. Tsai, "Statistical Modeling for Degradation Data," ed: Springer, 2017.

[9] W. Nelson, "Accelerated Life Testing: Statistical Models Test Plan and Data Analysis," *J. Wiley, New York, USA,* 1990.

[10] C. J. Lu and W. O. Meeker, "Using degradation measures to estimate a time-to-failure distribution," *Technometrics,* vol. 35, pp. 161-174, 1993.

[11] W. Q. Meeker, L. A. Escobar, and C. J. Lu, "Accelerated degradation tests: modeling and analysis," *Technometrics,* vol. 40, pp. 89-99, 1998.

[12] N. Gorjian, L. Ma, M. Mittinty, P. Yarlagadda, and Y. Sun, "A review on degradation models in reliability analysis," in *Engineering Asset Lifecycle Management*, ed: Springer, 2010, pp. 369-384.

[13] W. Meeker, Y. Hong, and L. Escobar, "Degradation Models and Analyses," *Encyclopedia of Statistical Sciences,* 2011.
[14] Z. Xu, Y. Hong, and R. Jin, "Nonlinear general path models for degradation data with dynamic covariates," *Applied Stochastic Models in Business and Industry,* vol. 32, pp. 153-167, 2016.
[15] Y. Shi, L. A. Escobar, and W. Q. Meeker, "Accelerated destructive degradation test planning," *Technometrics,* vol. 51, pp. 1-13, 2009.
[16] C.-C. Tsai, S.-T. Tseng, N. Balakrishnan, and C.-T. Lin, "Optimal design for accelerated destructive degradation tests," *Quality Technology & Quantitative Management,* vol. 10, pp. 263-276, 2013.
[17] H. Lim and B.-J. Yum, "Optimal design of accelerated degradation tests based on Wiener process models," *Journal of Applied Statistics,* vol. 38, pp. 309-325, 2011.
[18] S.-L. Jeng, B.-Y. Huang, and W. Q. Meeker, "Accelerated destructive degradation tests robust to distribution misspecification," *IEEE Transactions on Reliability,* vol. 60, pp. 701-711, 2011.
[19] X. Xiao and Z. Ye, "Optimal design for destructive degradation tests with random initial degradation values using the Wiener process," *IEEE Transactions on Reliability,* vol. 65, pp. 1327-1342, 2016.
[20] L. Hong, Z. Ye, and X. Zhao, "Investigating the necessity of acceleration in a degradation test," in *Industrial Engineering and Engineering Management (IEEM), 2016 IEEE International Conference on*, 2016, pp. 546-550.
[21] Y. Xie, C. B. King, Y. Hong, and Q. Yang, "Semiparametric Models for Accelerated Destructive Degradation Test Data Analysis," *Technometrics,* pp. 1-13, 2017.
[22] X. He and P. Shi, "Monotone B-spline smoothing," *Journal of the American Statistical Association,* vol. 93, pp. 643-650, 1998.
[23] A. K. Qin, V. L. Huang, and P. N. Suganthan, "Differential evolution algorithm with strategy adaptation for global numerical optimization," *IEEE Transactions on Evolutionary Computation,* vol. 13, pp. 398-417, 2009.
[24] D. Goldfarb and A. Idnani, "A numerically stable dual method for solving strictly convex quadratic programs," *Mathematical Programming,* vol. 27, pp. 1-33, 1983.
[25] M. C. Meyer, "A simple new algorithm for quadratic programming with applications in statistics," *Communications in Statistics-Simulation and Computation,* vol. 42, pp. 1126-1139, 2013.
[26] J. H. Friedman and B. W. Silverman, "Flexible parsimonious smoothing and additive modeling," *Technometrics,* vol. 31, pp. 3-21, 1989.
[27] X. He and P. Shi, "Bivariate tensor-product B-splines in a partly linear model," *Journal of Multivariate Analysis,* vol. 58, pp. 162-181, 1996.
[28] C. Kooperberg and C. J. Stone, "Logspline density estimation for censored data," *Journal of Computational and Graphical Statistics,* vol. 1, pp. 301-328, 1992.
[29] P. Shi and G. Li, "Optimal global rates of convergence of B-spline M-estimates for nonparametric regression," *Acta Mathematica, New Ser,* 1992.
[30] S. Portnoy, "Local asymptotics for quantile smoothing splines," *The Annals of Statistics,* vol. 25, pp. 414-434, 1997.
[31] C. J. Stone, "Optimal global rates of convergence for nonparametric regression," *The Annals of Statistics,* pp. 1040-1053, 1982.