



Alcoved Polytopes II

Thomas Lam and Alexander Postnikov

To the memory of Bertram Kostant

Abstract This is the second of two papers where we study polytopes arising from affine Coxeter arrangements. Our results include a formula for their volumes, and also compatible definitions of hypersimplices, descent numbers and major index for all Weyl groups. We give a q -analogue of Weyl's formula for the order of the Weyl group. For A_n , C_n and D_4 , we give a Gröbner basis which induces the triangulation of alcoved polytopes.

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1 Introduction

This is the second of two papers in which we investigate *alcoved polytopes* arising from affine Coxeter arrangements. Let $\Phi \subset V$ be an irreducible crystallographic root system and let W be the corresponding Weyl group. Associated to Φ is an infinite hyperplane arrangement known as the affine Coxeter arrangement. This hyperplane arrangement subdivides V into simplices of the same volume which are

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called *alcoves*. We define a *proper alcoved polytope* to be a convex polytope P which is the closure of a union of alcoves.

In [API], we studied these polytopes in the special situation of the root system $\Phi = A_n$. Two motivating examples for us were the *hypersimplices* and the *alcoved matroid polytopes*. Alcoved polytopes arising from other root systems have also been studied. Payne [Pay] showed that alcoved polytopes with vertices lying in the coweight lattice are normal and Koszul in classical types. Werner and Yu [WY] studied generating sets of alcoved polytopes. Fomin and Zelevinsky's *generalized associahedra* [FZ] are examples of polytopes which can be realized as alcoved polytopes.

We prove that the volume of an alcoved polytope P is given by

$$\text{Vol}(P) = \sum_{\tilde{w} \in W/C} I(P_{\tilde{w}}),$$

where W/C are certain cosets of the Weyl group, $P_{\tilde{w}}$ are certain alcoved polytopes and $I(P)$ denotes the number of integral coweights lying in P . The group $C \subset W$ was studied previously by Verma [Ver]. The order of the group C is equal to the index of connection of Φ . For the case $\Phi = A_{n-1}$, the group C is the cyclic group generated by the long cycle $(123 \cdots n)$, written in cycle notation.

Recall that the usual hypersimplex $\Delta_{k,n}$ has volume equal to the Eulerian number $A_{k,n-1}$. We define *generalized hypersimplices* Δ_k^Φ to be certain alcoved polytopes which generalize this property of $\Delta_{k,n}$ for each root system Φ . To this end we introduce the *circular descent number* $\text{cdes} : W \rightarrow \mathbb{Z}$, so that the volume of Δ_k^Φ counts the number of elements of W with a fixed circular descent number. We also introduce a *circular major map* $\text{cmaj} : W \rightarrow C$ that interacts in an interesting way with cdes . In particular $\{w \in W \mid \text{cmaj}(w) = \text{id}\}$ gives a set of coset representatives for W/C . In type A_{n-1} , the circular major map generalizes the major index of S_n , taken modulo n .

Weyl's formula for the order of the Weyl group W states that

$$|W| = f \cdot r! \cdot a_1 a_2 \cdots a_r,$$

where f is the index of connection, r is the rank of Φ and a_i are the coefficients of the simple roots in the highest root of Φ . We prove, using the geometry of alcoved polytopes, that

$$\sum_{w \in W} q^{\text{cdes}(w)} e^{\text{cmaj}(w)} = \left(\sum_{x \in C} e^x \right) \cdot A_r(q) \cdot [a_1]_q \cdots [a_r]_q,$$

where $A_r(q)$ is the usual Eulerian polynomial, and $[n]_q$ denotes the usual q -analogue. Here, $e^x \in \mathbb{Z}[C]$ lies in the group algebra of C .

Finally, in analogy with Sturmfels' triangulation of the hypersimplex in type A [Stu], we study the toric ideals I_P associated with an alcoved polytope P . When Φ is one of the root systems A_n, C_n, D_4 , we give a Gröbner basis \mathcal{G}_P for I_P which induces the triangulation of P into alcoves.

2 Root system notation

We recall standard terminology related to root systems, see [Hum] for more details. Let V be a real Euclidean space of rank r with nondegenerate symmetric inner product (λ, μ) . Let $\Phi \subset V$ be an irreducible *crystallographic root system* with choice of basis of *simple roots* $\alpha_1, \dots, \alpha_r$. Let $\Phi^+ \subset \Phi$ be the corresponding set of *positive roots* and let $\Phi^- = -\Phi^+$ be the set of *negative roots*. Then Φ is the disjoint union of Φ^+ and Φ^- . We will write $\alpha > 0$, for $\alpha \in \Phi^+$; and $\alpha < 0$, for $\alpha \in \Phi^-$. The collection of *coroots* $\alpha^\vee = 2\alpha/(\alpha, \alpha) \in V$, for $\alpha \in \Phi$, forms the *dual root system* Φ^\vee . The *Weyl group* $W \subset \text{Aut}(V)$ is generated by the reflections $s_\alpha : \lambda \mapsto \lambda - (\lambda, \alpha^\vee)\alpha$ with respect to roots $\alpha \in \Phi$. The Weyl group W is actually generated by simple reflections $s_i = s_{\alpha_i}$ subject to the Coxeter relations. The *length* $\ell(w)$ of an element $w \in W$ is the length of a shortest decomposition for w in terms of simple reflections. There is a unique element $w_\circ \in W$ of maximal possible length.

The *root lattice* $L = L(\Phi) \subset V$ is the integer lattice spanned by the roots $\alpha \in \Phi$. It is generated by the simple roots α_i . The *weight lattice* is defined by $\Lambda = \Lambda(\Phi) = \{\lambda \in V \mid (\lambda, \alpha^\vee) \in \mathbb{Z}, \text{ for all } \alpha \in \Phi\}$. The weight lattice Λ contains the root lattice L as a subgroup of finite index f . The quotient group Λ/L is isomorphic to the center of the universal simply-connected Lie group G^\vee associated with the root system Φ^\vee . The index $f = |\Lambda/L|$ is called the *index of connection*.

The *coroot lattice* is the integer lattice $L^\vee = L^\vee(\Phi)$ spanned by the coroots α_i^\vee . The *coweight lattice* Λ^\vee is the integer lattice defined by $\Lambda^\vee = \Lambda^\vee(\Phi) = \{\lambda \in V \mid (\lambda, \alpha) \in \mathbb{Z}, \text{ for all } \alpha \in \Phi\}$. Let $\omega_1, \dots, \omega_r \subset V$ be the basis dual to the basis of simple roots $\alpha_1, \dots, \alpha_r$, i.e., $(\omega_i, \alpha_j) = \delta_{ij}$. The ω_i are called the *fundamental coweights*. They generate the coweight lattice Λ^\vee .

Let $\rho = \omega_1 + \dots + \omega_r$. The *height* of a root α is the number (ρ, α) of simple roots that add up to α . Since we assumed that Φ is irreducible, there exists a unique *highest root* $\theta \in \Phi^+$ of maximal possible height. For convenience we set $\alpha_0 = -\theta$. Let $a_0 = 1$ and a_1, \dots, a_r be the positive integers given by $a_i = (\omega_i, \theta)$. In other words, $\theta = a_1\alpha_1 + \dots + a_r\alpha_r$, or, equivalently, $a_0\alpha_0 + a_1\alpha_1 + \dots + a_r\alpha_r = 0$. The *Coxeter number* is defined as $h = (\rho, \theta) + 1 = a_0 + a_1 + \dots + a_r$.

Lemma 2.1. *Let $\lambda \in \Lambda^\vee$ be an integral coweight and let $w \in W$ be a Weyl group element. Then*

$$w(\lambda) - \lambda \in L^\vee.$$

Proof. Since $L^\vee \subset \Lambda^\vee$ it suffices to check this for a simple reflection s_α . We compute $s_\alpha(\lambda) - \lambda = (\lambda, \alpha)\alpha^\vee \in L^\vee$. \square

3 The affine Weyl group and alcoved polytopes

The *affine Weyl group* W_{aff} associated with the root system Φ is generated by the reflections $s_{\alpha,k} : V \rightarrow V$, $\alpha \in \Phi$, $k \in \mathbb{Z}$, with respect to the affine hyperplanes

$$H_{\alpha,k} = \{\lambda \in V \mid (\lambda, \alpha) = k\}.$$

The coweight lattice Λ^\vee and coroot lattice L^\vee act on the space V by translations. We will identify Λ^\vee and L^\vee with these groups of translations. The Weyl group W normalizes these groups.

Lemma 3.1. [Hum] *The affine Weyl group W_{aff} is the semidirect product $W \ltimes L^\vee$ of the usual Weyl group W and the coroot lattice L^\vee .*

The connected components of the complement to these hyperplanes $V \setminus \bigcup H_{\alpha,k}$ are called *alcoves*. Let \mathcal{A} be the set of all alcoves. A *closed alcove* is the closure of an alcove. Each alcove A has the following form:

$$A = \{\lambda \in V \mid m_\alpha < (\lambda, \alpha) < m_\alpha + 1, \text{ for } \alpha \in \Phi^+\},$$

where $m_\alpha = m_\alpha(A)$ is a collection of integers associated with the alcove A .

Lemma 3.2. [Hum] *The affine Weyl group W_{aff} acts simply transitively on the collection \mathcal{A} of all alcoves.*

The fundamental alcove is the simplex given by

$$\begin{aligned} A_\circ &= \{\lambda \in V \mid 0 < (\lambda, \alpha) < 1, \text{ for } \alpha \in \Phi^+\} \\ &= \{\lambda \in V \mid (\lambda, \alpha_i) > 0, \text{ for } i = 1, \dots, r; \text{ and } (\lambda, \theta) < 0\} \\ &= \{x_1 \omega_1 + \dots + x_r \omega_r \mid x_1, \dots, x_r > 0 \text{ and } a_1 x_1 + \dots + a_r x_r < 1\} \\ &= \text{Convex Hull of the points } 0, \omega_1/a_1, \dots, \omega_r/a_r. \end{aligned} \tag{1}$$

Lemma 3.2 implies that all alcoves are obtained from A_\circ by the action of W_{aff} . In particular, all closed alcoves are simplices with the same volume. The closure of the fundamental alcove A_\circ is a fundamental domain of W_{aff} . Let $\mathcal{F} \supset \mathcal{A}$ be the set of all faces of alcoves of all dimensions. We will think of elements of \mathcal{F} as relatively open sets, so that the space V is the disjoint union of elements of \mathcal{F} .

Our main object is defined as follows.

Definition 3.3. An alcoved polytope P is a convex polytope in the space V such that P is a union of finitely many elements of \mathcal{F} . A proper alcoved polytope is an alcoved polytope of top dimension.

By the definition, each proper alcoved polytope comes equipped with a triangulation into closed alcoves. The following lemma is immediate from the definitions.

Lemma 3.4. *A bounded subset $P \subset V$ is an alcoved polytope if and only if P is the intersection of several half-spaces of the form $\{\lambda \in V \mid (\lambda, \alpha) \geq k\}$, for $\alpha \in \Phi$ and $k \in \mathbb{Z}$.*

Let (W, S) be a Coxeter group and $u, v \in W$. A path from u to v is a sequence $u = w_0 \rightarrow w_1 \rightarrow w_2 \rightarrow \dots \rightarrow w_s = v$ such that $w_{i+1} = w_i s$ for some simple reflection $s \in S$. A subset $K \subset W$ is called *convex* if for every $u, v \in K$, we have that any shortest path from u to v lies in K .

Proposition 3.5. *Let $P \subset V$ be a bounded subset which is a union of closed alcoves. Then P is a convex polytope if and only if one of the following conditions hold:*

- (1) *For any two alcoves $A, B \subset P$, any shortest path $A = A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_s = B$ lies in P . Here $A_i \in \mathcal{A}$ are alcoves and $A' \rightarrow A''$ means that the closures of the two alcoves A' and A'' share a facet.*
- (2) *The subset $W_P = \{w \in W_{\text{aff}} \mid w(A_{\circ}) \subset P\}$ of the affine Weyl group is a convex subset.*

Proof. Suppose P is a polytope and $A = A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_s = B$ is a shortest path with $A, B \in P$. Suppose that A_i lies in P but A_{i+1} lies outside of P . Then the hyperplane $H_{\alpha,k}$ which separates A_i and A_{i+1} must be a facet of P , so eventually the shortest path must cross $H_{\alpha,k}$ again. Suppose that A_j lies on the same side of $H_{\alpha,k}$ as A_{i+1} and A_{j+1} lies on the other side. Then reflecting the path $A_{i+1} \rightarrow A_{i+2} \rightarrow \dots \rightarrow A_j$ in the hyperplane $H_{\alpha,k}$, we get another path from A to B which is shorter. Conversely, suppose condition (1) holds but P is not convex. This implies that there are alcoves $A, B \in P$ and points $a \in A, b \in B$ such that the straight line \overline{ab} does not lie in P . Since P is compact we may assume a and b lie in the interior of A and B respectively. Thus without loss of generality, we may assume that the line \overline{ab} does not intersect any face of \mathcal{F} of codimension more than one. The sequence of alcoves obtained by travelling along \overline{ab} must be a shortest path. This is clear as every hyperplane $H_{\alpha,k}$ that \overline{ab} intersects separates A from B and so must be a separating hyperplane between some two alcoves A_i and A_{i+1} in any path from A to B .

Condition 2 follows immediately from translating the action of the simple generators of W_{aff} on alcoves. \square

Thus each alcoved polytope is of the form

$$P = \{\lambda \in V \mid k_{\alpha} \leq (\lambda, \alpha) \leq K_{\alpha}, \text{ for } \alpha \in \Phi^+\},$$

where $k_{\alpha} = k_{\alpha}(P)$ and $K_{\alpha} = K_{\alpha}(P)$ are two collections of integers indexed by positive roots $\alpha \in \Phi^+$.

Let $\Lambda^{\vee}/h = \{\lambda/h \mid \lambda \in \Lambda^{\vee}\}$ be the coweight lattice Λ^{\vee} shrunk h times. Recall that h denotes the Coxeter number.

We are grateful to Bertram Kostant for many helpful conversations over the years. We appreciate his attention to combinatorial aspects of representation theory. One little thing that we learned from him is that the center of mass of an alcove is the only point of the lattice Λ^{\vee}/h in the interior of this alcove.

Lemma 3.6. [Kos], see also [LP, Section 14] *For every alcove $A \in \mathcal{A}$, there is exactly one point of the lattice Λ^{\vee}/h in the interior of A . For the fundamental alcove A_{\circ} , we have $A_{\circ} \cap (\Lambda^{\vee}/h) = \{\rho/h\}$.*

Proof. Since the affine Weyl group acts simply transitively on \mathcal{A} and preserves the lattice Λ^{\vee}/h , it is enough to prove the claim for the fundamental alcove A_{\circ} . In the basis of fundamental coweights $\omega_1, \dots, \omega_r$, the fundamental alcove A_{\circ} is given by the inequalities $x_1, \dots, x_r > 0$ and $a_1x_1 + \dots + a_rx_r < 1$; and the lattice Λ^{\vee}/h is

given by $x_1, \dots, x_r \in \mathbb{Z}/h$. Recall that $h = a_0 + a_1 + \dots + a_r$. Then the intersection $A_\circ \cap (\Lambda^\vee/h)$ consists of a single point with coordinates $(1/h, \dots, 1/h)$. In other words, this intersection point is $(\omega_1 + \dots + \omega_r)/h = \rho/h$. \square

For $A \in \mathcal{A}$, we call the single element of $(\Lambda^\vee/h) \cap A$ the *central point* of the alcove A . Let $Z = (\Lambda^\vee/h) \setminus \bigcup H_{\alpha,k}$ be the set of central points of all alcoves; equivalently,

$$Z = \{\lambda \in V \mid h \cdot (\alpha, \lambda) \equiv 1, \dots, h-1 \pmod{h}, \text{ for } \alpha \in \Phi^+\}.$$

The set Z of central points is in one-to-one correspondence with the set \mathcal{A} of alcoves.

4 Weyl's formula for the order of the Weyl group

Let Vol be the volume form on the space V normalized by $\text{Vol}(A_\circ) = 1$. Then the volume of any alcove is 1 and the volume $\text{Vol}(P)$ of an alcoved polytope P is the number of alcoves in P . Equivalently, $\text{Vol}(P) = |P \cap Z|$.

Let Π be the alcoved polytope given by

$$\begin{aligned} \Pi &= \{\lambda \in V \mid 0 \leq (\lambda, \alpha_i) \leq 1, \text{ for } i = 1, \dots, r\} \\ &= \{x_1 \omega_1 + \dots + x_r \omega_r \mid 0 \leq x_i \leq 1, \text{ for } i = 1, \dots, r\}, \end{aligned}$$

i.e., Π is the parallelepiped generated by the fundamental coweights $\omega_1, \dots, \omega_r$. This polytope is a fundamental domain of the coweight lattice Λ^\vee . Since A_\circ is the simplex with the vertices $0, \omega_1/a_1, \dots, \omega_r/a_r$, we have

$$\text{Vol}(\Pi) = \text{Vol}(\Pi)/\text{Vol}(A_\circ) = r! a_1 \cdots a_r.$$

Thus the parallelepiped Π consists of $r! a_1 \cdots a_r$ alcoves.

Let H be the alcoved polytope given by

$$H = \{\lambda \in V \mid -1 \leq (\lambda, \alpha) \leq 1, \text{ for } \alpha \in \Phi^+\}.$$

The polytope H consists of all alcoves adjacent to the origin 0, i.e., it consists of the $|W|$ alcoves of the form $w(A_\circ)$, for $w \in W$. In particular, its volume is the order of the Weyl group: $\text{Vol}(H) = |W|$. Lemma 3.1 implies that the polytope H is a fundamental domain of the coroot lattice L^\vee . Thus $\text{Vol}(H)/\text{Vol}(\Pi) = |\Lambda^\vee/L^\vee| = f$ is the index of connection. This implies the well-known formula for the order of the Weyl group, see [Hum, 4.9]:

$$|W| = f \cdot r! \cdot a_1 \cdots a_r. \tag{2}$$

5 The group C

For an integral coweight $\lambda \in \Lambda^\vee$, the affine translation $A + \lambda$ of an alcove A is an alcove; and the affine translation $P + \lambda$ of an alcoved polytope P is an alcoved polytope.

Let us define the equivalence relation “ \sim ” on the affine Weyl group W_{aff} by

$$u \sim w \text{ if and only if } u(A_\circ) = w(A_\circ) + \lambda, \text{ for some } \lambda \in \Lambda^\vee,$$

where $u, w \in W_{\text{aff}}$. The relation “ \sim ” is invariant with respect to the left action of the affine Weyl group. According to Lemma 3.6, this equivalence relation can be defined in terms of central points of alcoves as

$$u \sim w \text{ if and only if } u(\rho/h) - w(\rho/h) \in \Lambda^\vee.$$

Let C be the subset of the usual Weyl group W given by

$$C = \{w \in W \mid w \sim 1\} = \{w \in W \mid w(\rho) - \rho \in h\Lambda^\vee\}.$$

Also let $C_{\text{aff}} = \{w \in W_{\text{aff}} \mid w \sim 1\}$. Actually, C is a subgroup in W and C_{aff} is a subgroup in W_{aff} . Indeed, $u \sim 1$ and $w \sim 1$ imply that $uw \sim u \sim 1$.

The coroot lattice L^\vee is a normal subgroup in C_{aff} . The group C_{aff} is the semidirect product $C \ltimes L^\vee$ and, thus, $C \simeq C_{\text{aff}}/L^\vee$.

Equivalence classes of elements of the Weyl group (respectively, the affine Weyl group) with respect to the relation “ \sim ” are exactly cosets in W/C (respectively, $W_{\text{aff}}/C_{\text{aff}}$). Since Π is a fundamental domain of the coweight lattice Λ^\vee and, for an alcove $A \in \mathcal{A}$, there is a translation $A + \lambda$ such that $A + \lambda = w(A_\circ)$, for some $w \in W$, we deduce that there are natural one-to-one correspondences between the followings sets:

$$W/C \simeq W_{\text{aff}}/C_{\text{aff}} \simeq \mathcal{A}/\Lambda^\vee \simeq \{\text{alcoves in } \Pi\}.$$

In particular, the number of cosets $|W/C|$ equals $\text{Vol}(\Pi) = |W|/f$ and, thus, the order of the group C is $|C| = f$.

There is a natural bijection $b : \Lambda^\vee \rightarrow C_{\text{aff}}$ given by $b(\lambda) = w$ whenever $w(A_\circ) = A_\circ + \lambda$. Notice that b may not be a homomorphism of groups. However the map $\bar{b} : \Lambda^\vee \rightarrow C$ given by the composition of b with the natural projection $C_{\text{aff}} \rightarrow C_{\text{aff}}/L^\vee \simeq C$ is a homomorphism. Indeed, let $\bar{b}(\lambda) = u$ and $\bar{b}(\mu) = w$. Then $u, w \in W$ are given by $u(A_\circ) \equiv A_\circ + \lambda \pmod{L^\vee}$ and $w(A_\circ) \equiv A_\circ + \mu \pmod{L^\vee}$. Then $uw(A_\circ) \equiv A_\circ + \lambda + u(\mu) \equiv A_\circ + \lambda + \mu \pmod{L^\vee}$. The last equation follows from Lemma 2.1. The kernel of the map \bar{b} is L^\vee . Thus \bar{b} induces the natural isomorphism of groups:

$$\Lambda^\vee/L^\vee \simeq C.$$

The group C is the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ in type A_n , a group of order 2 for types B_n , C_n and E_7 , a group of order 4 for type D_n , a group of order 3 for type E_6 , and trivial for E_8 , F_4 , and G_2 .

6 The statistic cdes

Let us say that a root $\alpha \in \Phi$ is an *inversion* of Weyl group element $w \in W$ if $w(\alpha) < 0$. Equivalently, a positive root α in an inversion of w if and only if $\ell(ws_\alpha) < \ell(w)$. Let us define

$$\text{inv}_\alpha(w) = \begin{cases} 0 & \text{if } w(\alpha) > 0, \\ 1 & \text{if } w(\alpha) < 0. \end{cases}$$

Lemma 6.1. *We have $\text{inv}_\alpha(w) = -m_\alpha(w^{-1}(A_\circ))$.*

Proof. Indeed, α is an inversion of w if and only if $(w(\alpha), \rho) = (\alpha, w^{-1}(\rho)) < 0$, that is, $-1 < (\lambda, \alpha) < 0$, for any $\lambda \in w^{-1}(A_\circ)$. \square

Let $d_i(w) = \text{inv}_{\alpha_i}(w)$, for $i = 0, \dots, r$. If $d_i(w) = 1$, we say that w has a descent at i .

Definition 6.2. Let $w \in W$. The *circular descent number* $\text{cdes}(w)$ is defined by

$$\text{cdes}(w) = \sum_{i=0}^r a_i d_i(w).$$

Note that $\text{cdes}(w)$ is always positive. Indeed if $d_i(w) = 0$ for $i \in [1, r]$, then we have $w = 1$ and $d_0(w) = 1$.

Define $\delta_w \in L^\vee$ for $w \in W$ by

$$\delta_w = \sum_{i=0}^r d_i(w) \cdot \omega_i$$

where for convenience we let $\omega_0 = 0$.

Lemma 6.3. *The coweight δ_w is the unique integral coweight such that $w^{-1}(A_\circ) + \delta_w \in \Pi$.*

Proof. Let $i \in [1, r]$. Then by Lemma 6.1, $-d_i(w) < (w^{-1}(\rho/h), \alpha_i) < 1 - d_i(w)$, so that $\lambda = w^{-1}(A_\circ) + \delta_w$ satisfies $0 \leq (\lambda, \alpha_i) \leq 1$. The coweight δ_w must be unique since adding or subtracting any fundamental coweight ω_i will cause λ to violate the inequality $0 < (\lambda, \alpha_i) < 1$. \square

We set $S = \{\alpha_0, \dots, \alpha_r\}$ and $S_i = \{\alpha_j \in S \mid a_j = i\}$. For convenience we set $J = \{j \in [0, r] \mid a_j = 1\}$.

Proposition 6.4. *We have the following equivalent descriptions of the group $C \subset W$.*

$$C = \{w \in W \mid \text{cdes}(w) = 1\} \quad (3)$$

$$= \{w \in W \mid w(S) = S\} \quad (4)$$

$$= \{w \in W \mid w(S_k) = S_k \text{ for all } k\}. \quad (5)$$

For any $j \in J$ there exist a unique Weyl group element $w^{(j)} \in C$ such that $w(\alpha_i) > 0$, for $i \neq j$ and $w(\alpha_j) < 0$.

Proof. Let $c \in C$. By Lemma 6.3 and the definition of C , we see that $c^{-1}(A_\circ) + \delta_c = A_\circ$. By Lemma 3.6, this implies that $c^{-1}(\rho/h) + \delta_c = \rho/h$. It is clear that $(\delta_c, \theta) = \text{cdes}(c) - d_0(c)$. We compute, using Lemma 6.1, that

$$d_0(c) - 1 \leq (c^{-1}(\rho/h), \theta) \leq d_0(c).$$

So summing, we have

$$\text{cdes}(c) - 1 \leq (\rho/h, \theta) \leq \text{cdes}(c)$$

which immediately implies that $\text{cdes}(c) = 1$ since $(\rho/h, \theta) = 1 - 1/h$. The converse follows in the same manner. This establishes the equality in (3).

Now suppose $c \in C$. We establish (4). By definition, $c(\alpha_j) < 0$ for some $j \in J$ and $c(\alpha_i) > 0$ for $i \neq j$. Let $S_{\neq j} = \{\alpha_i \in S \mid i \neq j\}$. We have $c^{-1}(A_\circ) + \omega_j = A_\circ$, and in particular, the set $\{0, \omega_1/a_1, \dots, \omega_r/a_r\}$ is sent to itself under the map $\lambda \mapsto c^{-1}(\lambda) + \omega_j$. Substituting this fact into $(c(\alpha_i), \omega_k/a_k) = (\alpha_i, c^{-1}(\omega_k/a_k))$ and noting that $\{\omega_1, \dots, \omega_n\}$ is a dual basis to $\{\alpha_1, \dots, \alpha_n\}$, we deduce that $c(S_{\neq j}) = S_{\neq 0}$, and $c(\alpha_j) = \alpha_0$. Thus (4) holds.

We get (5) from (4) by noting that up to scalar multiplication, the relation $\sum_i a_i \alpha_i = 0$ is the only linear dependence among the roots in S .

Conversely, (5) clearly implies (3) by the definition of cdes. The last statement of the proposition also follows from this discussion. \square

By property (3), we have $f = |C| = \#\{i \in [0, r] \mid a_i = 1\}$. These i 's correspond to minuscule coweights ω_i . Recall that a minuscule weight is one whose weight polytope has no internal weights.

We remark that the group C was previously studied by Verma [Ver] but not in the current context of the statistic cdes. The group C is related to the statistic cdes on the whole of W in an intimate way.

Theorem 6.5. *The statistic cdes is constant on the double cosets $C \backslash W / C$.*

Proof. Let $w \in W$ and $c \in C$ so that $c(\alpha_j) = \alpha_0$ for $j \in J$. We need to prove that $\text{cdes}(cw) = \text{cdes}(w) = \text{cdes}(wc)$. The latter equality is immediate from condition (5) of Proposition 6.4.

Let $\alpha \in S$ and let $\beta = w(\alpha) = b_1 \alpha_1 + \dots + b_r \alpha_r$. The b_i are either all positive or all negative. The element w has a descent at α if and only if $\beta < 0$. Now

$$cw(\alpha) = c(\beta) = b_1 c(\alpha_1) + \cdots + b_r c(\alpha_r). \quad (6)$$

If $b_j = 0$, then clearly $d_\alpha(w) = d_\alpha(cw)$. If $b_j \neq 0$, then we have a term of the form

$$b_j c(\alpha_j) = b_j \alpha_0 = -b_j(a_1 \alpha_1 + \cdots + a_r \alpha_r) \quad (7)$$

in equation (6). Since $-\alpha_0$ is the longest root, we have $|a_k| \geq |b_k|$. Thus substituting (7) into (6), we see that $b_j \neq 0$ implies that $d_\alpha(w) = 1 - d_\alpha(cw)$. Indeed $b_j \in \{0, 1, -1\}$, and we have $d_\alpha(w) - d_\alpha(cw) = b_j$.

Now we have

$$0 = a_0 w(\alpha_0) + \cdots + a_r w(\alpha_r),$$

and so expressing both sides in terms of the simple roots $\alpha_1, \dots, \alpha_r$, we see that the coefficient of α_j is 0. Write $b_j(\alpha)$ for the coefficient b_j in the proof earlier, obtaining the equality

$$0 = \sum_{i=0}^r a_i b_j(\alpha_i) = \sum_{i=0}^r (d_i(cw) - d_i(w)) = \text{cdes}(cw) - \text{cdes}(w). \quad \square$$

7 The map cmaj

Define the *circular major map* $\text{cmaj} : W \rightarrow C$ by

$$\text{cmaj}(w) = \bar{b}(\delta_w)$$

where \bar{b} is the isomorphism from Section 5.

Lemma 7.1. *The map cmaj satisfies $\text{cmaj}(c) = c$ for $c \in C$.*

Proof. By Lemma 6.3, we have $c(A_\circ) = A_\circ + c(\delta_c)$. But by Lemma 2.1, $c(\delta_c) = \delta_c \bmod L^\vee$, so that by definition $\bar{b}(\delta_c) = c$. \square

Theorem 7.2. *The map cmaj satisfies*

$$\text{cmaj}(c_1 w c_2) = c_1 w c_2^{\text{cdes}(w)}$$

for $c_i \in C$ and $w \in W$.

Proof. Let $w \in W$ and $c = w^{(j)} \in C$, so that $c^{-1}(\alpha_0) = \alpha_j$. Thus $c(\alpha_j) < 0$, so that $\text{cmaj}(c) = \omega_j$. We first consider $w c$. We have

$$w^{-1}(\rho/h) + \delta_w = \mu/h$$

for some $\mu \in \Lambda$ satisfying $0 < (\mu, \alpha_i) < h$ for $i \in [1, r]$. Applying c^{-1} on the left to both sides, we obtain

$$c^{-1} w^{-1}(\rho/h) + c^{-1}(\delta_w) = c^{-1}(\mu/h).$$

Now $0 < (c^{-1}(\mu/h), c^{-1}(\alpha_i)) < 1$ and as i varies, we obtain every simple root in the form $c^{-1}(\alpha_i)$ apart from $c(\alpha_0) = \alpha_j$ for some $j \in J$. But $-cdes(w) < (\mu/h, \alpha_0) < 1 - cdes(w)$, so we have $0 < (c^{-1}(\mu/h) + cdes(w) \cdot \omega_j, \alpha_j) < 1$.

Thus

$$\begin{aligned}\delta_{wc} &= c(\delta_w) + cdes(w) \cdot \omega_j \\ \text{cmaj}(wc) &\equiv \text{cmaj}(w) + cdes(w) \cdot \omega_j \pmod{L^\vee}.\end{aligned}$$

Hence $\text{cmaj}(wc) = \text{cmaj}(w)c^{cdes(w)}$. We have used Lemma 2.1.

Similarly,

$$c^{-1}(\rho/h) + \delta_c = \rho/h$$

gives

$$\begin{aligned}w^{-1}c^{-1}(\rho/h) + w^{-1}(\omega_j) &= w^{-1}(\rho/h) \\ w^{-1}c^{-1}(\rho/h) + w^{-1}(\omega_j) + \delta_w &= w^{-1}(\rho/h) + \delta_w = \rho/h.\end{aligned}$$

This implies that

$$\begin{aligned}\delta_{cw} &= w^{-1}(\omega_j) + \delta_w \\ \text{cmaj}(cw) &\equiv \text{cmaj}(w) + \omega_j \pmod{L^\vee}.\end{aligned}$$

Hence $\text{cmaj}(cw) = c \cdot \text{cmaj}(w)$. □

Theorem 7.2 shows that the map cmaj allows us to pick representatives for the right cosets W/C . For example $\{w \mid \text{cmaj}(w) = \text{id}\}$ is a set of right coset representatives. In type A , cmaj has an explicit representation theoretic meaning, see Theorem 11.1.

8 Relation between volumes and numbers of lattice points

Let P be an alcoved polytope, and let $A \in \mathcal{A}$ be an alcove. Let $k_\alpha = k_\alpha(P)$, $K_\alpha = K_\alpha(P)$, and $m_\alpha = m_\alpha(A)$, for $\alpha \in \Phi^+$ as in Section 3. Let us define the alcoved polytope $P_{(A)}$ as

$$P_{(A)} = \{\lambda \in V \mid k_\alpha - m_\alpha \leq (\lambda, \alpha) \leq K_\alpha - m_\alpha - 1, \text{ for all } \alpha \in \Phi^+\}.$$

The following claim follows directly from the definitions.

Lemma 8.1. *For P and A as above, the set $P_{(A)} \cap \Lambda^\vee$ of lattice points in $P_{(A)}$ is exactly the set of integral coweights $\lambda \in \Lambda^\vee$ such that $A + \lambda$ is an alcove in P .*

The lemma says that lattice points in $P_{(A)}$ are in one-to-one correspondence with alcoves in P that are obtained by affine translations of A .

For $w \in W$, the definition of the polytope $P_{(w)} = P_{(w(A_\circ))}$ can be rewritten as

$$P_{(w)} = \{\lambda \in V \mid k_\alpha + d_\alpha(w^{-1}) \leq (\lambda, \alpha) \leq K_\alpha + d_\alpha(w^{-1}) - 1, \text{ for all } \alpha \in \Phi^+\}.$$

We have used Lemma 6.1.

Notice that $P_{(A+\lambda)} = P_{(A)} - \lambda$. Thus the polytopes $P_{(A)} \equiv P_{(B)}$ are equivalent modulo affine translations by elements of Λ^\vee , whenever $A \equiv B \pmod{\Lambda^\vee}$. This implies that the polytope $P_{(\bar{w})} = P_{(w)}$ is correctly defined modulo affine translations by coweights $\lambda \in \Lambda$, where $w \in W$ is any representative of a coset $\bar{w} \in W/C$.

Let $I(P) = |P \cap \Lambda|$ be the number of lattice points in P . The following statement establishes a relation between the volume of an alcoved polytope and the numbers of lattice points in smaller alcoved polytopes.

Theorem 8.2. *Let P be an alcoved polytope. Then*

$$\text{Vol}(P) = \sum_{\bar{w} \in W/C} I(P_{(\bar{w})}).$$

Proof. According to Lemma 8.1, the total number of alcoves in P equals the sum of $I(P_{(A)})$ over representatives A of cosets \mathcal{A}/L^\vee . This is exactly the claim of the theorem. \square

9 Generalized hypersimplices

For $k = 1, \dots, h-1$, let us define the k -th *generalized hypersimplex* Δ_k^Φ as the alcove polytope given by

$$\Delta_k^\Phi = \{\lambda \in V \mid 0 \leq (\lambda, \alpha_i) \leq 1, \text{ for } i = 1, \dots, r; \text{ and } k-1 \leq (\lambda, \theta) \leq k\}.$$

In other words, the generalized hypersimplices are the slices of the parallelepiped Π by the parallel hyperplanes of the form $H_{\theta, k}$, for $k \in \mathbb{Z}$. Clearly, the first generalized hypersimplex is the fundamental alcove: $\Delta_1^\Phi = A_\circ$. Also the last generalized hypersimplex is the alcove given by $\Delta_{h-1}^\Phi = w_\circ(A_\circ) + \rho$, where $w_\circ \in W$ is the longest element in W .

Lemma 9.1. *Let $w \in W$. The polytope $(\Delta_k^\Phi)_{(w)}$ consists of a single point $\lambda \in \Lambda^\vee$, if $\text{cdes}(w^{-1}) = k$, and $(\Delta_k^\Phi)_{(w)}$ is empty, if $\text{cdes}(w^{-1}) \neq k$.*

Proof. By definition, the polytope $(\Delta_k^\Phi)_{(w)}$ is given by

$$(\Delta_k^\Phi)_{(w)} = \{\lambda \in \Lambda \mid d_i(w^{-1}) = (\lambda, \alpha_i), \text{ for } i = 1, \dots, r; k - d_0(w^{-1}) = (\lambda, \theta)\}.$$

The first r equations $d_i(w^{-1}) = (\lambda, \alpha_i)$ have a single solution $\lambda = \sum_{i \in D} \omega_i = \delta_{w^{-1}}$, where $D = \{i \mid d_{\alpha_i}(w^{-1}) = 1\}$. The last equation $k - d_0(w^{-1}) = (\lambda, \theta^\vee)$ for the point $\lambda = \sum_{i \in D} \omega_i$ says that $\text{cdes}(w) = k$. \square

Corollary 9.2. *All representatives w of a coset $\bar{w} \in W/C$ have the same generalized descent numbers $\text{cdes}(w^{-1})$.*

Proof. The polytopes $(\Delta_k^\Phi)_{(u)} \equiv (\Delta_k^\Phi)_{(w)}$ are equivalent modulo affine translations, whenever $\bar{u} = \bar{w}$ in W/C . \square

Define $\text{cdes}(\bar{w}^{-1}) = \text{cdes}(w^{-1})$, where $w \in W$ is any representative of a coset \bar{w} . Theorem 8.2 implies the following statement.

Theorem 9.3. *The volume $\text{Vol}(\Delta_k^\Phi)$ of k -th generalized hypersimplex Δ_k^Φ equals the number of cosets $\bar{w} \in W/C$ such that $\text{cdes}(\bar{w}^{-1}) = k$. Equivalently, $f \cdot \text{Vol}(\Delta_k^\Phi)$ equals the number of elements $w \in W$ such that $\text{cdes}(w^{-1}) = k$.*

Let $\mathcal{H}(b_1, \dots, b_r; k, K)$ be the *thick hypersimplex* given by

$$\{\lambda \in V \mid 0 \leq (\lambda, \alpha_i) \leq b_i, \text{ for } i = 1, \dots, r; \text{ and } k \leq (\lambda, \theta) \leq K\}.$$

Proposition 9.4. *We have*

$$\text{Vol}(\mathcal{H}(b_1, \dots, b_r; k, K)) = \sum_l \text{Vol}(\Delta_l^\Phi) \cdot I(\mathcal{H}(b_1 - 1, \dots, b_r - 1; l - K + 1, l - k)).$$

Proof. Let $\lambda \in \Lambda^\vee$ be in the interior of Δ_l^Φ and $\mu \in \Lambda^\vee$. Then $0 < (\lambda, \alpha_i) < 1$ for $i \in [1, r]$ and $l - 1 < (\lambda, \theta) \leq l$. Thus $\lambda + \mu \in \mathcal{H}(b_1, \dots, b_r; k, K)$ if and only if we have $0 \leq (\mu, \alpha_i) \leq b_i - 1$ and $l - K + 1 \leq (\mu, \theta) \leq l - k$. We conclude that for any alcove $A \in \Delta_l^\Phi$, we have $\mathcal{H}(b_1, \dots, b_r; k, K)_{(A)} = \mathcal{H}(b_1 - 1, \dots, b_r - 1; l - K + 1, l - k) \bmod \Lambda^\vee$. As l varies, we obtain a translate of $\mathcal{H}(b_1, \dots, b_r; k, K)_{(\bar{w})}$ for each coset \bar{w} exactly once in this form. \square

10 A q -analogue of Weyl's formula

Recall that for a permutation $w = w_1 \dots w_n$ in the symmetric group S_n , a *descent* is an index $i \in \{1, \dots, n-1\}$ such that $w_i > w_{i+1}$. Let $\text{des}(w)$ be the number of descents of $w \in S_n$. The n -th *Eulerian polynomial* $A_n(q)$ is defined as

$$A_n(q) = \sum_{w \in S_n} q^{\text{des}(w)+1},$$

for $n \geq 1$, and $A_0(q) = 1$. These polynomials can also be expressed as $A_n(q) = (1 - q)^{n+1} \cdot \sum_{k \geq 0} k^n q^k$. Let $[n]_q = (1 - q^n)/(1 - q)$ denote the q -analogue of an integer $n \in \mathbb{Z}$.

The group algebra $\mathbb{Z}[\Lambda^\vee/L^\vee]$ has a \mathbb{Z} -basis of formal exponents e^x , for $x \in \Lambda^\vee/L^\vee$, with multiplication $e^x \cdot e^y = e^{x+y}$. Let $\mathbb{Z}[q][\Lambda^\vee/L^\vee] = \mathbb{Z}[q] \otimes \mathbb{Z}[\Lambda^\vee/L^\vee]$. The following theorem generalizes Weyl's formula (2) for the order of the Weyl group.

Theorem 10.1. *The following identity holds in the group algebra $\mathbb{Z}[q][\Lambda^\vee/L^\vee]$:*

$$\sum_{w \in W} q^{\text{cdes}(w)} e^{\text{cmaj}(w)} = \left(\sum_{x \in \Lambda^\vee/L^\vee} e^x \right) \cdot A_r(q) \cdot [a_1]_q \cdots [a_r]_q.$$

In particular, we have the following identity for polynomials in $\mathbb{Z}[q]$:

$$\sum_{w \in W} q^{\text{cdes}(w)} = f \cdot A_r(q) \cdot [a_1]_q \cdots [a_r]_q.$$

We first establish the following generating function for the volumes of generalized hypersimplices.

Proposition 10.2. *The generating function for the volumes of generalized hypersimplices is given by*

$$\sum_{k=1}^{h-1} \text{Vol}(\Delta_k^\Phi) q^k = A_r(q) \cdot [a_1]_q \cdots [a_r]_q.$$

Proof. The union of the generalized hypersimplices Δ_k^Φ for $k = 1, 2, \dots, h-1$ is the fundamental parallelepiped Π . For a bounded polytope $P \subset V$, define the generating function

$$g_P(q) = \sum_k \text{Vol}(P \cap \{\lambda \in V \mid k-1 \leq (\lambda, \theta) \leq k\}) q^k \in \mathbb{R}[q^{\pm 1}].$$

Then $g_\Pi(q) = \sum_{k=1}^{h-1} \text{Vol}(\Delta_k^\Phi) q^k$. We note that if $(\lambda, \theta) = a \in \mathbb{Z}$, then $g_{P+\lambda}(q) = q^a g_P(q)$. Now set Ξ to be the parallelepiped spanned by the vectors $\omega_1/a_1, \dots, \omega_r/a_r$. Then Π is a union of translates of Ξ by integral linear combinations of the vectors ω_i/a_i , and we deduce that

$$g_\Pi(q) = g_\Xi(q) \cdot [a_1]_q \cdots [a_r]_q.$$

Since we are normalizing the fundamental alcove A_\circ with vertices $\omega_1/a_1, \dots, \omega_r/a_r$ to have $\text{Vol}(A_\circ) = 1$, it follows that $g_\Xi(q)$ is equal to the generating function of the normalized volumes of the usual hypersimplices:

$$g_\Xi(q) = \sum_{k=1}^r \text{Vol}([0, 1]^r \cap \{(x_1, \dots, x_r) \in \mathbb{R}^r \mid k-1 \leq x_1 + \cdots + x_r \leq k\}) q^k$$

which is well known to be equal to the Eulerian polynomial $A_r(q)$. This also follows from Theorem 9.3 (see Section 11) and is studied in detail in [API]. \square

Proof of Theorem 10.1. Using Theorem 7.2, we let $W' = \{w \in W \mid \text{cmaj}(w) = \text{id}\}$ be a set of left coset representatives for $C \setminus W$. Then $(W')^{-1}$ is a set of right coset representatives for W/C . We calculate

$$\begin{aligned}
& \sum_{w \in W} q^{\text{cdes}(w)} e^{\text{cmaj}(w)} \\
&= \sum_{w \in W'} q^{\text{cdes}(w)} \sum_{c \in C} e^{\text{cmaj}(c)} && \text{by Theorems 6.5 and 7.2,} \\
&= \sum_{w \in (W')^{-1}} q^{\text{cdes}(w^{-1})} \cdot \left(\sum_{x \in \Lambda^\vee / L^\vee} e^x \right) \\
&= \left(\sum_{k=1}^{h-1} \text{Vol}(\Delta_k^\Phi) q^k \right) \cdot \left(\sum_{x \in \Lambda^\vee / L^\vee} e^x \right) && \text{by Theorem 9.3,} \\
&= \left(\sum_{x \in \Lambda^\vee / L^\vee} e^x \right) \cdot A_r(q) \cdot [a_1]_q \cdots [a_r]_q && \text{by Proposition 10.2.}
\end{aligned}$$

□

Remark 10.3. We have

$$\sum_{w \in W} q^{\text{cdes}(w)} e^{\text{cmaj}(w)} = \sum_{w \in W} q^{\text{cdes}(w)} e^{\text{cmaj}(w^{-1})}.$$

This follows from the fact (Theorem 6.5) that cdes is constant on $C \backslash W / C$ double cosets. Each double coset is a disjoint union of left (resp. right) cosets $C \backslash W$ (resp. W / C) for which $e^{\text{cmaj}}(w)$ (resp. $e^{\text{cmaj}}(w^{-1})$) takes the values $(\sum_{x \in \Lambda^\vee / L^\vee} e^x)$.

Remark 10.4. It would be interesting to compare Theorem 10.1 with Stembridge and Waugh's Weyl group identity [SW].

The following question seems interesting.

Question 10.5. What is $\sum_{w \in W} x^{\text{cmaj}(w)} y^{\text{cmaj}(w^{-1})}$ in $\mathbb{Z}[\Lambda^\vee / L^\vee] \otimes \mathbb{Z}[\Lambda^\vee / L^\vee]$?

11 Example: type A

Let $\Phi = A_{n-1} \subset \mathbb{R}^n / \mathbb{R}(1, 1, \dots, 1)$ throughout this section. The simple roots are $\alpha_i = e_i - e_{i+1}$ where e_i are the coordinate basis vectors of \mathbb{R}^n . The longest root is $\theta = e_1 - e_n$ and we have $a_i = 1$ for $i \in [0, n]$. The Weyl group $W = S_n$ is the symmetric group on n letters and $\text{cdes}(w)$ is equal to the usual number of descents of w plus a descent at n if $w_n > w_1$. This is the reason for calling cdes the circular descent number. The group $C = \langle c = (123 \cdots (n-1)n) \rangle$ is generated by the long cycle. The fundamental coweights are given by $\omega_i = e_1 + e_2 + \cdots + e_i$ and one can check that $\delta_{c,i} = \omega_i$. Thus $\text{cmaj}(w) = c^{-\text{maj}(w) \bmod n}$ where $\text{maj}(w)$ denotes the usual major index of w . We can verify Proposition 7.2 directly: left multiplication by the long cycle c maps $w_1 w_2 \cdots w_n$ to $(w_1 + 1)(w_2 + 1) \cdots (w_n + 1)$ where ' $n + 1$ ' is identified with '1'. Right multiplication by c maps $w_1 w_2 \cdots w_n$ to $w_2 w_3 \cdots w_n w_1$.

The following theorem ([KW] and [EC2, Ex. 7.88]) suggests that the map cmaj may have an explicit representation theoretic interpretation. Let χ^λ denote the irreducible character of the symmetric group S_n labeled by a partition λ .

Theorem 11.1. *Let $C_n \subset S_n$ be a cyclic subgroup of order n . Let $\rho = \text{ind}_{C_n}^{S_n} e^{2\pi\sqrt{-1}j/n}$ be an induced character of S_n . Then we have*

$$\langle \rho, \chi^\lambda \rangle = \#\{\text{SYT}(T) \mid \text{sh}(T) = \lambda \text{ and } \text{maj}(T) \equiv j \pmod{n}\}.$$

Here a descent of a standard young tableau (SYT) T is an index i such that the box containing $i+1$ is to the southwest of the box containing i in T . The index $\text{maj}(T)$ is defined to be the sum of all the descents of T .

It is not hard to see that the polytopes $\Delta_k^{A_{n-1}}$ are affinely equivalent to the usual hypersimplices defined as the convex hull of the points ε_I where $\varepsilon_I = \sum_{i \in I} e_i$ and I varies over all k -subsets of $[n]$. The alcoved triangulation here is identical to the one studied in [API].

12 Example: Type C

Let $\Phi = C_n$ with $2n$ long roots $\pm 2e_i$ for $1 \leq i \leq n$ and $2n(n-1)$ short roots $\pm e_i \pm e_j$ for $1 \leq i \neq j \leq n$. A system of simple roots is given by $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n$. Then $\theta = 2e_1 = 2\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n$, so that $a_0 = a_n = 1$ and $a_i = 2$ for $1 \leq i \leq n-1$. The fundamental coweights are given by $\omega_1 = e_1, \omega_2 = e_1 + e_2, \dots, \omega_{n-1} = e_1 + \dots + e_{n-1}, \omega_n = 1/2(e_1 + \dots + e_n)$.

We identify the Weyl group W of type C_n with the group of signed permutations $w_1 w_2 \cdots w_n$ in the usual way: $w_i \in \pm\{1, 2, \dots, n\}$ and $|w_1||w_2| \cdots |w_n|$ is a usual permutation in S_n . For $i \in [1, n-1]$, a signed permutation $w = w_1 w_2 \cdots w_n$ has a descent at i if $w_i > w_{i+1}$, as usual. We have a descent at 0 if $w_1 > 0$ and a descent at n if $w_n < 0$. The group C has order two, with unique non-identity element $c = (-n - (n-1) - \dots - 2 - 1)$. The map $\text{cmaj} : W \rightarrow C$ is given by

$$\text{cmaj}(w) = \begin{cases} \text{id} & \text{if } w_n > 0 \\ c & \text{if } w_n < 0. \end{cases}$$

Theorem 10.1 states in this case that

$$\sum_{w \in W} q^{\text{cdes}(w)} e^{\text{cmaj}(w)} = (e^{\text{id}} + e^c) \cdot A_n(q) \cdot (1+q)^{n-1}.$$

13 Gröbner bases

In this section we will regularly refer to the results of the first paper in this series [API], in particular Appendix 19.

Let $\Phi \subset V$ be a fixed irreducible root system and P a proper alcoved polytope. We first note that the triangulations of alcoved polytopes are coherent.

Lemma 13.1. *Any polytopal subdivision arising from a hyperplane arrangement is coherent.*

Proof. As only finitely many hyperplanes will be involved in a triangulation or subdivision of a polytope, we may assume the set \mathcal{S} of hyperplanes is finite.

Pick a linear functional ϕ_H for each hyperplane $H \in \mathcal{S}$ such that H is given by the vanishing of ϕ_H . Then define the piecewise linear convex function $h : V \rightarrow \mathbb{R}$ by

$$h(v) = \sum_{H \in \mathcal{S}} |\phi_H(v)|.$$

It is clear that $h(v)$ is convex, since it is a sum of convex functions. The domains of linearity are exactly the regions determined by the hyperplane arrangement. Thus the subdivision of a polytope induced by a hyperplane arrangement is coherent. \square

We denote by N the set of vertices of the affine Coxeter arrangement. By [API, Theorem 19.1] the triangulation of P can be described by some appropriate term order on the polynomial ring

$$k[P] = k[x_a | a \in N \cap P].$$

Let us fix coordinates on V , so that all points in N have integer coordinates. Identify a vertex $a = (a_1, \dots, a_n) \in N$ with the coordinates $(a, 1) \in V \oplus \mathbb{R}$. Thus the triangulation is also equivalent to the reduced Gröbner basis \mathcal{G}_P of the toric ideal $I_P := I_{P \cap N}$ in the notation of [API, Appendix 19]. By our choice of coordinates this toric ideal is homogeneous.

In general, the Gröbner basis \mathcal{G}_P appears to be quite complicated, but many simplifications occur when N is a lattice. One can check directly that this is the case for the root systems A_n , C_n and D_4 .

We assume that Φ is one of these root systems from now on. Set $c_i = \frac{a_i}{d_i}$. Then N is spanned by the c_i . In this case, an alcove has normalized volume 1 with respect to N . Thus by [API, Proposition 19.2], \mathcal{G}_P has an initial ideal generated by square-free monomials.

Example 13.2. With the notation as in Section 12, the vertices N of the affine Coxeter arrangement of type C_n are exactly the points with all coordinates, either integers or half-integers. One can check that the lattice N is spanned by the vectors c_i .

Lemma 13.3. *Let $a, b \in N$. The midpoint $(a + b)/2$ is either a vertex or it lies on a unique edge, such that it is the midpoint of the two closest vertices lying on that edge.*

Proof. Suppose $c = (a + b)/2$ is not in N . The closed fundamental alcove $\overline{A_\circ}$ is the convex hull of the points c_i and 0. Thus there is a (affine) Weyl group element σ which takes the midpoint $c = (a + b)/2$ into $\overline{A_\circ}$. Since $\sigma(a)$ and $\sigma(b)$ are both integral linear combinations of the c_i , it is clear that $\sigma(c)$ must have the form $\frac{c_i + c_j}{2}$ or $\frac{c_i}{2}$. In the first case, $\sigma(c)$ lies on the edge given by the intersection of the hyperplanes $H_{\alpha_k, 0}$ for $k \neq i, j$ satisfying $k \in [1, r]$ and $H_{\theta, 1}$. In the second case the edge is given by the hyperplanes $H_{\alpha_k, 0}$ for $k \neq i$. Thus c is the midpoint of $\sigma^{-1}(c_i)$ and $\sigma^{-1}(c_j)$, or the midpoint of $\sigma^{-1}(c_i)$ and $\sigma^{-1}(0)$. \square

In the first case of Lemma 13.3, we set $u(a, b) = v(a, b) = (a + b)/2$. In the second case we set $u(a, b)$ and $v(a, b)$ to be the two closest vertices on the edge containing $(a + b)/2$.

Example 13.4. For type A_{n-1} , we can describe the vertices $u(a, b)$ and $v(a, b)$ in the following explicit manner ([API]). Let I, J be two k -element multi-subsets of $[n]$. Let $a_1 \leq a_2 \leq \dots \leq a_{2k}$ be the increasing rearrangement of $I \cup J$. We define two k -element multi-subsets $U(I, J)$ and $V(I, J)$ by $U(I, J) = \{a_1, a_3, \dots, a_{2k-1}\}$ and $V(I, J) = \{a_2, a_4, \dots, a_{2k}\}$. For a k -element multi-subset I , we let $a_I \in \mathbb{R}^n$ be the (integer) vector with j -th coordinate $(a_I)_j$ equal to the number of occurrences of $\{1, 2, \dots, j\}$ in I . Then one can check that $u(a_I, a_J)$ and $v(a_I, a_J)$ are exactly $a_{U(I, J)}$ and $a_{V(I, J)}$.

Lemma 13.5. *Suppose $a, b \in P$ are vertices of the affine Coxeter arrangement, where P is a proper alcoved polytope. Then the vertices $u(a, b)$ and $v(a, b)$ are also in P .*

Proof. As P is convex, $c = (a + b)/2 \in P$. Assume now that c is not a vertex and suppose $u(a, b) \notin P$. Then there exists some root α and some integer k , so that $H_{\alpha, k}$ separates $u(a, b)$ and c . Here we pick $H_{\alpha, k}$ so that it may go through c but not through $u(a, b)$. The intersection of $H_{\alpha, k}$ and the edge segment joining $u(a, b)$ to $v(a, b)$ is a vertex of the affine Coxeter arrangement. But this is impossible, since there are no vertices lying between $v(a, b)$ and $u(a, b)$. \square

Define a marked set \mathcal{G}_P of elements which lie in I_P as follows:

$$\mathcal{G}_P = \{x_{\underline{a}x_b} - x_{u(a, b)}x_{v(a, b)}\}, \quad (8)$$

where a, b range over pairs of unequal vertices in P . The main result of this section is the following theorem.

Theorem 13.6. *Let Φ be one of the root systems A_n , C_n or D_4 and P a proper alcoved polytope. Then there exists a term order \prec_P such that the quadratic binomials \mathcal{G}_P form a (reduced) Gröbner basis of the toric ideal I_P with respect to \prec_P , such that the underlined monomial is the leading term.*

Proof. By Lemma 13.5, the binomials in \mathcal{G}_P do indeed make sense and since $a + b = u(a, b) + v(a, b)$, they lie within I_P .

By Lemma 13.1, the triangulation is coherent and is given by the domains of linearity of the piecewise-linear function h . The same function h gives a weight vector ω as described in [API, Appendix]. By [API, Theorem A.1] the weight vector ω induces a term order \prec_P such that $\Delta_{\prec_P}(I_P) = \Delta_\omega$ (we have also used the fact that the triangulation is unimodular, and [API, Proposition A.2]).

Now let $a, b \in P$ be vertices of the affine arrangement. If $x_a x_b \neq x_{u(a, b)} x_{v(a, b)}$, then clearly a and b do not belong to the same simplex of the triangulation of P . Thus $x_a x_b$ belongs to the Stanley-Reisner ideal of the alcoved triangulation of P while $x_{u(a, b)} x_{v(a, b)}$ does not. This implies that under \prec_P , the underlined terms in the basis above are exactly the leading terms. In other words, the set \mathcal{G}_P is *marked coherently*.

Finally we check that \mathcal{G}_P is indeed a Gröbner basis of I_P under \prec . Since \mathcal{G}_P is marked coherently, it follows that the reduction of any polynomial modulo \mathcal{G}_P is Noetherian (that is, it terminates). It is clear that a monomial $x_{p_1} \dots x_{p_k}$ cannot be reduced further under \mathcal{G}_P if and only if all the p_i belong to the simplex of the triangulation. Thus every monomial can be reduced via \mathcal{G}_P to a standard monomial and hence $\text{in}_\prec(\mathcal{G}_P)$ generates $\text{in}_\prec(I_P)$.

The fact that this Gröbner basis is reduced is clear. \square

Corollary 13.7. *Let Φ be one of the root systems A_n , C_n or D_4 and $P \subset V$ be a convex polytope with vertices among the vertices of the affine Coxeter arrangement. Then P is alcoved if and only if the conclusion of Lemma 13.5 holds.*

Proof. “Only if” is the content of Lemma 13.5. For the other direction, we note that the quadratic binomials \mathcal{G}_P can be defined by (8). There is some large alcoved polytope Q which contains P and since $\mathcal{G}_P \subset \mathcal{G}_Q$, this allows us to conclude that \mathcal{G}_P is marked coherently. And so there is a term order \prec_P which selects the marked monomial as the leading monomial in \mathcal{G}_P . It is easy to check that \mathcal{G}_P is the Groebner basis of I_P with respect to \prec_P , and the standard monomials are exactly given by monomials corresponding to faces of alcoves. Thus we obtain an alcoved triangulation of P . \square

Naturally associated to the ideal I_P is the projective algebraic variety Y_P defined as

$$Y_P = \text{Proj}(k[x_a | a \in \mathcal{P}]/I_P).$$

This is the projective toric variety associated to the polytope P . The following corollary is immediate from Theorem 13.6 and [API, Proposition A.2].

Corollary 13.8. *Let Φ be one of the root systems A_n , C_n or D_4 and P a proper alcoved polytope. Let Y_P be the projective toric variety defined by I_P . Then Y_P is projectively normal and its Hilbert polynomial is equal to the Ehrhart polynomial of P (with respect to N).*

This should be compared with the work of Payne [Pay], who showed, in types A , B , C , and D that alcoved polytopes whose vertices lie in the coweight lattice are normal with respect to the coweight lattice.

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