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Root system chip-firing

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Abstract. Propp recently introduced a variant of chip-firing on the infinite path where the chips are given distinct integer labels and conjectured that this process is confluent from certain (but not all) initial configurations of chips. Hopkins, McConville, and Propp proved Propp's confluence conjecture. We recast this result in terms of root systems: the labeled chip-firing game can be seen as a process which allows replacing an integer vector λ by $\lambda+\alpha$ whenever λ is orthogonal to α, for α a positive root of a root system of Type A. We give conjectures about confluence for this process in the general setting of an arbitrary root system. We show that the process is always confluent from any initial point after modding out by the action of the Weyl group (an analog of unlabeled chip-firing in arbitrary type). We also study some remarkable deformations of this process which are confluent from any initial point. For these deformations, the set of weights with given stabilization has an interesting geometric structure related to permutohedra. This geometric structure leads us to define certain "Ehrhart-like" polynomials that conjecturally have nonnegative integer coefficients.

Keywords: chip-firing, Abelian Sandpile Model, confluence, root systems, permutohedra, Ehrhart polynomials

1 Introduction

This extended abstract summarizes our work on an extension of the chip-firing game to "other Cartan-Killing types" [8, 9]. All proofs are contained in those papers.

Chip-firing, as introduced by Björner, Lovász, and Shor [5], is a certain (solitaire) game played on a graph. The states of this game are configurations of chips on the vertices of the graph. A vertex which has at least as many chips as neighbors is said to be *unstable*. At any moment, we can *fire* an unstable vertex, which sends one chip from that vertex to each of its neighbors. And we can keep firing chips in this way until we reach a *stable*

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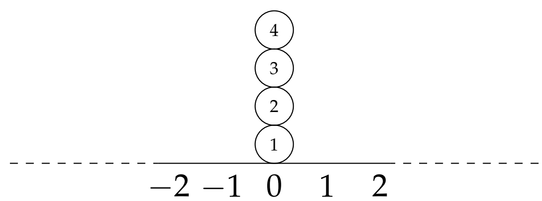
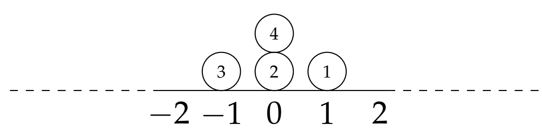
configuration, where all vertices are stable. A fundamental result of Björner–Lovász–Shor is that this process is *confluent*: either we keep firing forever, or we reach a unique stable configuration that does not depend on which unstable vertices we chose to fire. As it turns out, this chip-firing process is essentially the same as the *Abelian Sandpile Model*, originally introduced by the physicists Bak, Tang, and Wiesenfeld [3] and subsequently developed by Dhar [7].

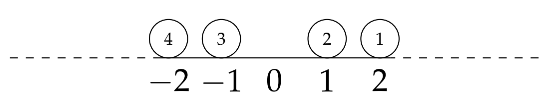
Björner, Lovász, and Shor were motivated to define chip-firing on an arbitrary graph by an earlier paper of Anderson et al. [1] which studied the special case of chip-firing on a line, i.e., on an infinite path graph. Because we identify the vertices of this infinite path graph with integers, we denote it by $\mathbb{Z}$. Inspired by this initial setting, Jim Propp recently introduced a version of *labeled* chip-firing on a line. The states of the labeled chip-firing process are configurations of distinguishable chips with integer labels $1, 2, \ldots, n$ on $\mathbb{Z}$. For example, with $n = 4$, the following is such a configuration:



The firing moves consist of choosing two chips that occupy the same vertex and moving the chip with the lesser label one vertex to the right and the chip with the greater label one vertex to the left. For example, if we chose to fire chips ① and ③ in the previous configuration that would lead to:



One can perform these firing moves until no two chips occupy the same spot. Propp conjectured that if one starts with an even number of chips at the origin, then this process is confluent and in particular the chips always end up in sorted order. For example, if we continue firing the four chips above, we necessarily will end up at:



It is easy to see that the labeled chip-firing process is not confluent if the initial number of chips is odd (e.g., three), and thus confluence is a more subtle question for labeled chip-firing than for classical chip-firing. Propp's sorting conjecture was recently proved by Hopkins, McConville, and Propp [10].

The following reformulation of labeled chip-firing is crucial as a starting point for our work. For any configuration of n labeled chips, if we define $v := (v_1, v_2, \ldots, v_n) \in \mathbb{Z}^n$ by

$$v_i := \text{the position of the chip } ⓘ,$$

then, for $1 \leq i < j \leq n$, we are allowed to fire chips ⓘ and ⓙ in this configuration as long as v is orthogonal to $e_i - e_j$; and doing so replaces the vector v by $v + (e_i - e_j)$. (Here $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{Z}^n$.) We use $v \to v + (e_i - e_j)$ to denote this firing move and hence think of the labeled chip-firing process as a binary relation defined on the lattice $\mathbb{Z}^n$.

But observe that the collection of vectors $e_i - e_j$ for $1 \leq i < j \leq n$ is exactly the set of positive roots Φ^+ of the root system Φ of Type A_{n-1}. Therefore, we might wonder what happens when we replace this collection of vectors by the positive roots Φ^+ of an arbitrary root system Φ (and replace the lattice $\mathbb{Z}^n$ by the weight lattice of Φ). We call this the *central-firing process* for Φ, because it allows firing when our vector lies on the Coxeter arrangement of Φ, which is a central hyperplane arrangement.

We also study some remarkable deformations of the central-firing process, which we call *interval-firing processes*. These interval-firing processes turn out to be confluent from any initial weight. Moreover, for these interval-firing processes, the set of weights with a given stabilization has an interesting geometric structure related to permutohedra. And, what is more, we show that there exist certain *Ehrhart-like polynomials* that count the number of weights with given stabilization as our deformation parameter varies.

2 Central-firing

Before formally defining the central-firing process, which as explained above generalizes labeled chip-firing on a line to other types, we need to briefly review some notation related to root systems and to binary relations.

For basics on root systems, consult [6]. Fix Φ, an irreducible crystallographic root system in an r-dimensional Euclidean vector space V with standard inner product $\langle \cdot, \cdot \rangle$. For any root $\alpha \in \Phi$, we use $\alpha^\vee := \frac{2}{\langle \alpha, \alpha \rangle} \alpha$ to denote the corresponding coroot. We use W to denote the Weyl group of Φ. We use $Q := \mathbb{Z}[\Phi]$ to denote the root lattice of Φ, and $P := \{v \in V \colon \langle v, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$ to denote the weight lattice of Φ. We fix a choice of positive roots Φ^+ and hence also a sequence of simple roots $\alpha_1, \ldots, \alpha_r$ and of fundamental weights $\omega_1, \ldots, \omega_r$.

For $\to$ a (binary) relation on a finite set X, we say that $x \in X$ is *stable* if there does not exist any $y \in X$ with $x \to y$. We use $\xrightarrow{*}$ to denote the reflexive transitive closure of $\to$. We say that $\to$ is *confluent from* $x \in X$ if for all $y_1, y_2 \in X$ such that $x \xrightarrow{*} y_1$ and $x \xrightarrow{*} y_2$, there exists some $y_3 \in X$ for which $y_1 \xrightarrow{*} y_3$ and $y_2 \xrightarrow{*} y_3$. We say that $\to$ is *terminating* if there does not exist an infinite sequence $x_0 \to x_1 \to x_2 \to \cdots$. If $\to$ is terminating and is confluent from $x \in X$, then there is a unique stable $y \in X$ with $x \xrightarrow{*} y$ and we call y the *stabilization* of x. We say that $\rightarrow$ is *confluent* if it is confluent from every $x \in X$.

Our goal is to study confluence and stabilizations for certain "chip-firing like" relations defined in terms of root systems. The first such relation is the central-firing process.

Definition 2.1. The *central-firing process* for Φ is the relation $\underset{0}{\rightarrow}$ defined on the weight lattice by $\lambda \underset{0}{\rightarrow} \lambda + \alpha$ for $\lambda \in P$ and $\alpha \in \Phi^+$ whenever λ is orthogonal to α.

As explained in the introduction, central-firing for $\Phi = A_{n-1}$ is the same as labeled chip-firing of n labeled chips on $\mathbb{Z}$. In fact, for any classical type (i.e., A_n, B_n, C_n, or D_n), central-firing can be described in terms of labeled chip configurations on $\mathbb{Z}$.

It is not hard to show that $\underset{0}{\rightarrow}$ is always terminating. The question we focus on is:

Question 2.2. For any root system Φ and any weight $\lambda \in P$, when is the central-firing process confluent from λ?

As discussed above, central-firing is confluent from some but not all initial weights. It follows from the work of [10] that central-firing is confluent from the origin for $\Phi = A_{n-1}$ when $n \equiv 0 \mod 2$, and for $\Phi = B_n$ for all values of n. We experimentally observed that central-firing is confluent from the origin for $\Phi = C_n$ when $n \equiv 1, 2 \mod 4$ and for $\Phi = D_n$ when $n \equiv 3 \mod 4$. Hence we see that Question 2.2 is subtle, and we have no conjectural answer in general. Nevertheless, we did observe some discernible root-theoretic phenomena in investigating Question 2.2: e.g., to first order, it seems that central-firing is confluent from a weight $\lambda \in P$ when λ is not equivalent to the *Weyl vector* $\rho := \sum_{i=1}^{r} \omega_i$ modulo the root lattice Q. In [9] we put forth a complete conjectural classification of confluence when λ is either zero or a fundamental weight.

In marked contrast to the subtly of Question 2.2, it turns out that if we mod out by the action of the Weyl group, then central-firing becomes confluent from all initial weights. Specifically, if $\rightarrow$ is a relation on a set X and a group G acts on X, then $\rightarrow$ descends to a relation, which we also denote by $\rightarrow$, on the orbits X/G: we have $G.x \rightarrow G.y$ if there is some $x' \in G.x$ and $y' \in G.y$ with $x' \rightarrow y'$. With this notation, we have the following:

Theorem 2.3. *The relation $\underset{0}{\rightarrow}$ on P/W is confluent (and terminating).*

Modding out by the action of the Weyl group $W = S_n$ for $\Phi = A_{n-1}$ is the same as forgetting the labels of the chips in the labeled chip-firing process. Thus Theorem 2.3 gives a generalization of *unlabeled* chip-firing on a line to any type. We note that recently Benkart, Klivans, and Reiner [4] also studied a generalization of unlabeled chip-firing to other types (*Cartan matrix chip-firing*), but in fact their generalization is different from ours: e.g., for Type A_{n-1} ours corresponds to chip-firing of n chips on the infinite path graph, whereas theirs corresponds to chip-firing of any number of chips on an n-cycle.

However, we can recover the Cartan matrix chip-firing of [4] by taking a certain $k \to \infty$ limit of the interval-firing processes described in the next section.

For *simply laced* Φ we can describe central-firing modulo the Weyl group as a certain numbers game on the Dynkin diagram of Φ, as we now explain.

Definition 2.4. Let Γ be a simply laced Dynkin diagram with vertex set $[r] := \{1, 2, \ldots, r\}$. Let $\gamma : [r] \to \mathbb{Z}_{\geq 0}$ be an assignment of nonnegative integers to the vertices of Γ. Consider an application of the following sequence of steps to γ:

(1) choose a *zero connected component* X of γ, that is, a connected component of the induced subgraph of Γ with vertex set $\{i \in [r] : \gamma(i) = 0\}$;

(2) complete (in a unique way) X to an affine Dynkin diagram $\widetilde{X}$ with vertex set $X \cup \{0\}$;

(3) for every edge $\{0, i\}$ of $\widetilde{X}$, increase $\gamma(i)$ by 1;

(4) for every vertex $j \notin X$ that is adjacent to a vertex $i \in X$, decrease $\gamma(j)$ by 1.

We denote the resulting assignment of integers by γ' and say that γ' is obtained from γ *via a UCF (="unlabeled central-firing") move along* X. We write $\gamma \xrightarrow[UCF]{} \gamma'$.

Example 2.5. Let us illustrate this definition by an example for a Dynkin diagram of Type E_7. Consider an assignment γ shown in Figure 1 (top). It has two zero connected components: X_1 of Type D_5 and X_2 of Type A_1. Applying a UCF move to γ along X_1 (resp., along X_2) produces assignments γ_1' (resp., γ_2') shown in Figure 1 (middle-left), resp., (middle-right). Note that γ_1' has a zero connected component of Type A_5 that contains X_2, and similarly, γ_2' has a zero connected component of Type E_6 that contains X_1. Applying another UCF move along the corresponding zero connected component of γ_1' (resp., of γ_2') actually produces the same result γ'' shown in Figure 1 (bottom).

Theorem 2.6. *Let Φ be a simply laced root system and Γ its Dynkin diagram. For each Dynkin diagram assignment $\gamma : [r] \to \mathbb{Z}_{\geq 0}$, define the corresponding weight $\lambda(\gamma) := \sum_{i=1}^{r} \gamma(i)\omega_i$. Then we have that $\gamma \xrightarrow[UCF]{} \gamma'$ if and only if $W.\lambda(\gamma) \xrightarrow[0]{} W.\lambda(\gamma')$.*

3 Interval-firing: confluence

We now introduce some remarkable deformations of central-firing. Continue to fix Φ as in the previous section. Recall that we have a firing move $\lambda \xrightarrow[0]{} \lambda + \alpha$ for $\alpha \in \Phi^+$ whenever λ is orthogonal to α; equivalently, $\lambda \xrightarrow[0]{} \lambda + \alpha$ whenever $\langle \lambda, \alpha^\vee \rangle = 0$. The deformations we study involve changing the set of inner products $\langle \lambda, \alpha^\vee \rangle$ at which we allow firing to be some larger interval.

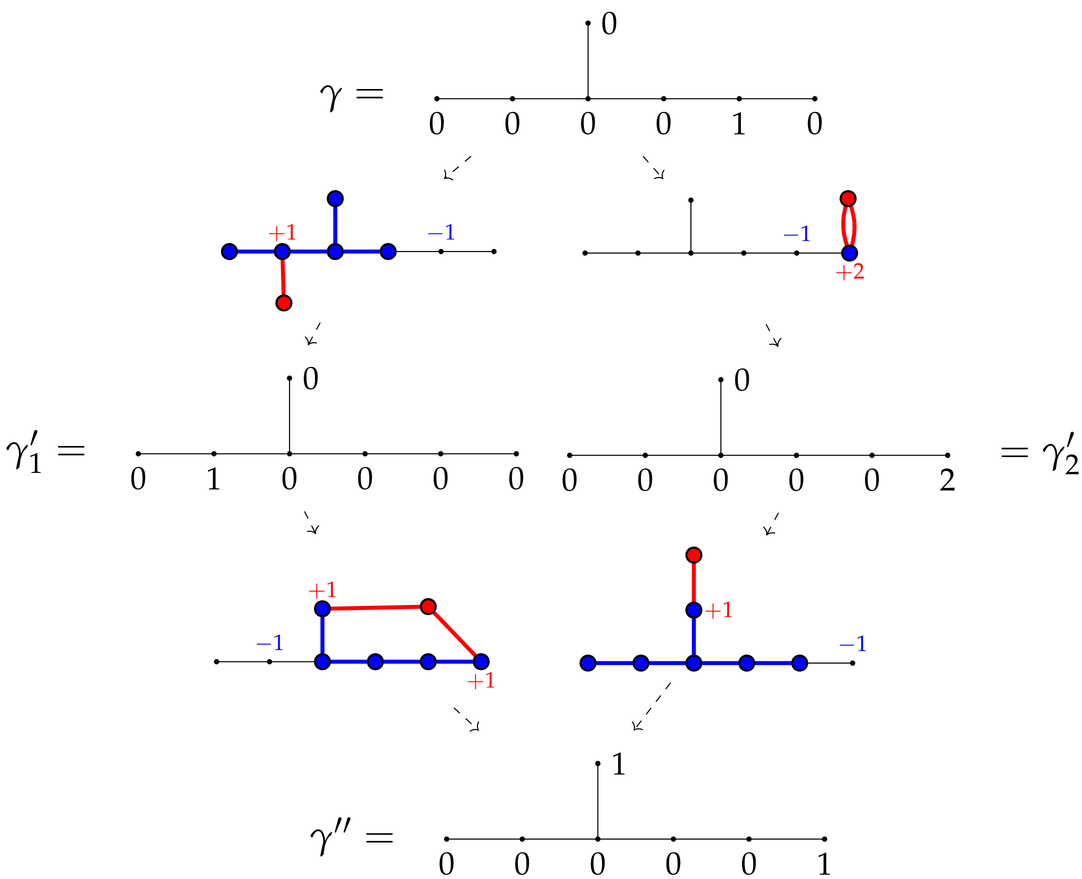


Figure 1: Applying UCF moves to the Dynkin diagram of E_7 (see Example 2.5). For each move, the component X is shown in blue, the extra node 0 of $\widetilde{X}$ is shown in red, changes from step (3) are shown in red, and changes from step (4) are shown in blue.

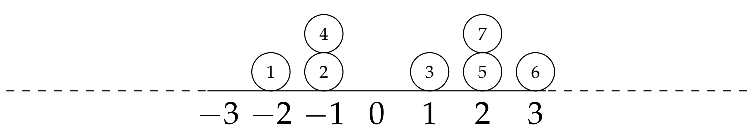
Definition 3.1. Fix a nonnegative integer $k \in \mathbb{Z}_{\geq 0}$. The *symmetric interval-firing process* and the *truncated interval-firing process* are the relations $\xrightarrow[\text{sym},k]{}$ and $\xrightarrow[\text{tr},k]{}$ on the weight lattice defined by

$$\lambda \xrightarrow[\text{sym},k]{} \lambda + \alpha \text{ whenever } \langle \lambda, \alpha^\vee \rangle + 1 \in \{-k, -k+1, \ldots, k\};$$
$$\lambda \xrightarrow[\text{tr},k]{} \lambda + \alpha \text{ whenever } \langle \lambda, \alpha^\vee \rangle + 1 \in \{-k+1, -k+2, \ldots, k\},$$

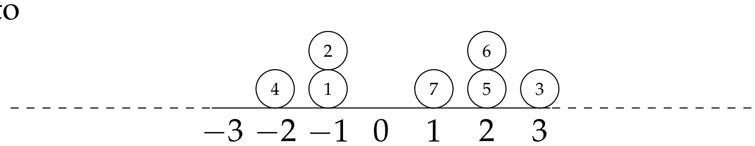
for $\lambda \in P$ and $\alpha \in \Phi^+$.

The symmetric interval-firing process is so called because the symmetric closure of this relation is W-invariant; the truncated interval-process is so called because the interval defining it has length one less than that of the symmetric process. These two processes should be seen as analogous to the (extended) $\Phi^\vee$-Catalan and (extended) $\Phi^\vee$-Shi hyperplane arrangements [12, 2].

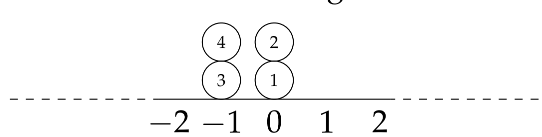
Figure 2 depicts the two interval-firing processes for $\Phi = A_2$ and $k = 0, 1, 2$. In this figure we draw an arrow from λ to μ to denote a firing move $\lambda \to \mu$, and we show only a finite portion of the weight lattice near the origin. Let us mention that, in addition to this "geometric picture," in Type A we can also think of interval-firing in terms of labeled chip configurations on $\mathbb{Z}$ via the same correspondence between configurations of n chips and vectors in $\mathbb{Z}^n$ discussed in the introduction (but with different firing moves). For example, consider symmetric interval-firing with $k = 0$. This corresponds to the labeled chip-firing process that allows the transposition of the chips $\textcircled{i}$ and $\textcircled{j}$ with $i < j$ whenever $\textcircled{i}$ is one position to the left of $\textcircled{j}$. It is immediately apparent that this process is confluent; for instance, the configuration of labeled chips



$\xrightarrow[\text{sym},0]{}$-stabilizes to



The next case to consider is truncated interval-firing with $k = 1$. This corresponds to the labeled chip-firing process that allows both the transposition moves from the symmetric $k = 0$ case, and the usual labeled chip-firing moves from the central-firing case. The reader can verify that for instance that the configuration



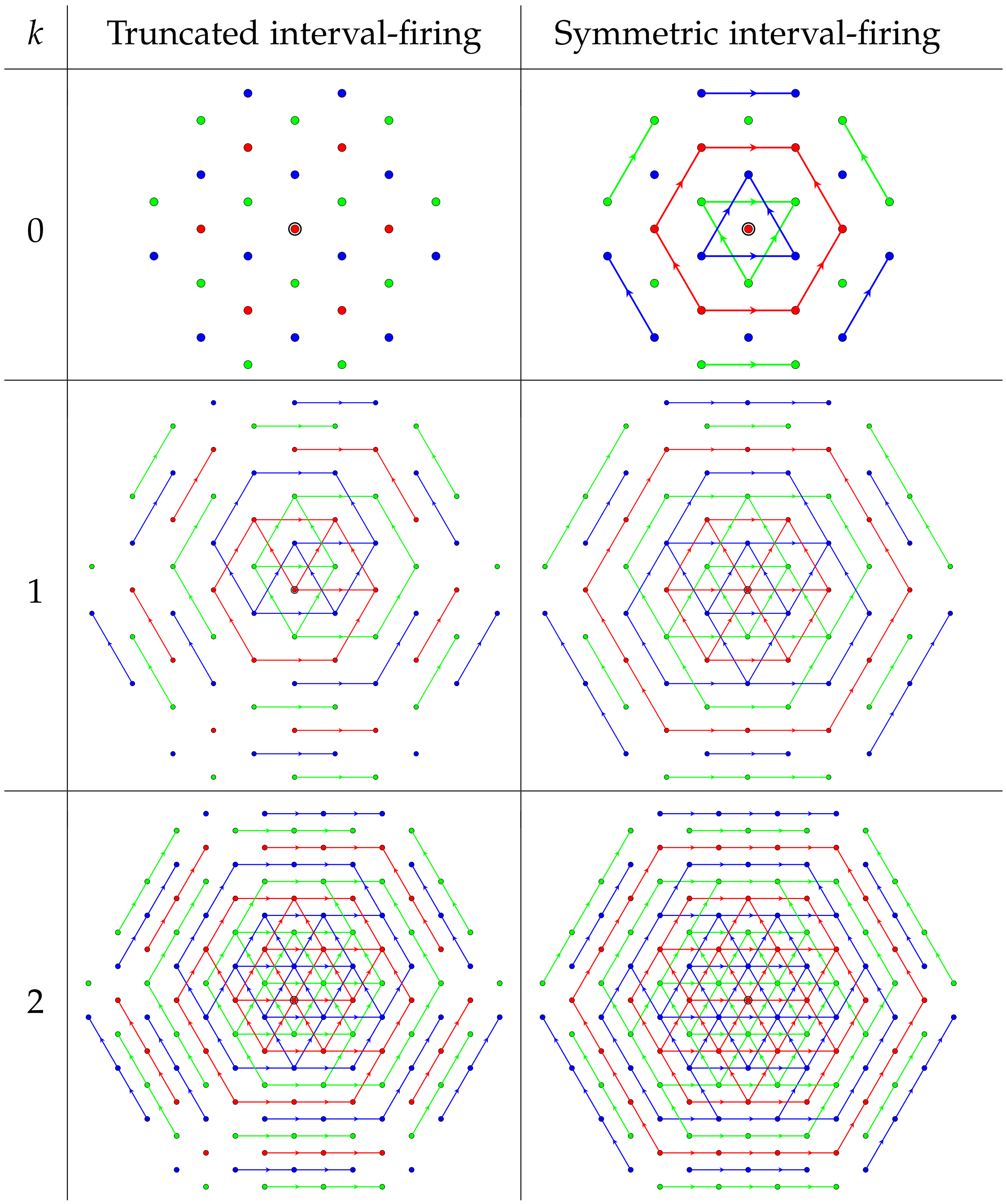
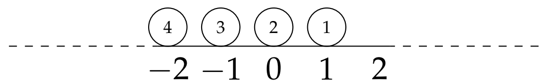


Figure 2: The interval-firing processes for $\Phi = A_2$ and $k = 0, 1, 2$.

$\xrightarrow[\mathrm{tr},1]{}$-stabilizes to



Here it is less obvious that confluence holds. But in fact, in [8] we prove the following:

Theorem 3.2. *For any $k \in \mathbb{Z}_{\geq 0}$, both the symmetric and truncated interval-firing processes are confluent (and terminating).*

The proof of Theorem 3.2 goes through the geometric picture. A key step in the proof is a formula for *traverse lengths of permutohedra*, which we now briefly explain.

For $\lambda \in P$, we define the *permutohedron* $\Pi(\lambda)$ by $\Pi(\lambda) := \mathrm{ConvexHull}\, W(\lambda)$, and we use $\Pi^Q(\lambda) := \Pi(\lambda) \cap (Q + \lambda)$ to denote the (root) lattice points in $\Pi(\lambda)$. For $\alpha \in \Phi$, an *α-string of length ℓ* is a set $\{\mu, \mu - \alpha, \ldots, \mu - \ell\alpha\}$ for some $\mu \in P$. An *α-traverse in* $\Pi(\lambda)$ is a maximal (by containment) α-string contained in $\Pi^Q(\lambda)$. The *traverse length of* $\Pi(\lambda)$ *in direction* α, denoted $\mathbf{l}_\lambda(\alpha)$, is defined to be the minimum length of an α-traverse in $\Pi(\lambda)$.

The intuition is that a minimum length α-traverse in $\Pi(\lambda)$ should be realized by an edge of $\Pi(\lambda)$. This intuition turns out almost to be correct, save for one class of exceptions which we call "funny" weights.

Definition 3.3. If Φ is simply laced, then there are no funny weights. So suppose that Φ is not simply laced. Then there is a unique long simple root α_l and short simple root α_s with $\langle \alpha_l, \alpha_s^\vee \rangle \neq 0$. We say the dominant weight $\lambda = \sum_{i=1}^r c_i \omega_i \in P$ is *funny* if $c_s = 0$, $c_l \geq 1$, and $c_i \geq c_l$ for all i such that α_i is long.

For a dominant weight $\lambda = \sum_{i=1}^r c_i \omega_i \in P$, define $\mathbf{m}_\lambda(\alpha) := \min(\{c_i \colon \alpha \in W(\alpha_i)\})$. Note that $\mathbf{m}_\lambda(\alpha)$ is the minimum length of an edge of $\Pi(\lambda)$ in direction α. Then:

Theorem 3.4. *For a dominant weight $\lambda \in P$, we have*

$$\mathbf{l}_\lambda(\alpha) = \begin{cases} \mathbf{m}_\lambda(\alpha) - 1 & \text{if } \alpha \text{ is long and } \lambda \text{ is funny,} \\ \mathbf{m}_\lambda(\alpha) & \text{otherwise.} \end{cases}$$

The connection between traverse lengths and interval-firing is the following simple lemma, which gives an alternate description of traverse length:

Lemma 3.5. *For any $\lambda \in P$, we have $\mathbf{l}_\lambda(\alpha) = \min(\{\langle \mu, \alpha^\vee \rangle \colon \mu \in \Pi^Q(\lambda),\ \mu + \alpha \notin \Pi^Q(\lambda)\})$.*

The proof of Theorem 3.2 then roughly follows the following plan: we use Lemma 3.5 to show that our interval-firing process gets "trapped" inside of certain permutohedra whose traverse lengths we know, thanks to Theorem 3.4, are large; then we show that there is only one stable point inside these permutohedra that the process could possibly terminate at; hence this process (which is easily seen to be terminating) must in fact be confluent.

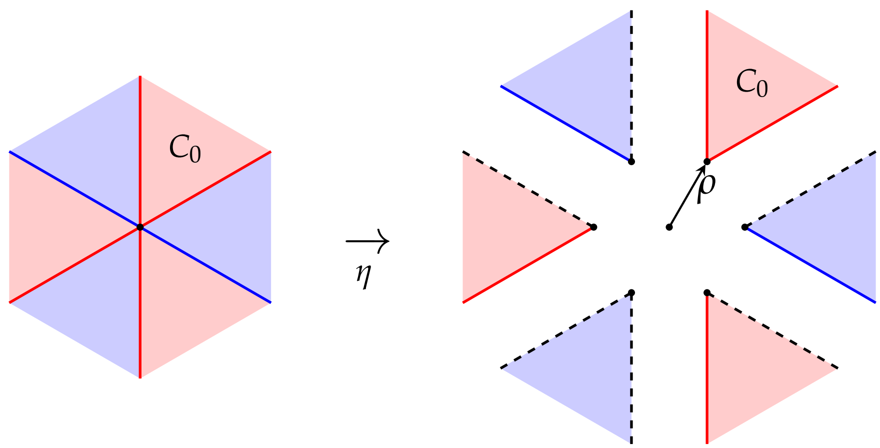


Figure 3: A graphical depiction of the piecewise-linear map η.

4 Interval-firing: stabilizations

Having established that the two interval-firing processes are always confluent from all initial weights, we might then want to compute, or at least in some sense study, stabilizations for these processes.

In order to study stabilizations, the first thing we need is a way to consistently describe the stable points of these processes across all values of the deformation parameter k. To do that, we define the piecewise-linear map $\eta\colon P \to P$ by $\eta(\lambda) := \lambda + w_\lambda(\rho)$, where $w_\lambda \in W$ is the minimum length Weyl group element such that $w_\lambda^{-1}(\lambda)$ is dominant. Figure 3 gives a graphical depiction of η: as we can see, this map "dilates" space by translating the Weyl chambers radially outwards; a point not inside any chamber travels in the same direction as the chamber closest to the fundamental chamber C_0 among those chambers whose closure the point lies in. The point of η is the following:

Lemma 4.1. *For any $k \in \mathbb{Z}_{\geq 0}$, the stable points of the symmetric interval-firing process are*

$$\{\eta^k(\lambda)\colon \lambda \in P \text{ such that } \langle \lambda, \alpha^\vee \rangle \neq -1 \text{ for all } \alpha \in \Phi^+\}$$

and the stable points of the truncated interval-firing process are $\{\eta^k(\lambda)\colon \lambda \in P\}$.

Thanks to Lemma 4.1, we can define the stabilization maps $s_k^{\mathrm{sym}}, s_k^{\mathrm{tr}}\colon P \to P$ by

$$s_k^{\mathrm{sym}}(\mu) = \lambda \Leftrightarrow \text{ the } \xrightarrow[\mathrm{sym},k]{}\text{-stabilization of } \mu \text{ is } \eta^k(\lambda);$$

$$s_k^{\mathrm{tr}}(\mu) = \lambda \Leftrightarrow \text{ the } \xrightarrow[\mathrm{tr},k]{}\text{-stabilization of } \mu \text{ is } \eta^k(\lambda).$$

One can observe already in Figure 2 that the set $(s_k^{\mathrm{sym}})^{-1}(\lambda)$ of weights with symmetric interval-firing stabilization $\eta^k(\lambda)$ seems to "dilate" as we scale k, and ditto for truncated interval-firing. Thus, in analogy with Ehrhart theory, we are motivated to prove the existence of *Ehrhart-like polynomials*, polynomials $L^{\mathrm{sym}}_{\lambda}(k)$ and $L^{\mathrm{tr}}_{\lambda}(k)$ for which

$$\#(s^{\mathrm{sym}}_{k})^{-1}(\lambda) = L^{\mathrm{sym}}_{\lambda}(k);$$
$$\#(s^{\mathrm{tr}}_{k})^{-1}(\lambda) = L^{\mathrm{tr}}_{\lambda}(k),$$

for all $k \in \mathbb{Z}_{\geq 0}$. Note that (except when λ is zero or a minuscule weight) the set of weights with stabilization $\eta^{k}(\lambda)$ is not in general the set of lattice points in a convex polytope, or indeed any convex set. Nevertheless, in [8] we do prove the following:

Theorem 4.2. *For all Φ and all $\lambda \in P$, the symmetric Ehrhart-like polynomial $L^{\mathrm{sym}}_{\lambda}(k)$ exists and has integer coefficients.*

Theorem 4.3. *For simply laced Φ and all $\lambda \in P$, the truncated Ehrhart-like polynomial $L^{\mathrm{tr}}_{\lambda}(k)$ exists and has integer coefficients.*

Moreover, based on computational evidence, we put forward the following striking positivity conjecture:

Conjecture 4.4. *For all Φ and all $\lambda \in P$, both the symmetric and truncated Ehrhart-like polynomials $L^{\mathrm{sym}}_{\lambda}(k)$ and $L^{\mathrm{tr}}_{\lambda}(k)$ exist and have* **nonnegative** *integer coefficients.*

The “half” of Conjecture 4.4 concerning the symmetric polynomials has very recently been proven by the second and fourth authors [11]. The truncated “half” remains open.

We establish Theorem 4.2 via some basic Ehrhart theory of zonotopes. We deduce Theorem 4.3 from the following lemma relating symmetric and truncated stabilizations (which requires the simply laced assumption for technical reasons):

Lemma 4.5. *For any Φ, $\mu \in P$, and $k \in \mathbb{Z}_{\geq 0}$, we have $s^{\mathrm{sym}}_{k}(\mu) = s^{\mathrm{sym}}_{0}(s^{\mathrm{tr}}_{k}(\mu))$. For simply laced Φ, $\mu \in P$, and $k \in \mathbb{Z}_{\geq 0}$, we have $s^{\mathrm{tr}}_{k+1}(\mu) = s^{\mathrm{tr}}_{1}(s^{\mathrm{sym}}_{k}(\mu))$.*

Lemma 4.5 has the following intriguing corollary. Suppose that Φ is simply laced. By iteratively applying Lemma 4.5, we see that $s^{\mathrm{sym}}_{k}(\mu) = (s^{\mathrm{sym}}_{1})^{k}(\mu)$ for all $k \geq 1$. But we also know thanks to Theorem 4.2 that $\#(s^{\mathrm{sym}}_{k})^{-1}(\lambda) = L^{\mathrm{sym}}_{\lambda}(k)$. Hence for all $\lambda \in P$ and all $k \geq 1$, we have

$$\#((s^{\mathrm{sym}}_{1})^{k})^{-1}(\lambda) = L^{\mathrm{sym}}_{\lambda}(k).$$

In other words, we have a map $f\colon X \to X$ from some discrete set to itself such that the sizes $\#(f^{k})^{-1}(x)$ of fibers of iterates of this map are given by polynomials (in k) for every point $x \in X$. This is a very special property for a self-map of a discrete set to have. Figure 4 depicts $s^{\mathrm{sym}}_{1}\colon P \to P$ for $\Phi = A_1$, and records the relevant polynomials in this one-dimensional example.

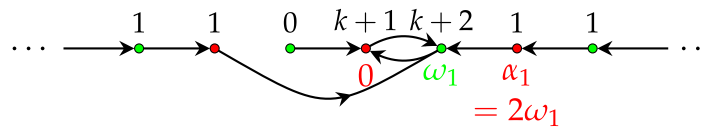


Figure 4: The map $s_1^{\mathrm{sym}}: P \to P$ for $\Phi = A_1$. We write $L_\lambda^{\mathrm{sym}}(k)$ above each weight $\lambda \in P$.

Acknowledgements

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