Cooling Codes: Thermal-Management Coding for High-Performance Interconnects

Yeow Meng Chee, *Senior Member, IEEE*, Tuvi Etzion, *Fellow, IEEE*, Han Mao Kiah, and Alexander Vardy, *Fellow, IEEE*

Dedicated to the memory of Solomon W. Golomb (1932–2016)

***Abstract*— High temperatures have dramatic negative effects on interconnect performance and, hence, numerous techniques have been proposed to reduce the power consumption of on-chip buses. However, existing methods fall short of fully addressing the thermal challenges posed by high-performance interconnects. In this paper, we introduce new efficient coding schemes that make it possible to directly control the *peak temperature* of a bus by effectively cooling its hottest wires. This is achieved by avoiding state transitions on the hottest wires for as long as necessary until their temperature drops off. We also reduce the *average power consumption* by making sure that the total number of state transitions on all the wires is below a prescribed threshold. We show how each of these two features can be coded for separately or, alternatively, how both can be achieved at the same time. In addition, *error-correction* for the transmitted information can be provided while controlling the peak temperature and/or the average power consumption. In general, our cooling codes use $n > k$ wires to encode a given k-bit bus. One of our goals herein is to determine the minimum possible number of wires n needed to encode k bits while satisfying any combination of the three desired properties. We provide full theoretical analysis in each case. In particular, we show that $n = k+t+1$ suffices to cool the t hottest wires, and this is the best possibility. Moreover, although the proposed coding schemes make use of sophisticated tools from combinatorics, discrete geometry, linear algebra, and coding theory, the resulting encoders and decoders are fully practical. They do not require significant computational overhead and can be implemented without sacrificing a large circuit area.**

***Index Terms*— Cooling codes, low-power codes, partial spreads, set systems, thermal-management coding.**

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Y. M. Chee and H. M. Kiah are with the School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 639798 (e-mail: ymchee@ntu.edu.sg; hmkiah@ntu.edu.sg).

T. Etzion was with the School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore. He is now with the Computer Science Department, Technion–Israel Institute of Technology, Haifa 32000, Israel (e-mail: etzion@cs.technion.ac.il).

A. Vardy was with the School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore. He is now with the Department of Electrical and Computer Engineering and the Department of Computer Science and Engineering, University of California at San Diego, San Diego, CA 92093 USA (e-mail: avardy@ucsd.edu).

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I. Introduction

POWER and heat dissipation limits have emerged as a first-order design constraint for chips, whether targeted for battery-powered devices or for high-end systems. With the migration to process geometries of 65 nm and below, power dissipation has become as important an issue as timing and signal integrity. Aggressive technology scaling results in smaller feature size, greater packing density, increasing microarchitectural complexity, and higher clock frequencies. This is pushing chip level power consumption to the edge. It is not uncommon for on-chip hot spots to have temperatures exceeding 100°C, while inter-chip temperature differentials often exceed 20°C.

Power-aware design alone is not sufficient to address this thermal challenge, since it does not directly target the spatial and temporal behavior of the operating environment. For this reason, thermally-aware approaches have emerged as one of the most important domains of research in chip design today.

High temperatures have dramatic negative effects on circuit behavior, with interconnects being among the most impacted circuit components. This is due, in part, to the ever decreasing interconnect pitch and the introduction of low-k dielectric insulation which has low thermal conductivity. For example, as shown in [3], the Elmore delay [15] of an interconnect increases 5% to 6% for every 10°C increase in temperature, whereas the leakage current grows exponentially with temperature. Therefore, minimizing the temperature of interconnects is of paramount importance for thermally-aware design.

A. Related Work

Numerous encoding techniques have been proposed in the literature [4], [9], [10], [31], [38], [43], [44], [46], [48], [49], [54], [55] in order to reduce the overall power dissipation consumption of both on-chip and off-chip buses. It is known to be of importance from practical point of view for over thirty years and optimization of the related integrated circuits were considered by the electric companies [20]. It is well established [11], [34], [45], [50], [54], [55] that bus power is directly proportional to the product of line capacitance and the average number of signal state transitions on the bus wires. Thus the general idea is to encode the data transmitted over the bus so as to reduce the average number of transitions. For example, the "bus-invert" code of [49] potentially complements the data on all the wires, according to the Hamming distance between consecutive transmissions, thereby ensuring that the total number of state transitions on n bus wires never exceeds $n/2$. Unfortunately, encoding techniques designed to minimize power consumption, do not directly address peak temperature minimization. In order to reduce the temperature of a wire, it is not sufficient to minimize its average switching activity. Rather, it is necessary to control the *temporal distribution of the state transitions* on the wire. To reduce the peak temperature of an interconnect, it is necessary to exercise such control for *all* of its constituent wires.

In [54], [55] the authors propose a *thermal spreading* approach. They present an efficient encoding scheme that evenly spreads the switching activity among all the bus wires, using a simple architecture consisting of a shift-register and a crossbar logic. This is designed to avoid the situation where a few wires get hot while the majority are at a lower temperature. This spreading approach is further extended in [9] and [43] using on-line monitoring of the switching activity on all the wires. Thermal spreading can be regarded as an attempt to control peak temperature indirectly, by equalizing the distribution of signal transitions over all the wires.

Finally, analysis from information theory point of view, which is highly related to our work, including solutions with data compression are given in several papers, e.g. [30], [40], [47].

B. Our Contributions

As technology continues to scale, existing methods may fall short of fully addressing the thermal challenges posed by high-performance interconnects in deep submicron (DSM) circuits. In this paper, we introduce new efficient coding schemes that simultaneously control both the *peak temperature* and the *average power consumption* of interconnects.

The proposed coding schemes are distinguished from existing state-of-the-art by having some or all of the following features:

A. *We directly control the peak temperature of a bus by effectively cooling its hottest wires. This is achieved by avoiding state transitions on the hottest wires for as long as necessary until their temperature decreases.*

B. *We reduce the overall power dissipation by guaranteeing that the total number of transitions on the bus wires is below a specified threshold in every transmission.*

C. *We combine properties* **A** *and/or* **B** *with coding for improved reliability (e.g., for low-swing signaling), using existing error-correcting codes.*

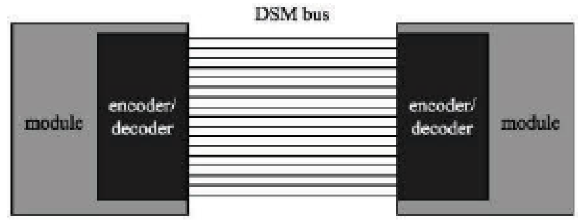

Fig. 1. Block diagram of the proposed bus architecture.

To achieve these desirable features, we propose to insert at the interface of the bus specialized circuits implementing the encoding and decoding functions, denoted herein by $\mathcal{E}$ and $\mathcal{D}$, respectively. This is illustrated in Figure 1. The various coding schemes introduced in this paper employ tools from various fields such as combinatorics, graph theory, block designs, discrete geometry, linear algebra, and the theory of error-correcting codes. Nonetheless, in each case the *resulting encoders/decoders $\mathcal{E}$ and $\mathcal{D}$ are efficient:* they do not require significant computational overhead and can be implemented without sacrificing a large circuit area. This is especially true for Property **A**, where the complexity of encoding and decoding scales linearly with the number of wires.

We consider both adaptive and nonadaptive (memoryless) coding schemes. The advantage of nonadaptive schemes is that they are easier to implement and do not require memory. The disadvantage is that it is not possible to implement Property **A** with nonadaptive encoding. For this reason, most of the coding schemes developed in this paper will be adaptive, based on the idea of *differential encoding*. Notably, however, all of our schemes require the encoder and decoder circuits to keep track of *only one* (the most recent) previous transmission.

Unlike the thermal spreading methods of [43], [54], and [55] that lead to irredundant coding schemes, the solutions we propose do introduce redundancy: we require $n > k$ wires to encode a given k-bit bus. A key consideration in this situation is the *area overhead due to the additional $n - k$ wires*. Therefore, it is important to determine the theoretically minimum possible number of wires n needed to encode k bits while satisfying the desired properties. We provide full theoretical analysis in each case. We moreover show that the number of additional wires required to satisfy Property **A** becomes negligible when k is large.

C. Thermal Model

Chiang *et al.* [11] came up with an analytic model that characterizes thermal effects due to Joule heating in high-performance Cu/low-k interconnects, under both steady-state and transient stress conditions. Shortly thereafter, Sotiriadis and Chandrakasan [45] gave a power dissipation model for DSM buses. These two models, accounting for thermal and power effects separately, were later unified and refined by Sundaresan and Mahapatra [50]. Finally, building upon this work, Wang, *et al.* [55] proposed a more accurate thermal-and-power model for DSM buses. In all these

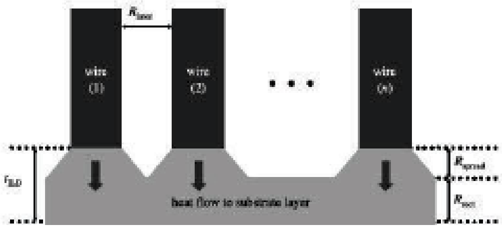

Fig. 2. Geometry used for calculating R_{spread}, R_{rect}, and R_{inter}.

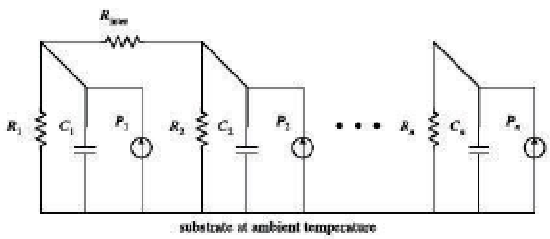

Fig. 3. Equivalent thermal RC-network for a n-bit bus.

papers, an n-bit bus (illustrated in Figure 2) is modeled in terms of the equivalent thermal-RC network in Figure 3. Sundaresan and Mahapatra [50] show that this thermal-RC network is governed by the following differential equations:

$$P_1 = C_1 \frac{\partial \theta_1}{\partial t} + \frac{\theta_1 - \theta_0}{R_1} + \frac{\theta_1 - \theta_2}{R_{\text{inter}}}, \quad (1)$$

$$P_n = C_n \frac{\partial \theta_n}{\partial t} + \frac{\theta_n - \theta_0}{R_n} + \frac{\theta_n - \theta_{n-1}}{R_{\text{inter}}}, \quad (2)$$

and

$$P_i = C_i \frac{\partial \theta_i}{\partial t} + \frac{\theta_i - \theta_0}{R_i} + \frac{2\theta_i - \theta_{i-1} - \theta_{i+1}}{R_{\text{inter}}}, \quad (3)$$

for $i = 2, 3, \ldots, n-1$, where P_i is the instantaneous power dissipated by wire i, C_i is the thermal capacitance per unit length of wire i, $R_i = R_{\text{spread}} + R_{\text{rect}}$ is the thermal resistance per unit length of wire i along the heat transfer path downwards, R_{inter} is the lateral thermal resistance used to account for the parallel thermal coupling effect between the wires, θ_i is the temperature of wire i, and θ_0 is the substrate ambient temperature.

In any bus model for which (1)–(3) hold, the temperature of a wire will increase whenever the wire undergoes a state transition; conversely, in the absence of state transitions, the temperature will gradually decrease. We let σ_i denote the switching activity of wire i, which is the number of times the wire changes state. Then the power dissipated by a bus is determined by its *total switching activity* $\sigma_1 + \sigma_2 + \cdots + \sigma_n$.

In order to directly control the peak temperature of a bus by avoiding transitions on its hottest wires, we need to know *which wires are the hottest* at every transmission. There are two general ways to obtain this information. We can use an analytical model [50], such as (1)–(3), to estimate the current temperatures of the wires. For each wire, such an estimate can be implemented with a counter that is incremented on transition and decremented on non-transition, where the precise magnitude of the increments/decrements is determined by the model. Alternatively, we can have actual temperature sensors for each wire. For DSM buses, accurate temperature sensing can be implemented using, for example, ring oscillators [13]. As shown in [13], sensors based on ring oscillators provide a resolution of 1°C while consuming an active power of only 65–112μW.

D. Organization

The rest of this paper is organized as follows. We begin in the next section with a precise formulation of the coding problems that result from the thermal-management features we propose to implement — namely, Properties **A**, **B**, and **C**. In Section III, we present a nonadaptive coding scheme that combines Property **B** (reducing the average power dissipation) with the thermal spreading approach of [55]. Our constructions in Section III are based on the notions of *anticodes* and *quorum systems*, and use key results from the theory of combinatorial designs. Section IV is devoted to Property **A**: we show how state transitions on the t hottest wires can be avoided by using only $t+1$ additional bus lines. This optimal construction is based on combining *differential coding* with the notion of *spreads* and *partial spreads* in projective geometry. The optimal construction can be applied when $t+1 \leqslant (n+1)/2$. When $t+1 > (n+1)/2$ we use another technique from the theory of error-correcting codes to construct efficient codes. The designed codes can be viewed as sunflowers, while the partial spreads, are also sunflowers, and hence can be also viewed as a special case of these codes. The technique used is generalized with the notion of generalized Hamming weights. In Section V, we show how Properties **A** and **B** can be all achieved *at the same time*. That is, we design coding schemes that simultaneously control peak temperature and average power consumption in every transmission. For this purpose, we present three types of constructions. The first construction is based upon the *Baranyai theorem* on complete hypergraph decomposition into pairwise disjoint perfect matchings. The second construction is based on concatenation of low weights codes based on appropriate non-binary dual codes or non-binary partial spreads. The third construction is the previous sunflower construction, which also satisfy Property **B**. Section VI is devoted to codes which satisfy Property **C** simultaneously with either Property **A** or Property **B** or both, i.e. we add also correction for possible transmission errors on the bus wires. The constructions in this section will also be of two types. The first type of constructions is based on resolutions in block design. The second type of constructions will be to employ the previously given constructions, where our set of transitions is restricted to the set of codewords in a given error-correcting code. In all these sections our bounds and constructions are applied for infinite sets of parameters, but there is no asymptotic analysis in any of these cases. The asymptotic analysis is postponed to Section VII, where the asymptotic behavior of our codes is analyzed. In particular we analyze area overhead of our constructions and prove that when k is large enough, the additional number of wires required to satisfy the desired properties is negligible. Finally, Section VIII summarizes our comprehensive work and presents a brief discussion of possible directions for future research. We would like to remark that this work is mainly of theoretical nature. A follow up work will present more constructions, especially for practical parameters. Examples and numerical experiments will be given and also comparison between the various constructions with emphasis on practical parameters. It will be illustrated and discussed how practical our methods and constructions are.

II. Problem Formulation and Preliminaries

Let us now elaborate upon Properties **A**, **B**, **C** introduced in the previous section. For each of these properties, we will characterize the performance of the corresponding coding scheme by a *single integer parameter*. All of our coding schemes will use $n > k$ wires to encode a k-bit bus. We assume that communication across the bus is synchronous, occurring in clocked cycles called *transmissions*. This leads to the following definition.

Definition 1: Consider a coding scheme for communication over a bus consisting of n wires. Let t, w, e be positive integers less than n. We say that the coding scheme has

Property A(t): *if every transmission does not cause state transitions on the t hottest wires;*

Property B(w): *if the total number of state transitions on all the wires is at most w, in every transmission;*

Property C(e): *if up to e transmission errors (*0 *received as* 1*, or* 1 *received as* 0*) on the n wires can be corrected.*

We presume that, at the time of transmission, it is known which t wires are the hottest; Property **A**(t) *is required to hold assuming that* ***any*** *t wires can be designated as the hottest.*

The values of t, w, e are *design parameters*, to be determined by the specific thermal requirements of specific interconnects. The proposed coding schemes will work for various values of t, w, and e. Nevertheless, it might be helpful to think of t as a small constant, since significant reductions in the peak temperature can be achieved by cooling only a few of the hottest wires. Thus, the most important values of t are small ones, say, less than half of the bus wires. But, our constructions in the following sections will consider also solutions for large values of t, specifically, any value of t. Similarly, w is also usually small since large w means a large number of state transitions on all the wires, which might result in too many hot wires. Finally, we also expect e to be small, especially as it must be smaller than $w/2$ as otherwise we won't be able to correct the errors.

Codes which satisfy Properties **A**(t),**B**(w), and **C**(e) simultaneously in every transmission, will be called (n, t, w, e)***-low-power error-correcting cooling codes*** or (n, t, w, e)-LPECC codes for short. When a nonempty meaningful subset of the three properties (property **C**(e) is not interesting alone in our context) will be satisfied, only the parameters and description related to this subset of properties will remain in the name. For example, (n, t, e)-LPEC codes stands for (n, t, e)-low-power error-correcting codes. Six such nonempty subsets exist and for each one we suggest coding schemes and related codes. It the conclusion of Section VIII a pointer will be given, where each one of these six subsets was considered.

We view the collective state of the n wires during each transmission as a binary vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. The set of all such binary vectors is the *Hamming n-space* $\mathcal{H}(n) = \{0, 1\}^n$. We will identify $\mathcal{H}(n)$ with the vector space $\mathbb{F}_2^n$. Given any $\mathbf{x}, \mathbf{y} \in \mathbb{F}_2^n$, the *Hamming distance* $d(\mathbf{x}, \mathbf{y})$ is the number of positions where $\mathbf{x}$ and $\mathbf{y}$ differ. The *Hamming weight* of a vector $\mathbf{x} \in \mathbb{F}_2^n$, denoted $\mathrm{wt}(\mathbf{x})$, is the number of nonzero positions in $\mathbf{x}$.

Conventionally, a binary *code* $\mathbb{C}$ *of length* n is simply a subset of $\mathbb{F}_2^n$. The elements of $\mathbb{C}$ are called *codewords*. Given a code $\mathbb{C}$, its *minimum distance* $d(\mathbb{C})$ and *diameter* $\mathrm{diam}(\mathbb{C})$ are defined as follows:

$$d(\mathbb{C}) \stackrel{\text{def}}{=} \min_{\mathbf{x},\mathbf{y}\in\mathbb{C}} d(\mathbf{x}, \mathbf{y}) \quad \text{and} \quad \mathrm{diam}(\mathbb{C}) \stackrel{\text{def}}{=} \max_{\mathbf{x},\mathbf{y}\in\mathbb{C}} d(\mathbf{x}, \mathbf{y}).$$

Later, in Sections IV and V, we will need to modify and generalize this conventional definition of binary codes in an important way. This modification will be needed for codes which satisfy Property **A**(t).

III. Nonadaptive Low-Power Codes

The encoding schemes considered in this section belong to the *nonadaptive* kind, in that the choice which codeword to transmit across the bus in the current transmission does not depend on codewords that have been transmitted earlier. Such coding schemes are also known as *memoryless*. The advantage of nonadaptive schemes is that they are simpler to implement: they do not need a continuously changing data model, and they do not require memory to track the history of previous transmissions.

In the nonadaptive case, an *n-bit coding scheme* for a source $\mathcal{S} \subseteq \mathbb{F}_2^k$ is a triple $\mathcal{E} = \langle \mathbb{C}, \mathcal{E}, \mathcal{D} \rangle$, where

1) $\mathbb{C}$ is a binary code of length n,
2) $\mathcal{E}: \mathcal{S} \to \mathbb{C}$ is a bijective map called an *encoding function*,
3) $\mathcal{D}: \mathbb{C} \to \mathcal{S}$ is a bijective map called a *decoding function*, such that $\mathcal{D}\big(\mathcal{E}(\boldsymbol{u})\big) = \boldsymbol{u}$ for all $\boldsymbol{u} \in \mathcal{S}$.

Encoding and decoding circuits that implement $\mathcal{E}$ and $\mathcal{D}$ are inserted at the interface of the bus (see Figure 1).

Suppose $\boldsymbol{u}, \mathbf{v} \in \mathcal{S}$ are two words that are to be communicated across the bus during consecutive transmissions. In the absence of a coding scheme, the total switching activity of the bus is then given by $|\{i : u_i \neq v_i\}|$. This is precisely the Hamming distance $d(\boldsymbol{u}, \mathbf{v})$, which could be as high as k. If an n-bit coding scheme is used, then $\mathbf{x} = \mathcal{E}(\boldsymbol{u})$ and $\mathbf{y} = \mathcal{E}(\mathbf{v})$ are transmitted instead. The resulting total switching activity of the bus is therefore $d(\mathbf{x}, \mathbf{y})$, which is upper bounded by $\mathrm{diam}(\mathbb{C})$.

It follows that the coding scheme *satisfies Property* **B**(w) *if and only if* $\mathrm{diam}(\mathbb{C}) \leqslant w$. As the power consumption of a bus is directly related to its total switching activity, we call such a code $\mathbb{C}$ an (n, w)***-low-power code*** ((n, w)-LP code for short).

In this section, we are interested in (n, w)-LP codes that also achieve low peak temperatures by *spreading the switching activities* among the bus wires as uniformly as possible. In doing so, we are following the analysis of [10], [54], and [55] and the resulting *thermal spreading* approach [10], [43], [54]. In order to quantify the thermal spreading achieved by a given coding scheme $\mathcal{E} = \langle \mathbb{C}, \mathcal{E}, \mathcal{D} \rangle$, let us treat the source $\mathcal{S}$ as a random variable taking on values in $\mathbb{F}_2^k$, and assume that $\mathcal{S}$ is uniformly distributed. This is a common assumption in bus analysis — see, for example, [45]. With this assumption, the expected switching activity of wire i is given by

$$\mu_i = \frac{1}{|\mathcal{S}|^2} \sum_{\boldsymbol{u},\boldsymbol{v} \in \mathcal{S}} \left| \mathcal{E}(\mathbf{u})_i - \mathcal{E}(\mathbf{v})_i \right| = \frac{2r_i(|\mathbb{C}| - r_i)}{|\mathcal{S}|^2} \qquad (4)$$

where r_i is the number of codewords $(x_1, x_2, \ldots, x_n) \in \mathbb{C}$ such that $x_i = 1$. If $\mu_1, \mu_2, \ldots, \mu_n$ are all equal, we say that the code $\mathbb{C}$ is ***thermal-optimal***, since the expected switching activities of the bus wires are then uniformly distributed. This leads to the following problem:

$$\text{Given } n \text{ and } w\text{, determine the maximum size of a thermal-optimal } (n, w)\text{-low-power code} \qquad (5)$$

The size of $\mathbb{C}$ is important because we wish to minimize the *area overhead* introduced by our coding scheme. This overhead is largely determined by the number $n - k$ of additional wires that we need to encode a given source $\mathcal{S} \subseteq \mathbb{F}_2^k$. Clearly, to encode such a source, we need a code $\mathbb{C}$ with $|\mathbb{C}| \geqslant 2^k$.

It is easy to see from (4) that $\mu_1, \mu_2, \ldots, \mu_n$ are all equal if and only if $r_1, r_2, \ldots, r_n$ are all equal. Hence in a thermal-optimal code $\mathbb{C}$, the number of codewords $(x_1, x_2, \ldots, x_n) \in \mathbb{C}$ having $x_i = 1$ is the same for all i. Such codes are said to be ***equireplicate*** in the combinatorics literature. To construct such codes, we will need tools from the theory of set systems as was suggested by Chee *et al.* [10].

A. Set Systems

Given a positive integer n, the set $\{1, 2, \ldots, n\}$ is abbreviated as $[n]$. For a finite set X and $k \leqslant |X|$, we define

$$2^X \stackrel{\text{def}}{=} \left\{ A : A \subseteq X \right\} \text{ and } \binom{X}{k} \stackrel{\text{def}}{=} \left\{ A \in 2^X : |A| = k \right\}$$

A *set system of order n* is a pair $(X, \mathcal{A})$, where X is a finite set of *n points* and $\mathcal{A} \subseteq 2^X$. The elements of $\mathcal{A}$ are called *blocks*. A set system $(X, 2^X)$ is a *complete set system*. The *replication number* of $x \in X$ is the number of blocks containing x. A set system is *equireplicate* if its replication numbers are all equal.

There is a natural one-to-one correspondence between the Hamming n-space $\mathbb{F}_2^n$ and the complete set system $([n], 2^{[n]})$ of order n. For a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_2^n$, the *support of* $\mathbf{x}$ is defined as

$$\text{supp}(\mathbf{x}) \stackrel{\text{def}}{=} \left\{ i \in [n] : x_i \neq 0 \right\}$$

With this, the positions of vectors in $\mathbb{F}_2^n$ correspond to points in $[n]$, each vector $\mathbf{x} \in \mathbb{F}_2^n$ corresponds to the block supp($\mathbf{x}$), and $d(\mathbf{x}, \mathbf{y}) = |\operatorname{supp}(\mathbf{x}) \triangle \operatorname{supp}(\mathbf{y})|$, where $\triangle$ stands for the symmetric difference. It follows from the above that there is a 1-1 correspondence between the set of all codes of length n and the set of all set systems of order n. Thus we may speak of the *set system of a code* or the *code of a set system*.

B. Thermal-Optimal Low-Power Codes

The set system $([n], \mathcal{A})$ of a thermal-optimal (n, w)-low-power code is defined by the following properties:

1) $|A_1 \triangle A_2| \leqslant w$ for all $A_1, A_2 \in \mathcal{A}$, and
2) $([n], \mathcal{A})$ is equireplicate.

It follows that our problem in (5) can be recast as an equivalent problem in extremal set systems, as follows:

$$\text{Given } n \text{ and } w\text{, determine } T(n, w)\text{, the maximum size of an equireplicate set system } (X, \mathcal{A}) \text{ of order } n \text{ such that } |A_1 \triangle A_2| \leqslant w \text{ for all } A_1, A_2 \in \mathcal{A} \qquad (6)$$

If the equireplication condition is removed, the resulting set system is known as an *anticode of length n and diameter w*. Hence, thermal-optimal low-power codes are equivalent to *equi- replicate anticodes*. Anticodes in general, and the size of anticodes of maximum size have been a subject of intensive research in coding theory, see [1], [2], [8], [16], [35], [42] and references therein.

The determination of equireplicate anticodes of maximum size appears to be a new problem, also to the combinatorics and coding theory communities. However, the maximum size of an anticode has been completely determined by Kleitman [29], and even earlier by Katona [27], in a different but equivalent setting. Thus the following theorem is from [27] and [29].

Theorem 1: Let $\mathfrak{T}(n, w)$ be the maximum number of blocks in a set system $([n], \mathcal{A})$ with $|A_1 \triangle A_2| \leqslant w$ for all $A_1, A_2 \in \mathcal{A}$. Then

$$\mathfrak{T}(n, w) = \begin{cases} \sum_{i=0}^{w/2} \binom{n}{i} & \text{if } w \equiv 0 \pmod 2 \\ \binom{n-1}{\frac{w-1}{2}} + \sum_{i=0}^{\frac{w-1}{2}} \binom{n}{i} & \text{if } w \equiv 1 \pmod 2. \end{cases}$$

For all even w, an extremal set system $([n], \mathcal{A})$ with $\mathfrak{T}(n, w)$ blocks is given by:

$$\mathcal{A} = \bigcup_{i=0}^{w/2} \binom{[n]}{i}. \qquad (7)$$

If w is odd, let x be any fixed element of $[n]$. Then an extremal set system $([n], \mathcal{A})$ is given by:

$$\mathcal{A} = \bigcup_{i=0}^{\frac{w-1}{2}} \binom{[n]}{i} \bigcup \left\{ A \cup \{x\} : A \in \binom{[n]\backslash\{x\}}{\frac{w-1}{2}} \right\}. \qquad (8)$$

We observe here that when w is even, the extremal set system in Theorem 1 is equireplicate. It consists of all the vectors of length n and weight at most $w/2$. Hence, we have the following result, which solves (5) and (6) for even w.

Corollary 1:

$$T(n, w) = \sum_{i=0}^{w/2} \binom{n}{i} \quad when\ w \equiv 0 \pmod 2.$$

The situation when w is odd is much more difficult. The set system in (8) is not equireplicate. In particular, we do not know if there exists an equireplicate anticode of order n and diameter w having size $\mathfrak{T}(n, w)$. Hence, from Theorem 1 we can derive only that for all odd w, we have:

$$\sum_{i=0}^{\frac{w-1}{2}} \binom{n}{i} \leqslant T(n, w) \leqslant \binom{n-1}{\frac{w-1}{2}} + \sum_{i=0}^{\frac{w-1}{2}} \binom{n}{i}. \qquad (9)$$

The left hand side of the equation is obtained from a code which consists of all the vectors of length n and weight at most $(w-1)/2$. The right hand side is obtained from the upper bound on $\mathfrak{T}(n, w)$ given in Theorem 1.

The next three propositions establishes some exact values of $T(n, w)$ for odd w.

Corollary 2:

$$T(n, 1) = 1 \quad for\ n \geqslant 2.$$

Proof: Since the distance between two different vectors of length n and weight one is two, it follows that $T(n, 1) = 1$ when $n \geqslant 2$. A code with maximum size consists of the unique all-zero vector of length n. ∎

Proposition 1:

$$T(n, n-1) = 2^{n-1} \quad for\ n \geqslant 3.$$

Proof: When the distance between two codeword is at most $n-1$, the code cannot contain two complement codewords and hence its size is at most 2^{n-1}. For odd n, an equireplicate set system of size 2^{n-1} is obtained from all vectors of length n and even weight. For even n, we give the following construction (which also works for any odd $n \geqslant 5$ if induction is applied). Let $([n-1], \mathcal{A})$ be a set system which attains $T(n-1, n-2) = 2^{n-2}$ and each element is contained in 2^{n-3} blocks. We define the following set system $([n], \mathcal{B})$.

$$\mathcal{B} \overset{\text{def}}{=} \{X \cup \{n\} : X \in \mathcal{A}\} \bigcup \{X : X \in \mathcal{A}\}.$$

We claim that $\mathcal{B}$ is an equireplicate set system which attains $T(n, n-1) = 2^{n-1}$ and each element is contained in 2^{n-2} blocks of $\mathcal{B}$. Clearly, $|\mathcal{B}| = 2|\mathcal{A}| = 2^{n-1}$ and the fact that $\mathcal{A}$ does not contain complement blocks immediately implies that also $\mathcal{B}$ does not contain complement blocks. Finally, $\mathcal{A}$ is equireplicate set system of size 2^{n-2} and each element of $[n-1]$ is contained in 2^{n-3} blocks of $\mathcal{A}$. Therefore, each $i \in [n]$ is contained in 2^{n-2} blocks in $\mathcal{B}$. Hence, $\mathcal{B}$ is an equireplicate set system which attains $T(n, n-1) = 2^{n-1}$. ∎

Proposition 2:

$$T(n, 3) = n + 1 \quad for\ n \geqslant 5.$$

Proof: First note, that by Proposition 1 we have $T(4, 3) = 8$. Also, $T(n, 2) = \sum_{i=0}^{1} \binom{n}{i} = n + 1$ for all $n \geqslant 2$. Finally, it is easy to verify that $T(5, 3) = 6$, These facts will be used in the current proof that $T(n, 3) = n + 1$ if $n > 4$. Let $\mathbb{C}$ be the largest possible equireplicate anticode of length $n > 5$ and diameter 3.

Let $\boldsymbol{x}$ and $\boldsymbol{z}$ be two codewords of $\mathbb{C}$ such that $d(\boldsymbol{x}, \boldsymbol{z}) = 3$. W.l.o.g. $\boldsymbol{x}$ and $\boldsymbol{z}$ differ in the last three coordinates and the first $n-3$ coordinates in both have $x_1, x_2, \ldots, x_{n-3}$.

Let $\alpha\beta\gamma$ and $\bar{\alpha}\bar{\beta}\bar{\gamma}$ be the values of the last three coordinates in $\boldsymbol{x}$ and $\boldsymbol{z}$, respectively. There is no other codeword in $\mathbb{C}$ which ends with either $\alpha\beta\gamma$ or $\bar{\alpha}\bar{\beta}\bar{\gamma}$ since such a codeword should also start with $x_1, x_2, \ldots, x_{n-3}$, to avoid distance greater than 3 from either $\boldsymbol{x}$ or $\boldsymbol{z}$, and hence such a codeword will be equal to either $\boldsymbol{x}$ or $\boldsymbol{z}$. Since each one of the last three columns has at least one *zero* and at least one *one* and the anticode is equireplicate, it follows that the weight of a column is at least 1 and at most $|\mathbb{C}| - 1$. The Hamming distance of any two of 100, 010, 001, 111 is 2 and hence the prefixes of length $n-3$ related to the codewords ending with these suffices differ in at most one coordinate. Since anticode with diameter one has at most two codewords, it follows that all codewords with these suffices (if differ) have different values in exactly the same coordinate (in the prefix of length $n-3$). The same argument holds also for the codewords ending with 011, 101, 110, and 000 (they have the same values in $n-4$ out of the first $n-3$ coordinates). Note that since the suffix of either $\boldsymbol{x}$ or $\boldsymbol{z}$ is in $\{100, 010, 001, 111\}$ and the other suffiex is in $\{011, 101, 110, 000\}$, it follows that all the other $n-5$ coordinates (which don't have different values) of $\boldsymbol{x}$ and $\boldsymbol{z}$ have the same values for all the codewords. Since $n > 5$, it follows that each one of the $n-5$ coordinates (which exist) forms either a column of *zeroes* or a column of *ones*. If the column consists of *zeroes* we have a contradiction since the weight of the column is 0 which is smaller than 1. If the column consists of *ones* we have a contradiction since the weight of the column is $|\mathbb{C}|$ which is greater than $|\mathbb{C}| - 1$.

Thus, for $n > 5$ there are no two codewords for which the Hamming distance is three. and hence for $n > 6$, $T(n, 3) = T(n, 2) = n + 1$, which completes the proof. ∎

For other odd values of w, we start with the extremal anticode $\mathcal{A}$ of diameter $w-1$ in (7) and add blocks to $\mathcal{A}$ while maintaining the equireplication requirement. Such blocks must contain exactly $(w+1)/2$ points to make sure that their distance with the blocks of $\mathcal{A}$ with $(w-1)/2$ points, or less, will not exceed $w+1$. Any two blocks with $(w+1)/2$ points must intersect in at least one point as otherwise their distance will be $w+1$. Interestingly, these properties precisely define a *regular uniform quorum system* of rank $(w+1)/2$. Quorum systems have been studied extensively in the literature on fault-tolerant and distributed computing — see [53] for a recent survey. There are many types of such systems. For example, if $(w+1)/2 = q+1$, $n = q^2 + q + 1$, and q is a prime power, then an optimal such system consists of q^2+q+1 blocks which form a projective plane of order q [52, p. 224].

In other words

Proposition 3: If $w = 2q + 1$, q is a prime power, then

$$T(q^2+q+1, 2q+1) = \sum_{i=0}^{q} \binom{q^2+q+1}{i} + q^2 + q + 1.$$

For a proof of this proposition and other similar constructions, we refer the reader to the work in [12], where such systems are constructed from combinatorial designs.

IV. Cooling Codes

Unfortunately, it is not possible to satisfy Property $\mathbf{A}(t)$ with nonadaptive coding schemes, even for $t = 1$. Indeed, suppose we wish to avoid state transitions on just the one hottest wire, say wire i. If the encoder does not know the current state of wire i, the only way to guarantee that there is no state transition is to have $x_i = y_i$ for *any* two codewords $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$. Since *any of the n wires* could be the hottest, we must have $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$. Thus all codewords are the same, and no communication is possible.

In this section, we shall see that if the encoder and decoder keep track of just *one previous transmission* then Property $\mathbf{A}(t)$ can be satisfied for any t with only $t + 1$ additional wires if $2(t + 1) \leqslant n$, by using spreads and partial spreads, notions from projective geometry. If $t + 1 > n/2$ we propose a construction based on a sunflower for which the construction for $2(t + 1) \leqslant n$ can be viewed as a special case. Finally, we provide a road map for our best lower bounds on the size of (n, t)-cooling codes in general, and in particular when $1 \leqslant t < n \leqslant 100$.

A. Differential Encoding and Decoding

The main idea of our differential encoding method is to encode the data to be communicated across the bus in the *difference* between the current transmission and the previous one. Similar ideas have been used in digital communications and in magnetic recording, among other applications.

Why is differential coding useful? The most useful feature in our context is this. When we use the differential encoding method to transmit a codeword $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, there is a state transition on wire i if and only if $x_i = 1$, and the total number of transitions is precisely $\mathrm{wt}(\mathbf{x})$, the Hamming weight of $\mathbf{x}$. This makes it possible to reduce the area overhead significantly beyond the best overhead achievable with nonadaptive schemes. For example, under differential encoding, a code $\mathbb{C}$ satisfies Property $\mathbf{B}(w)$ — and so is an (n, w)-LP code — if and only if $\mathrm{wt}(\mathbf{x}) \leqslant w$ for all $\mathbf{x} \in \mathbb{C}$. It follows that the thermal-optimal (n, w)-LP code of maximum size is given by

$$J^+(n, w) \stackrel{\text{def}}{=} \left\{\mathbf{x} \in \{0, 1\}^n : \mathrm{wt}(\mathbf{x}) \leqslant w\right\} \tag{10}$$

This set, distinguished from the Johnson space defined by $J(n, w) \stackrel{\text{def}}{=} \{\mathbf{x} \in \{0, 1\}^n : \mathrm{wt}(\mathbf{x}) = w\}$, is clearly equireplicate and its size is much larger than the size of the largest anticode of diameter w (cf. Theorem 1), for both odd and even w.

B. Definition of Cooling Codes

Even under differential encoding, it is still not possible to satisfy Property $\mathbf{A}(t)$ with conventional binary codes. To see this, again suppose that we wish to avoid transitions on just the one hottest wire, say wire i. With differential encoding, in order to guarantee no state transitions on wire i while transmitting a codeword $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, we must have $x_i = 0$. But, once again, any of the n wires could be the hottest, which implies that $\mathbf{x} = \mathbf{0}$. Since this must hold for any codeword, it follows that $\mathbb{C} = \{\mathbf{0}\}$ and no communication is possible.

Consequently, we henceforth modify our notion of a code $\mathbb{C}$ as follows. The elements of $\mathbb{C}$ will be *sets of binary vectors* of length n, say $C_1, C_2, \ldots, C_M$. We will refer to $C_1, C_2, \ldots, C_M$ as *codesets*. We do not require these codesets to be of the same size, but we do require them to be disjoint: $C_i \cap C_j = \varnothing$ for all $i \neq j$. The elements of each codeset C_i will be called *codewords*. The goal is to guarantee that no matter which codeset C_i is chosen, for each possible designation of t wires as the hottest, there is at least one codeword in C_i with zeros on the corresponding t positions. This leads to the following definition.

*Definition 2: For positive integers n and $t < n$, an (n, t)-**cooling code** $\mathbb{C}$ of size M is defined as a set $\{C_1, C_2, \ldots, C_M\}$, where $C_1, C_2, \ldots, C_M$ are disjoint subsets of $\mathbb{F}_2^n$ satisfying the following property: for any set $\mathcal{S} \subset [n]$ of size $|\mathcal{S}| = t$ and for all $i \in [M]$, there exists a codeword $\mathbf{x} \in C_i$ with* $\mathrm{supp}(\mathbf{x}) \cap \mathcal{S} = \varnothing$.

Given the foregoing definition of cooling codes, we also need to modify our notions of an encoding function and a decoding function, introduced in Section III. As before, we assume that the data to be communicated across the bus is represented by a source $\mathcal{S}$ taking on some $M \leqslant 2^k$ values in $\mathbb{F}_2^k$. In contrast to Section III, we no longer need to assume that $\mathcal{S}$ is uniformly distributed — in fact, no probabilistic model for $\mathcal{S}$ is required. On the other hand, the input to the encoding function $\mathcal{E}$ now comprises, in addition to a word $\mathbf{u} \in \mathcal{S}$, also a set $\mathcal{S} \subset [n]$ of size t representing the positions of the t hottest wires. We let

$$\mathcal{C} :\stackrel{\text{def}}{=} C_1 \cup C_2 \cup \cdots \cup C_M.$$

Then the output of the encoding function $\mathcal{E}$ is a vector $\mathbf{x} \in \mathcal{C}$ such that $\mathrm{supp}(\mathbf{x}) \cap \mathcal{S} = \varnothing$. For every possible $\mathcal{S}$, the function $\mathcal{E}(\cdot, \mathcal{S})$ is a bijective map from $\mathcal{S}$ to $\mathbb{C}$. Since the codesets $C_1, C_2, \ldots, C_M$ are disjoint, this allows the decoding function $\mathcal{D}$ to recover $\mathbf{u} \in \mathcal{S}$ from the encoder output $\mathbf{x} \in \mathcal{C}$. We summarize the foregoing discussion in the next definition.

*Definition 3: For integers n and $t < n$, an (n, t)-**cooling coding scheme** for a source $\mathcal{S} \subseteq \mathbb{F}_2^k$ is a triple $\mathcal{E} = \langle \mathbb{C}, \mathcal{E}, \mathcal{D} \rangle$, where*

1) *The code $\mathbb{C}$ is an (n, t)-cooling code;*
2) *The encoding function $\mathcal{E}\colon \mathcal{S} \times \binom{[n]}{t} \to \mathcal{C}$ is such that for all $\mathcal{S} \subset [n]$ of size t and all $\mathbf{u} \in \mathcal{S}$, we have*

$$\mathrm{supp}\big(\mathcal{E}(\mathbf{u}, \mathcal{S})\big) \cap \mathcal{S} = \varnothing\ ;$$

$\mathcal{S}$	000	001	010	011	100	101	110	111
{1,2}	001011	001101	001000	000100	000010	000001	001001	000101
{1,3}	010001	010000	010111	000100	000010	000001	000011	000101
{1,4}	001011	010000	001000	011000	000010	000001	001001	011011
{1,5}	010001	010000	001000	000100	001100	000001	001001	000101
{1,6}	011010	010000	001000	000100	000010	010010	001010	011110
{2,3}	100000	100011	100111	000100	000010	000001	000011	000101
{2,4}	100000	100011	001000	100001	000010	000001	001001	101001
{2,5}	100000	001101	001000	000100	100100	000001	001001	101100
{2,6}	100000	101110	001000	000100	000010	000110	001010	101100
{3,4}	100000	010000	110000	100001	000010	000001	000011	110010
{3,5}	100000	010000	110000	000100	100100	000001	110101	000101
{3,6}	100000	010000	110000	000100	000010	010010	110110	110010
{4,5}	100000	010000	001000	100001	101000	000001	001001	101001
{4,6}	100000	010000	001000	011000	000010	010010	001010	110010
{5,6}	100000	010000	001000	000100	100100	010100	111100	101100

3) *The decoding function* $\mathcal{D}$: $\mathcal{C} \to \mathbb{S}$ *is such that for all* $\mathbf{u} \in \mathbb{S}$, *we have* $\mathcal{D}(\mathcal{E}(\mathbf{u}, \mathcal{S})) = \mathbf{u}$ *regardless of the value of* $\mathcal{S}$.

It follows immediately from Definition 3 that, under differential encoding, an (n, t)-cooling coding scheme satisfies Property $\mathbf{A}(t)$ by avoiding state transitions on the t hottest wires, which are represented by the subset $\mathcal{S}$.

Example: Consider $n = 6$ and $t = 2$. An $(6, 3)$-cooling code is given by the following eight codesets,

$$C_{000} = \{100000, 101011, 111010, 001011, 010001, 110001, 011010\},$$

$$C_{001} = \{010000, 100011, 011101, 110011, 111110, 101110, 001101\},$$

$$C_{010} = \{001000, 100111, 111000, 101111, 011111, 010111, 110000\},$$

$$C_{011} = \{000100, 100101, 011100, 100001, 111001, 111101, 011000\},$$

$$C_{100} = \{000010, 100100, 001110, 100110, 101010, 101000, 001100\},$$

$$C_{101} = \{000001, 010010, 000111, 010011, 010101, 010100, 000110\},$$

$$C_{110} = \{110110, 001001, 110101, 111111, 111100, 001010, 000011\},$$

$$C_{111} = \{011011, 110010, 101100, 101001, 011110, 000101, 110111\}.$$

We index the codesets by all three-bit messages.

To verify Property $\mathbf{A}(2)$, we explicitly describe the encoding function $\mathcal{E} : \mathbb{S} \times \binom{[6]}{2} \to \mathcal{C}$ via the following lookup table.

C. Bounds on the Size of Cooling Codes

In this subsection, we show that realizing an (n, t)-cooling coding scheme requires at least $t + 1$ additional wires. That is, the number of bits that can be communicated over an n-wire bus while satisfying Property $\mathbf{A}(t)$ is at most $k \leqslant n - t - 1$. In the next subsection, we will present a construction that achieves this bound. Herein, let us begin with the following lemma.

Lemma 1: Let $\mathbb{C}$ *be an* (n, t)*-cooling code of size* $|\mathbb{C}| = M$. *Then*

$$M \leqslant \frac{t!(n-t)!}{n!} \sum_{w=0}^{n-t} \binom{n}{w}\binom{n-w}{t} = 2^{n-t}. \tag{11}$$

Proof: For convenience, we will refer to sets $\mathcal{S} \subset [n]$ of size $|\mathcal{S}| = t$ as t*-subsets*. Given a t-subset $\mathcal{S}$ and a vector $\mathbf{x} \in \mathbb{F}_2^n$, we shall say that $\mathbf{x}$ *covers* $\mathcal{S}$ if $\text{supp}(\mathbf{x}) \cap \mathcal{S} = \varnothing$. Observe that a vector of weight w covers exactly $\binom{n-w}{t}$ different t-subsets. Therefore, the total number of t-subsets (counted with multiplicity) covered by *all* the vectors in $\mathbb{F}_2^n$ is given by

$$N(n, t) \stackrel{\text{def}}{=} \sum_{w=0}^{n-t} \binom{n}{w}\binom{n-w}{t}.$$

Now consider a codeset C in $\mathbb{C}$. By definition, for any t-subset $\mathcal{S}$, there is at least one codeword $\mathbf{x} \in C$ that covers $\mathcal{S}$. Hence, the total number of t-subsets (possibly counted with multiplicity) covered by all the codewords of C is at least $\binom{n}{t}$. Since this holds for each of the M codesets in $\mathbb{C}$, the total number of t-subsets (again, counted with multiplicity) covered by all the vectors in $\mathcal{C} = C_1 \cup C_2 \cup \cdots \cup C_M$ is at least $M\binom{n}{t}$. But, since the codesets $C_1, C_2, \ldots, C_M$ are disjoint and $\mathcal{C} \subseteq \mathbb{F}_2^n$, this number cannot exceed $N(n, t)$, i.e. $M\binom{n}{t} \leqslant N(n, t)$, and the lemma follows. ■

The proof of Lemma 1 can be shorten by considering any given t-subset of coordinates that must be zeroes in at least one codeword of each codeset. There are 2^{n-t} such codewords of length n and hence the maximum number of codesets is 2^{n-t}. Nevertheless, there is one advantage in presenting the longer proof. It follows from the proof of Lemma 1 that an (n, t)-cooling code $\mathbb{C}$ of size $|\mathbb{C}| = 2^{n-t}$ would be *perfect*. In such a code, the codesets $C_1, C_2, \ldots, C_M$ form a partition of $\mathbb{F}_2^n$ and each of these codesets is a perfect covering of $\binom{[n]}{t}$, i.e. all words with exactly t *zeroes*. Using these observations, we can prove that such cooling codes do not exist, unless $t = 1$ or $t \geqslant n - 1$.

Proposition 4: Perfect (n, t)*-cooling codes do not exist, unless* $t = 1$ *or* $t \geqslant n - 1$.

Proof: W.l.o.g. assume that the codeset C_1 contains the codeword $\mathbf{x}$ which starts with a *one* followed by $n - 1$ *zeroes*. The only words which are not covered by this codeword are all those words which start with a *zero* and have exactly $t - 1$ *zeroes* in the other $n - 1$ coordinates. Let $\mathbf{z}$ be one of these $\binom{n-1}{t-1}$ words. A codeword $\mathbf{y}$ which covers $\mathbf{z}$ must starts with a *zero*. If $\mathbf{y}$ has at least $t + 1$ *zeroes* then it covers words with t *zeroes* which are also covered by $\mathbf{x}$, and therefore C_1 won't be a perfect covering. Hence, C_1 contains $\mathbf{x}$ and the $\binom{n-1}{t-1}$ codewords, which start with a *zero*, and have exactly t *zeroes*.

Now, assume w.l.o.g. that C_2 contains the codeowrd which starts with $n - 1$ *zeroes* and ends with a *one*. With similar analysis as for C_1 this codeset contains the $\binom{n-1}{t-1}$ codewords, which end with a *zero*, and have exactly t *zeroes*.

Now, by this analysis, both C_1 and C_2 must contain the $\binom{n-2}{t-2}$ words which start and end with a *zero* and have exactly *t zeroes*. Therefore, if $\binom{n-2}{t-2} > 0$ a perfect code cannot exist. $\binom{n-2}{t-2} > 0$, unless $t = 1$ or $t \geqslant n-1$.

- If $t = 1$, then there exists a perfect $(n, 1)$-cooling code, with 2^{n-1} codesets, each one contains a pair of complement codewords.
- If $t = n$, then there exists a perfect (n, n)-cooling code, with one codeset which contains all codewords of length n.
- If $t = n-1$ then there exists a perfect $(n, n-1)$-cooling code, with two codesets, one contains the all-zero codeword and the second contains all codewords of weight *one*. All other words can be partitioned arbitrarily between these two codesets. ■

Lemma 1 and Proposition 4 imply the following result.

Corollary 3: If $1 < t < n-1$, then the size of any (n, t)-cooling code $\mathbb{C}$ is bounded by $|\mathbb{C}| \leqslant 2^{n-t} - 1$. Consequently, such a code cannot support the transmission of $n-t$ or more bits over an n-wire bus.

Denote the maximum size of an (n, t)-cooling code by $C(n, t)$. In the following subsection we will obtain lower bounds on $C(n, t)$. We start in this subsection with a few simple bounds and values of $C(n, t)$.

Corollary 4:

1) $C(n, 1) = 2^{n-1}$;
2) $C(n, n-1) = 2$;
3) *If* $2 \leqslant t \leqslant n-2$, *then* $C(n, t) \leqslant 2^{n-t} - 1$.

Proposition 5: If $2 \leqslant t \leqslant n-2$, then $C(n, t) \geqslant n - t + 1$.

Proof: For $i \in [n-t+1]$, let C_i contain the set of all binary words of length n and weight $n-t-i+1$, i.e. each codeword in C_i has $t+i-1 \geqslant t$ *zeroes*. Clearly, the code $\mathbb{C}$ with these codesets is an (n, t)-cooling code. ■

By Corollary 4 and Proposition 5 we have the following

Corollary 5: $C(n, n-2) = 3$.

D. Construction of Optimal Cooling Codes

In this subsection, we construct (n, t)-cooling codes that support the transmission of up to $n-t-1$ bits over an n-wire bus. By Corollary 3, such cooling codes are *optimal*. Our construction is based on the notion of *spreads* and *partial spreads* in projective geometry. We will give the related equivalent definition for vector spaces. In this section we need only spread and partial spread over $\mathbb{F}_2$, but we will define and discuss them over $\mathbb{F}_q$, as those will be required later in our exposition.

Loosely speaking, a partial τ-spread of the vector space $\mathbb{F}_q^n$ is a collection of disjoint τ-dimensional subspaces of $\mathbb{F}_q^n$. Formally, a collection $V_1, V_2, \ldots, V_M$ of τ-dimensional subspaces of $\mathbb{F}_q^n$ is said to be a *partial τ-spread of $\mathbb{F}_q^n$* if

$$V_i \cap V_j = \{\mathbf{0}\} \quad \text{for all } i \neq j, \tag{12}$$

$$\mathbb{F}_q^n \supseteq V_1 \cup V_2 \cup \cdots \cup V_M. \tag{13}$$

If the τ-dimensional subspaces form a partition of $\mathbb{F}_q^n$ then the partial τ-spread is called a τ-spread. It is well known that such τ-spreads exist if and only if τ divides n, in which case $M = (q^n-1)/(q^\tau-1) > q^{n-\tau}$. For the case where τ does not divide n, *partial τ-spreads* with $M \geqslant q^{n-\tau}$ have been constructed in [18, Th. 11]. Let $M_q(n, \tau)$ be the maximum size of a partial τ-spread. The value of $M_q(n, \tau)$ has been considered for many years in projective geometry, and a survey with the known results is given in [17]. Recently, there has been lot of activity and the question has been almost completely solved [32], [33], [37].

Theorem 2: Let $V_1, V_2, \ldots, V_M$ be a partial $(t+1)$-spread of $\mathbb{F}_2^n$, and define the code $\mathbb{C} = \{V_1^*, V_2^*, \ldots, V_M^*\}$, where $V_i^* = V_i \setminus \{\mathbf{0}\}$ for all i. Then $\mathbb{C}$ is an (n, t)-cooling code of size $M \geqslant 2^{n-t-1}$.

Proof: It is obvious from (12) that the M codesets of $\mathbb{C}$ are disjoint subsets of $\mathbb{F}_2^n$. It remains to verify that for any set $\mathcal{S} \subset [n]$ of size t, each of $V_1^*, V_2^*, \ldots, V_M^*$ contains at least one vector whose support is disjoint from $\mathcal{S}$. To this end, consider an arbitrary $(t+1)$-dimensional subspace V of $\mathbb{F}_2^n$, and suppose $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{t+1}\}$ is a basis for V. Let $\mathbf{v}_1', \mathbf{v}_2', \ldots, \mathbf{v}_{t+1}'$ denote the projections of the basis vectors on the t positions in $\mathcal{S}$. These $t+1$ vectors lie in a t-dimensional vector space — the projection of $\mathbb{F}_2^n$ on $\mathcal{S}$. Hence, these vectors must be linearly dependent, and there exist binary coefficients $a_1, a_2, \ldots, a_{t+1}$, not all zero, with $a_1\mathbf{v}_1' + a_2\mathbf{v}_2' + \cdots + a_{t+1}\mathbf{v}_{t+1}' = \mathbf{0}$. But then $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_{t+1}\mathbf{v}_{t+1}$ is a nonzero vector in V whose support does not include any of the positions in $\mathcal{S}$. As this holds for an arbitrary $(t+1)$-dimensional subspace, it must hold for each of the subspaces $V_1, V_2, \ldots, V_M$ in the partial spread. ■

Whether we start with a spread or a partial spread, the size of the code $\mathbb{C}$ constructed in Theorem 2 will usually be strictly larger than 2^{n-t-1}. We omit also the possibility for adding another codset which contains the all-zero codeword since the encoding of k-bit data generated by the source $\mathcal{S}$, requires only 2^k codesets. In order to use such a code to communicate $k = n - t - 1$ bits, one can choose a subset of $\mathbb{C}$, with 2^k codesets, arbitrarily. We illustrate this point in the next subsection. Nevertheless, sometimes we will be interested in the exact number of codesets to find bounds on $C(n, t)$, which will be discussed later in this section.

E. Efficient Encoding and Decoding of Cooling Codes

Several efficient algorithms for coding into spreads are known. In this subsection, we describe a particularly simple and powerful method that was originally developed by Dumer [14] in the context of coding for memories with defects. This method involves computations in the finite field $\mathbb{F}_{2^{(t+1)}}$. Since t is a small constant, such computations are very efficient. In fact, for most applications of cooling codes, $t \leqslant 7$ will be more than sufficient to cool the bus wires. Thus the proposed encoder $\mathcal{E}$ and decoder $\mathcal{D}$ work with bytes, or even nibbles, of data.

As in the previous subsection, we set $\tau = t + 1$. For simplicity and w.l.o.g. we assume that τ divides n and, hence, also $k = n - \tau$. Presented with a k-bit data word $\mathbf{u}$ generated by the source $\mathcal{S}$, the encoder $\mathcal{E}$ first partitions $\mathbf{u}$ into $m = k/\tau$ blocks $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$, each consisting of τ bits. We will refer to these blocks as τ-*nibbles* and think of them as elements in the finite field $\mathbb{F}_{2^\tau}$. The output of the encoder is the n-bit vector:

$$\mathcal{E}(\boldsymbol{u}, \mathcal{S}) = (\beta \boldsymbol{u}_1, \beta \boldsymbol{u}_2, \ldots, \beta \boldsymbol{u}_m, \beta) \in \mathbb{F}_2^n \tag{14}$$

for a carefully chosen nonzero element β of $\mathbb{F}_{2^\tau}$ that depends on both $\boldsymbol{u}$ and $\mathcal{S}$. Note that, given β, computing $\mathcal{E}(\boldsymbol{u}, \mathcal{S})$ amounts to k/τ multiplications in $\mathbb{F}_{2^\tau}$. The operation of the decoder $\mathcal{D}$ is equally simple. Given its input $\boldsymbol{x} = \mathcal{E}(\boldsymbol{u}, \mathcal{S})$, the decoder first partitions $\boldsymbol{x}$ into τ-nibbles $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{m+1}$ and reads off $\beta = \boldsymbol{x}_{m+1}$. The decoder then computes β^{-1} from β in $\mathbb{F}_{2^\tau}$, and recovers the original data word $\boldsymbol{u}$ as follows:

$$\mathcal{D}(\mathbf{x}) = (\beta^{-1}\mathbf{x}_1, \beta^{-1}\mathbf{x}_2, \ldots, \beta^{-1}\mathbf{x}_m) = (\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m).$$

Thus decoding amounts to one inversion and k/τ multiplications in $\mathbb{F}_{2^\tau}$. It remains to explain how β is computed.

Computing β from $\boldsymbol{u}$ and $\mathcal{S}$ is equivalent to solving a system of t linear equations in τ unknowns over $\mathbb{F}_2$. We illustrate this with the following example for the case $\tau = 3$.

Example: In this example, we will work with the finite field $\mathbb{F}_{2^3} = \{0, 1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6\}$, defined by $\alpha^3 = 1 + \alpha$. Notice that $1, \alpha, \alpha^2$ is a basis for $\mathbb{F}_{2^3}$ over $\mathbb{F}_2$. Suppose that the $t = 2$ hottest wires, presented to the encoder via the set $\mathcal{S}$, fall in the 3-nibbles $\boldsymbol{x}_i = \beta \boldsymbol{u}_i$ and $\boldsymbol{x}_j = \beta \boldsymbol{u}_j$ (if both fall in the same 3-nibble, the situation is similar). Let us write

$$\boldsymbol{u}_i = u_0 + u_1\alpha + u_2\alpha^2 \quad \text{and} \quad \boldsymbol{u}_j = u_0' + u_1'\alpha + u_2'\alpha^2.$$

Let us also write $\beta = b_0 + b_1\alpha + b_2\alpha^2$; our goal is to determine the $\tau = 3$ unknowns b_0, b_1, b_2 from the $t = 2$ constraints. Computing $(x_0, x_1, x_2) = \beta \boldsymbol{u}_i$ and $(x_0', x_1', x_2') = \beta \boldsymbol{u}_j$, we have:

$$x_0 = u_0b_0 + u_2b_1 + u_1b_2 \tag{15}$$
$$x_1 = u_1b_0 + (u_0{+}u_2)b_1 + (u_1{+}u_2)b_2 \tag{16}$$
$$x_2 = u_2b_0 + u_1b_1 + (u_0{+}u_2)b_2 \tag{17}$$
$$x_0' = u_0'b_0 + u_2'b_1 + u_1'b_2 \tag{18}$$
$$x_1' = u_1'b_0 + (u_0'{+}u_2')b_1 + (u_1'{+}u_2')b_2 \tag{19}$$
$$x_2' = u_2'b_0 + u_1'b_1 + (u_0'{+}u_2')b_2. \tag{20}$$

Equating any two of $x_0, x_1, x_2, x_0', x_1', x_2'$ in (15)–(20) to zero generates a system of $t = 2$ linear equations in the unknowns b_0, b_1, b_2 with coefficients determined by $u_0, u_1, u_2, u_0', u_1', u_2'$. Such a system can be easily solved by Gaussian elimination or, if necessary, more efficient methods. □

In general, the complexity of computing β from $\boldsymbol{u}$ and $\mathcal{S}$ is $O(t^3)$, which is very small for constant t. The overall complexity of encoding/decoding is *linear* in the number of wires n.

F. Cooling Codes for Large t

The cooling codes constructed based on spreads or partial spreads can be used only for small t, or more precisely when $t + 1 \leqslant n/2$ since subspaces of dimension greater that $t + 1$ have a nontrivial intersection if $t + 1 > n/2$. Fortunately, this is probably what is usually required for the design parameters of a thermal system. Nevertheless, we want to find efficient cooling codes also for larger values of t. Therefore, when $t + 1 > n/2$ we have to use another construction which generates cooling codes of a large size. In this subsection we present a construction which forms a sunflower whose heart of seeds (*kernel*) is a linear code $\mathbb{C}$ and his flowers are codes obtained by the sum of $\mathbb{C}$ and elements of a spread. The construction of subsection IV-D, which is based on a partial spreads can be viewed also as such a sunflower, where the kernel is a trivial linear code (only the all-zero codeword), and the flowers are the elements of the partial spread without the all-zero codeword.

For the new construction and for some constructions which follow we will use some basic and more sophisticated elements in coding theory which will be defined. First, an $[n, \kappa, d]$ code $\mathbb{C}$ over $\mathbb{F}_q$ is a κ-dimensional subspace of $\mathbb{F}_q^n$, with minimum Hamming distance d. This is the only concept we need for the basic construction. The more sophisticated elements will be needed for a generalization of the construction which will be given later.

Theorem 3: Let n, t, s, r, d be integers, such that $r + t \leqslant (n + s)/2$. If there exists a binary $[n, s, d]$ code and a binary $[n - t, r, d]$ code does not exist, then there exists an (n, t)-cooling code of size $M > 2^{n-t-r}$.

Proof: Let n, t, s, r, d be integers, such that $r + t \leqslant (n + s)/2$, K a binary $[n, s, d]$ code, and assume that a binary $[n-t, r, d]$ code does not exist. Let B be an $(n-s)$-dimensional subspace of $\mathbb{F}_2^n$ such that

$$K + B = \{a + b : a \in K,\ b \in B\} = \mathbb{F}_2^n.$$

The fact that $r+t \leqslant (n+s)/2$ implies that $r+t-s \leqslant (n-s)/2$ and hence there exists a partial $(r + t - s)$-spread of B whose size M is $M_2(n - s, r + t - s) \geqslant 2^{n-r-t}$. Let $V_1, V_2, \ldots, V_M$ be the subspaces of a related partial spread and construct the following M sets:

$$C_i = (V_i + K) \setminus K \quad \text{for } i \in [M].$$

Since $V_i \cap V_j$ is the null space for $i \neq j$, it follows that $(V_i + K) \cap (V_j + K) = K$ and hence $C_i \cap C_j = \varnothing$. Hence, we can take the C_i's as the codesets in a code $\mathbb{C}$. To prove that $\mathbb{C}$ is an (n, t)-cooling code it remains to be shown that for any given subset $S \in [n]$ of size t and a codeset C_i, $i \in [M]$, there exists a codeword $\boldsymbol{x} \in C_i$, such that $\text{supp}(\boldsymbol{x}) \cap S = \varnothing$.

The code K is an s-dimensional subspace, V_i in an $(r + t - s)$-dimensional subspace, and $K \cap V_i = \{\boldsymbol{0}\}$ (since $V_i \subset B$ and $K \cap B = \{\boldsymbol{0}\}$). Hence, $V_i + K$ is an $(r + t)$-dimensional subspace. Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{r+t}\}$ be a basis for $V_i + K$. Let $\boldsymbol{v}_1', \boldsymbol{v}_2', \ldots, \boldsymbol{v}_{r+t}'$ denote the projections of the basis vectors on the t positions in $\mathcal{S}$. These $r + t$ vectors lie in a t-dimensional vector space — the projection of $\mathbb{F}_2^n$ on $\mathcal{S}$. Hence, there exists an r-dimensional subspace U_i spanned by these $r + t$ basis vectors (which span $V_i + K$), such that for each $\boldsymbol{z} \in U_i$, we have $\text{supp}(\boldsymbol{z}) \cap \mathcal{S} = \varnothing$. We can remove the t coordinates which only have *zero* elements in U_i, from all the vectors of U_i, to obtain an $[n - t, r, \delta]$ code U_i'. Since an

$[n-t, r, d]$ code does not exist it follows that $\delta < d$. Hence, U_i contains a vector $\mathbf{x}$ which is not contained in K. Since $\mathbf{x} \in U_i$ and $\mathbf{x} \notin K$, it follows that $\mathbf{x} \in C_i$, and hence $\mathbb{C}$ is an (n,t)-cooling code and the proof has been completed. ■

Theorem 3 can be applied in various ways. For example, we derive the following result.

Corollary 6: If $n = 2^m$, $m > 1$, and $n/2 < t \leqslant (n+m-1)/2$, then there exists an (n,t)-cooling code of size $M > 2^{n-t-1}$.

Proof: Apply Theorem 3 with the $[n, m+1, n/2]$ first order Reed-Muller code as the kernel code K, i.e. $s = m+1$. If $r = 1$ then $r + t \leqslant 1 + (n+m-1)/2 \leqslant (n+m+1)/2 = (r+s)/2$. Clearly, an $[n-t, 1, n/2]$ code does not exist since $n-t < n/2$. Hence, by Theorem 3 we obtain an (n,t)-cooling code whose size is greater than 2^{n-t-1}. This code is optimal by Corollary 3. ■

Theorem 3 can be generalized by using the concept of generalized Hamming weights which was defined by Wei [56]. For this definition we generalize the notion of support from a word to a subcode. The *support* of a subcode $\mathbb{C}'$ of $\mathbb{C}$, $\text{supp}(\mathbb{C}')$ is defined as the set of coordinates on which $\mathbb{C}'$ contains codewords with nonzero coordinates, i.e.

$$\text{supp}(\mathbb{C}') \stackrel{\text{def}}{=} \left\{ i \in [n] : \exists (x_1, x_2, \ldots, x_n) \in \mathbb{C}',\ x_i \neq 0 \right\}.$$

Now, the rth *generalized Hamming weight*, $d_r(\mathbb{C})$, of $\mathbb{C}$ is defined as the minimum number of coordinates in $\text{supp}(\mathbb{C}')$, for an r-dimensional subcode $\mathbb{C}'$ of $\mathbb{C}$, i.e.

$$d_r(\mathbb{C}) \stackrel{\text{def}}{=} \min \left\{ |\text{supp}(\mathbb{C}')| : \mathbb{C}' \text{ is an } r\text{-dimensional subcode of } \mathbb{C} \right\}.$$

Theorem 4: If n, t, s, r, d are integers, such that $r + t \leqslant (n+s)/2$, and the following two requirements are satisfied:

(R1) *There exists a binary $[n, s, d]$ code K.*

(R2) *The rth generalized Hamming weight of K is larger than $n - t$.*

Then there exists an (n,t)-cooling code of size $M > 2^{n-t-r}$.

Proof: The proof is along the same lines as the proof of Theorem 3, up to the point in which an $[n-t, r, \delta]$ code U_i' is obtained from U_i. Since the rth generalized Hamming weight of K is larger than $n-t$, it follows that the r-dimensional subspace U_i contains a vector $\mathbf{x}$ which is not contained in K. Since $\mathbf{x} \in U_i$ and $\mathbf{x} \notin K$, it follows that $\mathbf{x} \in C_i$, and hence $\mathbb{C}$ is an (n,t)-cooling code and the proof has been completed. ■

Theorem 4 is stronger than and generalizes Theorem 3. Indeed, consider the $[n, s, d]$ code K in Theorem 3. If there does not exist an $[n-t, r, d]$ code, then the rth generalized Hamming weight of K must be larger than $n-t$. But, if for a given $[n, s, d]$ code K the rth generalized Hamming weight of K is larger than $n-t$, then an $[n-t, r, d]$ code might still exists.

An independent question is whether Theorem 4 can improve on the results implied by Theorem 3, or all the results that can be obtained from Theorem 4 can also be obtained from Theorem 3. The answer is that Theorem 4 improves on some of the results obtained by Theorem 3. This is demonstrated with the following examples.

The first interesting example is considered in the range $1 \leqslant t < n \leqslant 100$. Consider the following 8×20 parity-check matrix H over $\mathbb{F}_2$,

$$H = \begin{pmatrix} 1&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&1&1&1&1 \\ 0&1&0&0&0&0&0&0&1&1&0&0&0&0&1&1&0&0&1&1 \\ 0&0&1&0&0&0&0&0&1&1&1&1&0&0&0&1&0&1&0&1 \\ 0&0&0&1&0&0&0&0&0&0&1&1&0&1&1&1&1&0&0&1 \\ 0&0&0&0&1&0&0&0&0&1&0&1&0&1&1&0&1&1&0&1 \\ 0&0&0&0&0&1&0&0&0&1&1&0&0&1&0&1&0&1&1&0 \\ 0&0&0&0&0&0&1&0&1&1&0&1&1&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&1&1&0&1&1&1&0&1&0&1&1&0&0 \end{pmatrix}$$

One can verify that every six columns have rank at least 5. Therefore, there exists an $[n, n-8, d]$ linear code with $d_2 \geqslant 7$ for $n \in \{18, 19, 20\}$. Thus, by Theorem 4, there exists an $(n, n-6)$ cooling code of size greater than $2^4 = 16$ for $n \in \{18, 19, 20\}$.

On other hand, suppose that Theorem 3 is used to obtain an $(n, n-6)$-cooling code of size greater than 2^4 for $n \in \{18, 19, 20\}$. It is required by Theorem 3 that $t + r \leqslant (n+s)/2$, where $t = n-6$ and $r = 2$, and hence $s \geqslant n-8$. Using the online codetables.de [22], the best code with dimension $n-8$ for length $n \in \{18, 19, 20\}$ has minimum Hamming distance 4. But, there exists a $[6, 2, 4]$ code, and hence Theorem 3 cannot be applied.

Another example is derived from Feng *et al.* [19] who computed the following bounds for the generalized Hamming weights for cyclic codes of length 255.

(i) The primitive double-error-correcting BCH $[255, 239, 5]$ code has $d_9 \geqslant 18$. If $t = 238$, then by Theorem 4, there exists a $(255, 238)$-cooling code of size greater than 2^8. On other hand, suppose that Theorem 3 is used to obtain a $(255, 238)$-cooling code of size greater than 2^8. It is required by Theorem 3 that $t + r \leqslant (n+s)/2$, where $n = 255$, $t = 238$ and $r = 9$, and hence $s \geqslant 239$. Using the online codetables.de [22], the best code with dimension 239 for length 255 has minimum Hamming distance 5. But, there exists a $[17, 9, 5]$ code, and hence Theorem 3 cannot apply.

(ii) The reversible cyclic double-error-correcting BCH $[255, 238, 5]$ code has $d_9 \geqslant 19$ [19]. If $t = 237$, then by Theorem 4, there exists a $(255, 237)$-cooling code with size greater than 2^9.
On other hand, suppose that Theorem 3 is used to obtain a $(255, 237)$-cooling code of size greater than 2^9. It is required by Theorem 3 that $237 + 9 \leqslant (255+s)/2$ and hence $s \geqslant 237$. Using the online codetables.de [22], the best code with dimension 237 has minimum Hamming distance 6. But, there exists an $[18, 9, 6]$ code, and hence Theorem 3 cannot be applied.

G. Best Lower Bounds on $C(n,t)$

Now, we will summarize our constructions by providing our best lower bounds on $C(n,t)$ in general and when

$1 \leqslant t < n \leqslant 100$ in particular.

$$C(n,t)\begin{cases} = 2^{n-t}, & \text{if } t=1 \text{ or } t=n-1,\\ > 2^{n-t-1}, & \text{if } t+1 \leqslant n/2,\\ > 2^{n-t-2}, & \text{if } (n,t)\in\{(18,12),\ (19,13),(20,14)\},\\ > M_2(n-s, r+t-s), & \text{if } n,\ t,\ s, \text{ and } r \text{ appear in TABLE I in the Appendix, and } (n,t)\notin\{(18,12),(19,13),(20,14)\},\\ \geqslant n-t+1, & \text{otherwise.}\end{cases}$$

V. Low-Power Cooling Codes

In this section, we present coding schemes that satisfy Properties $\mathbf{A}(t)$ and $\mathbf{B}(w)$ simultaneously in every transmission. The corresponding codes are called ***(n,t,w)-low-power cooling codes*** (or (n,t,w)-LPC codes for short). We suggest two types of constructions. The first type is based on decomposition of the complete hypergraph into disjoint perfect matchings. The second type is based on cooling codes over GF(q), dual codes of $[n,\kappa,t+1]$ codes, MDS codes, spreads, $J^+(r,\omega)$, and concatenation codes. Finally, we will show that also the sunflower construction can be used to obtain (n,t,w)-LPC codes.

As before, we assume that the coding schemes constructed in what follows are augmented by differential encoding. Since the codes are also (n,t)-cooling codes, they conform to Definition 2. Thus a code $\mathbb{C}$ is a collection of codesets $C_1, C_2, \ldots, C_M$, which are disjoint subsets of $\mathbb{F}_2^n$. In order to satisfy Property $\mathbf{B}(w)$, the codesets of the code $\mathbb{C}$ must satisfy that

$$C_1, C_2, \ldots, C_M \subset J^+(n,w). \tag{21}$$

As shown in Section IV-A, this guarantees that the total number of state transitions on the n bus wires is at most w.

A. Decomposition of the Complete Hypergraph

The first construction applies the well known Baranyai's theorem [52, p. 536]. The theorem can be stated in terms of set systems, but it is more known in the context of the decomposition of complete hypergraph on n vertices. The hyperedges of the complete hypergraph consist of all subsets of w vertices. In other words, the set of vertices is $[n]$ and the set of edges consists of all w-subsets of $[n]$. If w divides n then the Baranyai's theorem asserts that the hyperedges of this complete hypergraph can be decomposed into pairwise disjoint perfect matchings, where each matching consists of disjoint hyperedges (two hyperedges do not contain the same vertex) and each vertex is contained in exactly one hyperedge of the matching. Clearly, each perfect matching contains n/w hyperedges, and such a decomposition contains $\frac{w}{n}\binom{n}{w} = \binom{n-1}{w-1}$ disjoint matchings. How can such a decomposition can be used to construct (n,t,w)-LPC codes? The answer lies on a right choice of n, e.g., if $n = w(t+1+\epsilon)$, where ϵ is a nonnegative integer, we can use each matching in such a decomposition as a codeset. To prove that such a code is an (n,t,w)-LPC code, we have to prove that it satisfies Property $\mathbf{A}(t)$. Given a set $\mathcal{S} \subset [n]$ of size t with the numbers of the t hottest wires, and a perfect matching C, the elements of $\mathcal{S}$ are contained in at most t hyperedges. But, since a matching contains $t+1+\epsilon \geqslant t+1$ hyperedges, it follows that there is at least one hyperedge in the matching which does not contain any element from $\mathcal{S}$. Such a hyperedge is the codeword $\mathbf{x} \in C$ for which $\text{supp}(\mathbf{x}) \cap \mathcal{S} = \varnothing$. The most effective sets of parameters on which this construction can be applied are when $\epsilon = 0$, i.e. $(w(t+1), t, w)$-LPC code. Clearly, we can add codesets with codewords whose weight is less than w. When n is large, the contribution of such codesets is minor. When n is small such a contribution can be important. For example, if $n = 12$, $w = 3$, and $t = 3$, then there are 55 codesets if we restrict ourself only to codewords of weight 3. If we use all words of weight at most 3, then we can have a code with 81 codesets. The advantage is reduced for larger n — for example, if $n = 21$, $w = 3$ and $t = 6$. There are 190 codesets if only codewords of weight 3 are used. If there are codewords of weight at most 3, then we can have a code with 224 codesets, which is less dramatic improvements compared to the previous example.

What about encoding and decoding of this code? Here lies the big disadvantage of this method. Unless the parameters are relatively small there is no known efficient encoding and decoding algorithms. Anyway, in practice, usually small parameters are used and for many of these sets of parameters this method (code) is probably the most effective one.

B. Constructions Based on Cooling Codes Over $\mathbb{F}_q$

The low-power codes and the cooling codes are all binary codes, as a codeword indicates which transitions should be made on the bus-wires during the transmission. Hence, there is no use for non-binary codes as thermal codes. Nevertheless, non-binary codes might be useful in constructions of binary codes as will be demonstrated in this subsection.

*Definition 4: For positive integers n and $t < n$, and for a prime power q, an **$(n,t)_q$-cooling code** $\mathbb{C}$ of size M is defined as a set $\{C_1, C_2, \ldots, C_M\}$, where $C_1, C_2, \ldots, C_M$ are disjoint subsets of $\mathbb{F}_q^n$ satisfying the following property: for any set $\mathcal{S} \subset [n]$ of size $|\mathcal{S}| = t$ and for all $i \in [M]$, there exists a codeword $\mathbf{x} \in C_i$ with $\text{supp}(\mathbf{x}) \cap \mathcal{S} = \varnothing$.*

Similarly to Theorem 2 we can prove that

Lemma 2: Let $V_1, V_2, \ldots, V_M$ be a partial $(t+1)$-spread of $\mathbb{F}_q^n$, and define the code $\mathbb{C} = \{V_1^*, V_2^*, \ldots, V_M^*\}$, where $V_i^* = V_i \backslash \{\mathbf{0}\}$ for all i, $i = 1, 2, \ldots, M$. Then $\mathbb{C}$ is an $(n,t)_q$-cooling code of size $M \geqslant q^{n-t-1}$.

Theorem 2 can be applied when $t+1 \leqslant n/2$. When $t+1 > n/2$ we can generalize Theorem 3 to obtain $(n,t)_q$-cooling codes. The generalization is straightforward and hence it will be omitted.

A third construction, which is not a generalization of previous constructions, for $(n,t)_q$-cooling codes is based on the cosets of a dual code for a linear code over $\mathbb{F}_q$, whose minimum Hamming distance is at least $t+1$. We start with the related definitions and properties. For more information and proofs of the claims given, the reader can consult with [35].

Each $[n,\kappa,d]$ code $\mathbb{C}$ over $\mathbb{F}_q$ induces a partition of $\mathbb{F}_q^n$, where $\mathbb{C}_{\mathbf{z}}$, $\mathbf{z} \in \mathbb{F}_q^n$, is a part in this partition if

$$\mathbb{C}_{\mathbf{z}} = \{\mathbf{z} + \mathbf{c} : \mathbf{c} \in \mathbb{C}\}.$$

Each such part is called a *coset of* $\mathbb{C}$ and this partition contains $q^{n-\kappa}$ pairwise disjoint cosets.

Each $[n,\kappa,d]$ code $\mathbb{C}$ over $\mathbb{F}_q$ has a *dual code* $\mathbb{C}^\perp$, which is the dual subspace of $\mathbb{C}$. $\mathbb{C}^\perp$ is an $[n,n-\kappa,d']$ code.

A $\lambda \cdot q^t \times n$ matrix $\mathcal{A}$ with elements from $\mathbb{F}_q$ is called an *orthogonal array* with *strength* t if each t-tuple over $\mathbb{F}_q$ appears exactly λ times in each projection on any t columns of $\mathcal{A}$. The dual code $\mathbb{C}^\perp$ of an $[n,\kappa,d]$ code $\mathbb{C}$, is an orthogonal array of strength $d-1$.

Finally, the *Singleton bound* for an $[n,\kappa,d]$ code over $\mathbb{F}_q$ asserts that $d \leqslant n-\kappa+1$. A code which attains this bound, with equality, is called a *maximum distance separable* code (an *MDS* code in short).

Lemma 3: If there exists an $[n,\kappa,t+1]$ *code over* $\mathbb{F}_q$*, then there exists an* $(n,t)_q$*-cooling code of size* q^κ.

Proof: Let $\mathbb{C}$ be an $[n,\kappa,t+1]$ code over $\mathbb{F}_q$ and $\mathbb{C}^\perp$ be its dual code. $\mathbb{C}^\perp$ is an orthogonal array of strength t, i.e., each t-tuple over $\mathbb{F}_q$ appears the same number of times, in each projection of t columns of $\mathbb{C}^\perp$. Since the size of $\mathbb{C}^\perp$ is $q^{n-\kappa}$, it follows that each such t-tuple (including the all-zero n-tuple) appears $q^{n-\kappa-t}$ times in each projection (note that $q^{n-\kappa-t} > 0$ by the Singleton bound). Since a coset of $\mathbb{C}^\perp$ is formed by adding a fixed vector of length n to all the codewords of $\mathbb{C}^\perp$, it follows that each coset is also an orthogonal array of strength t. Therefore, the code $\mathbb{C}^\perp$ and its $q^\kappa - 1$ cosets, can be taken as q^κ codesets, to form an $(n,t)_q$-cooling code of size q^κ. ∎

Lemma 3 has some interesting consequences. First, one might ask why the construction implied by Lemma 3 was not given in Section IV? The answer is very simple. The binary codes obtained by this construction are not good enough as the codes presented in the constructions of Section IV.

A second and more important observation from Lemma 3 is a construction of a large $(n,t)_q$-cooling code if we use an MDS code as the original code $\mathbb{C}$. Recall that an $[n,\kappa,d]$ code is an MDS code if and only if $d = n-\kappa+1$. Moreover, the dual code $\mathbb{C}^\perp$ of an MDS code is also an MDS code. Finally, there is a well known conjecture about the range in which MDS codes can exist and there are MDS codes for all parameters in this range.

Conjecture 1: If $d \geqslant 3$*, then there exists an* $[n,\kappa,d]$ *MDS code over* $\mathbb{F}_q$ *if and only if* $n \leqslant q+1$ *for all* q *and* $2 \leqslant \kappa \leqslant q-1$*, except when* q *is even and* $\kappa \in \{3, q-1\}$*, in which case* $n \leqslant q+2$.

Theorem 5: If $d \geqslant 3$*, then there exists an* $[n,\kappa,d]$ *MDS code over* $\mathbb{F}_q$ *if* $n \leqslant q+1$ *for all* q *and* $2 \leqslant \kappa \leqslant q-1$*, except when* q *is even and* $\kappa \in \{3, q-1\}$*, in which case* $n \leqslant q+2$..

Corollary 7: If $n \leqslant q+1$*, then there exist an* $(n,t)_q$*-cooling code of size* q^{n-t}.

We continue to present a construction which transform an $(n,t)_q$-cooling code into an (ns,t,nw)-LPC code. This construction will be called a concatenation construction since it perform concatenations of the elements in $J^+(s,w)$ implied by an $(n,t)_q$-cooling code.

Theorem 6: If $q \leqslant \sum_{i=0}^{w} \binom{s}{i}$ *and there exists an* $(m,t)_q$*-cooling code of size* M*, then there exists an* (ms,t,mw)*-LPC code of size* M.

Proof: Let ψ be an injection from $\mathbb{F}_q$ to $J^+(s,w)$ such that $\psi(0) = \mathbf{0}$. Let Ψ be an injection from $\mathbb{F}_q^m$ to ${\mathbb{F}_q}^{ms}$ defined by $\Psi(x_1,x_2,\ldots,x_m) = (\psi(x_1),\psi(x_2),\ldots,\psi(x_m))$.

Let $\mathcal{C} = \{C_1,C_2,\ldots,C_M\}$ be an $(m,t)_q$-cooling code of size M. Let $\mathcal{D} = \{D_1,D_2,\ldots,D_M\}$ the image of $\mathcal{C}$ under Ψ, i.e. $D_i = \{\Psi(\mathbf{x}) : \mathbf{x} \in C_i\}$, for $1 \leqslant i \leqslant M$. We claim that $\mathcal{D}$ is an (ms,t,mw)-LPC code.

The length of codewords in the codesets of $\mathcal{D}$ is clearly ms and since $\psi(x_i) \leqslant w$ for each $x_i \in \mathbb{F}_q$ it follows that the weight of a codeword, in a codeset, is at most mw. It remains to show that for any given set $\mathcal{S} \subset [ms]$ of size t, and a codeset D_i, there exists a codeword $\mathbf{y}$ in D_i such that $\text{supp}(\mathbf{y}) \cap \mathcal{S} = \varnothing$. Since $\mathcal{C}$ is an $(m,t)_q$-cooling code, it follows that for any set $\mathcal{S}' \subset [m]$ of size t, the codeset C_i contains a codeword $\mathbf{x} = (x_1,x_2,\ldots,x_m)$ such that $\text{supp}(\mathbf{x}) \cap \mathcal{S}' = \varnothing$. If we partition the set of ms coordinates of the codewords in $\mathcal{D}$ into m consecutive sets $\mathcal{L}_1,\mathcal{L}_2,\ldots,\mathcal{L}_m$, where $\mathcal{L}_1$ contains the first s coordinates, $\mathcal{L}_2$ the next s coordinates, and so on, then the elements in $\mathcal{S}$ are contained in $t' \leqslant t$ of these sets, say, $I_{j_1},I_{j_2},\ldots,I_{j_{t'}}$. If we define $\mathcal{S}'' = \{j_1,j_2,\ldots,j_{t'}\}$, then $\mathcal{S}'' \subset [m]$ is a set of size $t' \leqslant t$. Hence, since $\mathcal{C}$ is an $(m,t)_q$-cooling code, it follows that there exists a codeword $\mathbf{z} \in C_i$ such that $\text{supp}(\mathbf{z}) \cap \mathcal{S}'' = \varnothing$. Since $\psi(0) = \mathbf{0}$, it follows from the definition of Ψ that $\Psi(\mathbf{z}) \in D_i$ and $\text{supp}(\Psi(\mathbf{z})) \cap \mathcal{S} = \varnothing$, i.e. we can take $\mathbf{y} = \Psi(\mathbf{z})$ to complete the proof. ∎

Theorem 6 can be combined with Theorem 2 and Corollary 7 to obtain the following two results.

Corollary 8: If $q \leqslant \sum_{i=0}^{w'} \binom{s}{i}$ *and* $t+1 \leqslant m/2$*, then there exists an* (ms,t,mw')*-LPC code of size* $M > q^{m-t-1}$.

Corollary 9: If $q \leqslant \sum_{i=0}^{w'} \binom{s}{i}$ *and* $m \leqslant q+1$*, then there exists an* (ms,t,mw')*-LPC code of size* q^{m-t}.

C. Constructions Based on Sunflowers

We examine the code obtained by the sunflower construction of Theorem 3. Note, that in the proof of Theorem 3 the codeword $\mathbf{z}$ of the related codeset used to show that $\text{supp}(\mathbf{z}) \cap \mathcal{S} = \varnothing$ has weight less than d. Therefore, we can remove from the codesets all codewords of weight greater than $d-1$. Thus, the sunflower construction yields an $(n,t,d-1)$-LPC code if the related constraints are satisfied as follows.

Corollary 10: Let n,t,s,r,d *be integers, such that* $r+t \leqslant (n+s)/2$*. If there exists an* $[n,s,d]$ *code and an* $[n-t,r,d]$

linear code does not exist, then there exists an $(n, t, d-1)$-LPC code of size $M > 2^{n-t-r}$.

Note, that a similar theorem can be obtained by using the sunflower of Theorem 4 which is based on the generalized Hamming weights.

VI. Error-Correcting Thermal Codes

In this section, we construct thermal codes that satisfy Property $\mathbf{C}(e)$ and simultaneously Property $\mathbf{A}(t)$ or Property $\mathbf{B}(w)$. The idea will be to modify and generalize the constructions which were given in the previous sections, by adding error-correction into the constructions. We will present constructions in the same order in which they were presented so far in this work. First, we discuss (n, w, e)-LPEC codes, for both nonadaptive and adaptive schemes. We continue with (n, t, e)-ECC codes, and conclude with (n, t, w, e)-LPECC codes.

A. Adaptive and Nonadaptive Low-Power Codes

For codes which satisfy simultaneously Properties $\mathbf{B}(w)$ and $\mathbf{C}(e)$, we consider first nonadaptive codes along the lines discussed in Section III. The type of codes which were considered in Section III are anticodes and in particular equireplicate anticodes. For this purpose we will use again set systems and in particular a family of block design called Steiner systems. A *Steiner system* $S(r, w, n)$ is a collection $\mathcal{B}$ of w-subsets (called *blocks*) from the n-set $[n]$ such that each r-subset of $[n]$ is contained in exactly one block of $\mathcal{B}$. The blocks of such a set system can be translated into a binary code $\mathbb{C}$ of length n and constant weight w for the codewords. It is easy to verify that the code $\mathbb{C}$ has minimum Hamming distance $2(w-r+1)$, i.e. the code $\mathbb{C}$ can correct any $w-r$ errors and can detect any $w-r+1$ errors. The diameter of the related code is at most $2w$, and it is less than $2w$ if an only if there is a nonempty intersection between any two codewords. If we are not restricted to equireplicated then we can use constant weight codes instead of Steiner systems. Constant weight codes have many applications and hence they were intensively investigated throughout the years, e.g. [7] and references therein. A constant weight code can be equireplicate, but it does not have to be such a code. In this subsection we consider only equireplicate codes. A Steiner system is equireplicate, so such systems will be the basis of our construction. Information on the known Steiner systems can be found in the main textbooks on block designs, e.g. [5]. There are well-known necessary conditions for the existence of a Steiner system $S(r, w, n)$. For each i, $0 \leqslant i \leqslant r$ the number $\binom{n-i}{r-i}/\binom{w-i}{r-i}$ is an integer. It was recently proved in [24] and [28] that for any given $0 < r < w$, these necessary conditions are also sufficient, except for a finite number of cases.

As was discussed in Section III, to have at most w transitions on the bus wires in the nonadaptive scheme, the minimum distance of our code must be at most w, and each codeword must be of weight at most $w/2$ if w is even, and $(w+1)/2$ if w is odd. For simplicity we will assume that w is even. Now, assume that we want our code to have Property $\mathbf{C}(e)$. For this purpose we need a Steiner system $S(w/2-e, w/2, n)$. If $2e \leqslant w/2$ then we can add codewords of weight $w/2-2e$. The number of possible such codewords is negligible compared to the number of codewords with weight $w/2$ and hence we omit these possible codewords in our discussion. Similar codes can be constructed for other parameters. Assume there exists a system set $\mathcal{B}$ which is a Steiner system $S(w/2-e, w/2, n+1)$ on the point set $[n+1]$. Consider the set

$$\mathcal{B}' \stackrel{\text{def}}{=} \{X : X \in \mathcal{B},\ n+1 \notin X\} \bigcup \{X \setminus \{n+1\} \ :\ X \in \mathcal{B},\ n+1 \in X\}.$$

It is easy to verify that $\mathcal{B}'$ is an (n, w, e)-LPEC code. Moreover, this code is also equireplicate and can be used for a nonadaptive scheme.

In contrast to (n, w)-LP codes, where an optimal anticode $\mathbb{C}$ with codewords of weight $w/2$ implies that $\mathbb{C}$ has diameter w, the situation when we consider also Property $\mathbf{C}(e)$ can be slightly different. Clearly, the diameter of the code should be w, but the weight of a codeword can be larger than $w/2$. For example, if $w/2 = q$, $n = q^2+q+1$, and q is a prime power, then an optimal such system consists of q^2+q+1 blocks which form a projective plane of order q [52, p. 224]. Recall that such a structure was considered also in subsection III-B. The blocks of such a projective plane form a Steiner system $S(2, q+1, q^2+q+1)$, where any two distinct blocks intersect in exactly one point and hence the diameter of the related code is $2q = w$ as required. The related code can correct any $q-1$ errors and can detect any q errors.

Finally, for parameters where related Steiner systems do not exist or no efficient construction for such systems is known, we can use similar constant weight codes based on the rich literature of such codes.

For adaptive (n, w)-LPEC codes we use as in most of our exposition the differential encoding method. The codes which are used are Steiner systems as in the nonadaptive case. The only difference is that the weight of the codewords will be at most w and not $w/2$ or $(w+2)/2$ as in the nonadaptive case. The number of errors which are corrected is defined by the Steiner system as we discussed in this subsection.

B. Error-Correcting Cooling Codes

In this subsection we will adapt the construction based on spreads and the sunflower construction, given in Section IV, to form codes which satisfy Property $\mathbf{A}(t)$ and Property $\mathbf{C}(e)$. The idea is start with a binary $[n, \kappa, 2e+1]$ code $\mathbb{C}$ which corrects e errors. In $\mathbb{C}$ there exists at least one set S of κ coordinates whose projection on $\mathbb{C}$ is $\mathbb{F}_2^\kappa$. This set of coordinates is called a *systematic* set of coordinates. On this set of S coordinates, we either apply the spread construction or the sunflower construction to obtain a code which satisfies simultaneously Property $\mathbf{A}(t)$ and Property $\mathbf{C}(e)$.

For the construction which is based on a partial $(t+1)$-spread, we start with our favorite binary $[n, \kappa, 2e+1]$ code $\mathbb{C}$, where $\kappa \geqslant 2(t+1)$. In addition, we take a partial $(t+1)$-spread (or a $(t+1)$-spread if $t+1$ divides κ) of $\mathbb{F}_2^\kappa$. Since $\mathbb{C}$ has dimension κ, there exists at least one set of κ coordinates whose projection on $\mathbb{C}$ spans $\mathbb{F}_2^\kappa$. The partial $(t+1)$-spread can be formed on these coordinates. The codewords of $\mathbb{C}$ are partitioned into codesets related to this partial spread. Given any t coordinates, each codeset has at least one codeword with *zeroes* in these t coordinates since the partial spread has dimension $t+1$. This is proved exactly as in the proof of Theorem 2. Moreover, the code can correct at least e errors since the codesets are disjoint and all the codewords in the codesets are contained in the code $\mathbb{C}$. Thus, we have constructed an (n, t, e)-ECC code. We summary this construction with the following theorem.

Theorem 7: If there exists a binary $[n, \kappa, 2e+1]$ *code and* $\kappa \geqslant 2(t+1)$*, then there exists an* (n, t, e)*-ECC code of size* $M > 2^{\kappa-t-1}$.

The sunflower construction is adapted in a similar way to obtain an (n, t, e)-ECC code. We start with our favorite binary $[n, \kappa, 2e+1]$ code $\mathbb{C}$ and apply the sunflower construction on κ systematic coordinates in $\mathbb{C}$. Similarly to Theorem 3 we have the following theorem

Theorem 8: Let n, t, s, r, κ, d*, be integers such that* $r+t \leqslant (\kappa+s)/2$ *and there exists a binary* $[n, \kappa, 2e+1]$ *code. If there exists a* $[\kappa, s, d]$ *code and a binary* $[\kappa-t, r, d]$ *code does not exist, then there exists an* (n, t, e)*-ECC code of size* $M > 2^{\kappa-t-r}$.

Similarly, to Theorem 8 we can adapt the construction of Theorem 4 to obtain an (n, t, e)-ECC code.

C. Constructions of Low-Power Error-Correcting Cooling Codes

In this subsection we consider codes which satisfy all Properties $\mathbf{A}(t)$, $\mathbf{B}(w)$, and $\mathbf{C}(e)$ simultaneously. The related codes are (n, t, w, e)-LPECC codes. We suggest three methods to construct such codes, which reflect the three methods in Section V, where only Properties $\mathbf{A}(t)$ and $\mathbf{B}(w)$, without Property $\mathbf{C}(e)$, were considered. The first method, generalizes the decomposition of the complete hypergraph as described in subsection V-A, by considering resolvable Steiner systems. The second method is based on dual codes and concatenation modifies the construction in subsection V-B. The last construction is based on the sunflower construction similarly to subsection V-C.

Our first method is a generalization for the decomposition of the complete hypergraph into pairwise disjoint perfect matchings. For this generalization we consider the hyperedges as blocks in a system set, or more precisely in a block design. The related concepts in block design are *resolution* and *parallel classes*. A block design (set system) is said to be *resolvable* if the blocks of the design can be partitioned into pairwise disjoint sets, called *parallel classes*, where each class forms a partition of the point sets into pairwise disjoint blocks. The whole process is called *resolution*. By this definition, the decomposition of the complete hypergraph on n vertices and hyperedges of size w is a resolution of all w-subsets of $[n]$. To have low-power error-correcting cooling codes we will use resolutions of Steiner system. Recall that a Steiner system $S(r, w, n)$ is a binary code with minimum Hamming distance $2(w-r+1)$ and hence it can correct any $w-r$ errors and can detect any $w-r+1$ errors. Since we are interested in an (n, t, w, e)-LPECC code, we should start with a Steiner system $S(w-e, w, w(t+1))$ and partition it into pairwise disjoint Steiner systems $S(1, w, n)$ (which are the parallel classes). Such partitions of Steiner systems are known in several cases given as follows:

1) If $n \equiv 3 \pmod 6$ then there exists a resolvable $S(2, 3, n)$ [25].
2) If $n \equiv 4 \pmod{12}$ then there exists a resolvable $S(2, 4, n)$ [25].
3) If $n \equiv w (\text{mod } w(w-1))$ then for sufficiently large n there exist a resolvable $S(2, w, n)$ [41].
4) If $n \equiv 4$ or $8 (\text{mod } 12)$ then there exists a resolvable $S(3, 4, n)$. This was proved in [23] for all n, except for 23 cases which were completed in [26].
5) If q is a prime power then there exists a resolvable $S(2, q, q^2)$ derived from affine plane (which can be generated from a projective plane of order q which is equivalent to $S(2, q+1, q^2+q+1)$).

These resolvable Steiner systems imply the existence of the following (n, t, w, e)-LPECC codes:

Theorem 9: If $n = 3(t+1)$*, where* t *is even, then there exists an* $(n, t, 3, 1)$*-LPECC code of size larger than* $\frac{n-1}{2}$.

Proof: If t is even, then $n = 3(t+1) \equiv 3 (\text{mod } 6)$, and hence there exists a resolvable Steiner triple system of order n whose size is $\frac{n(n-1)}{2 \cdot 3}$, with $\frac{n-1}{2}$ parallel classes, each one of size $\frac{n}{3}$. The addition of the all-zero vector of length n in a new codeset enlarge the size of the code. ■

Similarly, we have

Theorem 10: If $n = 4(t+1) \equiv 4$ *or* $8 (\text{mod } 12)$*, then there exists an* $(n, t, 4, 1)$*-LPECC code of size larger than* $\frac{(n-1)(n-2)}{6}$.

Theorem 11: If $n = 4(t+1) \equiv 4 (\text{mod } 12)$*, then there exists an* $(n, t, 4, 2)$*-LPECC code of size* $\frac{(n-1)(n-2)}{12}$.

Theorem 12: If $n = w(t+1) \equiv w (\text{mod } w(w-1))$*, then for sufficiently large* n *there exists an* $(n, t, w, w-2)$*-LPECC code of size* $\frac{(n-1)(n-2)}{w(w-1)}$.

Theorem 13: If q *is a prime power, then there exists a* $(q^2, q-1, q, q-2)$*-LPECC code of size* q^2.

The constructions derived, in subsection V-B, from codes over $\mathbb{F}_q$ are based on the codes constructed in Corollaries 8 and 9. These corollaries are derived from the concatenation construction presented in Theorem 6. To adapt these results to form (n, t, w, e)-LPECC codes, we use constructions of two types. The first one starts with an error-correcting code $\mathbb{C}$ over $\mathbb{F}_q$, partition of the codewords of $\mathbb{C}$ by using a partial spread, and apply the concatenation construction on the codesets derived from the partial spread. The second type of construction is based on the (n, t, w)-LPC codes derived from dual codes of MDS codes (see Corollary 9).

For the first construction we start with an $[m, \kappa, 2e+1]$ code $\mathbb{C}$ over $\mathbb{F}_q$, such that $\kappa \geqslant 2(t+1)$. In addition, we take a partial $(t+1)$-spread of $\mathbb{F}_q^\kappa$. Since $\mathbb{C}$ has dimension κ, it follows that there exists at least one set of κ systematic coordinates whose projection spans $\mathbb{F}_q^\kappa$. The partial $(t+1)$-spread can be formed on these κ systematic coordinates. The codewords of $\mathbb{C}$ can be partitioned into codesets related to this partial spread and form a new code $\mathbb{C}'$. Given any set $\mathcal{S}$ of size t, each codeset has at least one codeword with *zeroes* in the t coordinates of $\mathcal{S}$, since the subspaces of the partial spread have dimension $t+1$. The proof is exactly as the proof of Theorem 2. Moreover, the code $\mathbb{C}'$ can correct e errors, since the codesets of $\mathbb{C}'$ are disjoint and all the codewords in the codesets are contained in the code $\mathbb{C}$ which can correct any e errors. Finally, let $q \leqslant \sum_{i=0}^{w'} \binom{s}{i}$, and we use the concatenation construction given in Theorem 6 and Corollary 8. Thus, we obtain an (ms, t, mw', e)-LPECC code. The size of the code $\mathbb{C}'$ depends on the largest dimension of an $[m, \kappa, 2e+1]$ code $\mathbb{C}$, subject to the requirement that $\kappa \geqslant 2(t+1)$. The resulting code $\mathbb{C}'$ will have size $q^{\kappa-t-1}$.

The second construction is an immediate consequence of Corollary 9. The minimum Hamming distance of the low-power cooling code obtained in Corollary 9 is the same as the minimum Hamming distance of the related MDS code of length m, $m \leqslant q+1$. Therefore, the size of the code is q^{m-t} and its minimum Hamming distance $t+1$.

Corollary 11: If $q \leqslant \sum_{i=0}^{w'} \binom{s}{i}$ and $m \leqslant q+1$, then there exists an $(ms, t, mw', \lfloor t/2 \rfloor)$-LPECC code of size q^{m-t}.

Finally, we want to adapt the sunflower construction to form an (n, t, w, e)-LPECC code. The idea is to use the arguments of Corollary 10, in the construction implied by Theorem 8, which yields the following result

Corollary 12: Let n, t, s, r, κ, d, be integers such that $r+t \leqslant (\kappa+s)/2$ and there exists a binary $[n, \kappa, 2e+1]$ code. If there exists a $[\kappa, s, d]$ code and a binary $[\kappa-t, r, d]$ code does not exist, then there exists an $(n, t, d-1, e)$-LPECC code of size $M > 2^{\kappa-t-r}$.

Similarly, to Corollary 12 we can adapt the construction of Theorem 4 to obtain an (n, t, w, e)-LPECC code.

VII. Asymptotic Behavior

In this section we will analyze the asymptotic behavior of the thermal codes which were constructed in the previous sections. We start in subsection VII-A where we consider only cooling codes. As was proved in Section IV, when $t+1 \leqslant n/2$ our codes which use only $t+1$ redundancy bits are optimal and the number of additional wires used is negligible when k or n are large enough. Hence, the interesting case is the asymptotic behavior when $t+1 > n/2$ and the sunflower construction is used. The case is even more complicated when we consider low-power cooling codes. Our methods with efficient encoding and decoding algorithms are not optimal, but this does not exclude the possibility of being asymptotically optimal. The asymptotic behavior of these codes will be considered in subsection VII-B. In addition, to find asymptotically good codes, we will consider in this subsection a new Gilbert-Varshamov type method, called the expurgation method.

A. Asymptotic Behavior of Cooling Codes

Let $B(n, d)$ be the largest dimension of a binary $[n, \kappa, d]$ code. For $0 < \delta,\ \tau < 1$, define

$$\beta(\delta) \stackrel{\text{def}}{=} \limsup_{n\to\infty} \frac{B(n, \lfloor \delta n \rfloor)}{n},$$

$$\varrho(\tau) \stackrel{\text{def}}{=} \limsup_{n\to\infty} \frac{\log_2 C(n, \lfloor \tau n \rfloor)}{n}.$$

By Lemma 1, we have that any (n, t)-cooling code of size M, satisfies $M \leqslant 2^{n-t}$. This immediately implies the following asymptotic upper bound on the rate of an (n, t)-cooling code.

Corollary 13: If $0 < \tau < 1$, then

$$\varrho(\tau) \leqslant 1 - \tau.$$

The following proposition for computing an asymptotic lower bound on the rate of an (n, t)-cooling code.

Proposition 6: Assume $0 < \epsilon < 1$ satisfies the following conditions:

$$\epsilon > \beta(\delta) \tag{22}$$

$$\tau + \epsilon(1-\tau) < \frac{1 + \beta(\delta(1-\tau))}{2}. \tag{23}$$

Then $\varrho(\tau) \geqslant (1-\tau)(1-\epsilon)$.

Proof: By the definition of β, there exists N_1 such that

$$B(M, \lfloor \delta M \rfloor) < \epsilon M, \tag{24}$$

for all $M \geqslant N_1$.

If $\delta' = \delta(1-\tau)$ and $0 < \epsilon_1 < 1$, then for any given large enough integer N_2, there exists an integer $N \geqslant N_2$, such that

$$B(N, \lfloor \delta' N \rfloor) \geqslant \beta(\delta')N - \epsilon_1 N. \tag{25}$$

Set $s = \lfloor \beta(\delta')N - \epsilon_1 N \rfloor$, $d = \lfloor \delta' N \rfloor$, $t = \lceil \tau N \rceil$, $M = N - t$, and $r = \lceil \epsilon M \rceil$. We claim that the conditions of Theorem 3 are met for these parameters.

Observe that

$$\begin{aligned} t + r &= \lceil \tau N \rceil + \lceil \epsilon M \rceil = \lceil \tau N \rceil + \lceil \epsilon(N-t) \rceil \\ &= \lceil \tau N \rceil + \lceil \epsilon(N - \lceil \tau N \rceil) \rceil \\ &\leqslant (\tau N + 1) + (\epsilon(1-\tau)N + 1) \\ &= \tau N + \epsilon(1-\tau)N + 2. \end{aligned} \tag{26}$$

On the other hand,

$$\begin{aligned} \frac{N+s}{2} &= \frac{N + \lfloor \beta(\delta')N - \epsilon_1 N \rfloor}{2} \\ &\geqslant \frac{N + \beta(\delta')N - \epsilon_1 N - 1}{2}. \end{aligned} \tag{27}$$

Now, let

$$N_2 \geqslant \frac{5/2}{(1 + \beta(\delta') - \epsilon_1)/2 - (\tau + \epsilon(1-\tau))}. \tag{28}$$

Furthermore, since (23) holds, it follows that there exists $0 < \epsilon_2 < 1$, such that for all $0 < \epsilon_1 < \epsilon_2$ the denominator in the right hand side of (28) is strictly positive. Hence, since $N > N_2$, we have that

$$\left((1+\beta(\delta') - \epsilon_1)/2 - (\tau + \epsilon(1-\tau))\right) N \geqslant 5/2,$$

or,

$$\tau N + \epsilon(1-\tau)N + 2 \leqslant \frac{N + \beta(\delta')N - \epsilon_1 N - 1}{2}. \quad (29)$$

Combining (29) with Inequalities (26) and (27), we have that $t + r \leqslant (N+s)/2$, and the first condition of Theorem 3 is satisfied.

Next, since $s \leqslant \beta(\delta')N - \epsilon_1 N$, it follows from (25) that there exists an $[N, s, d]$ code, and the second condition of Theorem 3 is satisfied. Finally, observe that

$$M = N - t \geqslant N - (\tau N + 1) = (1-\tau)N - 1. \quad (30)$$

If we choose

$$N_2 \geqslant \frac{N_1 + 1}{1-\tau}, \quad (31)$$

then since $N \geqslant N_2$, it follows by (31) that $N(1-\tau) \geqslant N_1 + 1$ which implies by (30) that $M \geqslant N_1$. Thus, from (24), we infer that $B(M, \lfloor \delta M \rfloor) < \epsilon M \leqslant r$. Since $\lfloor \delta M \rfloor \leqslant \lfloor \delta(1-\tau)N \rfloor = d$, it follows that an $[N-t, r, d]$ code does not exist, and the third condition of Theorem 3 is satisfied.

In summary, to satisfy the three conditions of Theorem 3, we need N_2 to satisfy (28) and (31). Additionally, to guarantee that $N \geqslant N_0$ we require that $N_2 \geqslant N_0$ and hence we have that

$$N_2 = \max\left\{N_0, \frac{N_1+1}{1-\lambda}, \frac{5/2}{(1+\beta(\delta')-\epsilon_1)/2 - (\lambda + \epsilon(1-\lambda))}\right\}.$$

Therefore, by Theorem 3 there exists an (N, t)-cooling code of size greater than 2^{N-t-r}. Now,

$$\begin{aligned} N - t - r &= N - \lceil \tau N \rceil - \lceil \epsilon M \rceil \\ &= N - \lceil \tau N \rceil - \lceil \epsilon (N - \lceil \tau N \rceil) \rceil \\ &\geqslant N - \tau N - \epsilon N - \epsilon \tau N - 2 \\ &= N(1-\tau)(1-\epsilon) - 2. \end{aligned}$$

This implies that there exists an (N, t)-cooling code of size greater than $2^{(1-\lambda)(1-\epsilon)N-2}$. Since $t = \lceil \lambda N \rceil \geqslant \lfloor \lambda N \rfloor$, it follows that there also exists an $(N, \lfloor \lambda N \rfloor)$-cooling code of size greater than $2^{(1-\lambda)(1-\epsilon)N-2}$. Thus,

$$\frac{\log_2 C(N, \lfloor \lambda N \rfloor)}{N} \geqslant (1-\lambda)(1-\epsilon) - \frac{2}{N},$$

which implies that $\varrho(\tau) \geqslant (1-\tau)(1-\epsilon)$. ■

To apply Proposition 6 we have to use the best known lower bound on $\beta(\cdot)$ in (22) and the best known upper bound on $\beta(\cdot)$ in (23). For lower bound we will use the Gilbert-Varshamov bound [21], [51]. For upper bound we will use the McEliece-Rodemich-Rumsey-Welch (MRRW) bound [36]. Specifically, we have

1) For the lower bound, the Gilbert-Varshamov bound implies that $\beta(\delta) \geqslant 1 - H(\delta) + o(1)$, where $H(\cdot)$ is the binary entropy function, given by

$$H(x) \stackrel{\text{def}}{=} -x \log_2 x - (1-x)\log_2(1-x).$$

2) As for the upper bound, the MRRW bound [36, eq. 1.4] implies that

$$\begin{aligned} \beta(\delta) &\leqslant \text{MRRW}(\delta) \\ &= \min_{0 \leqslant u \leqslant 1-2\delta} 1 + g\left(u^2\right) - g\left(u^2 + 2\delta u + 2\delta\right), \end{aligned} \quad (32)$$

where $g(x) = H((1-\sqrt{1-x})/2)$.

Hence, we have the following corollary.

Corollary 14: Suppose that $0 < \epsilon, \delta, \tau < 1$ *satisfy the following conditions:*

$$\epsilon > \text{MRRW}(\delta), \quad (33)$$

$$\tau + \epsilon(1-\tau) < \frac{2 - H(\delta(1-\tau))}{2}. \quad (34)$$

Then $\varrho(\tau) \geqslant (1-\tau)(1-\epsilon)$.

Proof: Since $\epsilon > \text{MRRW}(\delta)$ by (33) and $\beta(\delta) \leqslant \text{MRRW}(\delta)$ by (32), it follows that $\epsilon > \beta(\delta)$ and hence Inequality (22) is satisfied.

From the Gilbert-Varshamov bound, we have that $\beta(\delta(1-\tau)) \geqslant 1 - H(\delta(1-\tau)) + o(1)$. Hence, we have

$$\begin{aligned} \tau + \epsilon(1-\tau) &< \frac{2 - H(\delta(1-\tau))}{2} \\ &= \frac{1 + (1 - H(\delta(1-\tau)))}{2} \\ &\leqslant \frac{1 + \beta(\delta(1-\tau))}{2}. \end{aligned}$$

Therefore, Inequality (23) is satisfied.

Since the conditions in Proposition 6 are satisfied, it follows from the proposition that $\varrho(\tau) \geqslant (1-\tau)(1-\epsilon)$. ■

Now, fix $0 < \tau < 1$ and in what follows, we compute a lower bound on $\varrho(\tau)$ implied by Corollary 14. Define the set

$$E(\tau) \stackrel{\text{def}}{=} \{\epsilon : \text{there exists a } \delta \text{ such that } \epsilon, \delta \text{ satisfy (33) and (34)}\}.$$

This definition implies that by Corollary 14, for all $\epsilon \in E(\tau)$, we can have that $\varrho(\tau) \geqslant (1-\tau)(1-\epsilon)$. Maximizing the right hand side, we obtain

$$\varrho(\tau) \geqslant (1-\tau)(1-\epsilon^*), \quad (35)$$

where ϵ^* is the value of ϵ for which $(1-\tau)(1-\epsilon)$ is maximized when ϵ^* is substituted for ϵ. For the next computation we define $E^*(\tau) \stackrel{\text{def}}{=} \epsilon^*$. Via numerical computations, we have that

$$E^*(\tau) = \begin{cases} \approx 0, & \text{if } 0 < \tau \leqslant 0.687 \\ \text{between 0 and 1}, & \text{if } 0.687 \leqslant \tau \leqslant 0.737 \\ \approx 1 & \text{if } 0.737 \leqslant \tau < 1. \end{cases}$$

The plot of the lower bound (35) for $\varrho(\tau)$ is given in Figure 4. Without giving the formal analyze we also add a curve for the

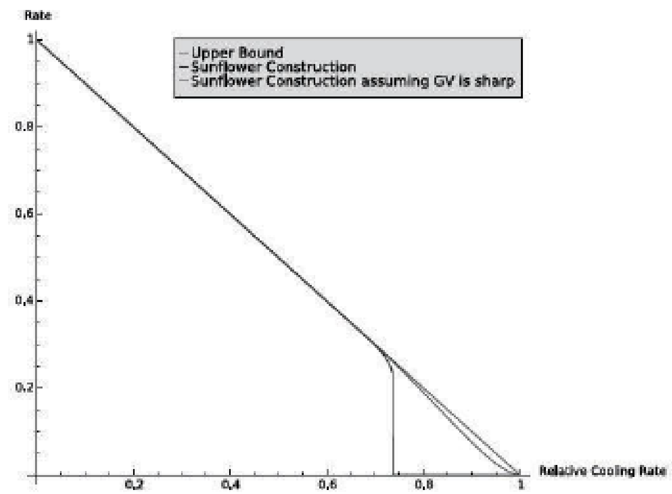

Fig. 4. Asymptotic Rates for Cooling Codes.

case that the Gilbert-Varshamov bound is tight (at least in the binary case).

Finally, we obtain the following result which is consistent with the figure, and proves that for $\tau \leqslant 0.687$ our constructions are asymptotically optimal.

Corollary 15: If $\tau \leqslant 0.687$, then $\varrho(\tau) = 1 - \tau$.

Proof: If $\tau \leqslant 0.687$, then the value of $1 - H((1-\tau)/2)/(1-\tau)$ is strictly positive. If we set $\delta = 1/2$, then (33) and (34) are reduced to

$$\epsilon > 0 \tag{36}$$

$$\tau + \epsilon(1-\tau) < \frac{2 - H((1-\tau)/2)}{2}. \tag{37}$$

As a consequence, $E(\tau)$ contains the interval $(0, 1 - H((1-\tau)/2)/(1-\tau))$. Hence, the expression $(1-\tau)(1-\epsilon)$ is maximized for ϵ which tends to 0. Therefore, $\varrho(\tau) \geqslant 1 - \tau$ and since by Corollary 13, we have $\varrho(\tau) \leqslant 1 - \tau$, it follows that $\varrho(\tau) = 1 - \tau$. ■

B. Asymptotic Behavior of Low-Power Cooling Codes

The asymptotic analysis when the codes have Property $\mathbf{B}(w)$ in addition to Property $\mathbf{A}(t)$ is more complicated, needless to say that our methods are slightly less efficient compared to the methods used to construct codes when only $\mathbf{A}(t)$ is satisfied. Let $C(n, t, w)$ be the largest size of an (n, t, w)-LPC code and for a fixed $0 < \tau < 1$ and a fixed $0 < \omega < 1$, consider the asymptotic rate

$$\varrho(\tau, \omega) \stackrel{\text{def}}{=} \limsup_{n\to\infty} \frac{\log_2 C(n, \lfloor \tau n \rfloor, \lfloor \omega n \rfloor)}{n}.$$

If $t = o(n)$ and $0 < \omega < 1$, then we claim that Corollary 8 provides a family of $(n, t, \lfloor \omega n \rfloor)$-LPC codes whose rates approach at least $H(\omega)$. Formally, we claim that if $\mathbb{C}_n$ is such an $(n, t, \lfloor \omega n \rfloor)$-LPC code, then

$$\lim_{n\to\infty} \frac{\log_2 |\mathbb{C}_n|}{n} \geqslant H(\omega). \tag{38}$$

To prove (38), consider any given $\epsilon > 0$, a prime power q and an integer $s > 1$ such that

$$q \leqslant \sum_{i=0}^{\lfloor \omega s \rfloor} \binom{s}{i} \text{ and } \frac{\log_2 q}{s} \geqslant H(\omega) - \epsilon.$$

For each n, set $m = \lfloor n/s \rfloor$. Since $t = o(n)$, it follows that $t + 1 \leqslant m/2$ for sufficiently large n. Applying Corollary 8, we obtain an $(n, t, \lfloor \omega n \rfloor)$-LPC code $\mathbb{C}_n$ of size at least q^{m-t-1}. Now, we have that

$$\begin{aligned}\lim_{n\to\infty} \frac{\log |\mathbb{C}_n|}{n} &\geqslant \lim_{n\to\infty} \frac{\log_2 q}{s}\left(1 - \frac{t+1}{m}\right)\\ &\geqslant (H(\omega) - \epsilon) \cdot \lim_{n\to\infty}\left(1 - \frac{t+1}{m}\right)\\ &= H(\omega) - \epsilon,\end{aligned}$$

and hence $\lim_{n\to\infty} \frac{\log |\mathbb{C}_n|}{n} \geqslant H(\omega)$, where the last equality follows since $t = o(n)$.

The next theorem describes a very simple construction which will be called in the sequel the *expurgation construction.*

Theorem 14: If $M > \sum_{i=w+1}^{n} \binom{n}{i}$ and an $(n, t)_q$-cooling code $\mathbb{C}$ of size M exists, then there exists an (n, t, w)-LPC code of size at least $M - \sum_{i=w+1}^{n} \binom{n}{i}$.

Proof: The theorem follows as an immediate consequence implied by removing all codewords of weight larger than w from all the codesets of $\mathbb{C}$. ■

Corollary 16: If $\tau \leqslant 0.687$, $\omega \geqslant 1/2$, and $H(\omega) < 1 - \tau$, then $\varrho(\tau, \omega) \geqslant 1 - \tau - o(1)$.

Proof: By the sunflower construction (see Theorem 3) there exists a family of (n, t)-cooling codes whose size is at least $2^{n(1-\tau-\epsilon')}$, where $t = \lfloor \tau n \rfloor$, $r = \lfloor \epsilon' n \rfloor$, and $\epsilon' = o(1)$ (see subsection VII-A). It is well known by using the Stirling's approximation [35, p. 310] that if $w = \lfloor \omega n \rfloor$, then $\sum_{i=w+1}^{n} \binom{n}{i} \leqslant 2^{nH(\omega)}$.

Now, applying the expurgation construction on a code from this family yields an (n, t, w)-LPC code whose size is at least $2^{n(1-\tau-\epsilon')}\left(1 - 2^{n(H(\omega)-1+\tau+\epsilon')}\right)$. In other words, the rate, $\varrho(\tau, \omega)$, of this family of (n, t)-cooling codes is at least

$$(1 - \tau - \epsilon') + \frac{\log\left(1 - 2^{n(H(\omega)-1+\tau+\epsilon')}\right)}{n},$$

which is at least $1 - \tau - \epsilon'$. ■

Clearly, $\varrho(\tau, \omega) \leqslant \varrho(\tau)$ and hence combining Corollaries 13 and 16 implies that

Corollary 17: If $\tau \leqslant 0.687$, $\omega \geqslant 1/2$, and $H(\omega) < 1 - \tau$, then $\varrho(\tau, \omega) = 1 - \tau$.

Unfortunately, Theorem 14 is nonconstructive and the domain of (τ, ω), where Corollary 16 is applicable, is limited. In the sequel we will consider other parameters outside this domain. It should be no surprise that in most cases, we observed that codes obtained by the sunflower construction have larger size than those obtained by the expurgation construction. Nevertheless, in certain instances, the LPC codes obtained from Theorem 14 has a larger size as compared to LPC codes resulting from the sunflower construction. For such an example, consider $t = 1$, $w = \frac{2}{3}(n-1)$, where 3 divides $n-1$. By Proposition 4, there exists an $(n, 1)$-cooling code of size 2^{n-1}. Theorem 14 yields an $(n, 1, \frac{2}{3}(n-1))$-LPC code of size at least $2^{n-1} - \sum_{i=w+1}^{n} \binom{n}{i}$. Note that

$$2^{n-1} - \sum_{i=w+1}^{n} \binom{n}{i} \geqslant 2^{n-1} - 2^{nH(2/3)} \geqslant 2^{n-1} - 2^{0.92n}. \quad (39)$$

By the Griesmer bound [35, p. 546], there is no $[n-1, 2, \frac{2}{3}(n-1)+1]$ code and hence Theorem 3 yields an $(n, 1, \frac{2}{3}(n-1))$-LPC code of size $M_2(n, 3) + 1$.

Since

$$\frac{5}{14} 2^n > 2^{0.92n} + 1 \text{ for } n \geqslant 19,$$

it follows that $2^{n-1} - 2^{0.92n} > M_2(n, 3) + 1$, and hence the expurgation construction yields an (n, t, w)-LPC code of larger size, compared to the code obtained by the sunflower construction, in this case.

Recall, that Corollary 10 is used to apply the sunflower construction and obtain $(n, t, d-1)$-LPC codes of size greater than $M_2(n-s, r+t-s) > 2^{n-t-r}$, where r is given in Corollary 10. Furthermore, for $0 < \tau, \ \omega < 1$, let $\delta = \omega/(1-\tau)$ and define $\epsilon = \epsilon(\tau, \delta)$ as

$$\epsilon(\tau, \delta) \stackrel{\text{def}}{=} \inf\{\epsilon : \epsilon \text{ satisfy (33) and (34)}\}.$$

With this setting, Corollary 14 implies that $\varrho(\tau, \omega) \geqslant (1-\tau)(1-\epsilon)$, and by Corollary 15 and its proof we have

Corollary 18: If $\tau \leqslant 0.687$ and $\omega \geqslant (1-\tau)/2$, then $\varrho(\tau, \omega) = 1 - \tau$.

We continue to examine the asymptotic behavior of (n, t, w)-LPC codes constructed from cooling codes over $\mathbb{F}_q$ with concatenation.

Corollary 19: For given $0 < \tau, \ \omega < 1$, suppose that there exists a prime power q and an integer $s > 1$ such that

$$q \leqslant \sum_{i=0}^{\lfloor \omega s \rfloor} \binom{s}{i} \quad \text{and} \quad \tau s \leqslant \frac{1}{2}. \quad (40)$$

Then $\varrho(\tau, \omega) \geqslant (1 - \tau s)\frac{\log_2 q}{s}$.

Proof: For a given n, set $m = \lceil \frac{n}{s} + \frac{1}{\tau s} \rceil$ and $t = \lfloor \tau n \rfloor$. Since $\tau s \leqslant 1/2$, if follows that

$$t + 1 \leqslant \tau n + 1 = \left(\frac{n}{s} + \frac{1}{\tau s}\right) \tau s \leqslant m/2.$$

Hence, by Corollary 8, there exists an $(ms, t, m\lfloor \omega s \rfloor)$-LPC code of size at least q^{m-t-1}. Therefore, we have that

$$\begin{aligned}
\varrho(\tau, \omega) &\geqslant \lim_{ms \to \infty} \frac{m-t-1}{ms} \log_2 q \\
&= \frac{\log_2 q}{s} \lim_{ms \to \infty} \frac{sm - st}{ms} \\
&= \frac{\log_2 q}{s} \lim_{ms \to \infty} \left(1 - \frac{t}{m}\right) \\
&= \frac{\log_2 q}{s} \lim_{ms \to \infty} \left(1 - \frac{\tau n}{n/s + 1/(\tau s)}\right) \\
&> \frac{\log_2 q}{s} \lim_{ms \to \infty} \left(1 - \frac{\tau n}{n/s}\right) \\
&= \frac{\log_2 q}{s}(1 - \tau s).
\end{aligned}$$

■

Note, that to apply Corollary 19 we have to be careful in choosing q and s such that equation (40) is satisfied.

VIII. Conclusion and Future Research

High temperatures have dramatic negative effects on interconnect performance and, hence, it is important to suggest techniques to reduce the power consumption of on-chip buses. We have suggested coding techniques to balance and reduce the power consumption of on-chip buses. Our codes can have three features (properties). A code is a "cooling code" if there are no transitions on the t hottest wires (Property $\mathbf{A}(t)$). A code has "low-power" if it reduces the number of transitions on the bus wires to at most w transitions (Property $\mathbf{B}(w)$). A code is "error-correcting" if it can correct any e errors of bus transitions (Property $\mathbf{C}(e)$). A code can have some of these properties. Six subsets out of the possible eight subsets of these properties are interesting in our context and they are considered in the following sections and subsections:

1) (n, w)-low-power codes ((n, w)-LP codes) were considered in Section III and subsection IV-A.
2) (n, t)-cooling codes were considered in Section IV.
3) (n, t, w)-low-power cooling codes ((n, t, w)-LPC codes) were considered in Section V.
4) (n, w, e)-low-power error-correcting codes ((n, w, e)-LPEC codes) were considered in subsection VI-A.
5) (n, t, e)-error-correcting cooling codes ((n, t, e)-ECC codes) were considered in subsection VI-B.
6) (n, t, w, e)-low-power error-correcting cooling codes ((n, t, w, e)-LPECC codes) were considered in subsection VI-C.

Our cooling codes without error-correction are optimal when $t+1 \leqslant n/2$ and can be proved to be optimal also in some cases when $t + 1 > n/2$. In all these cases the redundancy of these codes is $t + 1$. This redundancy, compared to the related best known error-correcting code, is also obtained for error-correcting cooling codes. Finally, the asymptotic analysis shows that in most cases our codes are asymptotically optimal.

For the combination of low-power cooling code with or without error-correction, our codes fall short of the known upper bounds and closing this gap is one of the problems for future research. In these cases, we would like to see efficient encoding and decoding algorithms. We would also like to improve our bounds and have asymptotic optimal codes for the case were $\tau = \frac{t}{n} > 0.687$. Finally, we would like to see more cases where sunflower construction with the generalized Hamming weights implies a large code. As we mention in the Introduction, a follow up work will present more construction, especially for practical parameters. Examples and numerical experiments will be given and also comparison between the various constructions with emphasis on practical parameters. It will be illustrated and discussed how practical our methods and constructions are.

APPENDIX

TABLE I
ADMISSIBLE PARAMETERS FOR SUNFLOWER CONSTRUCTIONS

n	t	r	s	d
3	2	1	1	3
6	3	1	2	4
7	3	1	1	7
7	4	1	3	4
8	4	1	2	5
8	5	1	4	4
9	4	1	1	9
9	5	1	3	5
10	5	1	2	6
10	6	2	6	3
11	5	1	1	11
11	6	1	3	6
11	7	2	7	3
12	6	1	2	8
12	7	1	4	6
12	8	2	8	3
13	6	1	1	13
13	7	1	3	7
13	8	2	7	4
13	9	2	9	3
14	7	1	2	9
14	8	1	4	7
14	9	2	8	4
14	10	2	10	3
15	7	1	1	15
15	8	1	3	8
15	9	1	5	7
15	10	2	9	4
15	11	2	11	3
16	8	1	2	10
16	9	1	4	8
16	10	2	8	5
16	11	2	10	4
17	8	1	1	17
17	9	1	3	9
17	10	1	5	8
17	11	2	9	5
17	12	2	11	4
18	9	1	2	12
18	10	2	9	6
18	11	1	6	8
18	12	3*	12	4
18	13	2	12	4
19	9	1	1	19
19	10	1	3	10
19	11	2	7	8
19	12	1	7	8
19	13	3*	13	4
19	14	2	13	4
20	10	1	2	13
20	11	1	4	10
20	12	2	8	8
20	13	1	8	8
20	14	3*	14	4
20	15	2	14	4
21	10	1	1	21
21	11	1	3	12
21	12	1	5	10
21	13	2	9	8
21	14	1	9	8
21	15	3	15	4
21	16	2	15	4
22	11	1	2	14
22	12	1	4	11
22	13	2	8	8
22	14	2	10	8
22	15	1	10	8
22	16	3	16	4
22	17	2	16	4
23	11	1	1	23
23	12	1	3	12
23	13	1	5	11
23	14	2	9	8
23	15	2	11	8
23	16	1	11	8
23	17	3	17	4
23	18	2	17	4
24	12	1	2	16
24	13	1	4	12
24	14	2	8	8
24	15	2	10	8
24	16	2	12	8
24	17	1	12	8
24	18	3	18	4
24	19	2	18	4
25	12	1	1	25
25	13	1	3	14
25	14	1	5	12
25	15	2	9	8
25	16	2	11	8
25	17	2	13	6
25	18	2	15	4
25	19	3	19	4
25	20	2	19	4
26	13	1	2	17
26	14	1	4	13
26	15	1	6	12
26	16	2	10	8
26	17	2	12	8
26	18	2	14	6
26	19	2	16	5
26	20	3	20	4
26	21	2	20	4
27	13	1	1	27
27	14	1	3	15
27	15	1	5	13
27	16	1	7	12
27	17	2	11	8
27	18	2	13	8
27	19	2	15	6
27	20	2	17	5
27	21	3	21	4
27	22	2	21	4
28	14	1	2	18
28	15	1	4	14
28	16	2	8	11
28	17	2	10	10
28	18	2	12	8
28	19	2	14	8
28	20	2	16	6
28	21	2	18	5
28	22	3	22	4
28	23	2	22	4
29	14	1	1	29
29	15	1	3	16
29	16	1	5	14
29	17	1	9	11
29	18	2	11	9
29	19	2	13	8
29	20	2	15	7
29	21	2	17	6
29	22	2	19	5
29	23	3	23	4
29	24	2	23	4
30	15	1	2	20
30	16	1	4	16
30	17	1	6	14
30	18	2	10	11
30	19	2	12	9
30	20	2	14	8
30	21	2	16	7
30	22	2	18	6
30	23	2	20	5
30	24	3	24	4
30	25	2	24	4
31	15	1	1	31
31	16	1	3	17
31	17	1	5	16
31	18	2	9	12
31	19	2	11	11
31	20	2	13	9
31	21	2	15	8
31	22	2	17	7
31	23	2	19	6
31	24	2	21	5
31	25	3	25	4
31	26	2	25	4
32	16	1	2	21
32	17	1	4	16
32	18	1	6	16
32	19	2	10	12
32	20	2	12	10
32	21	2	14	8
32	22	2	16	8
32	23	3	20	6
32	24	2	20	6
32	25	2	22	5
32	26	3	26	4
32	27	2	26	4
33	16	1	1	33
33	17	1	3	18
33	18	1	5	16
33	19	2	9	12
33	20	2	11	12
33	21	2	13	10
33	22	2	15	8
33	23	2	17	8
33	24	3	21	6
33	25	2	21	6
33	26	2	23	5
34	17	1	2	22
34	18	1	4	17
34	19	1	6	16
34	20	2	10	12
34	21	2	12	12
34	22	2	14	10
34	23	2	16	8
34	24	2	18	8
34	25	3	22	6
34	26	2	22	6
35	17	1	1	35
35	18	1	3	20
35	19	1	7	16
35	20	1	7	16
35	21	2	11	12
35	22	2	13	11
35	23	2	15	10
35	24	2	17	8
35	25	2	19	8
35	26	3	23	6
35	27	2	23	6
36	18	1	2	24
36	19	1	4	18
36	20	1	6	16
36	21	2	10	15
36	22	2	12	12
36	23	2	14	11
36	24	2	16	10
36	25	2	18	8
36	26	2	20	8
36	27	3	24	6
36	28	2	24	6
37	18	1	1	37
37	19	1	3	20
37	20	1	5	18
37	21	2	9	15
37	22	2	11	13
37	23	2	13	12
37	24	2	15	10
37	25	2	17	9
37	26	2	19	8
37	27	2	21	7
37	28	3	25	6
37	29	2	25	6
38	19	1	2	25
38	20	1	4	20
38	21	1	6	18
38	22	2	10	14
38	23	2	12	13
38	24	2	14	12
38	25	2	16	10
38	26	2	18	9
38	27	2	20	8
38	28	2	22	7
38	29	3	26	6
38	30	2	26	6
39	19	1	1	39
39	20	1	3	22
39	21	1	5	19
39	22	2	9	16
39	23	2	11	14
39	24	2	13	12
39	25	2	15	12
39	26	2	17	10
39	27	2	19	9
39	28	2	21	8
39	29	2	23	7
39	30	3	27	6
39	31	2	27	6
40	20	1	2	26
40	21	1	4	20
40	22	2	10	16
40	23	2	12	14
40	24	2	14	12
40	25	2	16	11
40	26	2	18	10
40	27	2	20	9
40	28	2	22	8
40	29	2	24	7
40	30	3	28	6
40	31	2	28	6
41	20	1	1	41
41	21	1	3	23
41	22	1	5	20
41	23	2	9	16
41	24	2	11	16
41	25	2	13	14
41	26	2	15	12
41	27	2	17	11
41	28	2	19	10
41	29	2	21	9
41	30	2	23	8
41	31	2	25	7
41	32	3	29	6
41	33	2	29	6
42	21	1	2	28
42	22	1	4	22
42	23	1	6	20
42	24	2	10	16
42	25	2	12	15
42	26	2	14	13
42	27	2	16	12
42	28	2	18	12
42	29	2	20	10
42	30	3	24	8
42	31	2	24	8
42	32	2	26	7
42	33	3	30	6
42	34	2	30	6
43	21	1	1	43
43	22	1	3	24
43	23	1	5	21
43	24	1	7	20
43	25	2	11	16
43	26	2	13	15
43	27	2	15	13
43	28	2	17	12
43	29	2	19	12
43	30	2	21	10
43	31	3	25	8
43	32	2	25	8
43	33	2	27	7
43	34	3	31	6
43	35	2	31	6
44	22	1	2	29
44	23	1	4	25
44	24	1	6	21
44	25	2	10	17
44	26	2	12	16
44	27	2	14	15
44	28	2	16	13
44	29	2	18	12
44	30	2	20	12
44	31	2	22	10
44	32	3	26	8
44	33	2	26	8
44	34	2	28	7
44	35	3	32	6
44	36	2	32	6
45	22	1	1	45
45	23	1	3	25
45	24	1	5	22
45	25	2	9	18
45	26	2	11	16
45	27	2	13	16
45	28	2	15	14
45	29	2	17	12
45	30	2	19	12
45	31	2	21	12
45	32	2	23	10
45	33	3	27	8
45	34	2	27	8
45	35	2	29	7
45	36	3	33	6
45	37	2	33	6
46	23	1	2	30
46	24	1	4	24
46	25	1	6	22
46	26	2	10	18
46	27	2	12	16
46	28	2	14	16
46	29	2	16	14
46	30	2	18	12
46	31	2	20	12
46	32	2	22	12
46	33	2	24	10
46	34	3	28	8
46	35	2	28	8
46	36	2	30	7
46	37	3	34	6
46	38	2	34	6
47	23	1	1	47
47	24	1	3	26
47	25	1	5	24
47	26	1	7	22
47	27	2	11	18
47	28	2	13	16
47	29	2	15	16
47	30	2	17	14
47	31	2	19	12
47	32	2	21	12
47	33	2	23	12
47	34	2	25	9
47	35	3	29	8
47	36	2	29	8
47	37	2	31	7
47	38	3	35	6
47	39	2	35	6
48	24	1	2	32
48	25	1	4	26
48	26	1	6	24
48	27	1	8	22
48	28	2	12	17
48	29	2	14	16
48	30	2	16	16
48	31	2	18	13
48	32	2	20	12
48	33	2	22	12
48	34	2	24	12
48	35	2	26	9
48	36	3	30	8
48	37	2	30	8
48	38	3	32	6
48	39	3	36	6
48	40	2	36	6
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TABLE I

(Continued.) ADMISSIBLE PARAMETERS FOR SUNFLOWER CONSTRUCTIONS

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TABLE I
(Continued.) ADMISSIBLE PARAMETERS FOR SUNFLOWER CONSTRUCTIONS

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99	61	2	27	27
99	62	2	29	26
99	63	2	31	25
99	64	2	33	24
99	65	2	35	24
99	66	3	39	20
99	67	3	41	20
99	68	3	43	20
99	69	3	45	20
99	70	2	45	20
99	71	2	47	20
99	72	3	51	16
99	73	4	55	14
99	74	4	57	14
99	75	3	57	14
99	76	5	63	12
99	77	4	63	12
99	78	4	65	11
99	79	5	69	10
99	80	4	69	10
99	81	4	71	9
99	82	6	77	8
99	83	8	83	6
99	84	7	83	6
99	85	6	83	6
99	86	5	83	6
99	87	5	85	5
99	88	7	91	4
99	89	6	91	4
99	90	5	91	4
100	50	1	2	66
100	51	1	4	52
100	52	1	6	49
100	53	1	8	48
100	54	2	12	42
100	55	2	14	40
100	56	2	16	38
100	57	2	18	36
100	58	2	20	34
100	59	2	22	32
100	60	2	24	32
100	61	2	26	30
100	62	2	28	27
100	63	2	30	26
100	64	2	32	25
100	65	2	34	24
100	66	2	36	23
100	67	3	40	20
100	68	3	42	20
100	69	3	44	20
100	70	3	46	20
100	71	2	46	20
100	72	3	48	20
100	73	3	52	16
100	74	4	56	14
100	75	4	58	14
100	76	3	58	14
100	77	5	64	12
100	78	4	64	12
100	79	4	66	11
100	80	5	70	10
100	81	4	70	10
100	82	4	72	9
100	83	6	78	8
100	84	8	84	6
100	85	7	84	6
100	86	6	84	6
100	87	5	84	6
100	88	5	86	5
100	89	7	92	4
100	90	6	92	4
100	91	5	92	4

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Yeow Meng Chee (SM'08) received the B.Math. degree in computer science and combinatorics and optimization and the M.Math. and Ph.D. degrees in computer science from the University of Waterloo, Waterloo, ON, Canada, in 1988, 1989, and 1996, respectively.

Currently, he is a Professor at the Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore. Prior to this, he was Program Director of Interactive Digital Media R&D in the Media Development Authority of Singapore, Postdoctoral Fellow at the University of Waterloo and IBMs Zürich Research Laboratory, General Manager of the Singapore Computer Emergency Response Team, and Deputy Director of Strategic Programs at the Infocomm Development Authority, Singapore.

His research interest lies in the interplay between combinatorics and computer science/engineering, particularly combinatorial design theory, coding theory, extremal set systems, and electronic design automation.

Tuvi Etzion (M'89–SM'94–F'04) was born in Tel Aviv, Israel, in 1956. He received the B.A., M.Sc., and D.Sc. degrees from the Technion - Israel Institute of Technology, Haifa, Israel, in 1980, 1982, and 1984, respectively.

From 1984 he held a position in the Department of Computer Science at the Technion, where he now holds the Bernard Elkin Chair in Computer Science. During the years 1985-1987 he was Visiting Research Professor with the Department of Electrical Engineering - Systems at the University of Southern California, Los Angeles. During the summers of 1990 and 1991 he was visiting Bellcore in Morristown, New Jersey. During the years 1994-1996 he was a Visiting Research Fellow in the Computer Science Department at Royal Holloway University of London, Egham, England. He also had several visits to the Coordinated Science Laboratory at University of Illinois in Urbana-Champaign during the years 1995-1998, two visits to HP Bristol during the summers of 1996, 2000, a few visits to the Department of Electrical Engineering, University of California at San Diego during the years 2000-2017, and several visits to the Mathematics Department at Royal Holloway University of London, during the years 2007-2017.

His research interests include applications of discrete mathematics to problems in computer science and information theory, coding theory, network coding, and combinatorial designs.

Dr. Etzion was an Associate Editor for Coding Theory for the IEEE TRANSACTIONS ON INFORMATION THEORY from 2006 till 2009. From 2004 to 2009, he was an Editor for the *Journal of Combinatorial Designs*. From 2011 he is an Editor for *Designs, Codes, and Cryptography*, and from 2013 an Editor for *Advances of Mathematics in Communications*.

Han Mao Kiah received his Ph.D. degree in mathematics from Nanyang Technological University (NTU), Singapore, in 2014. From 2014 to 2015 he was a Postdoctoral Research Associate at the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign. Currently he is a Lecturer at the School of Physical and Mathematical Sciences, NTU, Singapore. His research interests include combinatorial design theory, coding theory, and enumerative combinatorics.

Alexander Vardy (S'88–M'91–SM'94–F'99) was born in Moscow, U.S.S.R., in 1963. He earned his B.Sc. (summa cum laude) from the Technion, Israel, in 1985, and Ph.D. from the Tel-Aviv University, Israel, in 1991. During 1985–1990 he was with the Israeli Air Force, where he worked on electronic counter measures systems and algorithms. During the years 1992 and 1993 he was a Visiting Scientist at the IBM Almaden Research Center, in San Jose, CA. From 1993 to 1998, he was with the University of Illinois at Urbana-Champaign, first as an Assistant Professor then as an Associate Professor. Since 1998, he has been with the University of California San Diego (UCSD), where he is the Jack Keil Wolf Endowed Chair Professor in the Department of Electrical and Computer Engineering and the Department of Computer Science. While on sabbatical from UCSD, he has held long-term visiting appointments with CNRS, France, the EPFL, Switzerland, the Technion, Israel, and Nanyang Technological University, Singapore.

His research interests include error-correcting codes, algebraic and iterative decoding algorithms, lattices and sphere packings, coding for storage systems, cryptography and computational complexity theory, as well as fun math problems.

He received an IBM Invention Achievement Award in 1993, and NSF Research Initiation and CAREER awards in 1994 and 1995. In 1996, he was appointed Fellow in the Center for Advanced Study at the University of Illinois, and received the Xerox Award for faculty research. In the same year, he became a Fellow of the David and Lucile Packard Foundation. He received the IEEE Information Theory Society Paper Award (jointly with Ralf Koetter) for the year 2004. In 2005, he received the Fulbright Senior Scholar Fellowship, and the Best Paper Award at the IEEE Symposium on Foundations of Computer Science (FOCS). In 2017, his work on polar codes was recognized by the the IEEE Communications & Information Theory Societies Joint Paper Award.

During 1995–1998, he was an Associate Editor for Coding Theory and during 1998–2001, he was the Editor-in-Chief of the IEEE TRANSACTIONS ON INFORMATION THEORY. From 2003 to 2009, he was an Editor for the *SIAM Journal on Discrete Mathematics*. He is currently serving on the Executive Editorial Board for the IEEE TRANSACTIONS ON INFORMATION THEORY. He has been a member of the Board of Governors of the IEEE Information Theory Society during 1998–2006, and again during 2011–2017.