The Holographic Correspondence : Origins

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Abstract. Over the past several decades, developments in String Theory has led to a radically different description of gravity. In this description, gravity emerges as an approximate description of a quantum field theory in lower number of dimensions. The latter quantum theory is similar to those used to describe the other fundamental interactions, defined on a non-dynamical space-time. This talk traces the conceptual basis of such a description at a fairly non-technical level.

1. Introduction

It is a remarkable fact that we are now able to trace most forces of nature to several fundamental interactions and we understand how these interactions work to a very large extent. Three of these forces are very similar. Electromagetism, Weak Interactions and Strong Interactions are described very well by a kind of quantum field theory called "gauge theory". They can be thought of being mediated by particles which are the "force carriers" : photons for electromagnetism, W and Z bosons for weak interactions and gluons for strong interactions. These particles propagate in space-time which is *fixed* - the space-time is non-dynamical.

The fourth force is gravity. The nature of this force appears to be quite different from the others. In fact as Einstein showed - in a sense gravity is not a force of the usual type. Rather it is a manifestation of a curvature of space-time itself. Objects move on such a curved space-time as usual along the path of least action - but this path (a geodesic) appears to be curved, so we can alternatively describe this due to the action of a force. Gravity is then the dynamics of space-time itself - ripples of space-time are gravitational waves. This key difference could be at the root of our troubles in trying to make gravity consistent with quantum mechanics.

Finally, there are two huge elephants in the room : Dark Matter and Dark Energy - these account for 96 % of the energy of the known universe, and we know almost nothing about these.

This talk is about a modern point of view in which gravity is not really a fundamental force. Rather it is a kind of approximate description of something more fundamental. This fundamental theory is a quantum field theory defined on a fixed space-time background and lives in a *lower* number of dimensions - as if it were a hologram of the bulk. This is called the "Holographic Correspondence": at this time there are several concrete realizations of this idea. It has often been conjectured that at very short distances the notions of space and time break down and need to be replaced by some other mathematical structures. The holographic correspondence provides a way to find what this structure could be.

2. Strings in QCD

The origins of this remarkable correspondence can be traced to a rather different area of physics : the theory of strong interactions. Before the correct theory of strong interactions - quantum chromodynamics (QCD) - was found, it was realized that hadrons like mesons and baryons can be considered as quarks held together with *strings*. With the advent of QCD it became clear that these strings are really chromoelectric flux tubes. However, there was a puzzle. Consider for example a usual quantum theory of point particles. These particles are not really point-like : rather vacuum polarization effects render them a finite size. Nevertheless in theories like quantum electrodynamics we can consider e.g. an electron as a point particle since these vacuum polarization effects are proportional to powers of the dimensionless coupling, the fine structure constant $\alpha = e^2/(\hbar c)$. In QED the fine structure constant is a small number - so these effects are small. Indeed in the limit of an infinitesimally small α an electron can be genuinely considered as a point particle. Thus, if QCD can be described in terms of strings, there should be a small number in QCD which controls the interaction between strings. In the limit where this number vanishes, these strings would be infinitely thin and non-interacting. The puzzle was that there is no dimensionless number in QCD : if we for the moment ignore the quarks, QCD becomes pure Yang Mills theory with a gauge group $SU(3)$. This theory has no free parameter - it only has a dynamically generated scale. So it appears that there should be no useful way to consider this as a theory of strings !

2.1. The Large N limit

In a landmark work, 't Hooft solved this puzzle by considering a generalization of QCD in which the gauge group is $SU(N)$, and showed that in this theory the efffective strings interact with a strength $1/N$. Therefore in the $N = \infty$ these strings are free and thin.

More precisely, starting with the usual Yang-Mills action

$$S = \frac{1}{4g_{YM}^2} \int d^4 x \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the field strength of the $SU(N)$ gauge field $A_\mu = (A_\mu)_{ij}, i, j = 1 \cdots N$ and g_{YM} is the coupling. The large N limit is defined as

$$N \to \infty \quad g_{YM} \to 0 \quad \lambda \equiv g_{YM}^2 N = \text{fixed} \tag{2}$$

To see how $1/N$ can appear as a coupling constant of the effective theory of invariants, let us consider the problem of quantum mechanics of a particle moving in a central potential in N space dimensions. In cartesian coordinates $x_i, i = 1 \cdots N$ the hamiltonian is

$$H = \frac{1}{2} \sum_i -\frac{\partial^2}{\partial x_i^2} + V(|x|) \tag{3}$$

This problem has a $O(N)$ symmetry of rotations among the x^i. Suppose we are interested only in the *singlet* wave functions. It is clearly convenient to go to polar coordinates $(r, \theta_1, \cdots \theta_{N-2}, \phi)$ and rescale r to have

$$r^2 = \frac{1}{N} \sum_i (x^i)^2 \tag{4}$$

The singlet wave function $\Psi(r)$ is only a function of r. Generalizing a standard procedure in 3 dimensions we define a new wavefunction $\chi(r)$

$$\Psi(r) = r^{-(N-1)/2} \chi(r) \tag{5}$$

which now satisfies

$$\left[-\frac{1}{2N^2}\frac{d^2}{dr^2} + \frac{(N-1)(N-3)}{8N^2r^2} + V(r)\right]\chi(r) = \frac{E}{N}\chi(r) \qquad (6)$$

This is exactly like the eigenvalue equation of a one dimensional Schrodinger problem. Something interesting happens in the large N limit. The one dimensional problem now has a N independent effective potential

$$V_{eff}(r) = \frac{1}{8r} + V(r) \qquad (7)$$

The only place where N appears is the kinetic term of the radial hamiltonian, where it plays the role of $1/\hbar$. Therefore $1/N$ plays the role of a coupling in this effective theory of radial variables. The $N \to \infty$ limit is the *classical limit*. In fact the spectrum of this theory in this limit can be obtained from a classical calculation based on the potential (7).

This general pattern continues to work for any large-N quantum field theory : $N = \infty$ is a classical limit, and this can be seen by re-writing the theory in terms of appropriate invariant variables. The problem is that it is not possible to perform this change of variables for most interesting theories, like QCD. For pure $SU(N)$ Yang-Mills theories, the appropriate gauge invariant variables are Wilson loops

$$W(C) = \mathrm{Tr}P\exp[i\int_C A \cdot dl] \qquad (8)$$

the integral is over the closed loop C. At large N it is easy to see that these factorize, i.e.

$$< W(C_1)W(C_2) >=< W(C_1) >< W(C_2) > +O(1/N^2) \qquad (9)$$

which is characteristic of a classical limit.

2.2. Matrix Quantum Mechanics

In the early 1980's the problem of obtaining an effective action for Wilson loops was a popular activity, and people started to do "warm up problems". One of them later turned out to be of great importance in the development of the ideas behind the holographic correspondence.

The basic fields of $SU(N)$ Yang-Mills theory are matrix valued fields $(A_\mu)(\vec{x},t)_{ij}$ which are functions of the spatial coordinates $\vec{x}$ and time t. Consider a toy model obtained from this by forgetting about spatial dependence and restricting to only one Lorentz component. The dynamical variables of this toy model would be a single matrix $M_{ij}(t)$ - i.e. quantum mechanics of a single hermitian matrix. The $U(N)$ invariant hamiltonian would be of the form

$$H = -\sum_{ij}\frac{\partial^2}{\partial M_{ij}\partial M_{ij}} + V(M) \qquad (10)$$

where the potential $V(M)$ is invariant, e.g. containing sum of terms like $\mathrm{Tr}(M^n)$. We want to study the dynamics of this theory in the singlet sector. Just as in the single particle quantum mechanics problem we therefore need to go over to invariant variables. In this case the invariant variable is the density of eigenvalues

$$\rho(x) \equiv \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda_i - x) \qquad (11)$$

This is the analog of the radial variable in the single particle quantum mechanics problem discussed in the previous subsection. When N is large $\rho(x)$ is a smooth function restricted

by the condition $\int dx \rho(x) = 1$. In the singlet sector quantum mechanics of a $N \times N$ matrix, $M_{ij}(t)$ can be described by the quantum dynamics of $\rho(x,t)$. It turns out that one can perform this change of variables to ρ and one can write down the hamiltonian for ρ and its canonically conjugate momentum π_ρ. In this hamiltonian $1/N^2$ appears as a Planck constant, just as $1/N$ appeared as the Planck constant for the single particle quantum mechanics problem. Therefore in the large N expansion one can treat the theory in a semiclassical fashion. One finds the classical solution of the equation of motion, and treats the fluctuation around the classical solution quantum mechanically - just as one would perform a WKB expansion in usual quantum mechanics.

Motivated by developments in String Theory, this problem was revisited in the early 1990's. The result was rather surprising : the theory of the fluctuations of ρ turned out to be a quantum field theory in $1+1$ dimensions, with x playing the role of a space dimension. There is a completely equivalent description of the quantum mechanics of a single matrix with N^2 degrees of freedom - a $0+1$ dimensional theory in terms of a single field in $1+1$ dimensions. The matrix quantum mechanics of course did not have any "space". The theory of the density of eigenvalues had a "space" - this came from the mathematical space of eigenvalues, but rather surprisingly this behaves as a real space for all purposes. In fact this $1+1$ dimensional theory is the field theory of strings in $1+1$ dimensions, whose only propagating mode is in fact a massless scalar - this is now identified with $\rho(x,t)$.

String Theory is an ultraviolet completion of General Relativity. The quantized quadrapole moment of an oscillating closed string is a graviton. However, the normal modes of $1+1$ dimensional string theory does not have dynamical gravity in the sense that there are no gravitational waves in these low dimensions. However there are static gravitational forces - which in this theory have a finite range. It turns out that these gravitational forces can be deciphered from the theory of eigenvalues.

This is the holographic correspondence in its simplest form : a theory with N^2 degrees of freedom with no "space" and just time is equivalent to a theory in $1+1$ dimension and the latter theory does contain gravity. In this case, the latter theory can be explicitly constructed starting from the matrix model. One might hope that there is a similar story in higher dimensions : however in higher dimensions we have an infinite number of matrices (one matrix at every point) which do not commute - so they cannot be simultaneously digaonalized and there is no simple way to rewrite the theory in terms of invariant variables. In fact the invariant variables are Wilson loops - these are stringlike objects - and that is where we started the story anyway. Therefore if we could derive an effective theory of Wilson loops, one would get a theory of strings in higher number of dimensions - and if we start with the right large-N theory this string theory will contain gravity. That would be a description of gravity in terms of a conventional quantum field theory. The lesson of the lower dimensional example is that this theory of gravity necessarily lives in higher number of dimensions.

Despite many attempts, writing down a field theory of Wilson loops starting from Yang-Mills theory turned out to be impossible. Progress came in the mid 1990's from a rather different direction - the physics of black holes.

3. Black Holes and Strings

In classical General Relativity a black hole is an object which is surrounded by a *horizon*. Nothing can come out of the horizon. In fact once inside the horizon, any object will inevitably fall into the singularity characterized by large space-time curvatures. An important aspect of the horizon which will be important in the following is that any signal emitted from a point near the horizon and travelling to the asymptotic region far away suffers a huge redshift. As the emission point approaches the horizon, this redshift becomes infinite. This is also the reason why an infalling object appears to an asymptotic observer to slow down infinitely fast as it

approaches the horizon.

While the horizon is certainly a point of no return, the local curvature and tidal forces near the horizon for a very massive black hole are small. This prompted Hawking to apply standard quantum field theory methods to processes near the horizon. The result was surprising : due to particle creation, a black hole was found to emit radiation, which is thermal to a very good approximation. This tied up with earlier observations due to Bekenstein that black holes should be associated with an entropy and a temperature.

The entropy and temperature of every black hole have universal geometric expressions. The entropy (called the Bekenstein-Hawking entropy) is given by

$$S_{BH} = \frac{A_H}{4G_N} \tag{12}$$

in natural units, where A_H is the area of the horizon and G_N is the Newton's constant. This relationship holds for all black holes in all dimensions. The temperature (as e.g. obtained from the properties of Hawking radiation) is given by

$$T_H = \frac{\kappa}{2\pi} \tag{13}$$

where κ is the surface gravity at the horizon. Finally the decay rate of the black hole into some scalar with energy ω is given by

$$\Gamma(\omega) = \frac{\sigma(\omega)}{e^{\frac{\omega}{T_H}} - 1} \tag{14}$$

where $\sigma(\omega)$ is the corresponding classical absorption cross-section by the black hole. For minimally coupled massless scalars, a general theorem ensures that in the limit of small ω one has $\sigma(\omega) \to A_H$.

Bekenstein pushed this even further : he argued that if we consider a theory containing dynamical gravity in a large box, the *maximum* possible entropy of *anything* inside the box is bounded by the surface area of the box - rather than the volume of the box

$$S \leq \frac{A_{boundary}}{4G_N} \tag{15}$$

This is strange, since usually we learn that entropy is extensive and should be proportional to volume. In the early 1990's 't Hooft and Susskind provided a radical interpretation of this fact : they argued that a gravitational theory in a large region of space has a completely equivalent description in terms of a non-gravitational theory which lives on the boundary. Since there is no dynamical gravity in the latter theory, its entropy should be proportional to *its* "volume" - which is of course the boundary area of the gravitational theory. This would provide a natural explanation of Bekenstein's bound. However it took a few more dramatic developments to realize this idea concretely.

3.1. Microscopic origin of Black Hole Thermodynamics and Strings

Usually thermodynamics has a basis in Statistical Mechanics of microscopic constituents. The thermal nature of black holes indicates that there should be a microscopic origin as well. For example, the black hole entropy should be related to the number of states (at a given energy) by Boltzmann formula

$$S = \log(\Omega(E)) \tag{16}$$

However, no hair theorems in General Relativity tell us that for a given mass, charge and angular momentum there is basically one state of a black hole. Therefore in this framework the

appearance of a large entropy appears puzzling. Some physicists believed that maybe there is no microscopic origin of black hole thermodynamics. This would mean that there is a loss of information at the fundamental level - something which is clearly inconsistent with quantum mechanics. In particular this would mean that time evolution in the presence of gravity and black holes is not unitary. Developments in String Theory led to a rather different direction.

Originally String Theory was described as a theory of infinitely thin strings, just as quantum field theory is thought of as a theory of point particles. However particles are not the only excitations in a typical quantum field theory - there are extended objects like strings, membranes etc. which appear as states. In a similar fashion a String Theory is not just a theory of strings. Rather there are other extended objects of different dimensionalities - like membranes (or 2-branes) which are two dimensional. Consistent string theories live in ten dimensions, six of which are supposed to be curled up into tiny compact spaces - so there are many possibilities (3-branes, 4 branes, $\cdots$ 9-branes). Indeed, there are such objects. One class of these are called D-branes of different dimensionalities. In superstring theories N such D-branes parallel to each other can form a threshold bound state. The remarkable thing about these is that their low energy dynamics is described by a $SU(N)$ gauge theory which lives on the worldvolume of these D-branes.

On the other hand, D-branes have a tension and therefore they should produce gravitational fields. When the number of D-branes are large, this gravitational field is macroscopic, just like the gravitational field of a star. Under suitable circumstances these D-branes will collapse under mutual gravitational attraction and form black holes.

D-branes can therefore provide a microscopic description of black holes, pretty much like a collection of electrons can provide a microscopic description of an object with a large charge. In this latter case, we know in principle how to describe the collection of electrons and use this to describe e.g. the motion of a test charge. However when the number of such electrons become huge, this is not a useful way to approach the problem. Rather it is much better to describe the collection of electrons as a classical charge distribution, use Maxwell's equations to determine the fields produced and then use the fields to study the motion of the test charge. Similarly, if we consider a collection of D-branes there is microscopic description in terms of a non-abelian gauge theory on the worldvolume - and if we knew well how to calculate in this theory we would have known in principle to use this to study other objects near these D-branes. However as the number of these branes become very large, this strategy is senseless. Instead we should describe this by the gravitational field produced - which we should obtain as solutions to the equations of Einstein's General Relativity. The latter is what we would normally call a black hole. The D-branes carry charges for some gauge fields - so these black holes are charged.

Just as we can calculate the entropy of a box of gas by counting the number of states of molecules which make up the gas, we should be able to calculate the entropy of a black hole by counting the number of states of the collection of D-branes. This would provide a microscopic basis for black hole thermodynamics which otherwise looks strange. The problem is that in the regime of parameters in which the collection of D-branes look like a large classical black hole, the Yang-Mills theory describing the dynamics of D-branes is strongly coupled - and in general we do not know how to calculate things in strongly coupled field theories. There are exceptions : these are field theories which have enough supersymmetry - in this case certain quantities can be calculated even in strong coupling. The corresponding black hole is an *extremal* black hole - these are black holes with maximal charge for a given mass. They have zero temperature but nevertheless large horizon area and therefore large Bekenstein-Hawking entropy. In the microscopic description in terms of a gauge theory on the branes, this is a ground state in the given charge sector - but a highly degenerate ground state. It turns out that for a large class of such black holes, the ground state degeneracy can be computed reliably even when the theory on the brane is strongly coupled. The logarithm of the number of states is found to be in precise

agreement with the Bekenstein-Hawking entropy. This classic result of Strominger and Vafa which builds on earlier work by Sen was the first indication that these D brane configurations are indeed the microscopic description of black holes.

4. The AdS/CFT correspondence

Extremal black holes do not emit Hawking radiation since they have a zero temperature. To understand the nature of Hawking radiation, we need to go off extremality. While it is difficult to deal with black holes which are far from extremality, it is possible to work with near-extremal black holes which have a small temperature. Once again microscopic calculations show that the temperature and the entropy are precisely reproduced.

In the microscopic picture, Hawking radiation corresponds to annihilation of the quanta of the D-brane gauge theory into a mode, while absorption corresponds to the reverse process. The relation (14) is then a statement of detailed balance. This corss-section was computed in the microscopic theory using standard field theory methods and the result was in precise agreement with the grey body factors obtained by a classical absorption calculation.

Maldacena interpreted the agreement of the absorption calculations in the microscopic theory and in classical General Relativity to be indicative of a *duality* of these two descriptions. This is most transparent when one considers a stack of N D-branes which are three dimensional extended objects, $D3$ branes. The microscopic theory for this is a $3+1$ dimensional supersymmetric gauge theory with a gauge group $SU(N)$. The gravitational description of this stack yields a space-time metric

$$ds^2 = (1 + \frac{R^4}{r^4})^{-1/2}[-dt^2 + d\vec{x}^2] + (1 + \frac{R^4}{r^4})^{1/2}[dr^2 + r^2 d\Omega_5^2] \tag{17}$$

where the branes are extended in the directions labelled by the three coordinates $\vec{x}$. The coordinate r is the radial coordinate in the space transverse to the branes : together with five angles this forms the six dimensional transverse space. $d\Omega_5^2$ is the line element on a unit 5-sphere. The length scale R is given by

$$R^4 = 4\pi l_s^4 g_s N \tag{18}$$

Here l_s is the string length, and g_s is the dimensionless string coupling. This is related to the ten dimensional Newton constant by

$$G_{10} = 8\pi l_s^8 g_s^2 \tag{19}$$

Maldacena's argument went as follows. Consider the classical absorption of a graviton by this stack of D3 branes. Now think of dividing the spacetime into two parts : a part which is close to the horizon, and a part which is far from the horizon. In each of these regions we can consider gravitons - the near horizon gravitons and the asymptotic gravitons. One can now think of the absorption process as a conversion of the asymptotic gravitons into near horizon gravitons. The absorption cross-section turns out to poroportional to the square of the energy -this means that at low energies these two regions decouple.

The same absorption cross-section results from a calculation in the microscopic theory. In this calculation absorption may be thought of a conversion of asymptotic gravitons into the excitations of the brane system - the latter being described by the supersymmetric Yang-Mills theory. The agreement of the two calculations then suggest that these excitations must be an alternative description of the near horizon gravitons. Actually there is a little bit more. Since we are working with a string theory, these gravitons are modes of closed strings. Normally all the higher stringy modes are massive particles and decouple at low energies. However when the string lives in a region close to the horizon this is not quite true - because of the large redshift all these modes appear as low energy modes to the asymptotic observer. So the conjecture is that

the near-horizon strings provide an alternative and equivalent description of the gauge theory modes on the D-branes.

In the coordinate system used in (17) the horizon is at $r = 0$. However this is not a point. If we expand the metric in a power series in R/r we get the near-horizon metric as

$$ds^2 = \frac{r^2}{R^2}[-dt^2 + d\vec{x}^2] + R^2\frac{dr^2}{r^2} + R^2 d\Omega_5^2 \qquad (20)$$

In flat space the radii of spheres surrounding a point approach zero as we approach the point. This is not what happens here. Rather, the radius of the 5-spheres approach a constant, R -the geometry resembles a throat. The other five directions $(t, \vec{x}, r)$ decouple from the sphere - they form what is called an anti-De Sitter space-time - a spacetime which has constant negative curvature, $1/R^2$. The space-time is then written as $AdS_5 \times S^5$. On the other hand, the theory on the 3 branes is a theory which is invariant under conformal transformations - a conformal field theory. This is why this correspondence is called the AdS/CFT correspondence. The correspondence relates a $3+1$ dimensional quantum field theory with no dynamical gravity with a $9+1$ dimensional theory with dynamical gravity.

The two theories which are conjectured to be dual to each other have two dimensionless parameters each. The "bulk" theory in AdS has the string coupling g_s and the scale of the curvature of the space-time, R (as well as the radius of the S^5) in units of the string length l_s. On the Yang-Mills side there are two parameters, N and the Yang Mills coupling g_{YM}. The correspondence holds when

$$g_s = g_{YM} \qquad \left(\frac{R}{l_s}\right) = 4\pi g_{YM}^2 N \qquad (21)$$

The bulk theory is weakly coupled when $g_s = g_{YM}$ is small - this is when the theory is semiclassical. However the gravitational field is nontrivial when R/l_s is finite - from the second relation above this means that we need to take $N \to \infty$. Thus semiclassical string theory in $AdS_5 \times S^5$ is a good description of the Yang-Mills in the limit $N \to \infty$ and $g_{YM} \to 0$ with $g_{YM}^2 N = $ fixed. This is precisely 't Hooft's large N limit we discussed in previous sections. In this limit the real coupling of the Yang Mills theory is not g_{YM}^2, but $g_{YM}^2 N = \lambda$, which is called the 't Hooft coupling. And indeed in this limit we end up with a theory in higher number of dimensions. And, as we expected, this latter theory is a theory of strings. In this theory of strings, $1/N$ acts as a coupling constant.

If this was the end of the story, things would have been rather difficult. since we really do not know how to calculate things in the full string theory in such backgrounds. However (21) shows that when the 't Hooft coupling is large $g_{YM}^2 N \gg 1$, $R \ll l_s$. This means that the curvatures are much smaller than the string scale. This is the limit where we can ignore the fact that we are dealing with a theory of strings rather than usual quantum fields. The description is now in terms of *classical* field theory which contains gravity described by General Relativity. This, then, is the "holy grail" - we found a regime where a quantum field theory of the usual kind describes General Relativity in higher dimensions. Significantly this quantum field theory is necessarily strongly coupled.

The precise correspondence between the gauge theory and the theory containing gravity in this limit was found by Witten and by Gubser, Klebanov and Polyakov. In this correspondence, the gauge theory is defined on the boundary of AdS space-time - just as conjectured by 't Hooft and Susskind. This then is a concrete realization of the holographic correspondence.

5. Epilogue
While the AdS/CFT correspondence has provided major insight into many puzzling aspects of black holes, it is fair to say that the basic issue - the black hole information problem - has not

been completely understood. This is a very active field of research at this moment - and most discussions are in the context of this correspondence.

The holographic correspondence has been shown to work for a large variety of quantum field theories, most of which come from String Theory. However it is believed to be a lot more general. In fact it is most widely used in the other direction : to understand strongly coupled field theories which appear in particle physics, condensed matter physics and cold atom physics in terms of gravity. A key fact which makes this possible is that the gravitational dual of a finite temperature field theory happens to be a black hole inside AdS space-time. Field theories are notoriously difficult to study when they are strongly coupled. However if we assume that they have a gravitational dual, it is precisely in this strong coupling regime that the gravitational dual becomes classical. Therefore difficult questions in the field theory become questions in classical General Relativity. This has led to considerable insight into a large variety of phenomena, opening up a new way of thinking about old problems in hydrodynamics, superconductivity, quantum quench and thermalization.

6. Acknowledgements

Further Reading

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