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Planning facility location under generally correlated facility disruptions: Use of supporting stations and quasi-probabilities

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ABSTRACT

Many real-world service facilities are subject to probabilistic disruptions. Such disruptions often exhibit correlations that arise from shared external hazards or direct interactions among these facilities. This paper builds an overarching methodological framework for reliable facility location design under correlated facility disruptions. We first incorporate and extend the concepts of supporting station structure and quasi-probability from Li et al. (2013) and Xie et al. (2015), such that any correlated facility disruptions (positive and/or negative) can be equivalently represented by independent failures of a layer of properly constructed supporting stations, which are virtually added to the original facility system for capturing the effect of correlations among facilities. We then develop a compact mixed-integer mathematical model to optimize the facility location and customer assignment decisions in order to strike a balance between system reliability and cost efficiency. Lagrangian relaxation based algorithms, including modules for obtaining upper bound and lower bounds of relaxed subproblems, are proposed to effectively solve the optimization model. Numerical case studies are carried out to demonstrate the methodology, to test the performance of the framework, and to draw managerial insights.

1. Introduction

Facility location problems have been studied for decades in a variety of forms. Most classic models employ discrete optimization techniques to determine facility location and customer assignment decisions that minimize the total system costs or maximize the utility of these facilities (Drezner, 1995; Daskin, 2013). In recent years, natural and anthropogenic disasters repeatedly caused severe damages to built facilities and have resulted in catastrophic disasters. When a facility is disrupted, all customers originally assigned to this facility have to either be reassigned to a surviving alternative (and bear higher transportation costs), or lose service (and incur certain penalty costs). Ignoring the possibility of facility disruptions during system planning could lead to a suboptimal design that is vulnerable to even infrequent disruptions (Snyder and Daskin, 2005; Li and Ouyang, 2010).

In the reliable facility location literature, one stream of studies focused on design-related facility disruptions that can be prevented by fortification. Interdiction models were often used to identify critical components in an infrastructure system, and cost-effective fortification strategies were sought during facility location design. Church et al. (2004), for example,

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proposed two models for the r-interdiction median problem and the r-interdiction covering problem (which are variants to the p-median problem and the max covering problem, respectively), to identify the most critical facilities in a supply chain. Brown et al. (2005) described several bi-level and tri-level attacker-defender models to address system vulnerability and robustness issues in the contexts of electric power grids, subways, airports and other critical infrastructures. Scaparra and Church (2008b) reformulated the r-interdiction median problem (with fortification) as a maximal covering model with precedence constraints, and aimed at identifying a subset of facilities to fortify so as to best protect the system against the worst-case loss of non-fortified facilities. This model was later extended by Liberatore et al. (2011) to a stochastic version, which optimally allocated defense resources among facilities to minimize the worst-case impact of a randomized intentional disruption. Scaparra and Church (2008a) presented a bi-level formulation based on the classical p-median problem. The upper level problem involved decisions on which facilities to fortify in order to minimize the worst-case efficiency reduction due to loss of unprotected facilities, while the lower-level interdiction problem described the worst-case losses.

Another stream of research focused on modeling the expected consequences of location-specific facility disruptions. A comprehensive review can be found in Snyder (2006). Among a rich variety of efforts, Snyder and Daskin (2005) and Berman et al. (2009) formulated models where facilities are subject to independent disruptions with identical failure probabilities. More recently, a series of reliable location models were proposed to allow site-dependent disruption probabilities. Berman et al. (2007) provided a nonlinear mixed-integer programming formulation as well as an efficient heuristic solution approach. Cui et al. (2010) developed two distinct sets of models (discrete and continuous) and corresponding solution algorithms to allow the disruption probabilities to be site-dependent. O'Hanley et al. (2013) proposed a modeling technique based on a customized flow network to linearize the unreliable p-median facility location problems. Atamtürk et al. (2012) further presented reliable location-inventory models (which allowed facilities to be subject to failures due to inventory shortage) as well as an innovative conic programming solution approach. Zhang et al. (2016) considered the cost savings from inventory risk-pooling and economies of scale under homogeneous and heterogeneous disruption probabilities.

All these studies provided insightful knowledge on reliable location problems, but they held the assumption on independent facility disruptions, which may not always be realistic, as various types of connections and interactions may exist among built facilities in the real world. As a result, facility disruptions could be correlated when the facilities are exposed to shared hazards or mutual interactions. Such correlations could be positive or negative, or mixed. For example, adjacent facilities in a local geographical region are prone to simultaneous damage by a natural disaster (e.g., earthquake, hurricane, flooding). If one facility is known to have been disrupted by an earthquake, its neighboring facilities will bear a higher likelihood of being disrupted as well – this shows a positive correlation. The correlation can also be negative. Suppose multiple facilities along a river are all threatened by flooding. If one facility is known to have been disrupted by flooding, then its downstream peers become less likely to be disrupted due to the release of water pressure. Similar negative correlations may also exist when facilities compete for scarce resources.

Disruption correlation tends to have a strong impact on the performance of a reliable facility location design. Consider a simple network where two facilities A and B jointly serve one unit of demand from a customer. The costs for serving the demand from these two facilities are 10 and 20 units, respectively, and the penalty for not serving the demand is 100 units. When both facilities are perfectly reliable, the demand will obviously be served by A with a total cost of 10 units. When the facilities are subject to disruption, the demand will be served by A as long as A is functioning (i.e., event A), or by B if A is disrupted but B is functioning (i.e., event $\bar{A}B$), or the customer will bear the penalty if both A and B fail (i.e., event $\bar{A}\bar{B}$). In the case where A and B fail independently with an equal probability of 0.5, the expected service cost is $10 \times (0.5) + 20 \times (0.5 \times 0.5) + 100 \times (0.5 \times 0.5) = 35$ units. If the facility disruptions are positively correlated, say $P(AB) = P(\bar{A}\bar{B}) = 0.4, P(\bar{A}B) = P(A\bar{B}) = 0.1,$ the expected service cost becomes $10 \times (0.1 + 0.4) + 20 \times 0.1 + 100 \times 0.4 = 47$ units. If the facility disruptions are negatively correlated, say $P(AB) = P(\bar{A}\bar{B}) = 0.1, P(\bar{A}B) = P(A\bar{B}) = 0.4,$ the expected service cost becomes $10 \times (0.1 + 0.4) + 20 \times 0.4 + 100 \times 0.1 = 23$ units. Although the marginal failure probability of each facility remains 0.5 in all three cases, we can see that positive disruption correlation significantly increases the expected service cost, while negative correlation does the contrary. This simple example demonstrates the significant impact of correlations that has recently been reported in the existing literature (Li and Ouyang, 2010; Liberatore et al., 2012).

There are many ways to express the correlations among facility disruptions. Xie et al. (2015) showed equivalence of three general correlation representations using scenario, marginal, or conditional probabilities. Sometimes the correlations follow explicit physical laws and take special forms. For example, under certain natural disasters (such as earthquakes), the correlation between two candidate locations could sometimes be specified by a decaying probability of failure "contagion" (e.g., $e^{-distance}$) that depends on their relative *distance*. No matter how correlations are specified, however, a very straightforward modeling approach in the reliable facility location literature would involve some type of enumeration (or simulation and sampling) of an exponential number of random scenarios; this makes it computationally difficult to even just evaluate the performance of a given design. To the best of our knowledge, only a few efforts have been made to address correlated facility disruptions, either exactly or approximately (e.g., Liberatore et al. (2011); Li and Ouyang (2010); Lu et al. (2015); An et al. (2018)). In addition, Huang et al. (2010) addressed a variant of the p-center model in case of large-scale emergencies, where correlated disruption was introduced by allowing many facilities to become functionless simultaneously. Gueye and Menezes (2015) considered a two-stage stochastic program model for a median problem under correlated facility disruptions, and asymptotic results were presented based on a scenario-based model formulation. Berman and Krass (2011) and Berman et al. (2013) introduced analytical approaches to help understand the effects of correlated failures in simpler spatial settings, e.g., along a line segment. Li et al. (2013) proposed a virtual station structure that transforms a facility network with correlated disruptions into an equivalent one with added virtual supporting stations, and the virtual stations were assumed to be subject to independent disruptions. While an optimization model was developed, it can only handle cases where facilities are positively correlated, and the station disruption probabilities are all identical. Based on Li et al. (2013), Xie and Ouyang (2019) presented an optimization framework for reliable facility location problem under the risk of network access failures, in which facilities are correlated as they share common access points. The concept of supporting station is used to represent the network access points and to describe the system behaviors. Recently, Xie et al. (2015) extended the network transformation theory in Li et al. (2013) by developing a recipe for any (positive, negative, or mixed) type of correlations to be transformed into an augmented network (with additional supporting stations). It was proven that with such a station structure, a system of interdependent facilities with generally correlated disruptions can be equivalently represented by one with independent stations, whereas each station fails with a "failure propensity" which "inherits all mathematical characteristics and properties of a failure probability except that we allow it to be larger than 1" (Xie et al., 2015).[1] How to optimally design the reliable location of service facilities under site-dependent failures and positive/negative/mixed correlations, however, remains an open and nontrivial question.

In light of these challenges, we build upon the idea of supporting station structure (Li et al., 2013; Xie et al., 2015), so as to address the reliable facility location problem with any pattern of facility disruption correlations. An additional layer of independent yet heterogeneous supporting stations are incorporated into the service design framework. These stations, when properly connected to (and shared by) the candidate facility locations, collectively dictate the functioning state of built facilities at these locations and exactly capture the effect of correlated facility disruptions. The probability of the virtual station state (called "propensity" in Xie et al. (2015), but will be now referred to as *quasi-probability* so as to be consistent with the physics literature) may exceed unity or be negative for the purpose of calculating the final probabilities of physical disruption scenarios of the facilities. Similar to Feynman (1987), we introduce these quasi-probabilities to quantify the imagined intermediary states of added virtual stations (whose values may exceed unity or be negative), so as to systematically calculate the ultimate probabilities of physical facility states (whose values are ensured to be within the conventional range [0, 1]); more about these properties will be explained in Section 2.2 and Proposition 5. As a result, the optimization model developed in this paper, which transfers correlated disruptions of facilities to independent disruptions of such stations, is capable of addressing the facility location problem equivalently. A compact mixed-integer mathematical model is proposed to determine the optimal facility location and customer assignment plans. Several customized solution approaches based on Lagrangian relaxation, with careful treatment of negative and mixed correlations, are also developed. Case studies involving multiple patterns of correlations are conducted to demonstrate the performance and applicability of our methodology. Managerial insights are drawn as well.

The remainder of the paper is organized as follows. Section 2 introduces the station-based mixed-integer mathematical model for the reliable facility location problem under correlated facility disruptions. Section 3 presents the customized solution approaches to efficiently solve the optimization model. In Section 4, a range of case studies involving multiple patterns of correlations are shown. Finally, Section 5 concludes the paper and discusses future research directions.

2. Model formulation

This section first presents the traditional scenario-based formulation of the reliable facility location problem under correlated facility disruptions. Then, after a brief introduction of using station structure to represent correlated facility disruptions (Xie et al., 2015), the scenario-based reliable facility location model is transformed into an equivalent station-based model.

2.1. Scenario-based formulation

We denote $\mathcal{I}$ as the set of discrete customers, and each customer $i \in \mathcal{I}$ has a demand μ_i. We define $\mathcal{J}$ to be the set of discrete candidate facility locations, and associate each location $j \in \mathcal{J}$ with a fixed facility cost f_j. The cost for a facility at location j to satisfy one unit of demand from customer i is denoted by d_{ij}.

Customers can go to candidate location $j \in \mathcal{J}$ for service if a facility is built and no disruption has occurred there. Under any realization of the facility states, each customer i seeks service by visiting the available and functioning facility that has the smallest transportation cost. Moreover, a penalty cost π_i per unit demand will be imposed if customer i does not receive any service. This situation occurs if no facility is reachable, or if the cost of serving customer i by the nearest available facility already exceeds π_i. We model this possibility by adding an "emergency" facility index by $j = 0$ with fixed cost $f_0 = 0$ and transportation costs $d_{i0} = \pi_i, \ \forall i \in \mathcal{I}$.

Let $\Omega = \{0, 1\}^{|\mathcal{J}|}$ be the set of all possible disruption scenarios/realizations if facilities were built at all candidate locations (including the "emergency" facility). For each $\omega \in \Omega$, which occurs with probability p_ω, we use parameter $\delta_{j\omega} = 1$ to indicate

[1] In theoretical physics and especially quantum mechanics, a similar concept is sometimes called "quasi-probability" (e.g., Dirac (1942); Feynman (1987)), where "conditional probabilities and probabilities of imagined intermediary states may be negative in a calculation of probabilities of physical events or states. ... The other possibility is that the situation for which the probability appears to be negative is not one that can be verified directly" (Feynman, 1987).

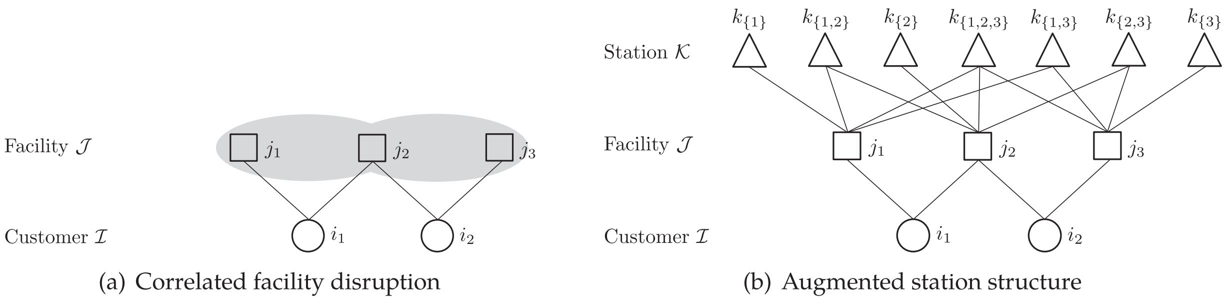


Fig. 1. Conceptual illustration of the station structure.

that the facility at j (if built) is functioning in scenario ω, or 0 otherwise. The emergency facility is assumed to be always functioning, i.e., $\delta_{0\omega} = 1, \forall \omega \in \Omega$.

We denote X_j and $Y_{ij\omega}$ as binary variables indicating whether a facility is built at location j, and whether customer i visits facility j in scenario ω, respectively. Specifically,

$$X_j = \begin{cases} 1 & \text{if a facility is built at location } j; \\ 0 & \text{otherwise.} \end{cases}$$

$$Y_{ij\omega} = \begin{cases} 1 & \text{if customer } i \text{ visits facility } j \text{ in scenario } \omega; \\ 0 & \text{otherwise.} \end{cases}$$

Then it is straightforward to see that the reliable facility location problem could be formulated as the following scenario-based formulation (RFL-SCE):

$$\text{(RFL-SCE)} \quad \min \sum_{j\in\mathcal{J}} f_j X_j + \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}\cup\{0\}} \sum_{\omega\in\Omega} \mu_i d_{ij} Y_{ij\omega} p_\omega \tag{1a}$$

$$\text{s.t.} \quad \sum_{j\in\mathcal{J}\cup\{0\}} Y_{ij\omega} = 1, \;\; \forall i \in \mathcal{I}, \omega \in \Omega, \tag{1b}$$

$$Y_{ij\omega} \le \delta_{j\omega} X_j, \;\; \forall i \in \mathcal{I}, j \in \mathcal{J}, \omega \in \Omega, \tag{1c}$$

$$X_j, Y_{ij\omega} \in \{0, 1\}, \;\; \forall i \in \mathcal{I}, j \in \mathcal{J} \cup \{0\}, \omega \in \Omega. \tag{1d}$$

The objective (1a) is the summation of the fixed facility costs and the expected transportation costs (including the penalty costs) across all possible facility disruption scenarios. Constraints (1b) enforce that in any disruption scenario $\omega \in \Omega$, each customer i is either assigned to a regular facility or assigned to the emergency facility. Constraints (1c) require each customer to be assigned to only a functioning open facility. Given a problem with correlated disruptions at $|\mathcal{J}|$ candidate locations, the total number of possible scenarios that need to be enumerated is $2^{|\mathcal{J}|}$. This implies that formulation (RFL-SCE), which is an integer program, requires an exponential number of variables and constraints; thus it is extremely difficult to solve, if not impossible. So in the next sections, we introduce a supporting station structure as well as an alternative station-based formulation that is more compact in size and can be solved more efficiently.

2.2. Supporting station structure

Given set $\mathcal{J}$, the set of all possible disruption scenarios Ω and their probabilities $\{p_\omega\}_{\omega\in\Omega}$ can be expressed in either of the following two mathematical representations (Xie et al., 2015): (i) *scenario representation* $\mathcal{S} = \{p_J^S\}_{\forall J\subseteq\mathcal{J}}$, where $p_J^S = p_\omega$ if ω satisfies $\delta_{j\omega} = 0, \forall j \in J$ & $\delta_{j\omega} = 1, \forall j \in \mathcal{J}\backslash J$, which denotes the probability for all locations in J to be disrupted while all others are functioning; and (ii) *marginal representation* $\mathcal{M} = \{p_J^M\}_{\forall J\subseteq\mathcal{J}}$, where $p_J^M = \sum_{\omega:\delta_{j\omega}=0,\forall j\in J} p_\omega$, which denotes the probability for all locations in J to be disrupted regardless of the states of all other locations. Here, the disruption of a candidate location $j \in \mathcal{J}$ implies that a facility will be disrupted if built at location j. In the rest of this paper, when we describe facility disruptions, we may use the phrases “disruptions of facilities” and “disruptions of candidate locations” interchangeably.

Two sets of facilities $J_1, J_2 \subseteq \mathcal{J}$ are independent if $p_{J_1}^M \cdot p_{J_2}^M = p_{J_1\cup J_2}^M$. In many real-world systems, facility disruptions exhibit spatial correlations (e.g., due to shared hazards), and there exist $J_1, J_2 \subseteq \mathcal{J}$ such that $p_{J_1}^M \cdot p_{J_2}^M \neq p_{J_1\cup J_2}^M$ while $J_1 \cap J_2 = \emptyset$. When facility disruptions are correlated as shown in Fig. 1(a), multiple facilities are subject to simultaneous impacts, and describing the correlation would typically require enumerating an exponential number of probabilities by specifying the scenario/marginal representation (e.g., (RFL-SCE) enumerates $2^{|\mathcal{J}|}$ scenarios). To circumvent this complexity, a virtual supporting station structure has been proposed to transform an arbitrary probabilistic representation of correlated facility disruptions into an equivalent representation with only independent disruptions (Li et al., 2013; Xie et al., 2015).

To build a station structure, a set of virtual stations, denoted by $\mathcal{K}$, are attached to the set of facilities, as shown in Fig. 1(b). A station $k \in \mathcal{K}$ could be connected to multiple facilities, and we use a binary parameter $l_{kj} = 1$ to indicate that facility j is connected to station k, or $l_{kj} = 0$ otherwise. We further assume that each station is associated with a non-negative site-dependent disruption quasi-probability $q_k \in [0, \infty)$ (which used to be referred to as "propensity" in Xie et al., 2015). The larger this quasi-probability value, the more damaging that station is to its connected facilities. When the disruption quasi-probability of a station exceeds 1, it simply means that the station's "functioning" probability is negative and it only causes pure damage to the connected facilities. This concept, similar to those in Feynman (1987), is used merely to quantify the imagined intermediary states of virtual stations in calculation of probabilities of states of physical facilities. Each station itself is also in a binary state: functioning or disrupted. The basic mechanism of the augmented facility-station system is defined as follows: *a facility remains operational if and only if at least one of its connected stations is functioning*. Hence the operating state of the facility system is determined collectively by the states of all stations. For example, in Fig. 1(b), j_1 and j_2 are disrupted and j_3 is functioning if and only if stations $k_{\{1\}}$, $k_{\{2\}}$, $k_{\{1,2\}}$, $k_{\{1,3\}}$, $k_{\{2,3\}}$, $k_{\{1,2,3\}}$ are all disrupted and $k_{\{3\}}$ is functioning. Here k_J is the station connected to all and only facility locations in J; i.e., $l_{k_J j} = 1, \forall j \in J$ and $l_{k_J j} = 0, \forall j \notin J$. Following this mechanism, we can ensure that the probability of a disruption scenario is equal to the product of q_k for each corresponding disrupted station k, and $(1 - q_k)$ for each functioning station k. For example, in Fig. 1(b), $p^S_{\{1,2\}} = q_{k_{\{1\}}} q_{k_{\{2\}}} q_{k_{\{1,2\}}} q_{k_{\{1,3\}}} q_{k_{\{2,3\}}} q_{k_{\{1,2,3\}}} (1 - q_{k_{\{3\}}})$. To summarize, per Proposition 1 of Xie et al. (2015), we have the following:

Proposition 1. *(Xie et al., 2015). For a given station structure $\mathcal{K}$ with $\{q_k\}_{\forall k \in \mathcal{K}}$, the scenario and marginal probabilistic disruption representations are formulated respectively as*

$$p^S_J = \sum_{J_1 : J \subseteq J_1} (-1)^{|J_1| - |J|} \left[\prod_{J_2 : J_2 \cap J_1 \neq \emptyset} q_{k_{J_2}} \right], \quad \forall J \subseteq \mathcal{J}, \tag{2}$$

$$p^M_J = \prod_{J_1 : J_1 \cap J \neq \emptyset} q_{k_{J_1}}, \quad \forall J \subseteq \mathcal{J}. \tag{3}$$

Based on the defined mechanism and Eqs. (2)–(3), any scenario or marginal probabilistic representation can be transformed back into an equivalent station structure representation based on Proposition 4 in Xie et al. (2015):

Proposition 2. *(Xie et al., 2015). For any $J \subseteq \mathcal{J}$, let $\bar{J} = \mathcal{J} \backslash J$, we can compute the disruption quasi-probability q_{k_J} of candidate station k_J as:*

$$q_{k_J} = \prod_{L : \bar{J} \subseteq L \subseteq \mathcal{J}} \left[\sum_{J_1 : L \subseteq J_1} p^S_{J_1} \right]^{(-1)^{|L| - |\bar{J}| + 1}} = \prod_{L : \bar{J} \subseteq L \subseteq \mathcal{J}} \left[p^M_L \right]^{(-1)^{|L| - |\bar{J}| + 1}}. \tag{4}$$

If $q_{k_J} \neq 1$, we add a station k and set its associated disruption quasi-probability to be q_{k_J}, and connect it to all and only facility locations in J; i.e., let $l_{kj} = 1, \forall j \in J$ and $l_{kj} = 0, \forall j \notin J$.

A recipe for systematically constructing the supporting station structure, as proposed in Xie et al. (2015), is presented in Appendix B.

2.3. Size of station structure

One may wonder how many stations will be needed to represent a complex correlated disruption profile. Theorem 1 in Xie et al. (2015) states that when the system is generally correlated, the number of needed stations $|\mathcal{K}|$ is comparable to the number of scenario probabilities $|\mathcal{S}|$ needed to describe the correlation. This statement is corroborated by the numerical examples in Xie et al. (2015).

While an exponential value of $|\mathcal{K}|$ with respect to the number of facilities $|\mathcal{J}|$ (as $|\mathcal{S}|$ could possibly be exponential) still makes the problem intractable (even with the introduction of the station structure), we shall note that there often are cases where $|\mathcal{S}|$ is exponential but $|\mathcal{K}|$ is polynomial (or even linear).

Consider the following example that can mimic many service systems in the real world. As shown in Fig. 2, suppose that $N \times M$ facilities (shown as squares) along a line are supported by surrounding stations (e.g., power plants, shown as triangles). The facilities are clustered into N groups, each with M facilities inside the group that are correlated with each other by sharing a set of supporting stations. In addition, in each group, L facilities near each side of the group boundary (i.e., called boundary facilities below) have shared supports with L boundary facilities in the adjacent group. In this system, any disruption scenario (i.e., a subset of facilities failing together) could potentially happen, and the probability of each scenario depends on the state of all facilities. As such, the number of scenario probabilities is $|\mathcal{S}| = 2^{NM}$. Yet, the relative independence between different facility groups (except for the boundary facilities) jointly show a largely "local" correlation pattern, and consequently, the number of stations needed to capture said correlation pattern is at most $|\mathcal{K}| = N(2^M - 1) + (N - 1)(2^{2L} - 2^L - 1)$, which is normally far smaller than $|\mathcal{S}|$. In particular, for the simplest case where

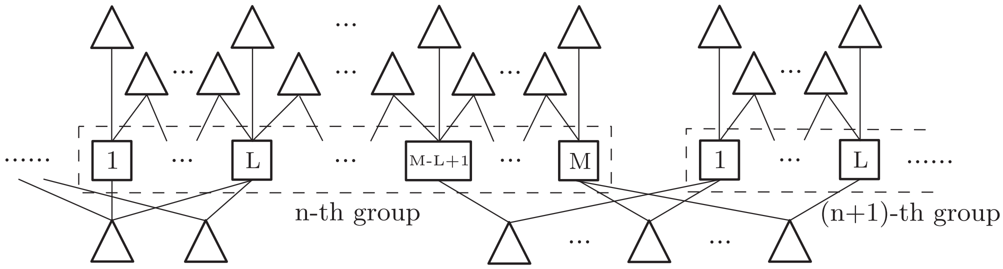


Fig. 2. Example to demonstrate the difference between $|\mathcal{K}|$ and $|\mathcal{S}|$.

$M = L = 1$, the number of facility disruption scenarios is 2^N (exponential) while the number of needed stations is only $2N - 1$ (linear).

As shown above, when the facility disruptions are "locally" correlated, $|\mathcal{K}|$ shall be far smaller than $|\mathcal{S}|$, and the difference between $|\mathcal{K}|$ and $|\mathcal{S}|$ grows sharply with the level of localness of the correlations (i.e., correlations being confined within a local area). In particular, if the facility system $\mathcal{J}$ could be partitioned into N mutually exclusive subsets $\{J_n\}_{n=1,2,\dots,N}$, such that the facilities within each subset J_i are correlated with one other, while facilities in different subsets are independent, the maximum number of needed stations is $|\mathcal{K}| \le \sum_{n=1}^{N} 2^{|J_n|}$, which is typically much smaller compared to the maximum number of scenarios that are used/needed to describe the correlation, $|\mathcal{S}| = 2^{|\mathcal{J}|}$. We state this obvious result in the following proposition without proof.

Proposition 3. *(Exclusive) If $\mathcal{J} = \cup_{n=1,2,\dots,N} J_n$ for some $N > 1$, such that for all $i = 1, 2, \dots, N$, the disruptions of all facilities in J_i are independent of those in $\mathcal{J} \backslash J_i$, then the maximum number of needed stations $|\mathcal{K}|$ and the number of scenarios $|\mathcal{S}|$ satisfy $|\mathcal{K}| \le \sum_{n=1}^{N} 2^{|J_n|}$ and $|\mathcal{S}| = 2^{\sum_{n=1}^{N} |J_n|}$, respectively, which further yields*

$$\frac{|\mathcal{K}|}{|\mathcal{S}|} \le \frac{\sum_{n=1}^{N} 2^{|J_n|}}{2^{\sum_{n=1}^{N} |J_n|}} \le \begin{cases} 1, & \text{if } N = 1 \text{ (globally correlated)}; \\ \frac{\sum_{n=1}^{N} 2^{|J_n|}}{2^{|\mathcal{J}|}}, & \text{if } 2 \le N \le |\mathcal{J}|/2 \text{ (locally correlated)}. \end{cases}$$

As an example, we consider a facility system $\mathcal{J} = \cup_{n=1,2,\dots,N} J_n$ where $J_i = \{3i-2, 3i-1, 3i\}$, and the disruptions of J_i and $\mathcal{J} \backslash J_i$ are independent. Each J_i has a system structure as shown in Fig. 1(a), and is subject to the scenario disruption profile in Table B.1. For this particular system $\mathcal{J}$, the total number of scenarios is $|\mathcal{S}| = \left(2^3\right)^N = 8^N$, while the number of stations is only $|\mathcal{K}| = 6N$, which is much smaller than $|\mathcal{S}|$. As such, the formulations we will present in later sections indicate that when $N = 4$, a scenario-based formulation would require at least $3N + 3N(3N+1) \cdot 8^N = 638988$ binary variables to describe the scenarios, while our proposed formulation will only need at most $3N + 3N(3N+1)(6N+1)6N = 93612$ binary variables.

Although Proposition 3 addresses the very special case where correlation among facilities can be divided into disjoint groups, similar relationships between $|\mathcal{S}|$ and $|\mathcal{K}|$ can be obtained for other cases. The following proposition, for example, shows that when the disruptions are positively correlated (per Definition 1 in Xie et al. (2015)), no station is needed to connect any two independent facilities.

Proposition 4. *If the facility disruptions are positively correlated per Definition 1 in Xie et al. (2015); i.e.,*

$$\prod_{L: J_1 \backslash \{j\} \subseteq L \subsetneq J \backslash \{j\}} (p_L^M)^{(-1)^{|L|-|J_1|}} \le \prod_{L: J_1 \subseteq L \subsetneq J} (p_L^M)^{(-1)^{|L|-|J_1|+1}}, \forall j \in J_1 \subseteq J \subseteq \mathcal{J},$$

then for any two facilities j_1 and j_2 that are independent of each other; i.e., $p_{\{j_1\}}^M \cdot p_{\{j_2\}}^M = p_{\{j_1, j_2\}}^M$, we have $q_{k_J} = 1$ for any facility set J that contains both j_1 and j_2. That is, no station is connected to both j_1 and j_2. Consequently, given pairwise-independent facility sets $\{J_i\}$, the process of computing station quasi-probabilities only needs to be conducted within each set J_i, and thus gives the overall computational complexity of $O(\sum_{i=1}^{N} 2^{|J_i|})$.

Proof. For any two facilities j_1 and j_2 that are independent of each other, (3) implies

$$p_{\{j_1\}}^M = \prod_{J: j_1 \in J} q_{k_J}, \quad p_{\{j_2\}}^M = \prod_{J: j_2 \in J} q_{k_J}, \quad p_{\{j_1, j_2\}}^M = \prod_{J: J \cap \{j_1, j_2\} \neq \emptyset} q_{k_J}.$$

Substituting these equations into $p_{\{j_1\}}^M \cdot p_{\{j_2\}}^M = p_{\{j_1, j_2\}}^M$ yields

$$\prod_{J: j_1 \in J, j_2 \in J} q_{k_J} = 1.$$

Proposition 5 in Xie et al. (2015) shows that, when facility disruptions are positively correlated, the disruption quasi-probability for any station $k \in \mathcal{K}$ satisfies $q_k \in [0, 1]$. Then, $\prod_{J: j_1 \in J, j_2 \in J} q_{k_J} = 1$ implies that $q_{k_J} = 1$ for any facility set J containing both j_1 and j_2. This completes the proof. □

Proposition 4 is quite revealing; it implies that the number of stations shall be limited if the correlations are local and positive (which often occurs in the real world). For example, consider a chain of N facilities located sequentially in a line, such that any two adjacent facilities are positively correlated while any non-adjacent facilities are mutually independent. In such a system, the maximum number of scenarios is $|\mathcal{S}| = 2^N$, while the maximum number of stations needed to express the correlation is only $|\mathcal{K}| = 2N - 1$, which is far smaller than $|\mathcal{S}|$ for all $N > 2$. In the most special case when facility disruptions are all independent to each other, there exist one-to-one connections between facilities and stations, which implies that $|\mathcal{S}| = 2^N$ and $|\mathcal{K}| = N$.

The supporting station structure of compact size is now ready to be integrated into a mathematical optimization framework for reliable facility location design under correlated facility disruptions. As we will soon see, the introduction of supporting stations helps avoid enumeration of an exponential number of disruption scenarios (which is necessary in the scenario-based formulation) and reduce the model complexity drastically.

2.4. Station-based formulation

In this section, we take advantage of the station structure to propose a new model formulation for the reliable facility location problem under correlated disruptions. Note that the station structure can be specified by $\{q_k\}_{\forall k\in\mathcal{K}}$ and $\{l_{kj}\}_{\forall j\in\mathcal{J},k\in\mathcal{K}}$, which are constructed from the recipe in Appendix B and used as input data for optimization. With the augmented facility-station system, each customer i now seeks service by visiting its most preferred facility j that is connected to a functioning station k (defined as a station-facility pair (k, j) in the rest of the paper, and a station-facility pair is disrupted if the involved station is disrupted), and if the station k becomes disrupted, the customer will visit the next most preferred pair option, which could be either the same facility j with another functioning station k', or a different facility j' with a different functioning station k'. We define R as the maximum number of regular station-facility pairs that a customer can visit, then each customer is assigned to a set of up to $(R-1)$ backup station-facility pairs as part of the service plan. We assume that all customers have full knowledge of the functioning status of station-facility pairs after the realization of disruption(s), and hence each customer directly visits its serviced station-facility pair that is functioning and has the smallest backup level. Specifically, a customer will visit its level-r station-facility pair if all its level-1, …, level-$(r-1)$ pair options have been disrupted. The transportation cost for station-facility pair (k, j) to satisfy one unit of demand from customer i is again denoted by d_{ij}. For all $i\in\mathcal{I}$, $j_1, j_2\in\mathcal{J}$, we let $c_{ij_1j_2} = 1$ if $d_{ij_1} \le d_{ij_2}$, or 0 otherwise.

Note that each customer can be assigned to at most R regular station-facility pairs. In addition to the dummy *emergency* facility (with index $j = 0$) defined earlier, we further add a dummy *emergency* station (with index $k = 0$) to allow the "penalty assignment", i.e., when a customer loses service. Note that $l_{00} = 1$ and $q_0 = 0$, and we set the corresponding transportation cost to be the penalty cost, i.e., $d_{ij}|_{j=0} = \pi_i, \forall i\in\mathcal{I}$. Typically, a customer shall be assigned to station-facility pair (0,0) at level $R+1$ as long as regular station-facility pairs are available for backup levels $1, 2, \dots, R$. However, if at some backup level $s\in\{1, 2, \dots, R\}$, customer i cannot receive service from any station-facility pair (k, j) at a cost less than π_i (per unit demand), it will choose to pay the penalty cost π_i, i.e., visit the emergency station-facility pair at level s.

We now need several sets of decision variables. Again, variables $\mathbf{X} := \{X_j\}_{j\in\mathcal{J}}$ denote the location decisions as in the scenario-based formulation. The assignment of customers to station-facility pairs at multiple backup levels is specified by $\mathbf{Y} := \{Y_{ikjr}\}_{i\in\mathcal{I},k\in\mathcal{K}\cup\{0\},j\in\mathcal{J}\cup\{0\},r\in\{1,2,\dots,R+1\}}$ where

$$Y_{ikjr} = \begin{cases} 1 & \text{if customer} i \text{is assigned to station-facility pair} (k, j) \text{at level } r; \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we define variables $\mathbf{Z} := \{Z_{ikjr}\}_{i\in\mathcal{I},k\in\mathcal{K}\cup\{0\},j\in\mathcal{J}\cup\{0\},r\in\{1,2,\dots,R+1\}}$ with $Z_{ikjr}\in\mathbb{R}$ denotes the quasi-probability for customer i to be assigned to station-facility pair (k, j) at level r. This could happen only when station k is functioning, station k and facility j are connected, and all the station-facility pairs assigned to customer i at levels $1, 2, \dots, r-1$ are unavailable. Therefore, the value of Z_{ikjr} depends on the assignments of customer i to station-facility pairs at levels $1, 2, \dots, r-1$ (i.e., $\{Y_{ihls}\}_{\forall(h,l),s=1,2,\dots,r-1}$) and the corresponding quasi-probabilities (i.e., $\{Z_{ihls}\}_{\forall(h,l),s=1,2,\dots,r-1}$).

The reliable facility-location problem under correlated facility disruptions can be formulated as the following station-based mixed-integer programming model (RFL-STA):

$$\text{(RFL-STA)} \quad \min \quad \sum_{j\in\mathcal{J}} f_j X_j + \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}\cup\{0\}} \sum_{j\in\mathcal{J}\cup\{0\}} \sum_{r=1}^{R+1} \mu_i d_{ij} Z_{ikjr} Y_{ikjr} \tag{5a}$$

$$\text{s.t.} \quad \sum_{r=1}^{R} Y_{ikjr} \le X_j, \;\; \forall i\in\mathcal{I}, j\in\mathcal{J}, k\in\mathcal{K}, \tag{5b}$$

$$Y_{ikjr} \le l_{kj}, \;\; \forall i\in\mathcal{I}, j\in\mathcal{J}\cup\{0\}, k\in\mathcal{K}\cup\{0\}, r = 1, 2, \dots, R+1, \tag{5c}$$

$$\sum_{j\in\mathcal{J}} \sum_{r=1}^{R} Y_{ikjr} \le 1, \;\; \forall i\in\mathcal{I}, k\in\mathcal{K}, \tag{5d}$$

$$\sum_{r=1}^{R+1} Y_{i00r} = 1, \ \forall i \in \mathcal{I}, \tag{5e}$$

$$\sum_{k\in\mathcal{K}} \sum_{j\in\mathcal{J}} Y_{ikjr} + \sum_{s=1}^{r} Y_{i00s} = 1, \ \forall i \in \mathcal{I}, r = 1, 2, \ldots, R+1, \tag{5f}$$

$$Y_{ik_1 j_1 r} \le \sum_{s=1}^{r-1} Y_{ik_2 j_2 s} + c_{ij_1 j_2} + 2 - l_{k_2 j_2} - \left[\frac{\sum_{h\in\mathcal{I}} \sum_{k\in\mathcal{K}} \sum_{s=1}^{R} Y_{hkj_2 s}}{|\mathcal{I}||\mathcal{K}|R} \right],$$
$$\forall i \in \mathcal{I}, j_1, j_2 \in \mathcal{J}, k_1, k_2 \in \mathcal{K}, 2 \le r \le R, \tag{5g}$$

$$Z_{ikj1} = l_{kj}(1 - q_k), \ \forall i \in \mathcal{I}, j \in \mathcal{J} \cup \{0\}, k \in \mathcal{K} \cup \{0\}, \tag{5h}$$

$$Z_{ikjr} = l_{kj}(1 - q_k) \cdot \sum_{k'\in\mathcal{K}} \sum_{j'\in\mathcal{J}} \frac{q_{k'}}{1 - q_{k'}} Z_{ik'j'(r-1)} Y_{ik'j'(r-1)},$$
$$\forall i \in \mathcal{I}, j \in \mathcal{J} \cup \{0\}, k \in \mathcal{K} \cup \{0\}, r = 2, 3, \ldots, R+1, \tag{5i}$$

$$X_j, Y_{ikjr} \in \{0, 1\}, \ \forall i \in \mathcal{I}, j \in \mathcal{J} \cup \{0\}, k \in \mathcal{K} \cup \{0\}, r = 1, 2, \ldots, R+1. \tag{5j}$$

The objective function (5a) presents the expected system cost including the fixed facility cost, the expected total transportation cost, and the expected penalty cost (associated with the dummy station-facility pair). Constraints (5b) and (5c) enforce that customers can only be assigned to station-facility pairs with open facilities. Constraints (5d) make sure that each customer's assignment will not involve any station at more than one backup level. Constraints (5e) postulate that each customer is assigned to the dummy emergency station-facility pair at a certain backup level, while constraints (5f) require that at each level r, a customer $i \in I$ is either assigned to a station-facility pair, or has been assigned to the dummy station-facility pair at an earlier level $s \le r$. Constraints (5g) enforce that a customer is always assigned to the closest functioning station-facility pair for service; i.e., for any $1 \le r \le R$ and two arbitrary station-facility pairs (k_1, j_1), (k_2, j_2) with $d_{ij_2} < d_{ij_1}$, if facility j_2 is built, and a customer i is assigned to (k_1, j_1) at level r, then i should have been assigned to (k_2, j_2) at some level $s < r$. Constraints (5g) ensure equivalence between (RFL-STA) and (RFL-SCE), as we will prove in C.1. Constraints (5h)–(5i) recursively define the values of Z_{ikjr} based on the mechanism of the station structure: at level $r > 1$, the value of Z_{ikjr} equals $(1 - q_k)q_{k'}/(1 - q_{k'})Z_{ik'j'r-1}$ if that customer i is assigned to (k', j') at level $r - 1$. Since quasi-probability q_k can take any nonnegative value in $[0, \infty)$, Z_{ikjr} can take any real value in $[\overline{M}_k, \widehat{M}_k]$, where

$$\overline{M}_k = \min_{\forall L\subseteq\mathcal{K}\setminus\{k\}} \left[(1 - q_k) \prod_{l\in L} q_l \right], \quad \widehat{M}_k = \max_{\forall L\subseteq\mathcal{K}\setminus\{k\}} \left[(1 - q_k) \prod_{l\in L} q_l \right], \quad \forall k \in \mathcal{K}.$$

Finally, Constraints (5j) define integrality of the variables.

We then show in Proposition 5 below that the above formulation (RFL-STA) correctly captures the key issues associated with the reliable location model under correlated disruptions. Similar to Proposition 1 in Cui et al. (2010), our station-based formulation (RFL-STA), with sufficiently large R, is equivalent to the scenario-based formulation (RFL-SCE).

Proposition 5. *When $R = |\mathcal{K}|$, the station-based formulation (RFL-STA) with station structure is guaranteed to yield exactly the same optimal objective value and optimal solutions as the scenario-based formulation (RFL-SCE).*

Proof. See Section C.1. □

If $R < |\mathcal{K}|$, the two formulations are not necessarily equivalent. However, when only a limited number of facilities are built in the optimal solution, if R is as large as the total number of all stations connected to the open facilities, the two formulations are equivalent. Furthermore, as the value of R only influences very high order terms in the formulation, even choosing an R value smaller than $|\mathcal{K}|$ would have only a small impact on the optimal location decisions.[2] More discussion on this choice can be found in Cui et al. (2010) and Section 4.1.

It is worth noting that when the facility disruptions are uncorrelated (no matter homogeneous or heterogeneous), each facility is connected to exactly one station, and vice versa. This implies that the virtual station structure is no longer needed (as we can consolidate each station-facility pair into a single facility) and index k can be removed from (RFL-STA). In such cases, our station-based model (RFL-STA) will degenerate to the (LRUFL) model in Cui et al. (2010).

[2] One of the side-effects of the approximate model formulation (with a small R) is that the customers do not necessarily go to the nearest operational facility, but rather they may, at least theoretically, go to a more reliable yet farther one so as to lower the risk of losing service completely.

Formulation (RFL-STA) is nonlinear because the objective and constraints (5i) contain nonlinear terms $Z_{ikjr}Y_{ikjr}$. However, since each $Z_{ikjr}Y_{ikjr}$ is a product of a bounded continuous variable and a binary variable, we can linearize it by applying a variant of the technique introduced by Sherali and Alameddine (1992), i.e., we replace each $Z_{ikjr}Y_{ikjr}$ by a new continuous variable W_{ikjr} and enforce their equivalence by adding the following four sets of constraints.

$$W_{ikjr} \le Z_{ikjr} + \overline{M}_k(Y_{ikjr} - 1), \tag{6a}$$

$$W_{ikjr} \ge Z_{ikjr} + \widehat{M}_k(Y_{ikjr} - 1), \tag{6b}$$

$$W_{ikjr} \le \widehat{M}_k Y_{ikjr}, \tag{6c}$$

$$W_{ikjr} \ge \overline{M}_k Y_{ikjr}. \tag{6d}$$

The model formulation is now transformed into the following:

$$\text{(LRFL-STA)} \quad \min \sum_{j\in\mathcal{J}} f_j X_j + \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}\cup\{0\}} \sum_{j\in\mathcal{J}\cup\{0\}} \sum_{r=1}^{R+1} \mu_i d_{ij} W_{ikjr} \tag{7a}$$

$$\text{s.t.} \quad (5b) - (5h), \tag{7b}$$

$$Z_{ikjr} = (1 - q_k) \sum_{k'\in\mathcal{K}} \sum_{j'\in\mathcal{J}} \frac{q_{k'}}{1 - q_{k'}} W_{ik'j'(r-1)},$$
$$\forall i \in \mathcal{I}, j \in \mathcal{J}\cup\{0\}, k \in \mathcal{K}\cup\{0\}, r = 2, 3, \ldots, R+1, \tag{7c}$$

$$(6a) - (6d), \ \forall i \in \mathcal{I}, j \in \mathcal{J}\cup\{0\}, k \in \mathcal{K}\cup\{0\}, r = 1, 2, \ldots, R+1, \tag{7d}$$

$$X_j, Y_{ikjr} \in \{0, 1\}, \ \forall i \in \mathcal{I}, j \in \mathcal{J}\cup\{0\}, k \in \mathcal{K}\cup\{0\}, r = 1, 2, \ldots, R+1. \tag{7e}$$

This mixed-integer linear program (LRFL-STA) could in theory be solved by commercial solvers such as CPLEX and Gurobi. However, the existence of station-facility pairs as well as their associated site-dependent disruption quasi-probability exacerbates the model complexity. In light of this, we develop customized solution approaches in the next section.

3. Solution approach

3.1. Lagrangian relaxation

We choose to relax constraints (5b) in (LRFL-STA) with Lagrangian multipliers $\{\lambda_{ikj}\}_{\forall i\in\mathcal{I},\forall k\in\mathcal{K},\forall j\in\mathcal{J}}$ and move them as penalty terms to the objective function. The objective function becomes

$$\min \sum_{j\in\mathcal{J}} \left(f_j - \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} \lambda_{ikj} \right) X_j + \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}\cup\{0\}} \sum_{j\in\mathcal{J}\cup\{0\}} \sum_{r=1}^{R+1} \mu_i d_{ij} W_{ikjr} + \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} \sum_{j\in\mathcal{J}} \lambda_{ikj} \sum_{r=1}^{R} Y_{ikjr}.$$

The above relaxation of the set of constraints (5b) essentially decouples the location and assignment variables **X** and **Y**. The remaining model can be decomposed into multiple disjoint parts. The part involving **X**,

$$\min_{X_j\in\{0,1\},\forall j} \sum_{j\in\mathcal{J}} \left(f_j - \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} \lambda_{ikj} \right) X_j,$$

can be solved by simple inspection; i.e., given any $\{\lambda_{ikj}\}$, we can easily find the optimal **X** as follows:

$$X_j = \begin{cases} 1 & \text{if } f_j - \sum_{i\in\mathcal{I}} \sum_{k\in\mathcal{K}} \lambda_{ikj} < 0; \\ 0 & \text{otherwise.} \end{cases}$$

We further notice that the remaining problem can be further separated into individual subproblems, one for each customer, as long as we relax the term $\sum_{h\in\mathcal{I}} \sum_{k\in\mathcal{K}} \sum_{s=1}^{R} Y_{hkj_2s}$ in (5g) by $\sum_{k\in\mathcal{K}} \sum_{s=1}^{R} Y_{ikj_2s}$. The relaxed subproblem (RFL-STA-SP$_i$) with respect to customer i is

$$\text{(RFL-STA-SP}_i\text{)} \quad \min \sum_{k\in\mathcal{K}\cup\{0\}} \sum_{j\in\mathcal{J}\cup\{0\}} \sum_{r=1}^{R+1} \mu_i d_{ij} W_{kjr} + \sum_{k\in\mathcal{K}} \sum_{j\in\mathcal{J}} \lambda_{kj} \sum_{r=1}^{R} Y_{kjr} \tag{8a}$$

$$\text{s.t.} \quad Y_{kjr} \le l_{kj}, \ \ \forall j \in \mathcal{J} \cup \{0\}, k \in \mathcal{K} \cup \{0\}, r = 1, 2, \ldots, R+1, \tag{8b}$$

$$\sum_{j \in \mathcal{J}} \sum_{r=1}^{R} Y_{kjr} \le 1, \ \ \forall k \in \mathcal{K}, \tag{8c}$$

$$\sum_{r=1}^{R+1} Y_{00r} = 1, \tag{8d}$$

$$\sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}} Y_{kjr} + \sum_{s=1}^{r} Y_{00s} = 1, \ \ \forall r = 1, 2, \ldots, R+1, \tag{8e}$$

$$Y_{k_1 j_1 r} \le \sum_{s=1}^{r-1} Y_{k_2 j_2 s} + c_{i j_1 j_2} + 2 - l_{k_2 j_2} - \frac{\sum_{k \in \mathcal{K}} \sum_{s=1}^{R} Y_{k j_2 s}}{|\mathcal{I}||\mathcal{K}|R}, \quad \forall i \in \mathcal{I}, j_1, j_2 \in \mathcal{J}, k_1, k_2 \in \mathcal{K}, 2 \le r \le R, \tag{8f}$$

$$Z_{kj1} = 1 - q_k, \ \ \forall j \in \mathcal{J}, k \in \mathcal{K}, \tag{8g}$$

$$Z_{kjr} = (1 - q_k) \sum_{k' \in \mathcal{K}} \sum_{j' \in \mathcal{J}} \frac{q_{k'}}{1 - q_{k'}} W_{j' k' (r-1)}, \quad \forall j \in \mathcal{J}, k \in \mathcal{K}, r = 2, 3, \ldots, R+1, \tag{8h}$$

$$(6a) - (6d), \tag{8i}$$

$$Y_{kjr} \in \{0, 1\}, \ \ \forall j \in \mathcal{J}, k \in \mathcal{K}, r = 1, 2, \ldots, R+1. \tag{8j}$$

Note that (RFL-STA-SP$_i$), although still a mixed-integer linear program, is much smaller in size than the original (LRFL-STA), and hence it can often be efficiently handled by commercial solvers like CPLEX. However, solving this subproblem repeatedly (for each customer, and across Lagrangian relaxation iterations) could pose as a computational burden. Thus, Section 3.3 further proposes an optional customized algorithm to solve (RFL-STA-SP$_i$).

The optimal objective values from the relaxed subproblems provide a lower bound to the original problem. Section 3.2 describes a heuristic to perturb the subproblem solutions in order to obtain a feasible solution to the original problem (which provides an upper bound). With the upper bound and lower bound, we use standard subgradient techniques (Fisher, 2004) to update the multipliers λ in the Lagrangian procedure; i.e.,

$$\lambda_{ikj}^{n+1} = \lambda_{ikj}^{n} + t_j^n \left(\sum_r Y_{ikjr}^n - X_j^n \right), \tag{9}$$

$$t_j^n = \frac{\xi^n (Z^* - Z_D(\lambda^n))}{\| \sum_r Y_{ikjr}^n - X_j^n \|^2}, \tag{10}$$

where λ_{ikj}^n represents a generic multiplier in the nth iteration, t^n is the step size, ξ^n is a scalar, and Z^* and $Z_D(\lambda^n)$ are the best upper bound and the current lower bound, respectively.

The above bounds, especially the lower bound, may be far from optimum (e.g., due to duality gaps from the relaxed constraints). If the Lagrangian relaxation algorithm fails to obtain a small enough gap in a certain number of iterations, we embed it into a branch-and-bound (B&B) framework to further reduce the gap. We construct a binary tree by branching on **X**. Specifically, among all unbranched variables, we select and branch on the one whose construction yields the least system cost. After building the branching tree, we run the Lagrangian relaxation algorithm at each node to determine the corresponding feasible solution and lower bound, and update them after finishing both child branches. While traversing the binary tree, depth-first search is found to perform slightly better than breadth-first or least-cost-first searches for small or moderate-sized instances (which are likely to be solved to optimality). However, if the instances are large, it is difficult to traverse the entire tree and completely close the gap. In such cases, least-cost-first search is preferable since it tends to yield a reasonably good lower bound before completely traversing the entire tree.

3.2. Upper bound

To obtain a good upper bound to the original model (RFL-STA), we first fix the facility location decisions from the relaxed subproblem. Then for each customer i, we sort all station-facility pairs associated with open facilities (i.e., pair (k, j) is considered if $X_j = 1, l_{kj} = 1$) in ascending order of (d_{ij}, p_k); (k_1, j_1) comes before (k_2, j_2) if $d_{ij_1} < d_{ij_2}$ or $d_{ij_1} = d_{ij_2}, q_{k_1} < q_{k_2}$. Then at every level r, we assign customer i to pair (k, j) with the smallest (d_{ij}, q_k) as long as i has never been assigned to any pair (k, j'), $\forall j'$ at levels $1, 2, \ldots, r-1$ before. The following two propositions state two properties of the optimal solution to (RFL-STA) and indicate that the feasible solution constructed from this heuristic approach is likely to be near optimum.

First, constraints (5g) ensure the following property, which we state without proof:

Proposition 6. *(**Property I**) In any optimal solution* $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ *to (RFL-STA), a customer will be assigned to backup station-facility pairs based on the corresponding distances; i.e., if* $Y_{ikjr} = 1$ *for some* i, k, j, r, *then* $X_{j'} = 0$ or $l_{k'j'} = 0$ or $\exists r' < r$ s.t. $Y_{ik'j'r'} = 1, \forall k', j'$ *with* $d_{ij'} < d_{ij}$.

Next, the following proposition reveals the relationship between assignment decisions and station disruption quasi-probabilities:

Proposition 7. *(**Property II**) In any optimal solution* $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ *to (RFL-STA), a customer will be assigned to backup station-facility pairs that involve the same facility based on the corresponding disruption quasi-probabilities of the associated stations; i.e., if* $Y_{ikjr} = 1$ *for some* i, k, j, r, *then* $l_{k'j} = 0$ *or* $\exists j', r' < r$ s.t. $Y_{ik'j'r'} = 1$, $\forall k'$ *with* $q_{k'} \le q_k$.

Proof. See Appendix C.3. □

Based on these two properties, given location decisions from the relaxed subproblem solutions, if R is sufficiently large, this heuristic yields the optimal customer assignments; otherwise, it can only guarantee feasible but not necessarily optimal assignments. Nevertheless, since the quasi-probabilities for using high-level back-ups (i.e., the product of multiple station disruption quasi-probabilities, which is equivalently the product of multiple facility disruption scenario probabilities) are often smaller by orders of magnitude, the solution given by this sorting/greedy heuristic shall be quite close to the optimal one.

3.3. Lower bound

As mentioned before, although the relaxed problem is separable by customer i, each subproblem is still combinatorial and the worst-case complexity is exponential. Therefore, in this section, we develop an algorithm which helps quickly find lower bounds to the relaxed subproblems (RFL-STA-SP$_i$).

Equations (8h) show that Z_{kjr} depends on $Z_{kj(r-1)}$ and $Y_{kj(r-1)}$, which builds connections across the decision variables and brings difficulty in solving subproblem (RFL-STA-SP$_i$). Instead of having Z_{kjr} directly in the formulation, we approximate them with fixed numbers.

Similar properties as those stated in Propositions 6 and 7 apply to (RFL-STA-SP$_i$), which suggest that certain customer-station-facility assignments would never appear in the optimal solution to (RFL-STA-SP$_i$). We summarize them into the following rules:

- **Rule 1** If customer i is assigned to two different facilities at some levels, then it will always be assigned to the closer facility at a lower level; i.e., $Y_{ik_1j_1r_1} = Y_{ik_2j_2r_2} = 1$, and $d_{ij_1} < d_{ij_2} \Rightarrow r_1 < r_2$;
- **Rule 2** If customer i is assigned to a facility j, then it will be assigned to all station-facility pairs associated with j, $\{(k, j)\}_{\forall k: l_{kj}=1}$ at consecutive backup levels as long as k has never been used at a lower level; i.e., let $K_j = \{k : l_{kj} = 1, Y_{ikls} = 0, \forall l \in \mathcal{J}, l \neq j, s < r\}$, then $Y_{ikjr} = 1$ for some $k, r \Rightarrow Y_{ik_1js} = Y_{ik_2j(s+1)} = \cdots = Y_{ik_nj(s+n-1)} = 1$ for some permutation $k_1, k_2, \ldots, k_n$ of K_j and some s;
- **Rule 3** If customer i is assigned to multiple station-facility pairs $(k_1, j), (k_2, j), \ldots, (k_n, j)$ that involve the same facility j, then these pairs should be used in ascending order of the involved station disruption quasi-probabilities; i.e., $Y_{ik_1jr} = Y_{ik_2j(r+1)} = \cdots = Y_{ik_nj(r+n-1)} = 1 \Rightarrow q_{k_1} \le q_{k_2} \le \cdots \le q_{k_n}$;

Based on these rules, we can set $Y_{kjr} = Z_{kjr} = 0$ for some (k, j, r) without affecting the optimal solution. For example, in the system shown in Fig. 1(b), if we assume that $d_{i_1j_1} < d_{i_1j_2}, q_{k_{\{1\}}} < q_{k_{\{1,2\}}} < q_{k_{\{2\}}} < q_{k_{\{1,2,3\}}} < q_{k_{\{1,3\}}} < q_{k_{\{2,3\}}}$, then we have: (i) $Y_{i_1k_{\{1,2\}}j_11} = Y_{i_1k_{\{1,2\}}j_13} = 0$ because j_1 should be used first if it is built, and $k_{\{1,2\}}$ should be used after $k_{\{1\}}$ and before $k_{\{1,3\}}$, $k_{\{1,2,3\}}$; (ii) $Y_{i_1k_{\{2\}}j_21} = Y_{i_1k_{\{2\}}j_23} = Y_{i_1k_{\{2\}}j_24} = Y_{i_2k_{\{2\}}j_26} = 0$ because $k_{\{2\}}$ should always be used after $k_{\{1,2\}}$, and also after $k_{\{1\}}$, $k_{\{1,2,3\}}$, $k_{\{1,3\}}$ if j_1 is used. After setting these Y_{kjr} and Z_{kjr} to be zero, we can use algorithm **LowerBound**(i), for each customer i, to construct lower bounds to Z_{kjr} and Z_{00r} as α_{kjr} and β_r, respectively, and relax (RFL-STA-SP$_i$) into the following (RFL-STA-RSP$_i$):

We replace Z_{kjr} and Z_{00r} respectively by their estimates α_{kjr} and β_r, and relax (RFL-STA-SP$_i$) into the following (RFL-STA-RSP$_i$):

$$\text{(RFL-STA-RSP}_i) \quad \min \sum_{k\in\mathcal{K}}\sum_{j\in\mathcal{J}}\sum_{r=1}^{R+1}\left(\mu_i d_{ij}\alpha_{kjr} + \lambda_{kjr}\right)Y_{kjr} + \sum_{r=1}^{R+1}\mu_i d_{i0}\beta_r Y_{00r} \tag{11a}$$

$$\text{s.t.} \quad Y_{kjr} \le l_{kj}, \quad \forall j \in \mathcal{J} \cup \{0\}, k \in \mathcal{K} \cup \{0\}, r = 1, 2, \cdots, R+1, \tag{11b}$$

$$\sum_{j \in \mathcal{J}} \sum_{r=1}^{R} Y_{kjr} \le 1, \quad \forall k \in \mathcal{K}, \tag{11c}$$

$$\sum_{r=1}^{R+1} Y_{00r} = 1, \tag{11d}$$

$$\sum_{k \in \mathcal{K}} \sum_{j \in \mathcal{J}} Y_{kjr} + \sum_{s=1}^{r} Y_{00s} = 1, \quad \forall r = 1, 2, \cdots, R+1, \tag{11e}$$

$$Y_{kjr} \in \{0, 1\}, \quad \forall j \in \mathcal{J}, k \in \mathcal{K}, r = 1, 2, \ldots, R+1. \tag{11f}$$

Proposition 8. *The solution to (RFL-STA-RSP$_i$) is a lower bound to the solution to (RFL-STA-SP$_i$).*

Proof. Let $\mathbf{Y}^*$, $\mathbf{Z}^*$ and $\mathbf{W}^*$ be the optimal solution to (RFL-STA-SP$_i$). We can construct (RFL-STA-RSP$_i$) from (RFL-STA-SP$_i$) in three sequential steps: (i) replace Z_{kjr} and Z_{00r} by α_{kjr} and β_r, respectively, and add constraints to set $Y_{kjr} = 0$ for some (k, j, r) pairs; (ii) remove constraints (8f)–(8i); and (iii) remove those constraints $Y_{kjr} = 0$ that were added in step (i), and instead, set the corresponding α_{kjr} to be sufficiently large. In step (i), we know that adding those $Y_{ikj} = 0$ does not change the optimal solution to (RFL-STA-SP$_i$), and based on the construction of α_{kjr} and β_r, we know $\alpha_{kjr} Y^*_{kjr}$ and $\beta_r Y^*_{00r}$ are lower bounds to Z^*_{kjr} and Z^*_{00r}, respectively. In step (ii), removing constraints obviously never increases the objective value of a minimization problem. Step (iii) just uses an alternative way to enforce the $Y_{kjr} = 0$ constraints; i.e., when the coefficients of those Y_{kjr} are set to be infinity, these variables cannot equal 1 at optimality (because a finite feasible solution is known to exist). Therefore, each of the three steps provides a lower bound to the model built in the previous step, hence the optimal objective value of (RFL-STA-RSP$_i$) is a lower bound to the optimal objective value of (RFL-STA-SP$_i$). This completes the proof. □

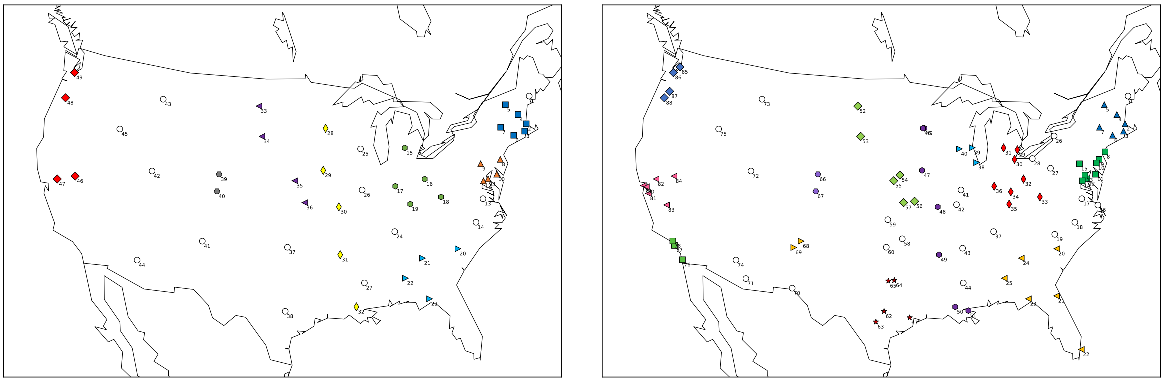
We observe that (RFL-STA-RSP$_i$) is a combinatorial generalized assignment problem, which can be solved by an adapted Hungarian algorithm as in Cui et al. (2010). (RFL-STA-RSP$_i$) aims at assigning one station-facility pair to each level (up to $R+1$) based on the updated coefficients associated with each Y_{kjr}, so as to minimize the total expected system cost. However, the actual maximum assignment level $R_{\max}$ (i.e., the largest r such that $Y_{kjr} = 1$ for some (k, j) pair) may be smaller than R due to lower cost associated with the emergency station-facility pair than all other remaining pairs at some level $r < R$. The main challenge is to identify the level that the emergency station-facility pair should be assigned to. As such, we enumerate $R_{\max}$ from 0 to R and for each $R_{\max}$, we fix $Y_{00,R_{\max}+1} = 1$ and $Y_{kjr} = 0, r > R_{\max} + 1$. In this way, the (RFL-STA-RSP$_i$) is simplified into a standard assignment problem that can be solved by conventional Hungarian algorithm. We solve (RFL-STA-RSP$_i$) and calculate the associated total cost for each enumeration of $R_{\max}$. By comparison, the value of $R_{\max}$ corresponding to the lowest total cost is the actual maximum assignment level $R_{\max}$. After fixing $R_{\max}$, It is worth noting that in the enumeration process, the assignment solutions to model with $R_{\max} = r$ can be used as a warm start to the model with $R_{\max} = r + 1$, which helps expedite the computation. Specifically, if the penalty cost d_{i0} (or say π_i) is sufficiently large, we only need to solve (RFL-STA-RSP$_i$) for one iteration, i.e., $R_{\max} = R$.

4. Case study

We apply the proposed model and solution algorithms to two sets of examples under different correlation patterns and parameter settings. The first set of examples demonstrate reliable facility system planning for the U.S. networks with 49 and 88 nodes under given correlation patterns; see Fig. 3.[3] The second set of examples are used to compare the results and computational performance of the scenario-based and station-based formulations. In all cases, the customer set I and facility candidate set J are set to be the same.

The proposed solution algorithms are programmed in C++ and run on a 64-bit Intel i7-3770 computer with 3.40 GHz CPU and 8 G RAM. The reformulated problem (RFL-STA-RSP$_i$) is solved by the Hungarian algorithm.

[3] The location data set is from Snyder and Daskin (2005) and can be accessed at http://www.lehigh.edu/~lvs2, all input data, including the correlation profile, will be available for download at the corresponding author's webpage https://yfouyang.cee.illinois.edu.



(a) 49-node network. (b) 88-node network.

Fig. 3. Input networks of the U.S. map with locations from 1990 census data.

Table 1
Scenario representation of correlation profile for the 49-node network.

J	p_J^S	J	p_J^S	J	p_J^S	J	p_J^S
Local 1		**Local 3**		**Local 5**		**Local 6**	
2	0.008	15	0.010	28	0.010	33	0.010
3	0.008	15,16	0.008	29	0.008	34	0.008
2,3	0.006	15,17	0.008	30	0.006	35	0.006
2,3,4	0.005	15,16,17	0.008	28,29	0.008	36	0.005
2,3,4,5	0.004	15,16,18	0.006	29,30	0.006	33,34	0.008
2,3,6	0.005	15,17,19	0.006	30,31	0.005	34,35	0.006
2,3,6,7	0.004	15,16,17,18	0.005	28,29,30	0.006	35,36	0.005
2,3,4,6	0.005	15,16,17,19	0.005	29,30,31	0.005	33,34,35	0.006
2,3,4,5,6	0.003	15,16,17,18,19	0.005	30,31,32	0.004	34,35,36	0.005
2,3,4,6,7	0.003	**Local 4**		28,29,30,31	0.005	33,34,35,36	0.004
2,3,4,5,6,7	0.002	20	0.010	29,30,31,32	0.003	**Local 8**	
Local 2		23	0.012	28,29,30,31,32	0.005	47	0.010
12	0.010	20,23	0.008	**Local 7**		48	0.005
11,12	0.010	20,21,23	0.006	39	0.010	49	0.005
10,11,12	0.008	20,22,23	0.005	40	0.010	46,47	0.010
9,11,12	0.008	20,21,22,23	0.004	39,40	0.010	48,49	0.010
9,10,11,12	0.008					47,48,49	0.005
8,9,10,11,12	0.006					46,47,48,49	0.005

4.1. U.S. network cases

We first test our methodology on the U.S. map: (i) a 49-node network with locations as the state capitals of the continental United States plus Washington, D.C.; and (ii) a 88-node network with the 49-node locations and 39 other largest cities in the United States.

Local disruption correlations are observed among the locations in each of the 8 local areas in the 49-node network (see different markers in Fig. 3(a)). Facility disruptions across these local areas, however, are assumed to be independent. The scenario-based correlation profile (as if obtained from historical observations) for each local area is presented in Table 1, in which the column with header "J" lists the facility locations disrupted in each scenario, and the column with header "p_J^S" presents the corresponding scenario probabilities.

The resulting station structure, including the set of facilities J_k connected to each station k as well as its disruption quasi-probability q_k, is listed in Table 2. The total number of stations is 75, which is much smaller than the total number of scenarios in the entire network $12 \times 7 \times 10 \times 7 \times 13 \times 11 \times 4 \times 8 \times 2^{14} = 4.41 \times 10^{11}$.

Similarly, the 88-node network as shown in Fig. 3(b) includes 13 local areas. The scenario-based disruption correlation profile for each local area is presented in Table 3, and the resulting station structure is listed in Table 4. Again, the total number of stations is only 129, while the total number of scenarios is 4.4×10^{18}.

We first test our model and algorithm for $R = 10, 15, 20, 75$ using the 49-node network with the system parameter values from Snyder and Daskin (2005), and $R = 10, 20, 30, 129$ using the 88-node network. For the 49-node network, in particular, we run three set of instances (49-I, 49-II, 49-III): (1) 49-I uses the exactly same parameter values as Snyder and Daskin (2005); (2) 49-II reduces the fixed facility costs to 1/3 of their original values in Snyder and Daskin (2005); and (3) 49-III doubles all the input scenario probabilities in Tables 1 and 3. In addition, We run another instance with independent

Table 2
Station disruption quasi-probabilities for the 49-node network.

k	J_k	q_K	k	J_k	q_k	k	J_k	q_k	k	J_k	q_k
1	7	0.4000	20	18,19	0.8696	39	28,31,32	0.9600	58	48,49	0.3333
2	6,7	0.5556	21	17,19	0.6250	40	28,30,31,32	0.9783	59	46,47,48	0.8000
3	5	0.4000	22	17,18,19	0.9946	41	28,29	0.6667	60	46,47,49	0.8000
4	5,7	0.9615	23	16,18	0.6250	42	28,29,30,31,32	0.0978	61	46,47,48,49	0.0938
5	5,6,7	1.0636	24	16,18,19	0.9946	43	36	0.4000	62	1	0.0200
6	4,5	0.5556	25	16,17,18,19	0.9758	44	35,36	0.5556	63	13	0.0200
7	4,5,7	1.0636	26	15,16,17,18,19	0.0610	45	34,35,36	0.6429	64	14	0.0200
8	4,5,6,7	1.0062	27	22	0.4000	46	33	0.4444	65	24	0.0200
9	3,4,5,6,7	0.8222	28	21	0.4444	47	33,36	1.0714	66	25	0.0200
10	2,4,5,6,7	0.8222	29	21,22	0.9783	48	33,35,36	1.0216	67	26	0.0200
11	2,3,4,5,6,7	0.0547	30	21,22,23	0.6970	49	33,34	0.6429	68	27	0.0200
12	8	0.4286	31	20,21,22	0.6571	50	33,34,36	1.0208	69	37	0.0200
13	8,10	0.6364	32	20,21,22,23	0.0502	51	33,34,35	0.7368	70	38	0.0200
14	8,9	0.6364	33	32	0.5000	52	33,34,35,36	0.1190	71	41	0.0200
15	8,9,10	0.8643	34	31,32	0.6250	53	39	0.5000	72	42	0.0200
16	8,9,10,11	0.8000	35	30,31,32	0.6667	54	40	0.5000	73	43	0.0200
17	8,9,10,11	0.0500	36	29,30,31,32	0.7059	55	39,40	0.0400	74	44	0.0200
18	18	0.5000	37	28	0.6250	56	46	0.5000	75	45	0.0200
19	19	0.5000	38	28,32	0.8889	57	46,47	0.5000			

Table 3
Scenario representation of correlation profile for the 88-node network.

J	p_J^S	J	p_J^S	J	p_J^S	J	p_J^S
Local 1		**Local 3**		**Local 4**		**Local 7**	
2	0.008	22	0.012	29	0.010	52	0.010
3	0.008	21,22	0.010	31	0.010	53	0.008
2,3	0.006	22,23	0.010	29,31	0.008	54,55	0.006
2,3,4	0.005	21,22,23	0.008	29,30,31	0.007	56,57	0.006
2,3,4,5	0.004	20,21,22,23	0.006	29,30,31,32	0.006	52,53	0.008
2,3,6	0.005	21,22,23,24	0.006	29,30,31,32,33	0.004	53,54,55	0.006
2,3,6,7	0.004	21,22,23,25	0.006	29,30,31,32,34	0.005	54,55,56,57	0.005
2,3,4,6	0.005	21,22,23,24,25	0.005	29,30,31,32,36	0.005	52,53,54,55	0.005
2,3,4,5,6	0.003	20,21,22,23,24,25	0.004	29,30,31,32,34,36	0.004	53,54,55,56,57	0.004
2,3,4,6,7	0.003	**Local 6**		29,30,31,32,33,34	0.003	52,53,54,55,56,57	0.003
2,3,4,5,6,7	0.002	45,46	0.010	29,30,31,32,33,34,36	0.003	**Local 8**	
Local 2		47	0.008	29,30,31,32,34,35,36	0.003	61	0.010
8	0.008	48	0.006	29,30,31,32,33,34,35,36	0.002	62	0.008
9,10	0.008	49	0.005	**Local 10**		63	0.008
11	0.008	50,51	0.004	68	0.010	64,65	0.008
8,9,10	0.006	45,46,47	0.008	69	0.010	62,63	0.006
9,10,11	0.006	47,48	0.006	68,69	0.020	61,62,63	0.005
8,9,10,11	0.005	48,49	0.005	**Local 11**		61,64,65	0.005
8,9,10,11,12	0.004	49,50,51	0.004	76	0.010	62,63,64,65	0.005
8,9,10,11,15	0.004	45,46,47,48	0.006	77,78	0.010	61,62,63,64,65	0.003
8,9,10,11,12,13,14	0.004	47,48,49	0.005	76,77,78	0.005	**Local 12**	
8,9,10,11,12,13,14,15	0.003	48,49,50,51	0.004	**Local 13**		79,80	0.008
Local 5		45,46,47,48,49	0.005	85	0.005	81	0.008
38	0.010	47,48,49,50,51	0.004	86	0.005	79,80,81	0.010
39	0.010	45,46,47,48,49,50,51	0.003	87	0.005	79,80,81,82	0.006
40	0.010	**Local 9**		88	0.005	79,80,81,83	0.005
38,39	0.008	66	0.010	85,86	0.010	79,80,81,82,83	0.004
39.40	0.008	67	0.010	87,88	0.010	79,80,81,82,84	0.004
38,39,40	0.005	66,67	0.010	85,86,87,88	0.005	79,80,81,82,83,84	0.003

facility disruptions using each of the network, with the same marginal facility probabilities as the correlated instances. The initial values of the Lagrangian multipliers are all set to be 0, and the Lagrangian relaxation/B&B procedure is executed to a tolerance of 0.5%, or up to 3600 seconds in CPU time. The algorithm performance is summarized in Table 5, and the optimal facility locations and initial customer assignments are shown in Fig. 4.

From Table 5 and Fig. 4, we can summarize the following observations:: (1) our model and algorithm perform very well, especially on the 49-node cases, solving the first set (i.e., case 49 -I) to less than 1% gap and most of the other two sets (i.e., 49-II, 49-III) to less than 5% gap within 1 h. For the larger 88-node cases, we can still obtain 7%-8% gap within 1 h computation time; (2) the maximum back-up level R does affect the location decision and optimal system cost when R is small; however, when R becomes large, the optimal facility locations are insensitive to it. This implies that we can set an arbitrary yet reasonably large value of R for many applications; (3) even with a sufficiently large R, i.e., $R = |\mathcal{K}|$, such that the

Table 4
Station disruption quasi-probabilities for the 88-node network.

k	J_k	q_k	k	J_k	q_k	k	J_k	q_k
1	7	0.4000	45	40	0.3846	89	68,69	0.0450
2	6,7	0.5556	46	38,39	0.5652	90	76	0.3333
3	5	0.4000	47	38,40	1.0903	91	77,78	0.3333
4	5,7	0.9615	48	39,40	0.5652	92	76,77,78	0.0450
5	5,6,7	1.0636	49	38,39,40	0.0970	93	84	0.4286
6	4,5	0.5556	50	50,51	0.3750	94	83	0.4286
7	4,5,7	1.0636	51	49,50,51	0.5714	95	83,84	0.9608
8	4,5,6,7	1.0062	52	48,49,50,51	0.6364	96	82,84	0.5833
9	3,4,5,6,7	0.8222	53	47,48,49,50,51	0.6875	97	82,83,84	0.9107
10	2,4,5,6,7	0.8222	54	45,46	0.4286	98	81,82,83,84	0.8000
11	2,3,4,5,6,7	0.0547	55	45,46,50,51	1.0980	99	79,80,82,83,84	0.8000
12	15	0.4286	56	45,46,49,50,51	1.0259	100	79,80,81,82,83,84	0.0500
13	13,14,15	0.6364	57	45,46,48,49,50,51	1.0127	101	87,88	0.3333
14	12,13,14	0.4286	58	45,46,47	0.6364	102	86,87,88	0.7500
15	12,13,14,15	1.2833	59	45,46,47,50,51	1.0275	103	85,87,88	0.7500
16	11,12,13,14,15	0.7692	60	45,46,47,49,50,51	1.0080	104	85,86	0.3333
17	9,10,11,12,13,14,15	0.7647	61	45,46,47,48	0.7333	105	85,86,88	0.7500
18	8,12,13,14,15	0.7692	62	45,46,47,48,50,51	1.0130	106	85,86,87	0.7500
19	8,11,12,13,14,15	0.8450	63	45,46,47,48,49	0.7895	107	85,86,87,88	0.1422
20	8,9,10,12,13,14,15	0.7647	64	45,46,47,48,49,50,51	0.1693	108	1	0.0200
21	8,9,10,11,12,13,14,15	0.0684	65	56,57	0.3750	109	16	0.0200
22	24,25	0.4000	66	54,55,56,57	0.5000	110	17	0.0200
23	20	0.4444	67	53,54,55,56,57	0.6154	111	18	0.0200
24	20,25	0.6000	68	52	0.4286	112	19	0.0200
25	20,24	0.6000	69	52,56,57	1.0370	113	26	0.0200
26	20,24,25	1.7857	70	52,54,55,56,57	1.0588	114	27	0.0200
27	20,23,24,25	0.7778	71	52,53	0.5844	115	28	0.0200
28	20,21,24,25	0.7778	72	52,53,56,57	1.0640	116	37	0.0200
29	20,21,23,24,25	0.8635	73	52,53,54,55	0.6667	117	41	0.0200
30	20,21,22,23,24,25	0.0670	74	52,53,54,55,56,57	0.1335	118	42	0.0200
31	35	0.4000	75	64,65	0.3750	119	43	0.0200
32	35,36	0.6250	76	62,63	0.3750	120	44	0.0200
33	34,35,36	0.6667	77	62,63,64,65	0.9275	121	58	0.0200
34	33	0.4000	78	61	0.3750	122	59	0.0200
35	33,35	1.0417	79	61,64,65	1.1228	123	60	0.0200
36	33,35,36	0.9600	80	61,63,64,65	0.7037	124	70	0.0200
37	33,34,35	0.7059	81	61,62,64,65	0.7037	125	71	0.0200
38	33,34,35,36	1.2143	82	61,62,63	1.0159	126	72	0.0200
39	32,33,34,35,36	0.8333	83	61,62,63,64,65	0.1086	127	73	0.0200
40	30,32,33,34,35,36	0.8400	84	66	0.5000	128	74	0.0200
41	30,31,32,33,34,35,36	0.8333	85	67	0.5000	129	75	0.0200
42	29,30,32,33,34,35,36	0.8333	86	66,67	0.0400			
43	29,30,31,32,33,34,35,36	0.0720	87	68	0.6667			
44	38	0.3846	88	69	0.6667			

station-based formulation is exactly equivalent to the scenario-based formulation, our algorithm still works quite effectively, as it is capable of providing solutions with a small optimality gap within a short amount of time; (4) by comparing the second set of 49-node cases (i.e., 49-II) under independent and correlated facility disruptions, we can see that ignoring correlations in reliable facility location models may lead to quite sub-optimal system design, i.e., solutions provided in the independent and correlated instances are different; (5) finally, the comparison between 49-I and 49-III, as expected, implies that doubling input disruption probabilities increases the overall system cost as well as the optimal number of facilities to be built.

4.2. Model comparison

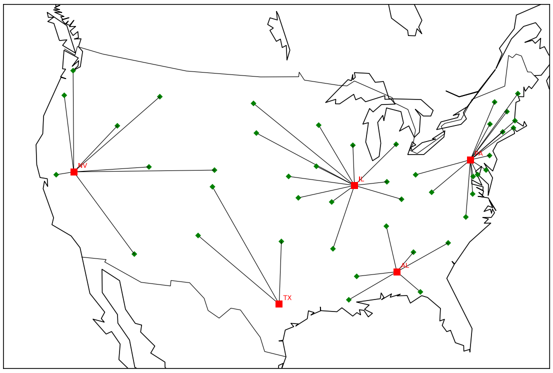
To better demonstrate the advantage of the proposed modeling framework, we further test both the scenario-based formulation (RFL-SCE) and station-based formulation (RFL-STA) on four networks, with 14, 17, 19, and 25 nodes, respectively. As shown in Fig. 5, each of the four networks is part of the 49-node network in Fig. 3, with the corresponding local correlation profiles presented in Table 1. For example, the three local areas in Fig. 5(a) correspond to Local 2, Local 3, and Local 4 in Table 1, respectively, with the location indices adjusted accordingly. Independent stations (those not appearing in Table 1) each fail independently with probability 0.02.

Table 6 compares the results from (RFL-SCE) and (RFL-STA). It can be observed that for each case, the solutions (i.e., the final UB and location decisions) from the two formulations are exactly the same. However, our station-based formulation can be solved to optimality much more quickly than the scenario-based formulation, especially when correlations are present. In

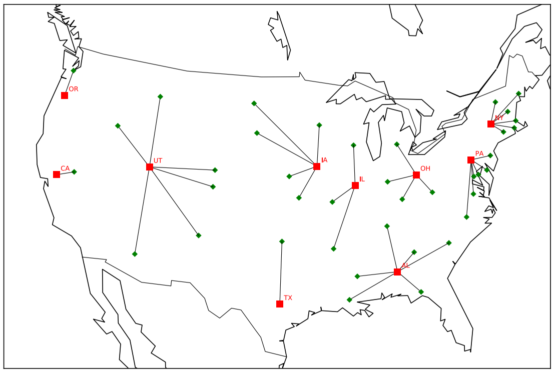
Table 5
Algorithm performance for the U.S. networks with 49 and 88 nodes.

Nodes	Pattern	R^a	Facility	Root UB	Root LB	Root gap (%)	Overall UB	Overall LB	Overall gap (%)	CPU time
49-I	Indp	5	9,22,26,38,46	891,150	860,404	3.450	887,868	883,507	0.491	96
	Corr	10	9,22,26,38,46	893,049	834,126	6.598	887,881	883,451	0.499	1557
		15	9,22,26,38,46	888,855	835,584	5.993	887,868	883,493	0.493	2373
		20	9,22,26,38,46	891,150	834,113	6.400	887,868	883,486	0.494	2568
		75	9,22,26,38,46	891,150	834,741	6.330	887,868	879,537	0.938	3600
49-II	Indp	5	7,9,16,22,26,36,38,47,48	586,886	572,538	2.445	586,194	583,291	0.495	654
	Corr	10	9,13,15,23,26,29,38,47,48	597,735	554,304	7.266	595,102	570,308	4.166	3600
		15	7,9,16,22,26,29,38,42,47,48	589,864	554,304	6.029	587,996	569,597	3.129	3600
		20	7,9,16,22,26,29,38,42,47,48	589,657	554,304	5.996	587,794	568,639	3.259	3600
		75	7,9,16,22,26,29,38,42,47,48	589,122	554,229	5.923	587,793	565,163	3.850	3600
49-III	Indp	5	9,17,22,36,38,46	922,190	862,314	6.493	911,476	906,932	0.499	521
	Corr	10	9,15,22,36,38,46	926,889	812,649	12.325	912,682	874,124	4.225	3600
		15	9,17,22,36,38,46	922,594	812,649	11.917	911,868	871,296	4.449	3600
		20	9,17,22,36,38,46	922,198	812,648	11.879	911,483	869,052	4.655	3600
		75	9,17,22,36,38,46	922,191	812,649	11.878	911,476	857,396	5.933	3600
88	Indp	5	10,25,29,39,57,61,71,83,87	1,242,190	1,200,420	3.363	1,242,200	1,221,060	1.702	3600
	Corr	10	10,25,29,39,57,61,71,83,87	1,254,600	1,112,030	11.364	1,254,600	1,150,340	8.310	3600
		20	10,25,29,39,57,61,71,83,87	1,242,200	1,115,950	10.163	1,242,200	1,150,620	7.372	3600
		30	10,25,29,39,57,61,71,83,87	1,244,970	1,107,200	11.066	1,242,200	1,150,830	7.355	3600
		129	10,25,29,39,57,61,71,83,87	1,244,970	1,102,360	11.455	1,242,200	1,140,060	8.223	3600

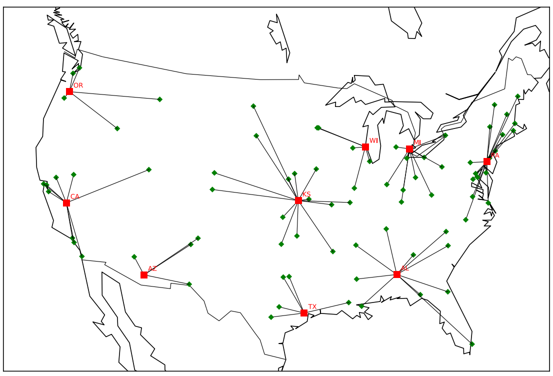
[a] In the case with independent facility disruptions, R refers to the maximum number of backup facilities (as in Cui et al. (2010)), while in the case with correlated facility disruptions, R corresponds to the maximum number of backup station-facility pairs.



(a) 49 nodes with input 1 (49-I)

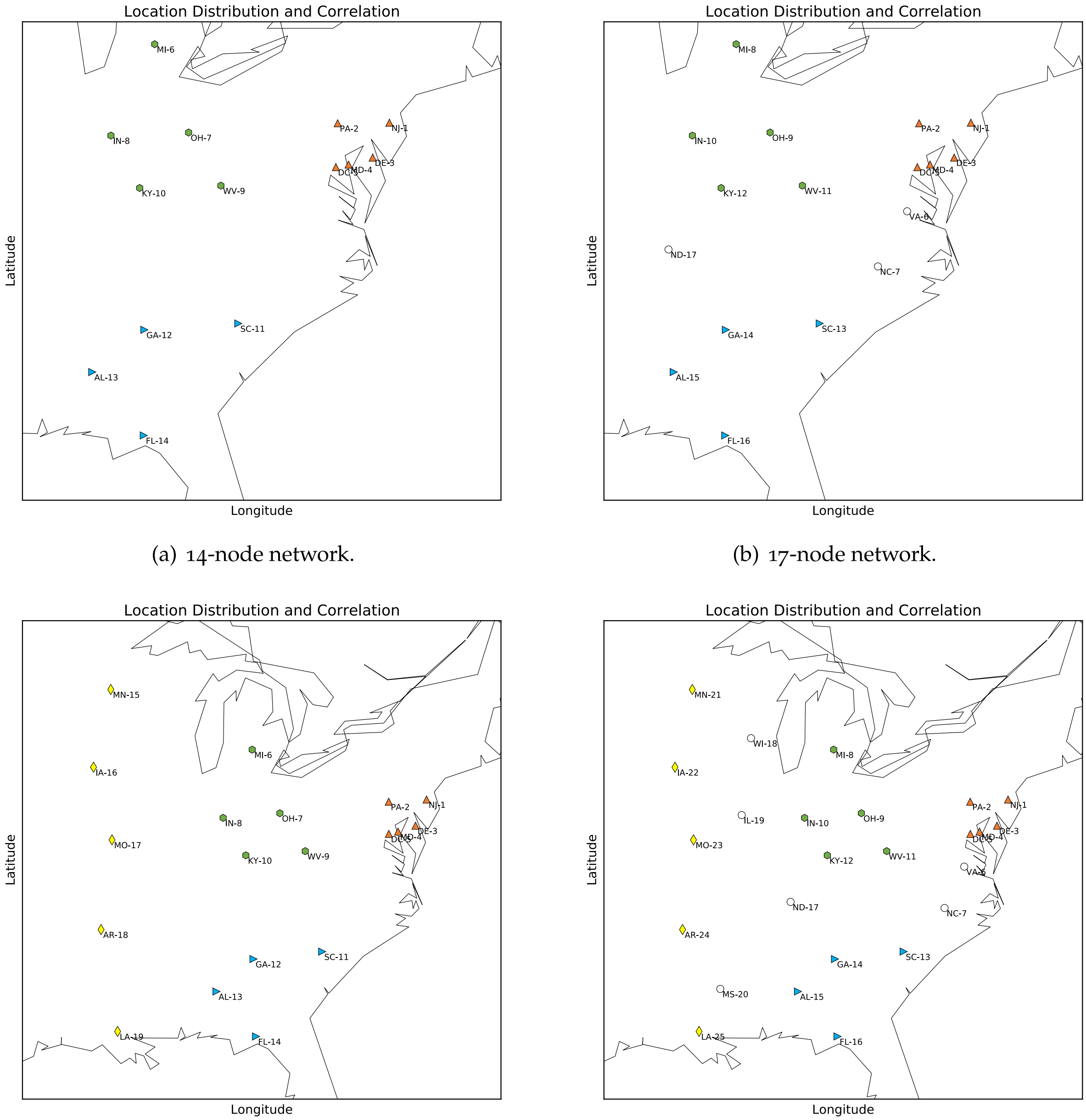


(b) 49 nodes with input 2 (49-II)



(c) 88 nodes

Fig. 4. Facility location solutions for U.S. networks with 49 and 88 nodes.



(a) 14-node network. (b) 17-node network.

(c) 19-node network. (d) 25-node network.

Fig. 5. Network setup and correlation pattern for the four cases.

Table 6
Performance comparison between (RFL-SCE) and (RFL-STA).

Pattern	# of	Scenario-based (RFL-SCE)			Station-based (RFL-STA)		
	nodes	UB	Facilities	Time (s)	UB	Facilities	Time (s)
No disruption	14	257,581	PA-2,MI-6,FL-14	0.1	257,581	PA-2,MI-6,FL-14	0.1
	17	304,372	PA-2,MI-8,AL-15	0.2	304,372	PA-2,MI-8,AL-15	0.3
	19	338,555	PA-2,MI-6,AL-13	0.4	338,555	PA-2,MI-6,AL-13	0.5
	25	426,942	PA-2,IN-10,AL-15	0.5	426,942	PA-2,IN-10,AL-15	0.9
Correlation	14	266,412	PA-2,MI-6,AL-13	0.5	266,412	PA-2,MI-6,AL-13	0.8
	17	313,426	PA-2,MI-8,AL-15	35.8	313,426	PA-2,MI-8,AL-15	1.6
	19	348,426	PA-2,IN-8,AL-13	252.2	348,426	PA-2,IN-8,AL-13	3.2
	25	–	–	–	436,833	PA-2,AL-15,IL-19	6.7

Algorithm 1 Construct α_{kjr} and β_r as lower bounds to Z_{kjr} and Z_{00r}, respectively, $\forall i \in \mathcal{I}$.

LowerBound(i)

```
for r do
  for (k, j) do
    \alpha_{kjr} = 0
    if l_{kj} = 1 then
      K_k^r = 0, prob = 1.0
      for k' do
        if k' \neq k, l_{k'j} = 1, q_{k'} < q_k then
          K_k^r = K_k^r + 1, prob = prob \cdot q_{k'}
        end if
      end for
      if K_k^r < r then
        minProduct = 1.0
        if K_k^r < r - 1 then
          for (k', j') do
            if j' \neq j, k' \neq k, d_{ij'} < d_{ij}, \alpha_{k'j',r-1-K_k^r} \in (0, minProduct) then
              minProduct = \alpha_{k'j',r-1-K_k^r}
            end if
          end for
        end if
        \alpha_{kjr} = q_k \cdot prob \cdot minProduct
      end if
    end if
  end for
end for
for r do
  \beta_r = 1.0, minProduct = 1.0
  for (k, j) do
    if l_{kj} = 1, \alpha_{kjr} \in (0, minProduct) then
      minProduct= \alpha_{kjr}
    end if
  end for
  \beta_r = minProduct
  for (k, j) do
    \alpha_{kjr} = \alpha_{kjr} \cdot (1 - q_k)/q_k
    if \alpha_{kjr} = 0 then
      \alpha_{kjr} = \infty
    end if
  end for
end for
```

particular, the scenario-based formulation cannot even provide a feasible solution to the 25-node network due to exhaustion of computer memory (by the excessive number of scenarios). This comparison verifies the correctness and effectiveness of our station-based formulation (RFL-STA), as well as its clear superiority over the traditional scenario-based formulation (RFL-SCE), even for moderate-sized problem instances. We can easily project that the advantage will be even bigger for larger applications.

5. Conclusion

In this paper, we focus on the problem of reliable facility location design under correlated facility disruptions. To handle the complexity associated with correlated facility disruptions, we incorporate and extent the idea of supporting station structure introduced in Li et al. (2013); Xie et al. (2015), and augment the original facility system by adding an additional layer of supporting stations. The stations, each being associated with a site-dependent disruption quasi-probability (which may take a value larger than 1), are assumed to function independently and can equivalently capture the effect of correlation among facility disruptions. We show a number of important properties of such a station structure. To optimize facility location design, we further show that the reliable facility location problem under correlated facility disruptions can be formulated into a compact mixed-integer mathematical model, which is equivalent to the traditional scenario-based formulation and is much more compact in size. The proposed model can be solved by customized Lagrangian relaxation algorithms (with customized modules for obtaining upper and lower bounds). Multiple case studies with various network settings and correlation patterns are conducted to test the performance and applicability of the methodology. Superiority of the proposed station structure has also been clearly demonstrated via numerical experiments.

The study can be further extended in several directions. First, in many real-world contexts, the reachability of facilities or the access distances could be affected by the facility/station disruptions as well. For example, debris from earthquakes or floods may block nearby roadway segments and change the shortest paths between points. Such complicating issues should be addressed in future studies. Second, real-world applications may involve specific types of correlation patterns from physical laws (e.g., decaying probability of failure “contagion”, or conventional correlation matrix). Specific model structure and insights might be available at those correlation patterns. Third, it will be beneficial to conduct a systematic complexity analysis on data input preparation, model formulation, and solution algorithm for the most general correlation patterns. We also plan to apply our methodology to more real-world cases, so as to help policy makers develop engineering and planning guidelines that will lead to more reliable and resilient systems. In light of the large size of many real-world applications, additional efforts might be put on developing even more sophisticated and effective solution algorithms.

Contribution statement

Reliable facility location models have been developed in a variety of forms to address probabilistic facility disruptions. Most of these existing models assume independent facility disruptions. In reality, however, disruptions could be correlated when the facilities are subject to shared hazards/mutual interactions, and such correlations could be positive or negative, or mixed. Facing the challenge of handling an exponential number of disruption scenarios even for a given location design, only a few very recent studies have incorporated correlated facility disruptions into the location design framework for very special correlation patterns. A systematic methodology framework is needed to design reliable facility locations under site-dependent and generally correlated facility disruptions.

In this paper, we extend the station structure approach in Li et al. (2013) and incorporate the idea of quasi-probability (Xie et al., 2015) to capture any pattern of facility disruption correlations (positive, negative, or mixed). An additional layer of supporting stations are added and properly connected to the facilities. These stations, each being associated with a site-dependent disruption quasi-probability, are assumed to function independently and can equivalently capture the correlations among facilities. With these newly added stations, we are able to develop a compact mixed-integer mathematical model to optimize the reliable facility location decisions. To hedge against the complexity associated with the model, Lagrangian relaxation based algorithms, including customized modules for obtaining upper bound and lower bounds of relaxed subproblems are developed. Multiple case studies with various types of correlated facility disruptions are carried out to demonstrate the applicability and performance of our model and algorithms. Managerial insights are also drawn.

Acknowledgments

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Appendix A Notation list

$\mathcal{I}$	Set of customers
$\mathcal{J}$	Set of candidate facility locations
$\mathcal{K}$	Set of virtual stations
Ω	Set of all possible facility disruption scenarios/realizations
$\mathcal{S}$	Scenario representation of facility disruptions
$\mathcal{M}$	Marginal representation of facility disruptions
R	Total number of regular station-facility pairs assigned to any customer ($R-1$ pairs as backup options)
μ_i	Demand of customer i
f_j	Fixed cost for building a facility at location j
d_{ij}	Cost for a facility at j to satisfy one unit of demand from customer i
$c_{ij_1j_2}$	Whether facility j_1 is closer to customer i than facility j_2, $c_{ij_1j_2}=1$ if $d_{ij_1}<d_{ij_2}$, 0 otherwise
l_{kj}	Whether facility j is connected to station k, $l_{kj}=1$ if they are connected, 0 otherwise
q_k	Quasi-probability for station k to be disrupted
k_J	Station connected to all and only facilities in set J
π_i	Penalty cost for customer i to lose one unit of demand
$\delta_{j\omega}$	Whether facility built at location j is functioning in scenario ω, $\delta_{j\omega}=1$ if it is functioning, 0 otherwise
p_ω	Probability for scenario ω to occur
p_J^S	Probability for all locations in J to be disrupted while all others are operating

p_J^M	Probability for all locations in J to be disrupted regardless of the states of all other locations
X_j	Whether a facility is built at location j, $X_j = 1$ if it is, 0 otherwise
$Y_{ij\omega}$	Whether customer i is assigned facility j in scenario ω, $Y_{ij\omega} = 1$ if it is, and 0 otherwise
Y_{ikjr}	Whether customer i is assigned to station-facility pair (k, j) at level r, $Y_{ikjr} = 1$ if it is, and 0 otherwise
Z_{ikjr}	Quasi-probability for customer i to be assigned to station-facility pair (k, j) at level r
W_{ikjr}	$W_{ikjr} = Z_{ikjr} Y_{ikjr}$
$\overline{M}_k$	Minimum value of $Z_{ikjr}, \forall i \in \mathcal{I}, j \in \mathcal{J} \cup \{0\}, r = 1, 2, \ldots, R+1$
$\widehat{M}_k$	Maximum value of $Z_{ikjr}, \forall i \in \mathcal{I}, j \in \mathcal{J} \cup \{0\}, r = 1, 2, \ldots, R+1$
λ_{ikj}^n	Lagragian multiplier in the nth iteration of the Lagrangian relaxation procedure
t_j^n	Step size in the nth iteration of the Lagrangian relaxation procedure
α_{kjr}	Approximation of Z_{kjr} for any customer i
β_r	Approximation of Z_{00r} for any customer i

Appendix B. Recipe for building station structure

This appendix presents the recipe in Xie et al. (2015) for building the station structure from the scenario representation of facility disruption correlations, i.e., $\mathcal{S} = \{p_J^S\}_{\forall J \subseteq \mathcal{J}}$.

Formula (4), which calculates station disruption quasi-probabilities from scenario probabilities, can be decomposed via two steps: (i) calculate marginal facility disruption probability p_J^M as follows

$$p_J^M = \sum_{J_1 : J \subseteq J_1} p_{J_1}^S; \tag{B.1}$$

and (ii) calculate station disruption quasi-probabilties as follows

$$q_{k_J} = \prod_{L : \bar{J} \subseteq L \subseteq \mathcal{J}} \left[p_L^M\right]^{(-1)^{|L|-|\bar{J}|+1}}. \tag{B.2}$$

To illustrate the idea, we use the simple three-facility system in Fig. 1(a)–(b) as an example. The scenario-based disruption probabilities are given as input data, as shown in Table B.1. The marginal facility disruption probabilities from (B.1) is as follows:

$$\begin{aligned}
p_{\{1\}}^M &= p_{\{1\}}^S + p_{\{1,2\}}^S + p_{\{1,3\}}^S + p_{\{1,2,3\}}^S = 0.60, \\
p_{\{2\}}^M &= p_{\{2\}}^S + p_{\{1,2\}}^S + p_{\{2,3\}}^S + p_{\{1,2,3\}}^S = 0.55, \\
p_{\{3\}}^M &= p_{\{3\}}^S + p_{\{1,3\}}^S + p_{\{2,3\}}^S + p_{\{1,2,3\}}^S = 0.50, \\
p_{\{1,2\}}^M &= p_{\{1,2\}}^S + p_{\{1,2,3\}}^S = 0.45, \\
p_{\{1,3\}}^M &= p_{\{1,3\}}^S + p_{\{1,2,3\}}^S = 0.40, \\
p_{\{2,3\}}^M &= p_{\{2,3\}}^S + p_{\{1,2,3\}}^S = 0.35, \\
p_{\{1,2,3\}}^M &= p_{\{1,2,3\}}^S = 0.30.
\end{aligned}$$

Then from (B.2), the disruption quasi-probabilities for the stations can be computed as

$$\begin{aligned}
q_{k_{\{1\}}} &= \frac{p_{\{1,2,3\}}^M}{p_{\{2,3\}}^M} = \frac{0.30}{0.35} = 0.86, \\
q_{k_{\{2\}}} &= \frac{p_{\{1,2,3\}}^M}{p_{\{1,3\}}^M} = \frac{0.30}{0.40} = 0.75, \\
q_{k_{\{3\}}} &= \frac{p_{\{1,2,3\}}^M}{p_{\{1,2\}}^M} = \frac{0.30}{0.45} = 0.67,
\end{aligned}$$

Table B.1
Different representations of the correlated facility disruptions for the 3-facility example.

Scenario representation	$p_{\{1\}}^S$ 0.05	$p_{\{2\}}^S$ 0.05	$p_{\{3\}}^S$ 0.05	$p_{\{1,2\}}^S$ 0.15	$p_{\{1,3\}}^S$ 0.10	$p_{\{2,3\}}^S$ 0.05	$p_{\{1,2,3\}}^S$ 0.30
Marginal representation	$p_{\{1\}}^M$ 0.60	$p_{\{2\}}^M$ 0.55	$p_{\{3\}}^M$ 0.50	$p_{\{1,2\}}^M$ 0.45	$p_{\{1,3\}}^M$ 0.40	$p_{\{2,3\}}^M$ 0.35	$p_{\{1,2,3\}}^M$ 0.30
Station representation	$q_{k_{\{1\}}}$ 0.86	$q_{k_{\{2\}}}$ 0.75	$q_{k_{\{3\}}}$ 0.67	$q_{k_{\{1,2\}}}$ 0.93	$q_{k_{\{1,3\}}}$ 0.95	$q_{k_{\{2,3\}}}$ 1.00	$q_{k_{\{1,2,3\}}}$ 0.79

$$q_{k_{\{1,2\}}} = \frac{p^M_{\{1,3\}} \cdot p^M_{\{2,3\}}}{p^M_{\{3\}} \cdot p^M_{\{1,2,3\}}} = \frac{0.40 \cdot 0.35}{0.50 \cdot 0.30} = 0.93,$$

$$q_{k_{\{1,3\}}} = \frac{p^M_{\{1,2\}} \cdot p^M_{\{2,3\}}}{p^M_{\{2\}} \cdot p^M_{\{1,2,3\}}} = \frac{0.45 \cdot 0.35}{0.55 \cdot 0.30} = 0.95,$$

$$q_{k_{\{2,3\}}} = \frac{p^M_{\{1,2\}} \cdot p^M_{\{1,3\}}}{p^M_{\{1\}} \cdot p^M_{\{1,2,3\}}} = \frac{0.45 \cdot 0.40}{0.60 \cdot 0.30} = 1.00,$$

$$q_{k_{\{1,2,3\}}} = \frac{p^M_{\{1\}} \cdot p^M_{\{2\}} \cdot p^M_{\{3\}} \cdot p^M_{\{1,2,3\}}}{p^M_{\{1,2\}} \cdot p^M_{\{1,3\}} \cdot p^M_{\{2,3\}}} = \frac{0.60 \cdot 0.55 \cdot 0.50 \cdot 0.30}{0.45 \cdot 0.40 \cdot 0.35} = 0.79.$$

In so doing, the entire marginal representation and associated station structure are obtained, as summarized in Table B.1.

Appendix C. Proof of propositions

C1. Proof of Proposition 5

Proof. We first map an optimal solution to (RFL-STA) to a feasible solution to (RFL-SCE). Let (**X, Y, Z**) be an optimal solution to (RFL-STA). We let $j(i,r) = j : Y_{ikjr} = 1, k(i,r) = k : Y_{ikjr} = 1, J(i,r) = \{j \in \mathcal{J} \cup \{0\} : j \neq j(i,r), \exists k, l \leq r-1, Y_{ikjl} = 1\}, \mathcal{R}_i = \{1\} \cup \{r > 1 : \exists j \neq j', k \neq k', \text{ s.t. } Y_{ikjr} = Y_{ik'j'r-1} = 1\}$, and for each $r \in \mathcal{R}$, we let $r_i(r) \in \{r' : r' \in \mathcal{R}_i, r' > r, r' \leq r'', \forall r'' > r\}$. We construct a solution $(\mathbf{X}', \mathbf{Y}')$ as follows

(i) $X'_j = X_j$;

(ii) $Y'_{ij\omega} = \begin{cases} 1, & \text{if } j = j(i,r) \text{ for some } r, \delta_{j\omega} = 1, \delta_{j'\omega} = 0, \forall j' \in J(i,r); \\ 0, & \text{otherwise.} \end{cases}$

By construction, $(\mathbf{X}', \mathbf{Y}')$ is a feasible solution to (RFL-SCE). In particular, for any customer $i \in \mathcal{I}$ and any scenario $\omega \in \Omega$, either there exists exactly one $j \in \mathcal{J}$ such that $j = j(i,r)$ for some $r, \delta_{j\omega} = 1, \delta_{j'\omega} = 0, \forall j' \in J(i,r)$, or there exists no $j \in \mathcal{J}$ such that $j = j(i,r)$ for some $r, \delta_{j\omega} = 1$. Hence, there exists exactly one $j \in \mathcal{J} \cup \{0\}$ such that $Y_{ij\omega} = 1, \forall i \in \mathcal{I}, \omega \in \Omega$, which implies that (1b) hold.

We next show that $(\mathbf{X}', \mathbf{Y}')$ achieves the same objective value as (**X, Y, Z**). We denote Φ(**X, Y, Z**) and $\Psi(\mathbf{X}', \mathbf{Y}')$ as the objectives of (RFL-STA) and (RFL-SCE), respectively, and $\Omega(i,r) = \{\omega \in \Omega : Y'_{ij(i,r)\omega} = 1\}$. We have the following result

$$\begin{aligned}
\Psi(\mathbf{X}', \mathbf{Y}') &= \sum_{j\in\mathcal{J}} f_j X'_j + \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}\cup\{0\}} \sum_{\omega\in\Omega} \mu_i d_{ij} Y'_{ij\omega} p_\omega \\
&= \sum_{j\in\mathcal{J}} f_j X_j + \sum_{i\in\mathcal{I}} \sum_{r\in\mathcal{R}_i} \mu_i d_{ij(i,r)} \sum_{\omega\in\Omega(i,r)} p_\omega \\
&= \sum_{j\in\mathcal{J}} f_j X_j + \sum_{i\in\mathcal{I}} \sum_{r\in\mathcal{R}_i} \mu_i d_{ij(i,r)} \sum_{J:J(i,r)\subseteq J, j(i,r)\notin J} p^S_J.
\end{aligned}$$

Applying Equations (6) in Xie et al. (2015) yields

$$\begin{aligned}
\sum_{J:J(i,r)\subseteq J, j(i,r)\notin J} p^S_J &= \sum_{J:J(i,r)\subseteq J, j(i,r)\notin J} \sum_{J_1:J\subseteq J_1} (-1)^{|J_1|-|J|} \left[\prod_{J_2:J_2\cap J_1\neq\emptyset} q_{k_{J_2}} \right] \\
&= \sum_{J:J(i,r)\subseteq J, j(i,r)\notin J} \sum_{J_1:J\subseteq J_1} (-1)^{|J_1|-|J|} \mathcal{A}(J_1) \\
&= \sum_{J:J(i,r)\subseteq J} C_J \mathcal{A}(J),
\end{aligned}$$

where $\mathcal{A}(J) = \prod_{J_2:J_1\cap J\neq\emptyset} q_{k_{J_1}}$ and C_J is the ultimate coefficient of $\mathcal{A}(J)$, which are

$$C_J = \begin{cases} 1, & \text{if } J = J(i,r); \\ -1, & \text{if } J = J(i,r) \cup \{j(i,r)\}; \\ \sum_{J'\subseteq J\setminus J(i,r)} (-1)^{|J\setminus J(i,r)|-|J'|} = \sum_{n=0}^{|J\setminus J(i,r)|} (-1)^n \binom{|J\setminus J(i,r)|}{n} = 0, & \text{otherwise .} \end{cases}$$

Therefore, we have

$$\Psi(\mathbf{X}', \mathbf{Y}') = \sum_{j\in\mathcal{J}} f_j X_j + \sum_{i\in\mathcal{I}} \sum_{r\in\mathcal{R}_i} \mu_i d_{ij(i,r)} \sum_{J:J(i,r)\subseteq J, j(i,r)\notin J} p^S_J$$

$$
\begin{aligned}
&= \sum_{j\in\mathcal{J}} f_j X_j + \sum_{i\in\mathcal{I}}\sum_{r\in\mathcal{R}_i} \mu_i d_{ij(i,r)}[\mathcal{A}(J(i,r)) - \mathcal{A}(J(i,r)\cup\{j(i,r)\})] \\
&= \sum_{j\in\mathcal{J}} f_j X_j + \sum_{i\in\mathcal{I}}\sum_{r\in\mathcal{R}_i} \mu_i d_{ij(i,r)}\left[\prod_{J:J\cap J(i,r)\neq\emptyset} q_{k_J} - \prod_{J:J\cap(J(i,r)\cup\{j(i,r)\})\neq\emptyset} q_{k_J}\right] \\
&= \sum_{j\in\mathcal{J}} f_j X_j + \sum_{i\in\mathcal{I}}\sum_{r\in\mathcal{R}_i} \mu_i d_{ij(i,r)} \sum_{l=r}^{r_i(r)-1}\prod_{l'=1}^{l-1} q_{k(i,l')}\left(1-q_{k(i,l)}\right) \\
&= \sum_{j\in\mathcal{J}} f_j X_j + \sum_{i\in\mathcal{I}}\sum_{r=1}^{R+1} \mu_i d_{ij(i,r)} \prod_{l=1}^{r-1} q_{k(i,l)}\left(1-q_{k(i,r)}\right) \\
&= \sum_{j\in\mathcal{J}} f_j X_j + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}\cup\{0\}}\sum_{k\in\mathcal{K}\cup\{0\}}\sum_{r=1}^{R+1} \mu_i d_{ij} Y_{ikjr} Z_{ikjr} \\
&= \Phi(\mathbf{X},\mathbf{Y},\mathbf{Z}),
\end{aligned}
$$

which implies that the optimal solution to (RFL-SCE) is a lower bound to (RFL-STA).

Conversely, we map an optimal solution to (RFL-SCE) to a feasible solution to (RFL-STA). Given an optimal solution (**X, Y**) to (RFL-SCE), without loss of generality, we assume that each customer always visits its closest open facility for service, and if there exist more than one facility with equal distance, we break the tie by choosing the facility based on index: let $J^* = \{j \in \mathcal{J} : X_j = 1\}$, for each customer i, let $j^i_1, j^i_2, \ldots, j^i_{|J^*|+1}$ be an ordering of the facilities in $J^*\cup\{0\}$ such that for all $2 \le r \le |J^*|+1$, $d_{ij^i_{r-1}} \le d_{ij^i_r}$ and if $d_{ij^i_{r-1}} = d_{ij^i_r}$, $j^i_{r-1} < j^i_r$. Since a facility is functioning if and only if at least one of its connected stations is operating, we let $K^* = \{k \in \mathcal{K} : \exists j \in J^*, l_{jk} = 1\}$, and $r^i_n = |\{k \in K^* : \exists n_1 < n, l_{j^i_{n_1}} = 1\}|$ be the total number of stations that are connected to at least one facility in $\{j^i_1, \ldots, j^i_{n-1}\}$, we know that facility j^i_n is visited by i if only if all facilities in $\{j^i_1, \ldots, j^i_{n-1}\}$ are unavailable (i.e, all the r^i_n stations are disrupted). Then we define two sequences of facilities and stations respectively as: (i) $j(i,1), j(i,2), \ldots, j(i,|K^*|+1)$ such that $j(i,r) \in \{j : r_j < r \le r_{j+1}\}$; and (ii) $k(i,1), k(i,2), \ldots, k(i,|K^*|+1)$ such that $k(i,r) \in K(i,r) = \{k : l_{j(i,r)k} = 1, l_{j(i,l)k} = 0, \forall l < r, j(i,l) \neq j(i,r)\}$, and if $K(i,r-1) = K(i,r), k(i,r-1) < k(i,r)$. We construct a solution (**X**′, **Y**′, **Z**′) as follows

(i) $X'_j = X_j$;

(ii) $Y'_{ikjr} = \begin{cases} 1, & \text{if } j = j(i,r), k = k(i,r), d_{ij} \le d_{i0}; \\ 0, & \text{otherwise;} \end{cases}$

(iii) $Z'_{ikjr} = \begin{cases} \left(1 - q_{k(i,r)}\right)\prod_{l=1}^{r-1} q_{k(i,l)}, & \text{if } j = j(i,r), k = k(i,r), d_{ij} \le d_{i0}; \\ 0, & \text{otherwise.} \end{cases}$

By examining the constraint sets in (RFL-STA), we observe that (**X**′, **Y**′, **Z**′) is a feasible solution to (RFL-STA). We next show that (**X**′, **Y**′, **Z**′) achieves the same objective value as (**X, Y**). We let $\mathcal{R}_i = \{r^i_1+1, r^i_2+1, \ldots, r^i_{|J^*|+1}+1\}$, for each $r \in \mathcal{R}_i$, we let $r_i(r) \in \{r' : r' \in \mathcal{R}_i, r' > r, r' \le r'', \forall r'' > r\}$, and $\Omega(i,r) = \{\omega \in \Omega : \delta_{j(i,r)\omega} = 1, \delta_{j(i,l)\omega} = 0, \forall l < r, j(i,l) \neq j(i,r)\}$

$$
\begin{aligned}
\Phi(\mathbf{X}',\mathbf{Y}',\mathbf{Z}') &= \sum_{j\in\mathcal{J}} f_j X'_j + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}\cup\{0\}}\sum_{k\in\mathcal{K}\cup\{0\}}\sum_{r=1}^{R+1} \mu_i d_{ij} Y'_{ikjr} Z'_{ikjr} \\
&= \sum_{j\in\mathcal{J}} f_j X_j + \sum_{i\in\mathcal{I}}\sum_{r=1}^{R+1} \mu_i d_{ij(i,r)} \prod_{l=1}^{r-1} q_{k(i,l)}\left(1-q_{k(i,r)}\right) \\
&= \sum_{j\in\mathcal{J}} f_j X_j + \sum_{i\in\mathcal{I}}\sum_{r\in\mathcal{R}_i} \mu_i d_{ij(i,r)} \sum_{l=r}^{r_i(r)-1}\prod_{l'=1}^{l-1} q_{k(i,l')}\left(1-q_{k(i,l)}\right) \\
&= \sum_{j\in\mathcal{J}} f_j X_j + \sum_{i\in\mathcal{I}}\sum_{r\in\mathcal{R}_i} \mu_i d_{ij(i,r)} \sum_{J:J(i,r)\subseteq J, j(i,r)\notin J} p^S_J \\
&= \sum_{j\in\mathcal{J}} f_j X_j + \sum_{i\in\mathcal{I}}\sum_{r\in\mathcal{R}_i} \mu_i d_{ij(i,r)} \sum_{\omega\in\Omega(i,r)} p_\omega \\
&= \sum_{j\in\mathcal{J}} f_j X_j + \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}\cup\{0\}}\sum_{\omega\in\Omega} \mu_i d_{ij} Y_{ij\omega} p_\omega \\
&= \Psi(\mathbf{X},\mathbf{Y}).
\end{aligned}
$$

Therefore, the optimal solution to (RFL-STA) is also a lower bound to (RFL-SCE), implying that the optimal solutions to (RFL-STA) and (RFL-SCE) are exactly the same. This completes our proof. □

C2. Example for the proof of Proposition 5

Proof. We present a small example to better illustrate the proof procedure in C.1. Considering the three-facility system presented in Appendix B, we first assume that the optimal solution to (RFL-STA) is (**X, Y, Z**) with

$$\begin{aligned}
X_1 &= X_3 = 1, X_2 = 0,\\
Y_{1k_{\{1\}}11} &= Y_{1k_{\{1,2\}}12} = Y_{1k_{\{1,3\}}13} = Y_{1k_{\{1,2,3\}}14} = Y_{1k_{\{3\}}35} = Y_{1k_{\{2,3\}}36} = Y_{1007} = 1,\\
Y_{2k_{\{1\}}11} &= Y_{2k_{\{1,2\}}12} = Y_{2k_{\{1,3\}}13} = Y_{2k_{\{1,2,3\}}14} = Y_{2k_{\{3\}}35} = Y_{2k_{\{2,3\}}36} = Y_{2007} = 1,\\
Y_{3k_{\{3\}}31} &= Y_{3k_{\{1,3\}}32} = Y_{3k_{\{2,3\}}33} = Y_{3k_{\{1,2,3\}}34} = Y_{3k_{\{1\}}15} = Y_{3k_{\{1,2\}}16} = Y_{3007} = 1,
\end{aligned}$$

and the optimal objective is

$$\begin{aligned}
\Phi(\mathbf{X},\mathbf{Y},\mathbf{Z}) &= f_1 + f_3\\
&\quad + \mu_1\big(d_{13}q_{k_{\{1\}}}q_{k_{\{1,2\}}}q_{k_{\{1,3\}}}q_{k_{\{1,2,3\}}}(1-q_{k_{\{3\}}}q_{k_{\{2,3\}}}) + \pi_1 q_{k_{\{1\}}}q_{k_{\{1,2\}}}q_{k_{\{1,3\}}}q_{k_{\{1,2,3\}}}q_{k_{\{3\}}}q_{k_{\{2,3\}}}\big)\\
&\quad + \mu_3\big(d_{31}q_{k_{\{3\}}}q_{k_{\{1,3\}}}q_{k_{\{2,3\}}}q_{k_{\{1,2,3\}}}(1-q_{k_{\{1\}}}q_{k_{\{1,2\}}}) + \pi_3 q_{k_{\{3\}}}q_{k_{\{1,3\}}}q_{k_{\{2,3\}}}q_{k_{\{1,2,3\}}}q_{k_{\{1\}}}q_{k_{\{1,2\}}}\big)\\
&\quad + \mu_2\big(d_{21}(1-q_{k_{\{1\}}}q_{k_{\{1,2\}}}q_{k_{\{1,3\}}}q_{k_{\{1,2,3\}}}) + d_{23}q_{k_{\{1\}}}q_{k_{\{1,2\}}}q_{k_{\{1,3\}}}q_{k_{\{1,2,3\}}}(1-q_{k_{\{3\}}}q_{k_{\{2,3\}}})\\
&\quad + \pi_2 q_{k_{\{1\}}}q_{k_{\{1,2\}}}q_{k_{\{1,3\}}}q_{k_{\{1,2,3\}}}q_{k_{\{3\}}}q_{k_{\{2,3\}}}\big)\\
&= f_1 + f_3 + \mu_1(0.2d_{13} + 0.4\pi_1) + \mu_2(0.4d_{21} + 0.2d_{23} + 0.4\pi_2) + \mu_3(0.1d_{31} + 0.4\pi_3)
\end{aligned}$$

Constructing from (**X, Y, Z**), a feasible solution to (RFL-SCE) is (**X**′, **Y**′, **Z**′) with

$$\begin{aligned}
X'_1 &= X'_3 = 1, X'_2 = 0,\\
Y'_{11\emptyset} &= Y'_{11\{2\}} = Y'_{11\{3\}} = Y'_{11\{2,3\}} = Y'_{13\{1\}} = Y'_{13\{1,2\}} = Y'_{10\{1,3\}} = Y'_{10\{1,2,3\}} = 1,\\
Y'_{21\emptyset} &= Y'_{21\{2\}} = Y'_{21\{3\}} = Y'_{21\{2,3\}} = Y'_{23\{1\}} = Y'_{23\{1,2\}} = Y'_{20\{1,3\}} = Y'_{20\{1,2,3\}} = 1,\\
Y'_{33\emptyset} &= Y'_{33\{1\}} = Y'_{33\{2\}} = Y'_{33\{1,2\}} = Y'_{31\{3\}} = Y'_{31\{2,3\}} = Y'_{30\{1,3\}} = Y'_{30\{1,2,3\}} = 1,
\end{aligned}$$

which gives the corresponding objective value as

$$\begin{aligned}
\Psi(\mathbf{X}',\mathbf{Y}',\mathbf{Z}') &= f_1 + f_3\\
&\quad + \mu_1\big(d_{13}(p_{\{1\}} + p_{\{1,2\}}) + \pi_1(p_{\{1,3\}} + p_{\{1,2,3\}})\big)\\
&\quad + \mu_3\big(d_{31}(p_{\{3\}} + p_{\{2,3\}}) + \pi_3(p_{\{1,3\}} + p_{\{1,2,3\}})\big)\\
&\quad + \mu_2\big(d_{21}(p_{\emptyset} + p_{\{2\}} + p_{\{3\}} + p_{\{2,3\}}) + d_{23}(p_{\{1\}} + p_{\{1,2\}}) + \pi_2(p_{\{1,3\}} + p_{\{1,2,3\}})\big)\\
&= f_1 + f_3 + \mu_1(0.2d_{13} + 0.4\pi_1) + \mu_2(0.4d_{21} + 0.2d_{23} + 0.4\pi_2) + \mu_3(0.1d_{31} + 0.4\pi_3)\\
&= \Phi(\mathbf{X},\mathbf{Y},\mathbf{Z}).
\end{aligned}$$

This implies that the optimal solution to (RFL-SCE) is no larger than Φ(**X, Y, Z**), thus is a lower bound to (RFL-STA).

Conversely, assuming that the optimal solution to (RFL-SCE) is (**X**′, **Y**′, **Z**′), we can construct (**X, Y, Z**) which is a feasible solution to (RFL-STA) (The detail is omitted here as it is very similar to the procedure described right above). Since $\Phi(\mathbf{X},\mathbf{Y},\mathbf{Z}) = \Psi(\mathbf{X}',\mathbf{Y}',\mathbf{Z}')$, the optimal solution to (RFL-STA) is no larger than Ψ(**X**′, **Y**′, **Z**′), thus is a lower bound to (RFL-SCE). Therefore, we can conclude that the optimal solutions to (RFL-SCE) and (RFL-STA) are exactly the same. □

C3. Proof of Proposition 7

Proof. Suppose, for a contradiction, that (**X, Y, Z**) is optimal to (RFL-STA) but violates Property II, i.e., there exist i, j, k_1, k_2, r such that $X_j = 1, l_{k_1 j} = l_{k_2 j} = 1, q_{k_1} \le q_{k_2}, Y_{ik_2 jr} = 1, Y_{ik_1 jr'} = 0, \forall r' < r$. We will show that by replacing k_2 with k_1 the objective of (RFL-STA) will decrease. We simply construct a different solution (**X**′, **Y**′, **Z**′) as follows:

(i) $X'_j = X_j$;

(ii) $Y'_{hkls} = \begin{cases} 1, & \text{if } (h,k,l,s) = (i,k_1,j,r);\\ 0, & \text{if } (h,k,l,s) = (i,k_2,j,r);\\ Y_{hkls}, & \text{otherwise;} \end{cases}$

(iii) $Z'_{hkls} = \begin{cases} \frac{1-q_{k_1}}{1-q_{k_2}} Z_{hk_2ls}, & \text{if } (h,k,l,s) = (i,k_1,j,r);\\ 0, & \text{if } (h,k,l,s) = (i,k_2,j,r);\\ \frac{q_{k_1}}{q_{k_2}} Z_{hkls}, & \text{if } s > r;\\ Z_{hkls}, & \text{otherwise.} \end{cases}$

By construction, $(\mathbf{X}', \mathbf{Y}', \mathbf{Z}')$ is a feasible solution to (RFL-STA). We use $\Phi(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ to denote the objective value of (RFL-STA) associated with $(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$, assume that $Y_{ik^{ir}j^{ir}r} = Y_{i00R^i} = 1$, it follows that

$$\Phi(\mathbf{X},\mathbf{Y},\mathbf{Z}) - \Phi(\mathbf{X}',\mathbf{Y}',\mathbf{Z}') = \mu_i d_{ij} Z_{k_2jr} + \sum_{s=r+1}^{R^i} \mu_i d_{il^{is}} Z_{k^{is}l^{is}s} - (\mu_i d_{ij} Z'_{k_1jr} + \sum_{s=r+1}^{R^i} \mu_i d_{il^{is}} Z'_{k^{is}l^{is}s})$$
$$= \frac{\mu_i Z_{k_2jr}}{1-q_{k_2}} \left[d_{ij}(1-q_{k_2}) + q_{k_2} \sum_{s=r+1}^{R^i} d_{il^{is}} \prod_{s'=r+1}^{s-1} q_{k^{is'}} (1-q_{k^{is}}) \right.$$
$$\left. - \left(d_{ij}(1-q_{k_1}) + q_{k_1} \sum_{s=r+1}^{R^i} d_{il^{is}} \prod_{s'=r+1}^{s-1} q_{k^{is'}} (1-q_{k^{is}}) \right) \right]$$
$$= \frac{\mu_i Z_{k_2jr}}{1-q_{k_2}} (q_{k_2} - q_{k_1}) \left(-d_{ij} + \sum_{s=r+1}^{R^i} d_{il^{is}} \prod_{s'=r+1}^{s-1} q_{k^{is'}} (1-q_{k^{is}}) \right)$$

As $q_{k^{iR^i}} = 0$, and $d_{il^{is}} \le d_{il^{i,s+1}}$ from Proposition 6, we have

$$\sum_{s=R_i-1}^{R^i} d_{il^{is}} \prod_{s'=r+1}^{s-1} q_{k^{is'}} (1-q_{k^{is}}) \ge d_{il^{iR^i-1}} \prod_{s'=r+1}^{R^i-2} q_{k^{is'}} (1-q_{k^{iR^i-1}}) + d_{il^{iR^i-1}} \prod_{s'=r+1}^{R^i-1} q_{k^{is'}}$$
$$= d_{il^{iR^i-1}} \prod_{s'=r+1}^{R^i-2} q_{k^{is'}}$$

By induction, we can conclude that

$$-d_{ij} + \sum_{s=r+1}^{R^i} d_{il^{is}} \prod_{s'=r+1}^{s-1} q_{k^{is'}} (1-q_{k^{is}}) \ge -d_{ij} + d_{il^{ir+1}} \ge 0.$$

Since $\frac{\mu_i Z_{k_2jr}}{1-q_{k_2}} \ge 0$, $q_{k_2} - q_{k_1} \ge 0$, we have $\Phi(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \ge \Phi(\mathbf{X}', \mathbf{Y}', \mathbf{Z}')$, which implies that $\Phi(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ is not optimal. This completes the proof. □

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