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# Time-Scale Transformed Iterative Learning Control for a Class of Nonlinear Systems With Uncertain Trial Duration

Patrick M. Sammons, David Hoelzle, and Kira Barton

***Abstract*— Iterative learning control (ILC) is a powerful tool for improving the tracking performance in systems characterized by trial-repetitive behavior, including exogenous signals such as references and disturbances, through repeated update of control signals based on previous error histories. The application of ILC algorithms to physical systems typically requires trial-to-trial invariance that is expressed as a set of assumptions on the operation of the process. Among these assumptions is that the duration of the system trajectory is trial-invariant. In some physical processes, however, the trial duration may be a function of system outputs or states that may force the trial to be either shorter or longer than expected. This results in situations in which the trial duration is unknown *a priori*, breaking the typical invariance assumptions, but in which updating the control input is still desired. In order to address this class of processes, the approach proposed here uses a time-scale transformation to nondimensionalize the trial duration. An ILC algorithm is then proposed on the nondimensional time scale, and it is shown that the output error converges asymptotically toward the origin. A learning operator design is proposed, which provides a direct tradeoff between convergence rate and steady-state error. Simulation results on a benchmark nonlinear system demonstrate the efficacy of the proposed approach.**

***Index Terms*— Iterative learning control (ILC), time-scale transformation, uncertain systems.**

## I. Introduction

ITERATIVE LEARNING CONTROL (ILC) is a feedforward control scheme that attempts to improve dynamic system tracking performance by utilizing historical input and output data. ILC is characterized by operation over two dimensions—a bounded interval (typically time or space) along which the nominal system dynamics that evolve can be either continuous or discrete and an unbounded discrete interval, termed the trial domain, along which the control input is updated. On each trial, the historical data are filtered through a control update law to generate control inputs for the next trial. In this way, repetitive disturbances can be rejected and tracking performance can be improved over conventional feedback controllers that do not have the ability to incorporate historical trial data.

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ILC has received considerable attention in the past several decades for applications ranging from robotics to chemical manufacturing processes [1]. The conventional implementation of an ILC algorithm in these applications requires the invocation of several invariance assumptions. In short, the system, which includes the plant, exogenous signals, trial duration, and initial conditions, is trial-invariant [2], [3]. While these assumptions seem restrictive at first glance, many physical systems and processes, during normal operation, satisfy these assumptions. However, there are also many systems that do not satisfy all of these assumptions despite operating in a repetitive, also termed trial-to-trial, manner. In these cases, analyses, and algorithms that leverage these analyses, are needed to understand the implications of relaxing these assumptions.

Relaxation of the constraint on matched initial conditions has seen some of the earliest attention as this is arguably the most difficult to completely satisfy in practice [4], [5]. Methods to treat systems with trial-varying plant dynamics have been proposed [6], [7]. Further, algorithms for tracking trial-varying references have also been proposed, for example, the internal model in [8], the basis function approach in [9], and the "library"-based approaches in [10]–[13]. In [12] and [13], a direct learning approach is taken where the control input is computed from a complete knowledge of the plant and the iteration-to-iteration feedback loop is broken. In addition, the goal of the methods proposed therein is to learn new input signals for different desired references, given a "library" of previous outputs and inputs. In [14], linear time-varying systems are treated in a similar fashion to [15] where sections of systems signals are zeroed out to ensure learning only on observed intervals. However, attempting to learn a single trajectory in the face of uncertain trial duration for nonlinear systems without requiring manual treatment of the control signal has received relatively little attention.

Repetitive systems with trial-varying trial duration occur in several applications. For example, in human gait, the duration of each gait cycle (with a complete gait cycle as the trial analog) can vary, in addition to the duration of the stance and swing sections of the gait cycle [16]. The uncertain trial length arising in human gait systems was treated using a discrete-time linear system lifted framework in [15] and [17]. In this approach, the lifted control update signal is manually set to zero when the trial duration is shorter than the maximum expected duration. A similar class of gait systems is approached using a repetitive control methodology, i.e., repetitive in the time-domain, in [18] where a time-sample scaling factor is used to correct for errors in cycle duration. Moore [19] and Moore and Mathews [20] proposed a discontinuous update law to toggle learning based on the trial duration and applied the algorithm to the Gas-Metal Arc Welding process to improve mass flow rate repeatability. Another example of uncertain trial length in manufacturing is electrohydrodynamic jet (e-jet) printing where the time-to-ejection and duration of ejection vary as a function of the input [21]. In addition to the application-oriented approaches above, other approaches treat the trial duration as a random variable [22], [23], and ensure convergence with respect to an expectation operator. In contrast to the approaches mentioned above, the algorithm proposed here addresses a more general class of systems, nonlinear control-affine systems with *a priori* unknown trial duration, via a time-scale transformation, inspired by [24] and [25] and similar to the approach in [18]. In contrast to [24], [25], however, here a distinct time-scale transformation is applied on each trial. Despite this, the simplicity of conventional proportional ILC update laws is retained. The time-scale transformation scales the output trajectory to a nondimensional trial duration over which the control law is updated. This leads to a robust control-oriented methodology with a tradeoff between tracking accuracy and system uncertainty, in the form of the uncertain trial duration. Existing literature regarding trial-dependent duration has addressed a wide range of systems. The focus of the work presented here, however, is on nonlinear systems where the trial duration is assumed unknown *a priori*, but bounded. Further, the proposed control update law does not require a manual restriction of the control input to the observed duration and is the only signal that undergoes an update or transform trial-to-trial.

The rest of the brief is structured as follows. First, in Section II, some preliminaries are given to aid in the development of the ILC method. In Section III, the statement and assumptions of the control problem are presented. Section IV presents the main result of the brief showing asymptotic convergence. In addition, Section IV provides detail regarding control design. In Section V, the proposed control methodology is applied to a benchmark nonlinear system in two simulation cases. Finally, a summary of the work, as well as conclusions and areas for future work, are given in Section VI.

## II. Preliminaries

This section summarizes some necessary preliminaries for the results presented later in this brief.

*Definition 1 (Lie Derivative):* Let $f : \mathbb{D} \to \mathbb{R}^n$ and $h : \mathbb{D} \to \mathbb{R}$ with $\mathbb{D} \subset \mathbb{R}^n$. The Lie Derivative of $h$ along $f$ is defined as

$$L_f h(x) = \frac{\partial h}{\partial x} f(x).$$

Repeated application of the Lie Derivative is denoted as

$$L_f^k h(x) = L_f L_f^{k-1} h(x) = \frac{\partial \left(L_f^{k-1} h(x)\right)}{\partial x} f(x)$$

with $L_f^0 h(x) = h(x)$. The Lie Derivative of $h$ first along $f$ then along $g : \mathbb{D} \to \mathbb{R}^n \times \mathbb{R}^m$ is defined as

$$L_g L_f h(x) = \frac{\partial (L_f h(x))}{\partial x} g(x).$$

The Lie Derivative has a rich history in nonlinear control systems. The reader is directed to other sources, for example [26], for a deeper treatment.

*Definition 2 (Relative Degree):* Let $f$, $h$, and $g$ be defined as above, where $\dot{x} = f(x) + g(x)u$ and $y = h(x)$. Then, the system has relative degree $\eta$ at $x = x_0$ if

1) $L_g L_f^k h(x) = 0$ for $k < \eta - 1$ and in a neighborhood of $x_0$.
2) $J(x) = L_g L_f^{\eta-1} h(x) \neq 0$ in a neighborhood of $x_0$.

*Definition 3 (Time-Weighted Norm [27]):* The time-weighted norm of $l \in \mathcal{X}$, or the $\omega$-norm, is defined as

$$\|l(t)\|_\omega = \sup_t \|l(t)\| e^{-\omega t}$$

where $\omega > 0$, $\|\cdot\|$ is any norm on $\mathcal{X}$, and $t \in [0, T]$. The $\omega$-norm is equivalent to the supremum norm

$$\|l(t)\|_\omega \le \|l(t)\|_\infty \le e^{\omega} \|l(t)\|_\omega.$$

*Definition 4 (Time-Scale Transformation):* A time-scale transformation $\mathcal{T} : [0, T] \to [0, 1]$ maps absolute time coordinates $t \in [0, T]$, $0 < T < \infty$, to a nondimensional unit interval $\tau$ and is written as

$$\tau = \mathcal{T}(t)$$

where $\tau \in [0, 1]$. A special case of the time-scale transformation is the constant scaling

$$\mathcal{T}(t) = \frac{t}{T}.$$

*Remark 1 (Time-Scale Transformation Derivatives):* For a time-scale transformation $\mathcal{T}$, transformed signal derivatives can be calculated from the absolute time-scale signal as

$$x^{(k)}(t) = \left(\frac{d\tau}{dt}\right)^k x^{(k)}(\tau)$$

where $x^{(k)}$ denotes the $k$th time derivative of $x$ for $k \in \mathbb{Z}_+$ and $(\cdot)^k$ denotes the exponentiation operation. For the special case of a constant time-scale transformation given above, signal derivatives of $x(\tau)$ with $t \in [0, T]$ are calculated as

$$x^{(k)}(\tau) = T^k x^{(k)}(t).$$

## III. Problem Formulation

Consider a nonlinear system of the form

$$\begin{aligned} \dot{x}_j(t) &= f(x_j(t)) + g(x_j(t))u_j(t) \\ y_j(t) &= h(x_j(t)) \end{aligned} \tag{1}$$

where $j \in \mathbb{Z}_+$ is the trial index, $t \in [0, T_j]$ is the trial time index, $T_j$ is the trial duration, $x \in \mathbb{R}^n$ is the state vector, $u \in \mathcal{U} \subset \mathbb{R}^m$ is the input vector, $\mathcal{U}$ is a bounded subset of $\mathbb{R}^m$, $y \in \mathbb{R}^p$ is the output vector, $f : \mathbb{R}^n \to \mathbb{R}^n$ is the state mapping, $g : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ is the input mapping, and $h : \mathbb{R}^n \to \mathbb{R}^p$ is the output mapping.

Let $\mathcal{T}_j(t) = (t/T_j)$ be a time-scale transformation that is applied on each trial. Then, under the time-scale transformation in conjunction with Remark 1, (1) is

$$\begin{aligned} \dot{x}_j(\tau) &= f_j(x_j(\tau)) + g_j(x_j(\tau))u_j(\tau) \\ y_j(\tau) &= h(x_j(\tau)) \end{aligned} \tag{2}$$

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where $f_j = T_j f$ and $g_j = T_j g$. Note that the output mapping $h$ is unchanged. Assume the following about the system.

A1. The trial duration on trial $j$ is unknown *a priori*, but bounded, $T_j \in [T_l, T_u]$ for all $j \in \mathbb{Z}_+$ and $0 < T_l < T_u < \infty$. The trial duration can be determined *a posteriori*.

A2. The mapping from input $u$ to output $y$ (through the mapping from $u$ to $x$) is one-to-one and there exists a control input $u_\infty$ such that

$$\begin{aligned}\dot{x}_\infty(t) &= f(x_\infty(t)) + g(x_\infty(t))u_\infty(t)\\ r(t) &= h(x_\infty(t))\end{aligned}$$

where $r$ is the system reference, $x_\infty$ is the reference state trajectory, $t \in [0, T_\infty]$, and $T_\infty$ is the known reference duration. The time-scale transformed reference system is

$$\begin{aligned}\dot{x}_\infty(\tau) &= f_\infty(x_\infty(\tau)) + g_\infty(x_\infty(\tau))u_\infty(\tau)\\ r(\tau) &= h(x_\infty(\tau))\end{aligned}$$

A3. The mappings in (1) are Lipschitz with respect to $x$ for all $t$, and where appropriate, Lipschitz in $T$ for all $T_j \in [T_l, T_u]$. That is, there exist constants $\rho_{f,x}$, $\rho_{f,T}$, $\rho_{g,x}$, and $\rho_{g,T}$ such that

$$\begin{aligned}\|T_1 f(x_1(\tau)) - T_2 f(x_2(\tau))\| &\le \rho_{f,x}\|x_1(\tau) - x_2(\tau)\|\\ &\quad + \rho_{f,T}|T_1 - T_2|\\ \|T_1 g(x_1(\tau)) - T_2 g(x_2(\tau))\| &\le \rho_{g,x}\|x_1(\tau) - x_2(\tau)\|\\ &\quad + \rho_{g,T}|T_1 - T_2|\end{aligned}$$

for all $x_i \in \mathbb{R}^n$, $\tau \in [0, 1]$, and $T_j \in [T_l, T_u]$. In addition, there exists a constant $\rho_{h,x}$ such that

$$\|h(x_1(\tau)) - h(x_2(\tau))\| \le \rho_{h,x}\|x_1(\tau) - x_2(\tau)\|.$$

A4. The relative degree of the nonlinear system is $\eta$ for all $t$

A5. The reference signal is continuously differentiable at least $\eta$ times.

Assumption A1 ensures that both the trial duration is nontrivial, i.e., $T_j > 0$, and that a trial occurs in finite time. In addition, a bounded trial duration, both above and away from zero, ensures that the time-scale transformation is bounded. Further, in order for a time-scale transformation to be possible using previous trial information, the trial duration must be accessible at the completion of a trial. Note, however, that it is not assumed that there is knowledge of the trial duration bounds. Assumption A2 ensures that a reference trajectory exists and that, if a control input is chosen correctly, it is possible to reconstruct the reference trajectory. Assumption 2 is necessary for the result provided here as convergence in the input space is used as an intermediate step to show convergence in the error space. While it may be possible to relax this assumption, it is not considered here. Knowledge of the reference trial duration follows from Assumption A1. In addition, knowledge of the reference trajectory implies knowledge of the reference trial duration, $T_\infty$. As mentioned in the introduction, applications where this hold include the droplet ejection dynamics in e-jet printing where the desired time to ejection (or interval between ejection events) is a design parameter, [21], [28], time between gait phases in human gait systems [18], and many motion control problems. Therefore, the reference duration is known *a priori*. Assumption A3 ensures both continuity of the mappings $f$, $g$, $h$, and the Lie derivatives of the system in (1). Continuous differentiability of $f$ and $g$ is not assumed. Finally, Assumptions A4 and A5 are required to iteratively apply the update law that is proposed in Section IV. Implicit in A4 is the fact that $h$ is continuously differentiable at least $\eta$ times

## IV. ILC Design and Analysis

In this section, the assumptions and problem formulation given above are leveraged to present an appropriate ILC update law and to show the convergence of the algorithm.

The unique challenge of attempting to implement conventional ILC on systems where the trial length is uncertain *a priori* is that control and error signals on different trials are defined over different intervals. Attempting to treat these systems in absolute time coordinates would require marking different sections of the output trajectories as suitable and not suitable for learning. However, this leads to a discontinuous update law and specific switching logic for the control signal. An alternative is to scale each trial duration to a common interval and implement the update algorithm on the common interval. Once the transformation to a common interval is performed, conventional ILC update laws can be leveraged as the transformed system is equivalently trial-invariant. This is the approach taken here.

Let $e_j(t) = r(t) - y_j(t)$ be the tracking error defined over $t \in [0, T_j]$. As above, the $i$th time derivative of $e_j$ is denoted by $e_j^{(i)}$. In this brief, an ILC update law of the form

$$u_{j+1}(\tau) = q_j(\tau) * \left(u_j(\tau) + \Gamma(\tau)e_j^{(\eta)}(\tau)\right) \tag{3}$$

where $\tau \in [0, 1]$, $q_j$ is a robustness filter, $\Gamma$ is the static learning gain, and $(\cdot)*(\cdot)$ represents the convolution operation, is considered. The filter $q_j$ is a common element of ILC laws that aim to remove unwanted frequency content from the control law. Note that while the learning gain $\Gamma$ is time-varying, there does not exist any dynamics between it and the error term $e_j^{(n)}$. For the sake of clarity and compactness in presentation, only the single-in, single-out (SISO) case is considered here. However, extension to the multi-in multi-out (MIMO) case is straightforward with appropriate modifications to the proof. Details of these modifications are provided in the Appendix. Below is the main result of the work.

*Theorem 1 (SISO Case, $p = m = 1$):* The error of the system (2) and (3) converges asymptotically to a ball centered at the origin, with respect to $j$, if

$$\sup_{\substack{\tau\in[0,1]\\ \forall j\in\mathbb{Z}_+}} \|q_j(\tau) * (1 - \Gamma(\tau)J(x_\infty))\| = \rho < 1 \tag{4}$$

where $J(x_\infty) = L_{g_\infty}L_{f_\infty}^{\eta-1}h(x_\infty(\tau))$.

*Proof:* First, note the error derivatives can be written as

$$\begin{aligned} e_j^{(\eta)}(\tau) &= r^{(\eta)}(\tau) - y_j^{(\eta)}(\tau) \\ &= \frac{d^{\eta-1}}{d\tau^{\eta-1}}\frac{d}{d\tau}r(\tau) - \frac{d^{\eta-1}}{d\tau^{\eta-1}}\frac{d}{d\tau}y_j(\tau) \\ &= \frac{d^{\eta-1}}{d\tau^{\eta-1}}\frac{dx}{d\tau}\frac{d}{dx}h(x_\infty(\tau)) - \frac{d^{\eta-1}}{d\tau^{\eta-1}}\frac{dx}{d\tau}\frac{d}{dx}h(x_j(\tau)) \\ &= \frac{d^{\eta-1}}{d\tau^{\eta-1}}L^1_{f_\infty}h(x_\infty) - \frac{d^{\eta-1}}{d\tau^{\eta-1}}L^1_{f_j}h(x_j) \\ &= L^\eta_{f_\infty}h(x_\infty) - L^\eta_{f_j}h(x_j) \\ &\quad + L_{g_\infty}L^{\eta-1}_{f_\infty}h(x_\infty)u_\infty - L_{g_j}L^{\eta-1}_{f_j}h(x_j)u_j \end{aligned} \tag{5}$$

where by the definition of relative degree, the input signal $u$ does not appear until the output has been differentiated $\eta$ times. Then, using (5) in (3), denoting $\delta u_j = u_j - u_\infty$ and dropping the $\tau$ argument for compactness, gives

$$\begin{aligned} &\delta u_{j+1} \\ &= q_j * \left(u_j + \Gamma e_j^{(\eta)}\right) - u_\infty \\ &= q_j * \left(u_j + \Gamma\left[L^\eta_{f_\infty}h(x_\infty) - L^\eta_{f_j}h(x_j)\right.\right. \\ &\qquad\left.\left. + L_{g_\infty}L^{\eta-1}_{f_\infty}h(x_\infty)u_\infty - L_{g_j}L^{\eta-1}_{f_j}h(x_j)u_j\right]\right) - u_\infty \\ &= q * \left(1 - \Gamma L_{g_\infty}L^{\eta-1}_{f_\infty}h(x_\infty)\right)(\delta u_j) \\ &\quad + q * \Gamma\left(L_{g_\infty}L^{\eta-1}_{f_\infty}h(x_\infty) - L_{g_j}L^{\eta-1}_{f_j}h(x_j)\right)u_j \\ &\quad + q * \Gamma\left(L^\eta_{f_\infty}h(x_\infty) - L^\eta_{f_j}h(x_j)\right) + (q-1) * u_\infty. \end{aligned} \tag{6}$$

Denote $\delta x_j = x_j - x_\infty$ and $\delta T_j = T_j - T_\infty$. Then, taking the norm of both sides of (6) gives

$$\begin{aligned} \|\delta u_{j+1}\| \le\ &\rho\|\delta u_j\| + c_1\gamma_{x,j}\|\delta x_j\| \\ &+ c_1\gamma_{T,j}|\delta T_j| + \sup_{\substack{\tau\in[0,1] \\ j\in\mathbb{Z}_+}} \|(q_j - 1) * u_\infty\| \end{aligned} \tag{7}$$

where $c_1 = \sup_{\tau,j}\|q_j * \Gamma\|$, $\gamma_{x,j} = m_{f,x} + m_{g,x}m_u$, $m_u = \sup_u\|u\|$, $\gamma_{T,j} = m_{f,T} + m_{g,T}m_u$, $m_{f,x}$ and $m_{f,T}$ are the Lipschitz constants of the Lie derivatives $L^\eta_f h(x)$ with respect to $x$ and $T$, respectively, and $m_{g,x}$ and $m_{g,T}$ are the Lipschitz constants of the Lie derivatives $L_g L^{\eta-1}_f h(x)$ with respect to $x$ and $T$, respectively. This follows from continuity of $f$, $g$, and $h$. Using the system dynamics (2), Grönwall's lemma [27], and the definition of the $\omega$-norm, $\delta x_j$ can be bounded by

$$\|\delta x_j\|_\omega \le \rho_x(c_2\|\delta u_j\|_\omega + \gamma_{T,\infty}|\delta T_j|) \tag{8}$$

where $\rho_x = \sup_\tau(1-e^{-\omega\tau})(\omega - \gamma_{x,\infty}(1-e^{-\omega\tau}))^{-1}$, $\gamma_{x,\infty} = m_{f,x} + m_{g,x}\|u_\infty\|$, $\gamma_{T,\infty} = m_{f,T} + m_{g,T}\|u_\infty\|$, and $c_2 = \|g(x)\|$. Using the definition of the $\omega$-norm in (7) with (8) gives

$$\begin{aligned} \|\delta u_{j+1}\|_\omega \le\ &\underbrace{(\rho + c_1\gamma_{x,j}\rho_x c_2)}_{\rho_u}\|\delta u_j\|_\omega \\ &+ \underbrace{(c_1\gamma_{x,j}\rho_x\gamma_{T,\infty} + c_1\gamma_{T,j})}_{\rho_T}|\delta T_j| \\ &+ \underbrace{\sup_j\|(q_j - 1) * u_\infty\|_\omega}_{\upsilon_\infty}. \end{aligned} \tag{9}$$

Since $\rho < 1$, there exists a value of $\omega$ sufficiently large such that $\rho_x \approx 0$ and thus $\rho_u < 1$. Then, it is easy to show that

$$\lim_{j\to\infty}\|\delta u_j\| \le \frac{\rho_T|T_u - T_l| + \upsilon_\infty}{1-\rho_u}. \tag{10}$$

Finally, from A2 and the system dynamics

$$\|e_j\|_\omega \le \rho_{h,x}\rho_x\left(c_2\|\delta u_j\|_\omega + \gamma_{T,\infty}|\delta T_j|\right)$$

and therefore

$$\begin{aligned} \lim_{j\to\infty}\|e_j\| \le\ &\rho_{h,x}\rho_x\left(c_2\frac{\rho_T}{1-\rho_u} + \gamma_{T,\infty}\right)|T_u - T_l| \\ &+ \rho_{h,x}\rho_x c_2\frac{\upsilon_\infty}{1-\rho_u}. \end{aligned} \tag{11}$$

□

The following remarks provide general insights into the proof presented above.

*Remark 2:* The result in Theorem 1 ensures only asymptotic convergence which is in contrast to many ILC schemes which show monotonic convergence to the origin. While it may be possible to achieve this result for the class of systems considered here, it is reserved for future work.

*Remark 3:* The final ball to which the tracking error converges, (11), contains numerous design and system parameters. However, clearer representation of the parameters that influence the magnitude of the converged error is given in (10). The parameters $\rho_u$, $\rho_T$, and $\upsilon_\infty$ are influenced by control design choices. The convergence parameter $\rho_u$ is a function of the learning gain $\Gamma$ through $\rho$, in addition to the $q$-filter through $c_1$. The parameter $\rho_T$ is also influenced by the $q$-filter and the learning gain $\Gamma$ through $c_1$. Finally, $\upsilon_\infty$ is explicitly a function of the $q$-filter. While the aforementioned parameters are influenced by control design, the remaining parameters in (11) are dependent on the system properties and are largely influenced by the magnitude of the operators $f$ and $g$, in addition, the Lipschitz constants of these parameters both with respect to $x$ and $T$. The final ball to which the error converges, however, is dependent on the choice of the time-weighted norm parameter $\omega$. Further, the time-weighted norm is upper bounded by the supremum norm, and, therefore, the ball given in (11) is a conservative upper bound on the converged error.

*Remark 4:* In the development of Theorem 1, it is assumed that the trial length is unknown but bounded on each trial by the constants $T_u$ and $T_l$. In some practical situations, it might be the case that as tracking error decreases, the bounds on the trial length may shrink. Further, in some cases, as will be demonstrated in the simulation section, the trial duration may converge to a trial-invariant value as tracking performance improves. Clearly then, from (11), tracking performance can be significantly improved in these cases. In addition, by examining (10) and (11), it can be seen that when the trial duration is known, i.e., $T_u = T_l$ and $T_\infty = T_j$, the associated terms on the right-hand side (RHS) disappear and, as expected, it is possible to recover zero-error convergence in the limit.

*Remark 5:* The second term on the RHS of (11) denotes the residual tracking error due to the filter $q$. Performance improvements can be made by designing $q$ such that it minimizes $\|q - 1\|$.

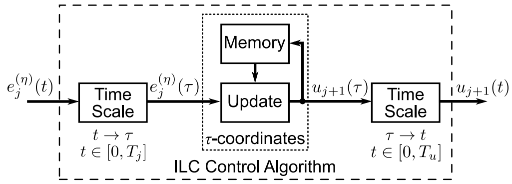


Fig. 1. Schematic of the proposed ILC update law implementation. Error signals are transformed into $\tau$-coordinates, the control law is updated for trial $j+1$, and the updated control law is transformed into the $T_u$ time-scale.

The following remarks and Lemma are specific properties that arise from the treatment of the ILC problem through the use of the time-scale transformation.

*Remark 6:* Typically, the filter $q$ is designed to be a low pass filter in order to attenuate unwanted high-frequency content from entering the control signal. When designing the filter $q$, a notion of the allowable bandwidth is needed. Since this would be done in practice based on the desired trial duration, and the time-scale transformation distorts frequency content, a method is needed to update the $q$ filter to ensure the proper bandwidth is retained. One method to retain the desired frequency domain properties of $q$ is to use the Laplace transform property

$$q_j(t) = q_j(T_j\tau) \leftrightarrow \frac{1}{T_j} Q_j\left(\frac{s}{T_j}\right) \tag{12}$$

where $Q(s) = \mathcal{L}\{q(t)\}$ is the frequency-domain representation of $q$.

With a known filter structure and the property pair given above, a simplification of Theorem 1 can be presented.

*Lemma 1:* Let $q$ be a filter with desired trial-invariant frequency response properties over the normalized interval $\tau \in [0, 1]$. The system (2) and (3) converges asymptotically to a ball centered at the origin, with respect to $j$, if

$$\sup_{\substack{\tau\in[0,1]\\ T\in[T_l,T_u]}} \left\| q_T(\tau) * \left(1 - \Gamma(\tau) L_{g_\infty} L_{f_\infty}^{\eta-1} h(x_\infty(\tau))\right)\right\| = \rho < 1 \tag{13}$$

where now the robustness filter is parameterized by $T$.

*Proof:* Follows directly from Theorem 1 and the property pair above. □

*Remark 7:* The implementation of the control law in (3) requires two time-scale transformations: one transformation from $t$ with trial duration $T_j$ to $\tau$-coordinates and a second transformation from $\tau$-coordinates to $t$ with trial duration $T_{j+1}$. However, as it is assumed that this is not known *a priori*, a logical estimate for the time scale is the reference time scale $T_\infty$. In nearly all practical situations, and ensured by Assumption A2 in Section III, $T_\infty$ is known. This estimate of $T_{j+1}$ is consistent with the result of Theorem 1. A schematic of the update law algorithm is shown in Fig. 1.

*Remark 8:* Examining (2) above reveals that the time-scaled system is in essence the original system with an uncertain gain on the state-to-state mapping $f$ and the input-to-state mapping $g$. Therefore, the ILC algorithm provided in this brief is equally effective for the original class of systems, i.e., of the form (17), with an unknown, but bounded gain

$$\begin{aligned}\dot{x}_j(t) &= K_j(f(x_j(t)) + g(x_j(t))u(t))\\ y(t) &= h(x_j(t))\end{aligned} \tag{14}$$

where $K_j \in [K_l, K_u]$ is a gain on the system for each trial, $t \in [0, T]$ and $T$ is the trial-invariant trial duration. By posing the same problem as being of trial-invariant duration and trial-varying gain, established analyses such as those in [29] can be applied.

### A. *q-Filter and Learning Operator Design*

There are two user-determined parameters in (3): the filter $q$ and the gain $\Gamma$. Following the proof for Theorem 1, the two controller parameters play different roles in the overall system performance. The learning gain $\Gamma$ determines the error convergence rate, while the filter $q$ determines which frequency ranges of the system should be learned. In conventional ILC, when certain conditions on the plant $P$ are met, the learning gain can be chosen to approximate the plant inverse such that $\|1-\Gamma P\|$ is as close to zero as possible [30]–[32]. In addition, the filter $q$ is chosen such that high-frequency content is attenuated. While the filter $q$ and the learning gain $\Gamma$ function in similar ways here, some guidelines are presented to aid in the selection of the learning gain $\Gamma$ and the robustness filter $q$.

The stability requirement and performance guarantees of Theorem 1 require that the condition in (4) be satisfied. However, in practice, it may be difficult to find a learning gain $\Gamma$ which satisfies this condition. A unique challenge with the proposed approach is the possibility for the desired reference signal to cause $J$ in (4) to become ill-conditioned. In this case, a pure "plant inverse" design $\Gamma(\tau) = J^{-1}(x_\infty)$ is not feasible or desired. In order to avoid inverting an ill-conditioned $J$, and to provide a systematic method for designing $\Gamma$ for the class of systems considered here, a choice for selecting the learning operator is

$$\Gamma(\tau) = (J^\intercal(x_\infty(\tau))J(x_\infty(\tau)) + \lambda^2 I)^{-1} J^\intercal(x_\infty(\tau)) \tag{15}$$

where $\lambda$ is a tuning parameter and $(\cdot)^\intercal$ is the transpose operator. The operator selection in (15) satisfies

$$\min_{\Gamma(\tau)}(\|I - \Gamma(\tau)J(x_\infty(\tau))\|^2 + \lambda^2\|\Gamma(\tau)\|) \tag{16}$$

and $\lambda$ determines the tradeoff between solution accuracy, i.e., minimization of (4), and the magnitude of the gain $\Gamma$. Note that the solution in (15) is termed the damped least squares solution [33], [34] and is a single step of the Levenberg–Marquardt algorithm.

Examining (10), it can be observed that the magnitude of the ball to which the input converges is, for sufficiently large $\omega$, proportional to the inverse of $1-\rho$. In the limit with $\|1 - \Gamma L_{g_\infty} L_{f_\infty}^{\eta-1} h(x_\infty)\| \approx 1$, the input converges slower to a ball of larger radius than if $\|1 - \Gamma L_{g_\infty} L_{f_\infty}^{\eta-1} h(x_\infty)\| \approx 0$. However, note that the uncertainty in the trial duration is proportional to the constant $c_1$ and thus the learning gain $\Gamma$ as in (10). Therefore, there exists a tradeoff between the rate at which the error converges and the magnitude of the converged error. A schematic of the convergence performance

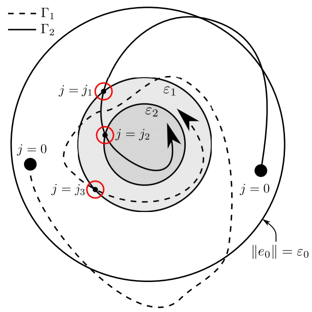


Fig. 2. Schematic of the convergence performance properties for two designs where $\Gamma_1 < \Gamma_2$ and $\varepsilon_2 < \varepsilon_1 < \varepsilon_0$. The convergence rate is faster for $\Gamma_2$ with $j_1 < j_2 < j_3$ than for $\Gamma_1$, in addition to converging to a smaller ball centered on the origin. Note that in both cases, the error may grow in magnitude as only asymptotic, not monotonic, convergence is guaranteed.

tradeoff is shown in Fig. 2 for a case where the trial duration uncertainty is small compared to the term $\upsilon_\infty$ for simplicity of presentation.

The selection of $q$ is more application-specific than the learning gain $\Gamma$, but the typical structure is that of a low-pass filter. In the framework presented here, the implementation of the filter $q$ is in $\tau$-coordinates. In many practical situations, there is knowledge of the desired frequency content to be learned in absolute time coordinates, i.e., $t \in [0, T_j]$. Therefore, a design procedure for $q$ is as follows. First, select the structure of $q$ such that it has unit gain

$$\int_{-\infty}^{\infty} q(t)dt = 1.$$

Then, set the application specific filter bandwidth with respect to absolute time coordinates. Finally, perform the transformation to $\tau$-coordinates using the property pair in Remark 6.

## V. Numerical Example

In this section, the ILC algorithm presented in the previous section is applied to a benchmark nonlinear system. Details of the system, the control design, and finally, the simulation results are provided.

### A. Benchmark Nonlinear System

Consider the following nonlinear mass-spring-damper system

$$\begin{bmatrix} \dot{x}_{1,j}(t) \\ \dot{x}_{2,j}(t) \end{bmatrix} = \begin{bmatrix} x_{2,j}(t) \\ -K(x_{1,j}(t)) - B(x_{2,j}(t)) \end{bmatrix} + \begin{bmatrix} 0 \\ F(x_{1,j}(t)) \end{bmatrix} u_j(t)$$
$$y_j(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} x_j(t) \quad (17)$$

where $t \in [0, T_j]$ is the trial duration (s), $x_j = [x_{1,j} \ x_{2,j}]^\intercal$ is the state vector, $u_j$ is the input signal, and

$$K(x(t)) = kx^5(t)$$
$$B(x(t)) = bx^2(t)\text{sgn}(x(t))$$
$$F(x(t)) = a + x(t)\text{sgn}(x(t)) \quad (18)$$

with $k = 1$, $b = 1$, and $a = 0.1$. While the system considered here is not explicitly modeling a physical system, it is similar in structure to many physically relevant systems such as the ejection dynamics in e-jet printing [28].

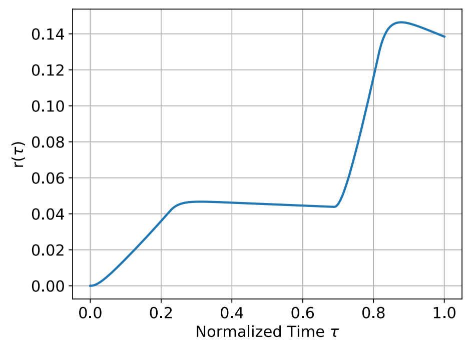


Fig. 3. Reference trajectory for each of the three simulation cases presented.

Two different simulation cases are considered here: a case where the trial duration is dictated by an output-dependent function without output noise and a case where the trial duration is dictated by the same output-dependent function with output noise. For both cases, the reference trajectory shown in Fig. 3 is used with $T_\infty = 4$. The $q-$filter is chosen as a causal, first order low-pass filter with a cutoff frequency of 4 kHz on the normalized trial duration. This cutoff frequency maps to 1 kHz on the reference trial duration $T_\infty$.

Examining the system in (17), it can be seen that the relative degree is $\eta = 1$ and the Lie Derivative is

$$J(x_\infty) = L_{g_\infty} L^0_{f_\infty} h(x_\infty) = T_\infty F(x_{1,\infty}). \quad (19)$$

*Remark 9:* In order to check stability and design a control gain using the procedure detailed in Section IV-A, knowledge of $x_\infty$ is needed. With Assumption 2, the nonlinear system is one-to-one, and it is possible to invert the system dynamics, given $r$, to find an estimate of the reference states. Then, the Lie derivative above can be calculated.

### B. Case 1

In the first simulation case, the system in (17) is treated as known with no uncertainty and the trial duration is determined by

$$T_j = \begin{cases} t_f & \inf\{t_f = t \mid y_j(t) \geq 1.10 r(t)\} \\ T_u & \text{otherwise} \end{cases} \quad (20)$$

where $t_f$ is the time instant when the inequality above is true. The system is simulated for $M = 1000$ trials with values of $\lambda = 0.1$, 0.5, and 1.0. These values of $\lambda$ are chosen to show the effects of the magnitude of $\rho$ in (4) on the error properties in the trial domain.

Fig. 4 shows the supremum norm of the tracking error as a function of trial for each value of $\lambda$ (recall Definition 3). As mentioned above, smaller values of $\lambda$, and in turn smaller values of $\rho$, yield faster convergence and tighter error bounds. This is demonstrated in the trial domain error plots in Fig. 4. For the smallest value of $\lambda = 0.1$ and thus the largest magnitude learning gain $\Gamma$, the convergence occurs in approximately five trials to a level of approximately $1.5 \times 10^{-3}$. Alternatively, for the largest magnitude of $\lambda = 1.0$, convergence does not occur until approximately 40 trials and only to a value of approximately $3 \times 10^{-3}$.

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.



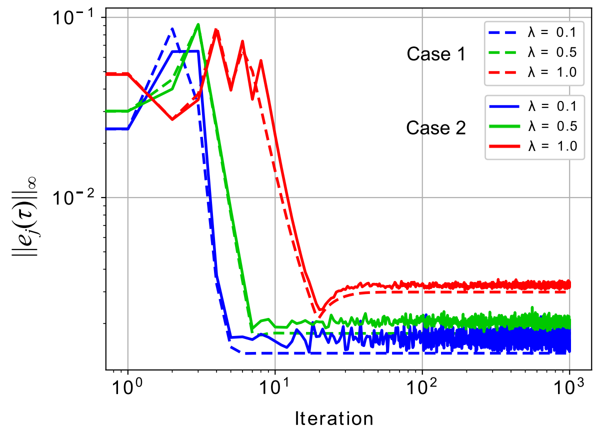


Fig. 4. Error magnitude as a function of trial for simulation cases 1 and 2.

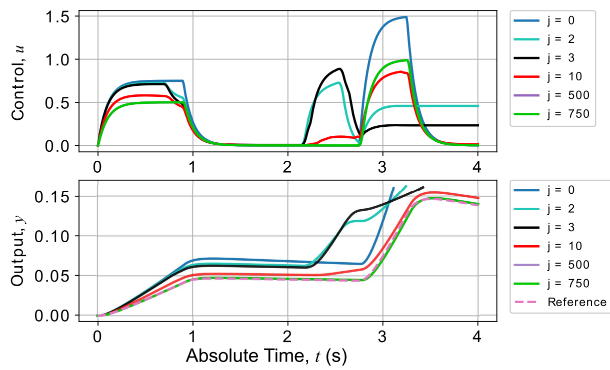


Fig. 5. Input $u$ and output $y$ signal's for selected trials for simulation case 1.

Selected input and output signals for Case 1 when $\lambda = 1.0$ are shown in Fig. 5. On trial $j = 0$, the initial input signal results in a large output signal that violates the inequality in (20). The ILC algorithm quickly updates the control law such that the peak magnitude has moved earlier in the control signal (approximately $t = 2.5$ s on both trials $j = 2$ and $j = 3$). By trial $j = 10$, the ILC algorithm has sufficiently learned the appropriate control law in order to closely track the desired reference. Past approximately trial $j = 40$, the ILC algorithm has converged and no further tracking performance is gained. This is evidenced by the error plot in Fig. 4 and the coincident control and output signals for trials $j = 500$ and $j = 750$ in Fig. 5.

### *C. Case 2*

In the second scenario, additive noise is included on the output channel and is drawn from a normal distribution with characteristics $\mathcal{N}(0, 1\times10^{-4})$. The trial duration is determined as in (20).

The trial domain error can again be observed in Fig. 4. The trends with respect to $\lambda$ follow the same as described for Case 1. Further, as is expected, the performance is slightly degraded by the additional process uncertainty. After convergence, each error curve in Case 2 is offset above the corresponding Case 1 error curve by approximately $5 \times 10^{-3}$. In Case 1, where no output noise is considered, the differentiation required to obtain the error derivative needed for (3) results in a relatively smooth signal. However, in this case, where output noise is considered, the differentiation operation results in a significantly noisier error derivative. Despite this, however, the ILC algorithm successfully overcomes both the uncertain trial duration and the output noise to achieve similar performance to the noiseless case.

## VI. Conclusion

Conventionally, implementation of ILC requires several assumptions on trial-to-trial invariance be satisfied. Among those is the assumption that the trial duration is constant. However, there exist some practical examples of systems that operate in a trial-to-trial manner where it is necessary to relax this assumption. Here, an ILC update algorithm, based on a time-scale transformation, is proposed to improve tracking performance for a class of nonlinear, control-affine systems where the trial duration is bounded, but unknown *a priori*. A criterion is presented which, when satisfied, ensures asymptotic convergence of the tracking error to a ball centered at the origin. The magnitude of the ball is a function of both the mappings in the nonlinear system and of the ILC update law parameters.

A series of simulation cases show the efficacy of the proposed solution. In two cases where the trial duration is determined by a function of the output, as tracking performance improves, the trial duration converges to a trial-invariant value. In these cases, higher learning gains yield faster convergence and lower tracking error. This agrees well with intuition and existing results of robust ILC. As compared to some other ILC methodologies where the trial duration is uncertain, the nonmonotone convergence of the approach taken must be accounted for carefully. However, the less restrictive system class and assumptions on trial duration greatly increase the applicability of the proposed approach. Practical implementation of the proposed methodology, through the convergence criterion, requires partial knowledge of the model and state trajectories. This is solved here through the invertibility of the output mapping. For higher relative degree systems, numerical differentiation is required to reconstruct the necessary error signals, but the noise introduced by this process can be mitigated through the use of the $q$-filter. Application of the ILC paradigm demonstrated in the simulations to a physical system is an area of future work.

## Appendix

*Theorem 2 (MIMO System, $p \leq m$):* Consider a control law of the form in (3) where now $\Gamma \in \mathbb{R}^{m\times p}$ is a static learning gain matrix and $e_j^{(\eta)} = \left[e_{1,j}^{\eta_1} e_{2,j}^{\eta_2} \cdots e_{m,j}^{\eta_m}\right]$ is an error vector corresponding to the vector relative degree of the system. The error converges asymptotically to a ball centered at the origin, with respect to $j$, if

$$\sup_{\substack{\tau\in[0,1]\\ \forall j\in\mathbb{Z}_+}} \|q_j(\tau) * (1 - \Gamma(\tau)J(x_\infty))\| = \rho < 1 \qquad (21)$$

where $J(x_\infty) = L_{g_\infty} L_{f_\infty}^{\eta-1} h(x_\infty(\tau))$.

*Proof:* Define the error derivatives as

$$e_{k,j}^{(i)}(\tau) = L_f^i h_k(x_\infty) - L_f^i h_k(x_j), \quad 0 \le i \le \eta_k - 1$$
$$e_{k,j}^{(i)}(\tau) = L_f^i h_k(x_\infty) - L_f^i h_k(x_j) + \left[L_{g_1} L_f^{i-1} h_k(x_\infty) \dots L_{g_p} L_f^{i-1} h_k(x_\infty)\right] u_\infty + \left[L_{g_1} L_f^{i-1} h_k(x_j) \dots L_{g_p} L_f^{i-1} h_k(x_j)\right] u_j. \quad (22)$$

Following similar arguments as in the proof of Theorem 1 by denoting again $\delta u_j = u_j - u_\infty$ and defining as in (6) with appropriate modifications for the error signals in (22) gives the result. □

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