Reliable Facility Location Design with Round-trip Transportation under Imperfect Information Part II: A Continuous Model

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Abstract

The Part I paper (Yun et al., 2018) of this study developed a discrete model and a customized Lagrangian relaxation algorithm for the reliable problem of facility locations considering round-trip transportation when customers are not aware of facility states in real time until they visit them on site. Since the investigated problem is an NP-hard problem, large-scale instances of this problem may not be solved efficiently by the discrete model. To address this issue, this paper proposes a counterpart continuous model to solve large-scale instances of the investigated problem. The continuous model assumes that all the settings are continuous and adopts the continuum approximation (CA) technique to obtain a near-optimum solution to this investigated problem. The CA technique also reveals theoretical insights into solution structures of each sub-problem on a customer's pattern of visiting facilities on a homogeneous plane. Numerical experiments find that the continuous model with the CA technique has superior computational efficiency for large-scale instances. The results of the case studies indicate that the proposed continuous model can obtain a near-optimum solution for the investigated location problem with heterogeneous settings and has a robust performance.

Keywords: reliability, imperfect information, facility location, continuum approximation, round-trip

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1. Introduction

Unexpected disruptive events due to anthropogenic (e.g., terrorist attacks, labor strikes, etc. (D'Amico, 2002; Schewe, 2004; Hirsch et al., 2015)) and natural (e.g., hurricanes, earthquakes, etc. (Godoy, 2007; Sharkey et al., 2015; Sheppard and Landry, 2016)) disasters have been observed to substantially impair infrastructure system performance. This highlights that considering facility reliability is importance in facility location problems to enhance system resilience. One recent concept in reliable location design is to have facilities backup one another in the event of disruptions. In this manner, multiple facilities are assigned to a customer to ensure that she can get backup service in any disruption scenario (Snyder and Daskin, 2005). Most reliable models for facility locations assume that customers can get the real-time information regarding facility states if they want, whereas several other models assume customers cannot obtain this information. The difference in these two assumptions is illustrated in the Part I paper (Yun et al., 2018).

In reliable facility location design, realizations of customer trips largely affect operation costs of systems. Many service systems need to account the round-trip (outbound and inbound) transportation cost for a customer when this customer accesses the service. Outbound transportation is realized when a customer travels to the service facility, and inbound transportation is realized when the customer returns to her initial location after the completion of the service. When customers have perfect information, outbound and inbound transportation costs are simply identical and there is usually no need to differentiate them in model formulation. However, these two costs are quite different in reliable facility location problems under imperfect information: While a customer may visit a series of facilities consecutively during the outbound trip (if the first several trials happen to hit disrupted facilities), the inbound trip always simply includes a direct trip from the last stop to her home. This difference was illustrated in a motivating example in the Part I paper.

A typical approach to reliable facility location problems is to develop mathematical programming models with discrete setting that can only numerically solve very limited-scale problem instances. To solve large-scale problem instances, the continuum approximation (CA) technique proposed by Daganzo and Newell (1986) has been employed in numerous location design studies. Recently, many studies have been performed using the CA technique to investigate large-scale instances of location problems. Please see the following section for a literature review on relevant studies. Despite these fruitful modeling developments, existing CA methods cannot overcome two challenges in the investigated problem, i.e., (i) the impact from imperfect information and (ii) significant structural difference between the investigated problem and those in the literature (e.g., the asymmetric inbound and outbound trips). To address this gap, we propose a new continuous model that adopts the CA technique to solve this complex location problem by decomposing it into homogenous sub-problems. Due to the above-mentioned challenges, it is even difficult to directly solve the optimal solution to a sub-problem. To circumvent the challenges, a novel solution approach is proposed to efficiently solve feasible and lower bound solutions to each sub-problem, instead of directly looking for the exact optimum. These two bounds have been shown very close to each other, which only adds negligible errors in the whole solution approach. This solution approach also reveals theoretical properties and analytical insights into the facility location in the corresponding geographic neighborhood. Integrating the solutions across all sub-problems yields a near-optimum solution to the investigated problem in the heterogeneous space.

Numerical examples are conducted to examine the performance of the proposed model and draw managerial insights. Compared with the discrete model in the Part I paper, the continuous model with the CA technique has superior efficiency and scalability for large-scale instances. Its simple structure also enables revealing analytical insights into the problem structure and solution optimality. Overall, this continuous model can solve this new reliable location problem to near-optimum. However, it should be noted that the solution obtained by the CA approach is not guaranteed to be the true optimum, and thus the counterpart discrete model proposed in the Part I paper is necessary to get a rigorous optimality bound or the optimal solution.

The structure of the paper is shown as follows. Section 2 reviews the literature referring to facility location problems. Section 3 presents the framework and the solution technique for the continuous model. Section 4 applies the continuous model with the CA techinque to solve instances and draws insights into the effects of the inbound trip and imperfect information. Section 5 presents a conclusion for the paper and points out future research directions.

2. Literature review

Facility location problems are classical strategic decision problems in many service systems drawing numerous attentions from researchers. Reviews by Drezner (1995) and Daskin (1995) summarize a number of classic models on facility locations in the last century. In recent decades, reliable facility location design has been extensively studied due to the frequent occurrence of facility disruptions. Most reliable facility location models are formulated as mathematical programming models with discrete settings. Commercial solvers or customized algorithms can solve these models numerically. Please see the Part I paper for a thorough review of these studies. However, discrete models may be compromised from excessive computational burdens when solving larger-scale location problems in some applications.

To complement discrete facility location models for better scalability, the CA technique proposed by Daganzo and Newell (1986) has been employed in numerous location design studies. Thorough reviews on the use of CA methods are summarized by Langevin et al. (1996) and Daganzo (2005) for deterministic facility location design problems. Ouyang et al. (2015) proposed a CA approach to investigate the facility location problem with elastic demand and traffic congestion. Li et al. (2016) investigated the system design problem for one-way electric vehicle (EV) sharing by developing a CA model. Wang et al. (2017) incorporated temporal dynamics on market growth to a CA model. Although the CA method has been successfully employed for various traditional deterministic facility location problems, it has seldom been applied to the reliable location design context. The CA model proposed by Cui et al. (2010) solved the reliable location design under site-dependent disruptions. Li and Ouyang (2010) introduced correlated probabilistic disruptions to the CA model for the reliable location problem. Li and Ouyang (2012) developed a CA model for the reliable sensor deployment problem along a single corridor. Wang and Ouyang (2013) proposed a CA scheme with the game theory to deal with spatial competition. Later, Wang et al. (2015) extended this work to competition between new and incumbent companies. Please see Ansari et al. (2018) for a recent review on this topic. These reliability studies have a common assumption that a customer gets the real-time information regarding facility states, and thus always visits the most convenient facility among all operating ones.

To the authors' knowledge, no study has employed the CA method to investigate reliable location design under imperfect information. Furthermore, the inbound trip that is important for deciding the customer's visiting sequence under imperfect information is also ignored in the previous studies. This causes a customer to travel far from her home, which may be unrealistic under imperfect information. Therefore, this paper studies a reliable facility location design considering round-trip transportation under imperfect information using a CA approach. This study also extends the depth of the research in the Part I paper.

3. Continuous model

In this section, we propose a continuous modeling approach. We first present the general problem formulation. Then, we propose a continuum approximation solution approach that yields an approximate solution to this continuous problem in an efficient manner.

3.1 Continuous problem formulation

We list key symbol definitions in Appendix A for the convenience. In a space $S \subseteq \mathbb{R}$, let $\lambda(x), \forall x \in S$ denote the customer demand density at location x. Numerous facilities will be constructed in space S to serve customers. Facilities are permitted to be built at any location $x \in S$. We assume that a facility built at location x (or facility x) is disrupted with probability $q(x)$, which is independent across the space. The set of constructed facilities is denoted by $\mathbf{x} := \{x_1, x_2, \cdots$ where N denotes the total facility numbers. We assume that customers cannot know facility states (i.e., whether they are disrupted) in any disruption scenarios before physically visiting the facilities. Thus, they visit facilities using a "trial-and-error" strategy, as illustrated in Figure 1, in which triangles denote the built facilities and the dot is a customer at location x (or customer x for short). Essentially, customer x tries a set of $R+1$ facilities according to a pre-specified order regardless of facility states. We denote the rth facility that customer x visits as $j_r(x|\mathbf{x})$, $\forall r \in \{0,1,2,\ldots$, where $\cdot|\mathbf{x}$ denotes that this item depends on location design $\mathbf{x}$. For notation convenience, we denote customer x's facility visiting sequence set as $J(x|\mathbf{x}) =: \{j_r(x|\mathbf{x})\}_{r=0,1,\cdots}$. When customer x does not obtain the service, a penalty cost $\varphi(x)$ is imposed assuming $\varphi(x) \gg \quad -x_j\| : \forall j \in J(x|\mathbf{x})\}$ for all $x \in S$. Therefore, customer x will always try to visit all the facilities in $J(x|\mathbf{x})$ if available before the penalty is imposed. A customer's trial of visiting these facilities ends either at the first operating facility to get the service or after trying the entire set of facilities without finding a functional one and thus receiving a penalty. After these moves, this customer returns to her home location where the trip began[1]. In each move, the travel cost is accounted by the Euclidean distance between the starting and ending locations of this move.

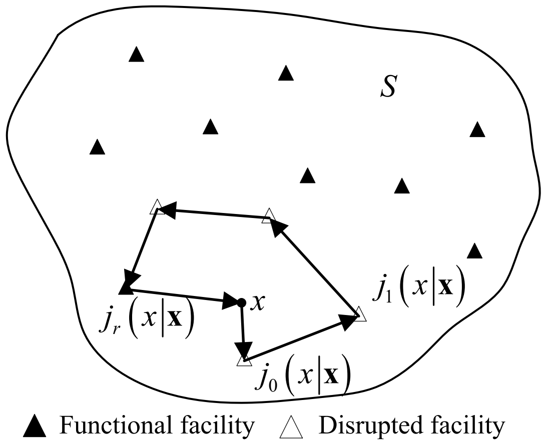


Figure 1 Illustration of customer travel sequence in a particular failure scenario.

In continuous settings, all parameters' values are set according to the location in the planning area. These settings can be converted to discrete settings for the discrete location model. First, we partition the planning area into numerous small cells. Second, we integrate the customer demand density in each cell as the customer demand for this cell. Third, we select the location, such as each cell center as usual, to place a customer and one candidate facility for each cell. Therefore, the size of the cell number generally decides the approaching degree of the optimal result between the continuous model and discrete model. However, the discrete settings can also be converted to the continuous setting, which is more complex than the inverse conversion. We can apply the approximation method proposed by Peng et al. (2014) to finish this conversion.

Now we investigate how to formulate the relevant cost component for a given $\mathbf{x}$. The opening cost for facility x is denoted by $f(x)$; then, the total facility opening cost is

$$C^{\mathrm{F}}(\mathbf{x}) := \sum_{x \in \mathbf{x}} f(x). \tag{1}$$

Let $d_r(x|J(x|\mathbf{x}))$ denote customer x's total travel distance given that she ends at her rank r facility given visiting sequence $J(x|\mathbf{x})$, which we further divide into two parts, outbound (departure) distance $d_r^{\mathrm{O}}(x|J(x|\mathbf{x}))$ and inbound (return) distance $d_r^{\mathrm{I}}(x|J(x|\mathbf{x}))$,

[1] To highlight the difference between our model and existing reliable location models with perfect information (Li et al., 2010), note that the perfect information counterpart instead assumes that a customer knows real-time information of facility states and thus always directly visits the closest functioning facility (if any) in just one move.

as formulated below:

$$d_r\left(x\middle|J(x\mid\mathbf{x})\right)=d_r^{\mathrm{O}}\left(x\middle|J(x\mid\mathbf{x})\right)+d_r^{\mathrm{I}}\left(x\middle|J(x\mid\mathbf{x})\right), \tag{2}$$

$$d_r^{\mathrm{O}}\left(x\middle|J(x\mid\mathbf{x})\right)=\left\|x-x_{j_0(x|\mathbf{x})}\right\|+\sum_{i=1}^{r}\left\|x_{j_{i-1}(x|\mathbf{x})}-x_{j_i(x|\mathbf{x})}\right\|, \tag{3}$$

$$d_r^{\mathrm{I}}\left(x\middle|J(x\mid\mathbf{x})\right)=\left\|x_{j_r(x|\mathbf{x})}-x\right\|. \tag{4}$$

Let $P_r\left(x\middle|J(x\mid\mathbf{x})\right)$ denote the probability that customer x get the service from facility $j_r\left(x\middle|\mathbf{x}\right)$ conditioned on facility visiting sequence $J(x\mid\mathbf{x})$, which occurs if facility $j_r\left(x\middle|\mathbf{x}\right)$ is operational and all facilities lower than rank r failed. Furthermore, let $\bar{P}\left(x\middle|J(x\mid\mathbf{x})\right):=1-\sum_{r=0}^{R}P_r\left(x\middle|J(x\mid\mathbf{x})\right)$ denote the probability that customer x finally obtains no service from any facility conditioned on facility visiting sequence $J(x\mid\mathbf{x})$, which apparently happens when all facilities in $J(x\mid\mathbf{x})$ are disrupted. Then, the expected total transportation cost is

$$C^{\mathrm{T}}(\mathbf{x}):=\int_{x\in S}\lambda(x)\begin{pmatrix}\sum_{r=0}^{R}d_r\left(x\middle|J(x\mid\mathbf{x})\right)P_r\left(x\middle|J(x\mid\mathbf{x})\right)\\ +d_R\left(x\middle|J(x\mid\mathbf{x})\right)\bar{P}\left(x\middle|J(x\mid\mathbf{x})\right)\end{pmatrix}dx. \tag{5}$$

The first term is the summation for the expected travel cost when customer x is served, and the last term formulates the travel cost when customer x is not successful in trying to obtain the service. When the trial is not successful, the expected total penalty cost is formulated by

$$C^{\mathrm{P}}(\mathbf{x}):=\int_{x\in S}\lambda(x)\varphi(x)\bar{P}\left(x\middle|J(x\mid\mathbf{x})\right)dx. \tag{6}$$

With these cost components, the studied reliable continuous location design problem under imperfect information with round-trip transportation (CRLP-IIRT) aims to get the minimum total system cost by selecting the optimal location $\mathbf{x}$ and the corresponding optimal facility visiting sequence $\left\{J(x\mid\mathbf{x})\right\}_{x\in S}$, as formulated below

$$\text{CRLP-IIRT:}\quad\min_{\mathbf{x}\in S,\left\{J(x|\mathbf{x})\right\}_{x\in S}}C(\mathbf{x}):=C^{\mathrm{F}}(\mathbf{x})+C^{\mathrm{T}}(\mathbf{x})+C^{\mathrm{P}}(\mathbf{x}). \tag{7}$$

3.2. Continuum approximation approach

It is difficult to directly solve the original CRLP-IIRT in the heterogeneous space. This section proposes a CA approach to achieve an approximate near-optimum solution. Essentially, each local neighborhood of space S can be approximately treated as an infinite homogeneous plane (IHP) that has homogenous settings everywhere and thus can be easily solved. If the original heterogeneous space varies mildly, the solution to IHP can approximately replace the optimal solution of this neighborhood in the original space. Furthermore, since a customer only visits a finite number of facilities in a local area of the space before opting to receive the penalty, the cost structures of distant areas will not be strongly coupled and thus, approximating each local neighborhood separately is a reasonable treatment. Finally, integrating the approximated solutions across the entire space will yield a near-optimum location design for the original space. In the following descriptions, Section 3.2.1 presents the method to solve a generic IHP problem, and Section 3.2.2 discusses the method of integrating the IHP solutions across the entire space into a solution to the original heterogeneous space.

3.2.1 IHP problem

In an IHP (i.e., $\mathbb{R}$), all parameters are set to be constant across all locations, i.e., $\varphi(x)=\varphi$, $f(x)=f$, $\lambda(x)=\lambda$, $q(x)=q$, $\forall x\in\mathbb{R}$. We define the initial service area as the area that a facility serves when all facilities are operating. Following previous work (Cui et al., 2010; Li and Ouyang, 2010), the IHP can be formed as a regular hexagonal tessellation by facilities' initial service areas. The center of each regular hexagon locates a facility. Toth (1959) proved that the hexagon tessellation is the optimal facility location layout for the classic location problem in IHP. We denote the size of initial service areas by A, and location decision $\mathbf{x}$ on the IHP now reduces to finding the optimal A value. Because all the hexagons are identical and the IHP has the transitional symmetry property, our analysis is centered at a generic hexagon service area, as illustrated in Figure 2.

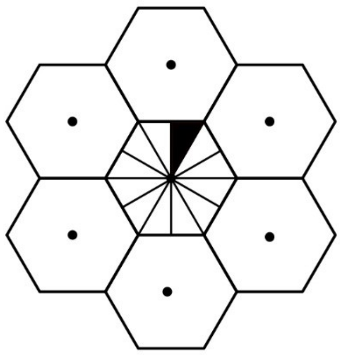


Figure 2 Analysis area illustration.

With these homogenous settings, optimizing objective function (7) for the IHP is equivalent to finding the optimal service area A and the facility visiting sequence for each customer to minimize its unit-area system cost, including facility opening cost C^{F}, penalty cost C^{P}, and transportation cost C^{T}. Based on transitional symmetry, C^{F} can easily be obtained as follows:

$$C^{\mathrm{F}}=f/A. \tag{8}$$

For customer x to receive the penalty, the customer must visit and find that all facilities in $J(x|\mathbf{x})$ are disrupted. This approach indicates that the penalty probability $\overline{P}(x|J(x|\mathbf{x}))$

is simply the probability that all its $R+1$ assigned facilities are disrupted. Since $q(x)=q$ everywhere and the failures are independent, $\bar{P}\left(x \middle| J\left(x|\mathbf{x}\right)\right)$ is independent of customer location x or visiting sequence $J\left(x|\mathbf{x}\right)$ and is equal to

$$\bar{P}\left(x \middle| J(x \mid \mathbf{x})\right) = q^{R+1}, \forall x \in \mathbb{R} \tag{9}$$

This leads to the formulation of C^{P} as follows:

$$C^{\mathrm{P}} = \lambda \varphi q^{R+1}. \tag{10}$$

Next, we formulate C^{T} for the IHP. Since the facility failures are i.i.d. across the IHP, the rank r service probability for each customer $x \in \mathbb{R}$, $P_r\left(x \middle| J(x \mid \mathbf{x})\right)$, is again independent of customer location x or visiting sequence $J\left(x|\mathbf{x}\right)$ and is identical to

$$P_r\left(x \middle| J(x \mid \mathbf{x})\right) := q^r\left(1-q\right), \quad \forall x \in \mathbb{R} \tag{11}$$

The challenge in our IHP analysis is that customers in the central service area do not have the same visiting sequence, which is different from the IHP analysis in the previous paper (Li and Ouyang, 2010). To overcome this challenge, the central hexagon is divided into twelve identical sectors, as illustrated in Figure 2. In each sector, customers can be treated as identical. Furthermore, due to the symmetric of regular hexagon, the cost for customers in these sectors are identical. Thus, we chose to analyze the upper-right sector (as highlighted in black in Figure 2) for the following analysis, which we denote by $\mathcal{T}$ with its area size $|\mathcal{T}|$.

Following Equations (2) and (4), C^{T} can be decomposed into outbound component C^{TO} and inbound component C^{TI} as formulated below:

$$C^{\mathrm{TO}} = \lambda \frac{\int_{x\in\mathcal{T}} \left(d^{\mathrm{O}}_{-}\left(x \middle| J(x \mid \mathbf{x})\right) q^{R+1} + \sum_{r=0}^{R} d^{\mathrm{O}}_r \left(x \middle| J(x \mid \mathbf{x})\right) q^r (1-q) \right) dx}{|\mathcal{T}|}, \tag{12}$$

$$C^{\mathrm{TI}} = \lambda \frac{\int_{x\in\mathcal{T}} \left(d^{\mathrm{I}}_{-}\left(x \middle| J(x \mid \mathbf{x})\right) q^{R+1} + \sum_{r=0}^{R} d^{\mathrm{I}}_r \left(x \middle| J(x \mid \mathbf{x})\right) q^r (1-q) \right) dx}{|\mathcal{T}|}. \tag{13}$$

Next, we investigate how to formulate the transportation costs. Note that $d^{\mathrm{O}}_r\left(x \middle| J\left(x|\mathbf{x}\right)\right)$

and $d_r^{\mathrm{I}}\left(x\middle|J\left(x\middle|\mathbf{x}\right)\right)$ for customer $x\in\mathcal{T}$ are essentially determined by service area A and customer x's visiting sequence $J(x|\mathbf{x})$. Because each customer x's optimal visiting sequence (OVS) is hard to be determined exactly, it is appealing to find a feasible visiting sequence for all customers in $\mathcal{T}$ that is close to the optimal solution. We denote this sequence as a near-optimum visiting sequence (NOVS). In order to construct the NOVS, we first index all facilities (or hexagons), as illustrated by Figure 3. We index the central facility with 0, and then index all the remaining facilities in a spiral pattern with integers sequentially increasing from 1. The NOVS assumes that customers in area $\mathcal{T}$ visit these facilities in a sequence corresponding to their indexes, i.e.,

$$J^{\mathrm{NO}}\left(x\mid\mathbf{x}\right)=\left\{j_r^{\mathrm{NO}}\left(x\mid\mathbf{x}\right)=r\right\}_{\forall r=0,1,\cdots}\quad\forall x\in\mathcal{T}\tag{14}$$

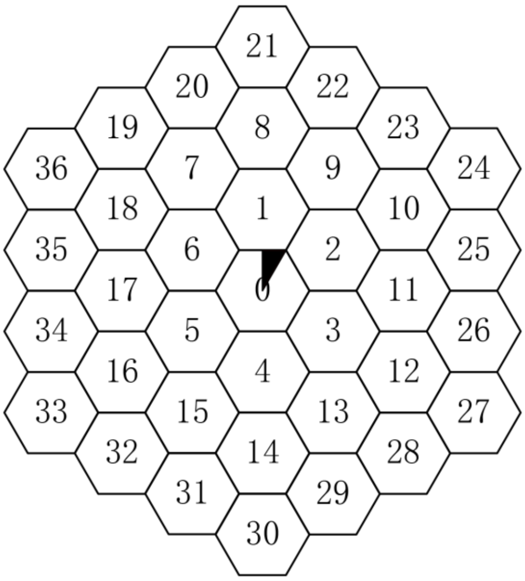


Figure 3 Indexing facilities and analysis areas.

With the NOVS, C^{TO} and C^{TI} can be simplified as

$$C^{\text{TO-NO}}=\lambda\left(d_R^{\text{O-NO}}\left(A\right)q^{R+1}+\sum_{r=0}^{R}d_r^{\text{O-NO}}\left(A\right)q^r\left(1-q\right)\right),\tag{15}$$

$$C^{\text{TI-NO}}=\lambda\left(d_R^{\text{I-NO}}\left(A\right)q^{R+1}+\sum_{r=0}^{R}d_r^{\text{I-NO}}\left(A\right)q^r\left(1-q\right)\right),\tag{16}$$

where $d_r^{\text{O-NO}}\left(A\right)$ and $d_r^{\text{I-NO}}\left(A\right)$ are the average outbound and inbound transportation distances, respectively, across all customers in $\mathcal{T}$ under the NOVS when they finally visit facility r before giving up. Note that they are both determined by the A value. Here, we append the superscripts with –NO to highlight that the corresponding terms are associated with the near-optimum (NO) solution of the NOVS and will be differentiated from those in the exact optimal solution with the OVS. Now, we investigate how to formulate $d_r^{\text{O-NO}}(A)$ and $d_r^{\text{I-NO}}(A)$. The average distance from facility r to customer x can be calculated by the method proposed in Appendix B. The results show that the average distance between facility r and customers in $\mathcal{T}$ is proportional to $A^{1/2}$ with a coefficient determined by r, which we denote by β_r. With this, inbound travel distance $d_r^{\text{I-NO}}(A)$, which is identical to the average distance from facility r to the customers in $\mathcal{T}$, is formulated as

$$d_r^{\text{I-NO}}(A) = \beta_r A^{1/2}. \tag{17}$$

During the outbound travel, the average distance from customer x to facility 0 is equal to $\beta_0 A^{1/2}$. When $r > 0$, the distance from the facility of rank $(r-1)$ to the facility of rank r is always equal to $\left(\frac{4}{3}\right)^{1/4} A^{1/2}$. Therefore, the outbound travel distance $d_r^{\text{O-NO}}(A)$ can be formulated as

$$d_r^{\text{O-NO}}(A) = \beta_0 A^{1/2} + r\left(\frac{4}{3}\right)^{1/4} A^{1/2}. \tag{18}$$

Therefore, the near-optimum solution to C^{T} with the NOVS is formulated as follows:

$$C^{\text{T-NO}} = \lambda A^{1/2}\left(\left(\beta_0 + R\left(\frac{4}{3}\right)^{1/4} + \beta_R\right)q^{R+1} + \sum_{r=0}^{R}\left(\beta_0 + r\left(\frac{4}{3}\right)^{1/4} + \beta_r\right)q^r(1-q)\right). \tag{19}$$

The following analysis aims to obtain an optimality gap between this near-optimum solution and the optimal C^{T} under the OVS. Although the optimal solution of C^{T} remains unknown, we can construct a lower bound to the optimal C^{T} as follows. If the near-optimum solution is very close to the lower bound, we believe that this solution well approximates the exact optimal solution and call it a near-optimum solution. Thus, we use the near-optimum solution in the analysis instead of the optimal solution. We then construct a lower bound solution to outbound transportation cost C^{TO} alone through the following analysis, simply denoted as $C^{\text{TO-NO}}$ in the following proposition.

Proposition 1. $C^{\text{TO-NO}}$ (defined in Equation (15)) is no greater than C^{TO} (defined in Equation (12)) for any feasible visiting sequence.

Proof. For a customer $x \in \mathcal{T}$, let $J'(x|\mathbf{x}) = \{j'_r(x|\mathbf{x})\}$ denote an arbitrary feasible visiting sequence, and then $\{d_r^{\mathrm{O}}(x|J'(x|\mathbf{x}))\}$ shall be the corresponding outbound travel distances at all ranks. Similarly, let $\{d_r^{\mathrm{O}}(x|J^{\mathrm{NO}}(x|\mathbf{x}))\}$ denote the corresponding outbound travel distances with the NOVS. It is easy to see that since the closest facility to customer x is facility 0, we have $d_0^{\mathrm{O}}(x|J^{\mathrm{NO}}(x|\mathbf{x})) \le d_0^{\mathrm{O}}(x|J'(x|\mathbf{x}))$. Furthermore, at a rank $r \ge 1$, customer x has to travel from one facility to another, and it is easy to note that the distance from facility $r-1$ to facility r is the minimum distance between any two different facilities, i.e., $d_r^{\mathrm{O}}(x|J^{\mathrm{NO}}(x|\mathbf{x})) \le d_r^{\mathrm{O}}(x|J'(x|\mathbf{x})), \forall r = 1, \cdots$ With this, we obtain

$$
\begin{aligned}
& d_R^{\mathrm{O}}(x \mid J^{\mathrm{NO}}(x \mid \mathbf{x}))q^{R+1} + \sum_{r=0}^{R} d_r^{\mathrm{O}}(x \mid J^{\mathrm{NO}}(x \mid \mathbf{x}))q^r(1-q) \\
& \le d_R^{\mathrm{O}}(x \mid J'(x \mid \mathbf{x}))q^{R+1} + \sum_{r=0}^{R} d_r^{\mathrm{O}}(x \mid J'\ (x \mid \mathbf{x}))q^r(1-q).
\end{aligned}
\tag{20}
$$

Then, based on Equation (12), we conclude that the NOVS yields the minimum value for C^{TO}. This completes the proof.

Furthermore, when q is less than 0.5, the unit-area inbound travel cost is actually minimized when the assignment rank of a facility to a customer is consistent with the distance from this customer to this facility, i.e., rank-r facility to customer x is the rth nearest facility from customer x. We denote this minimum cost with $C^{\text{TI-MIN}}$. This is proven in the proposition below.

Proposition 2. When $q \le 0.5$, the inbound travel cost $C^{\text{TI-MIN}}$ resulting from the distance-based facility assignment is no greater than C^{TI} (defined in Equation (13)) with any feasible visiting sequence.

Proof. For each customer x, we denote their distance-based facility assignment by $J^{\mathrm{DIS}}(x|\mathbf{x}) = \{j_r^{\mathrm{DIS}}(x|\mathbf{x})\}$ such that

$$
D\left(x, j_0^{\mathrm{DIS}}(x|\mathbf{x})\right) \le D\left(x, j_1^{\mathrm{DIS}}(x|\mathbf{x})\right) \le \cdots \qquad j_R^{\mathrm{DIS}}(x|\mathbf{x})) \tag{21}
$$

and

$$
D\left(x, j_R^{\mathrm{DIS}}(x|\mathbf{x})\right) \le D(x, j), \forall j \in \mathbf{x} \setminus J^{\mathrm{DIS}}(x|\mathbf{x}), \tag{22}
$$

where function $D(x, j)$ denotes the distance from customer x to facility j. Let $J'(x|\mathbf{x}) = \{j'_r(x|\mathbf{x})\} \ne J^{\mathrm{DIS}}(x|\mathbf{x})$ denote an arbitrary feasible visiting sequence. Then, according to Equation (13), the inbound cost associated with $J^{\text{DIS}}(x|\mathbf{x})$ is

$$C^{\text{TI-MIN}} = \lambda\left(D\left(x, j_R^{\text{DIS}}(x|\mathbf{x})\right) q^{R+1} + \sum_{r=0}^{R} D\left(x, j_r^{\text{DIS}}(x|\mathbf{x})\right) q^r (1-q) \right). \tag{23}$$

Furthermore, the inbound cost associated with $J'(x|\mathbf{x})$ is

$$C^{\text{TI}'} = \lambda\left(D\left(x, j'_R(x|\mathbf{x})\right) q^{R+1} + \sum_{r=0}^{R} D\left(x, j'_r(x|\mathbf{x})\right) q^r (1-q) \right). \tag{24}$$

We show that $J'(x|\mathbf{x})$ can be made equal to $J^{\text{DIS}}(x|\mathbf{x})$ after numerous adjustment steps, and each change can only bring down the cost of $C^{\text{TI}'}$. First if the set of facilities in $J'(x|\mathbf{x})$ is not identical to $J^{\text{DIS}}(x|\mathbf{x})$, then at each adjustment step, we replace an element in $J'(x|\mathbf{x})\setminus\left(J'(x|\mathbf{x})\cap \quad \mathbf{x})\right)$ with an element in $J^{\text{DIS}}(x|\mathbf{x})\setminus\left(J'(x|\mathbf{x})\cap \quad \mathbf{x})\right)$. This substitution always reduces the cost of $C^{\text{TI}'}$ because of Equations (21) and (22). This process can be repeated until $J'(x|\mathbf{x})$ and $J^{\text{DIS}}(x|\mathbf{x})$ have the same set of facilities, i.e., the $R+1$ closest facilities to customer x. Then, if the order of $J'(x|\mathbf{x})$ is not the same as $J^{\text{DIS}}(x|\mathbf{x})$, at each adjustment step, we pick two ranks $r_1 < r_2$ such that $D\left(x, j'_{r_1}(x|\mathbf{x})\right) \geq D\left(x, j'_{r_2}(x|\mathbf{x})\right)$. Then, we swap the assignment ranks of these two facilities, and the change in $C^{\text{TI}'}$ is

$$\begin{cases} -\Delta d \cdot \left(q^{r_1} - q^{r_2}\right)(1-q), \text{if } r_2 < R; \\ -\Delta d \cdot q^{r_1}(1-q) + \Delta d q^{r_2}, \text{if } r_2 = R, \end{cases} \tag{25}$$

where $\Delta d > D\left(x, j'_{r_1}(x|\mathbf{x})\right) - D\left(x, j'_{r_2}(x|\mathbf{x})\right)$. This term is apparently non-positive in either case, since $q \leq 0.5$. By repeating this adjustment step properly, $J'(x|\mathbf{x})$ can always be made identical to $J^{\text{DIS}}(x|\mathbf{x})$. In all of the previous steps, $C^{\text{TI}'}$ never increases. Thus, the original $C^{\text{TI}'}$ is always no less than $C^{\text{TI-MIN}}$. This completes the proof.

Based on the results from Li and Ouyang (2010), the average distance between a generic central facility and all customers having this central facility for the rank-r assignment is proportional to $A^{1/2}$ and denoted by $\gamma_r A^{1/2}$. Because of transitional symmetry, $\gamma_r A^{1/2}$ will be identical to the average distance between a customer on the IHP and her rth nearest facility.

As previously mentioned, the average distance between customers in $\mathcal{T}$ and facility r is equal to $\beta_r A^{1/2}$, and we denote the average distance between customers in $\mathcal{T}$ and the rth nearest facility by $\gamma_r A^{1/2}$. From Figure 3, we can see that facility r is the rth nearest facility when $r \leq 2$ for all customers in $\mathcal{T}$. Therefore, parameter γ_r is equal to β_r when $r \leq 2$. When $r > 2$, we calculate the distance from each customer in $\mathcal{T}$ to the rth nearest facility by numerical integration and then obtain the value of parameter γ_r. Thus, the minimum value of the unit-area inbound cost $C^{\text{TI-MIN}}$ can be expressed by

$$C^{\text{TI-MIN}} = \lambda A^{1/2}\left(\left(\sum_{r=0}^{2}\beta_r q^r + \sum_{r=3}^{R}\gamma_r q^r\right)(1-q) + \gamma_R q^{R+1}\right). \tag{26}$$

Proposition 3. The sum of $C^{\text{TO-NO}}$ and $C^{\text{TI-MIN}}$, denoted by $C^{\text{T-LB}}$, is the lower bound to C^{T}.

Proof. $C^{\text{TO-NO}}$ is no greater than C^{TO} for all feasible visiting sequences. Therefore, $C^{\text{TO-NO}}$ is the lower bound for C^{TO}, i.e., $C^{\text{TO-NO}} \leq C^{\text{TO}}$. $C^{\text{TI-MIN}}$ is the minimum value of C^{TI}, i.e., $C^{\text{TI-MIN}} \leq C^{\text{TI}}$ for all feasible visiting sequences. Combining these two inequations, we can obtain a new inequation, $C^{\text{TO-NO}} + C^{\text{TI-MIN}} \leq C^{\text{TO}} + C^{\text{TI}}$. Because $C^{\text{TO}} + C^{\text{TI}}$ is equal to C^{T}, the new inequation leads to $C^{\text{T-LB}} = C^{\text{TO-NO}} + C^{\text{TI-MIN}} \leq C^{\text{T}}$ for all visiting sequences. This indicates that $C^{\text{T-LB}}$ is a lower bound to C^{T} and thus completes the proof.

According to Equations (15), (18) and (26), we can formulate $C^{\text{T-LB}}$ as follows:

$$C^{\text{T-LB}} = \lambda A^{1/2}\left(\begin{array}{l}\left(\beta_0 + R\left(\frac{4}{3}\right)^{1/4} + \gamma_R\right)q^{R+1} \\ +\left(\sum_{r=0}^{R}\left(\beta_0 + r\left(\frac{4}{3}\right)^{1/4}\right)q^r + \sum_{r=0}^{2}\beta_r q^r + \sum_{r=3}^{R}\gamma_r q^r\right)(1-q)\end{array}\right). \tag{27}$$

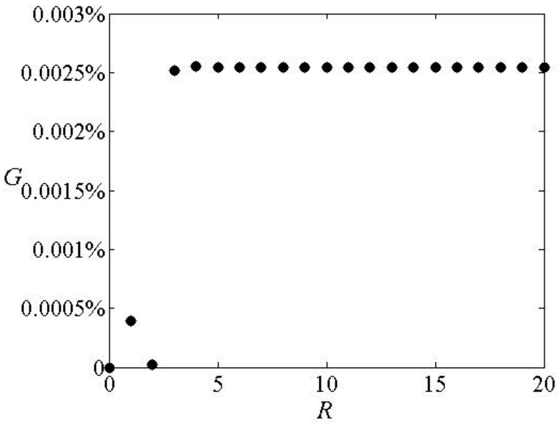
Now, we formulate the gap between $C^{\text{T-NO}}$ and $C^{\text{T-LB}}$ as follows:

$$G := \frac{C^{\text{T-NO}} - C^{\text{T-LB}}}{C^{\text{T-NO}}}. \tag{28}$$

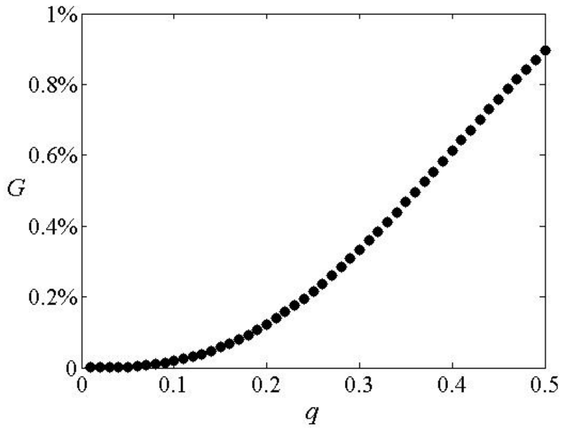
Based on the Equations (19) and (27), we obtain

$$G := \frac{(\beta_R - \gamma_R) q^{R+1} + \sum_{r=3}^{R} (\beta_r - \gamma_r) q^r (1-q)}{\left(\beta_0 + R\left(\frac{4}{3}\right)^{1/4} + \beta_R\right) q^{R+1} + \sum_{r=0}^{R} \left(\beta_0 + r\left(\frac{4}{3}\right)^{1/4} + \beta_r\right) q^r (1-q)} \times 100\%. \tag{29}$$

With Equation (29), we obtain the G value for any given R and q. Figure 4 plots the values of G for various R and q values. Figure 4(a) shows that G is no more than 0.003% because R ranges from 1 to 20 when $q = 0.05$. Figure 4(b) shows that G increases with the increase of q, but the maximum value remains below 1%. These small gap values are reasonable for engineering practices. Therefore, we can use the near-optimum solution in lieu of the exact optimal solution to calculate the unit-area transportation cost.



(a)



(b)

Figure 4 The gap between FS and LB, (a) $q = 0.05$; (b) $R = 5$.

Remark. As we know, to obtain the minimum inbound cost, each customer has her own visiting sequence $J^{\text{DIS}}(x|\mathbf{x}) = \{j_r^{DIS}(x|\mathbf{x})\}$. If we divide the central service area by the identical visiting sequence, we find the following feature. For a given value of R, we can always find one sub-area in the central service area in which the visiting sequences are identical for all customers. In our paper, all customers in $\mathcal{T}$ have the same visiting sequence when $R \leq 5$. When $R > 5$, we can calculate the average distance by numerical integration for each customer although they have different visiting sequences. However, this process involves extreme complexity, and the extra inbound cost will be small compared with the cost when

$R \le 5$. Therefore, we focus on the situation where $R \le 5$ in the following section unless we indicate otherwise.

From Equations (8), (10) and (19), the unit-area system cost for the IHP problem is

$$
\begin{aligned}
C(A) &:= C^{\mathrm{F}} + C^{\mathrm{P}} + C^{\mathrm{T}} = f/A + \lambda\varphi q^{R+1} \\
&+ \lambda A^{1/2}\left(\left(\beta_0 + R\left(\frac{4}{3}\right)^{1/4} + \beta_R\right)q^{R+1} + \sum_{r=0}^{R}\left(\beta_0 + r\left(\frac{4}{3}\right)^{1/4} + \beta_r\right)q^r(1-q)\right).
\end{aligned}
\tag{30}
$$

Note that A is the only one variable in the above problem.

Proposition 4. The function $C(A)$ is unimodal.

Proof. $\left(\left(\beta_0 + R\left(\frac{4}{3}\right)^{1/4} + \beta_R\right)q^{R+1} + \sum_{r=0}^{R}\left(\beta_0 + r\left(\frac{4}{3}\right)^{1/4} + \beta_r\right)q^r(1-q)\right)$ is represented by Q for simplification. The derivative of function $C(A)$ is

$$\dot{C} \qquad f/A^2 + \frac{1}{2}\lambda A^{-1/2}Q \tag{31}$$

Then, set $\dot{C}$, and we can obtain the unique result, which is shown as follows.

$$A = \left(\frac{2f}{\lambda Q}\right)^{2/3} \tag{32}$$

When $A > 0$, the function $\dot{C}$ is a continuous function. Therefore, the function $\dot{C}$ is less than 0 if $0 < A < (2f/\lambda Q)^{2/3}$ and greater than 0 if $A > (2f/\lambda Q)^{2/3}$. This indicates that the function $C(A)$ is unimodal and thus completes the proof.

With this proposition, we can easily obtain the optimal solution A^* by derivation, which is equal to $\left(2f \Big/ \lambda\left(\left(\beta_0 + R\left(\frac{4}{3}\right)^{1/4} + \beta_R\right)q^{R+1} + \sum_{r=0}^{R}\left(\beta_0 + r\left(\frac{4}{3}\right)^{1/4} + \beta_r\right)q^r(1-q)\right)\right)^{2/3}$.

Therefore, the near-optimum unit-area system cost $C(A^*)$ can be obtained.

3.2.2 CA approach for a heterogeneous space

This section discusses the application of the IHP results to the finite heterogeneous space S. We assume that all relevant parameters (i.e. $f(x)$, $\lambda(x)$, $\varphi(x)$ and $q(x)$) vary relatively mildly over the original heterogeneous space and the facility initial area $A(x)$ is far smaller than the size of S, i.e., $A(x) \ll \qquad \equiv S$. Instead of searching for discrete location $\mathbf{x}$, we try to find the optimal solution to $A(x) \in \mathbb{R}$ near each location x.

Since the facilities are densely located, we can ignore the boundary effect with substantially no effect on the total cost. Since all parameters vary mildly over space S, each neighborhood of $x \in S$ can be approximated as an IHP and can be simply solved by Equation (30). We define $C(x, A(x))$ as the cost per unit area near x, which can be formulated as

$$C(x, A(x)) := f(x)/A(x) + \lambda(x)\varphi(x)q^{R+1}(x) + \lambda(x)A^{1/2}(x) \cdot \left(\left(\beta_0 + R\left(\frac{4}{3}\right)^{1/4} + \beta_R\right)q^{R+1}(x) + \sum_{r=0}^{R}\left(\beta_0 + r\left(\frac{4}{3}\right)^{1/4} + \beta_r\right)q^r(x)\left(1 - q(x)\right)\right) \tag{33}$$

We can obtain the optimal service area $A^*(x)$ for each neighborhood x by solving $C(x, A(x))$. Then, the optimal system cost in the original finite heterogeneous plane can be approximated by

$$C^* = \int_{x \in S} C\left(x, A^*(x)\right) dx. \tag{34}$$

Since $\left[A^*(x)\right]^{-1}$ is the optimal facility density function, the total number of optimal facilities can be estimated by

$$N^* \approx \int_{x \in S} \left[A^*(x)\right]^{-1} dx. \tag{35}$$

The optimal solutions, $A^*(x)$ and N^*, can be used in the direct sweeping method proposed by Fan et al. (2018) to discrete facility locations. This method just require search the space once to obtains a discrete location solution, and the initial service area of each facility is very close to the value of $A^*(x)$. More implementation details are shown in the paper (Fan et al., 2018). The reference illustrates that the sweeping model can obtain the near-optimum total cost that approximates to that estimated by Equation (34).

4. Numerical examples

This section presents several numerical examples to illustrate the performance of the continuous model on the investigated location problem. Space S is now a $[0, s] \times [0, s]$ square for the convenience of comparison and scalability. The density function of customer demand is $\lambda(x) = \bar{\lambda}\left[1 + \tau_\lambda \cos\left(\omega\|x\|\right)\right]$. The cost for opening a facility at x is $f(x) = \bar{f}\left[1 + \tau_f \cos\left(\omega\|x\|\right)\right]$. The disruption probability of a facility at x is $q(x) = \bar{q}\left[1 + \tau_q \cos\left(\omega\|x\|\right)\right]$. $\tau_\lambda \in (-1, 1)$, $\tau_f \in (-1, 1)$ and $\tau_q \in (-1, 1)$ are the heterogeneity control parameters for $\lambda(x)$, $f(x)$ and $q(x)$ over S, respectively. Scalar ω is selected to normalize the average customer density, facility cost and disruption probability by scalar ω (e.g., $\int_S \lambda(x)dx = \bar{\lambda}$, $\int_S f(x)dx = \bar{f}$ and $\int_S q(x)dx = \bar{q}$). $\|x\|$ expresses the distance between the location x and the center of space S. Because $\varphi(x)$ does not influence the value of $A^*(x)$, for simplicity, we set $\varphi(x) = \varphi$ as a constant over the space S. We calculate C^* and N^* by Equations (34) and (35), respectively. The default values of parameters are set as $s = 1$, $\bar{f} = 4$, $\bar{\lambda} = 100$, $\bar{q} = 0.05$, $\tau_f = 0.1$, $\tau_\lambda = 0.5$, $\tau_q = 0.5$, $\omega = 11.73$, $\varphi = 1$, and $R = 3$.

First, we make a comparison between the continuous and discrete (introduced in the Part I paper) models. For comparison purposes, the continuous parameters should be converted into the discrete parameters. We partition the continuous space S into $n \times n$ identical square cells where n is an integer parameter for the space granularity. Set $\mathcal{J}$ contains all the square cells. Customer z_i and candidate facility z_j, where $i, j \in \mathcal{J}$, are located at the center of the corresponding cells, respectively. We set $f_j = f(z_j)$, $q_j = q(z_j)$ and $\varphi_i = \varphi$. Demand λ_i is equal to the total demand in this square cell and is approximately formulated as

$$\lambda_i = \lambda(z_i)\frac{|S|}{n^2}. \tag{36}$$

In the conversion process, we omit the transportation cost when the customer and candidate facility are located in the same cell. This omission will underestimate the total transportation cost. To compensate it, we set that the distance between the customer and candidate facility in the same cell is approximately equal to one quarter of the cell length.

In Table 1, C_D^* and N_D^* denote the best system cost and facility number for the discrete model, respectively. We use the percentage $\varepsilon = \left(C_D^* - C^*\right)/C_D^*$ to indicate the difference in the system cost between the continuous and the discrete models. Let t and t_D denote the solution times of the continuous and the discrete models, respectively. Table 1 shows how the solutions change with different cell numbers for both continuous and discrete models. We observe that C^* is less than C_D^* in most instances. This is because the transportation cost between the customer and the facility in the same cell is equal to approximately one quarter of the cell length in the discrete model, which may overestimate the transportation cost relative to the continuous model. As n^2 increases, C^* increases that is close to the exact integral value. Although ε fluctuates, the trend of ε is decreasing as n^2 increases. However, we can observe that t is smaller than t_D for all experiments, and the former increases almost linearly with the number of cells, yet the latter increases apparently super-linearly. We also see that the instances with large n^2 cannot be solved by the discrete model due to the memory limit. From the results, it is obvious that the continuous model has better scalability and can solve large-scale instances efficiently.

Table 1 Total system cost estimation for the continuous and discrete models.

n^2	C^*	N^*	t	C_D^*	N_D^*	t_D	ε (%)
49	56.4594	4.6404	0.000437	60.3250	5	82	6.41%
64	56.8256	4.6639	0.000959	60.4693	4	185	6.03%
81	57.0914	4.6806	0.000724	60.0849	5	402	4.98%
100	57.2926	4.6933	0.000867	59.0395	4	73	2.96%
121	57.448	4.7031	0.001067	60.1336	4	1336	4.47%
144	57.5695	4.7107	0.00141	59.9606	4	1800	3.99%
169	57.6658	4.7167	0.001396	59.8608	4	1800	3.67%
196	57.7434	4.7216	0.001687	59.2987	4	1247	2.62%
225	57.8069	4.7256	0.001834	---	---	---	---
400	57.9988	4.7375	0.0031	---	---	---	---
1600	58.1897	4.7493	0.01145	---	---	---	---
6400	58.2384	4.7523	0.045572	---	---	---	---
10000	58.2442	4.7526	0.080393	---	---	---	---
40000	58.2521	4.7531	0.316043	---	---	---	---

Now, we vary only s and compare the performances of the continuous and discrete models. Table 2 shows the results for several instances with various s values with $n^2 = 100$. In this table, we see that C^* and N^* increase linearly as s increases. However, the solution time t does not change obviously with increasing s, which is apparent because the discretization resolution remains the same. Correspondingly, C_D^* and N_D^* also increase as s increases. However, the solution time t_D remains approximately the same but is much greater than t. In this table, we also observe that although ε fluctuates, the absolute value of ε is still less than 5% as s increases, indicating that the total system costs from both the continuous and discrete models are similar, regardless of the space size and the customer density.

Table 2 Total system cost estimation for the continuous and discrete models with various s

s	C^*	N^*	t	C_D^*	N_D^*	t_D	ε (%)
0.6	16.5571	1.4323	0.000977	17.1244	2	724	3.31%
0.7	23.7018	2.0247	0.000844	23.8019	2	728	0.42%
0.8	33.3532	2.7939	0.000821	32.7050	4	73	-1.98%
0.9	44.9065	3.7044	0.000835	44.0375	4	32	-1.97%

1	57.2926	4.6933	0.000906	59.0395	4	73	2.96%
1.5	126.3304	10.3942	0.000913	127.9645	12	657	1.28%
1.6	145.3540	11.9367	0.000862	141.3677	11	624	-2.82%
1.7	165.7275	13.5736	0.000945	159.5829	13	625	-3.85%
1.8	186.5041	15.2567	0.000849	183.8099	14	590	-1.47%
1.9	206.8956	16.9451	0.000915	215.2402	18	649	3.88%

Next, we compare the performance of the CA approach for the imperfect information and perfect information conditions with the following experiments. With perfect information, a customer knows the status of all facilities and chooses the nearest functional facility to obtain the service. The optimal service area, optimal system cost and optimal facility number under perfect information are denoted by $A_{PI}^{*}(x)$, C_{PI}^{*} and N_{PI}^{*}, respectively. We use C_{II} to express the actual cost under imperfect information when $A_{PI}^{*}(x)$ is implemented. $\varepsilon_{PI} = \left(C^{*} - C_{PI}^{*}\right) / C^{*}$ denotes the difference in total system cost between perfect information and imperfect information. Table 3 shows the solutions of several problem instances with various $\bar{f}$ and $\bar{q}$ values under perfect information and imperfect information. In Table 3, we see that the optimal system cost with imperfect information is higher than that with perfect information in all experiments. The difference ε_{PI} increases with increasing $\bar{q}$ and is higher than 13.6% when $\bar{q} \geq 0.2$. The optimal facility number is also small in the perfect information condition. Thus, if the customer can obtain perfect information, the system cost can obviously decrease by adjusting the facility location. Therefore, one aim of technology development is to ensure that the customer can always obtain perfect information. Otherwise, facility locations should be designed more robustly to prevent imperfect information. If we omit the consideration of imperfect information in the design, the actual system cost C_{II} is higher than both C_{PI}^{*} and C^{*}.

Table 3 Total system cost estimation under imperfect information and perfect information.

#	$\bar{f}$	$\bar{q}$	N^{*}	N_{PI}^{*}	C^{*}	C_{PI}^{*}	C_{II}	ε_{PI} (%)
1	0.5	0.05	19.01	18.36	29.13	28.10	29.14	3.52
2	0.5	0.1	20.38	18.99	31.29	29.12	31.34	6.94
3	0.5	0.15	21.86	19.64	33.72	30.24	33.82	10.30
4	0.5	0.2	23.47	20.29	36.52	31.54	36.73	13.63
5	1	0.05	11.98	11.56	36.70	35.41	36.71	3.52
6	1	0.1	12.84	11.96	39.42	36.68	39.47	6.94
7	1	0.15	13.77	12.38	42.44	38.07	42.58	10.31

8	1	0.2	14.79	12.78	45.90	39.63	46.17	13.66
9	2	0.05	7.55	7.29	46.24	44.61	46.25	3.52
10	2	0.1	8.09	7.54	49.66	46.21	49.73	6.94
11	2	0.15	8.68	7.80	53.44	47.93	53.61	10.31
12	2	0.2	9.31	8.05	57.73	49.82	58.06	13.69
13	4	0.05	4.75	4.59	58.25	56.20	58.28	3.52
14	4	0.1	5.10	4.75	62.56	58.22	62.64	6.94
15	4	0.15	5.47	4.91	67.30	60.36	67.51	10.32
16	4	0.2	5.87	5.07	72.62	62.67	73.05	13.71

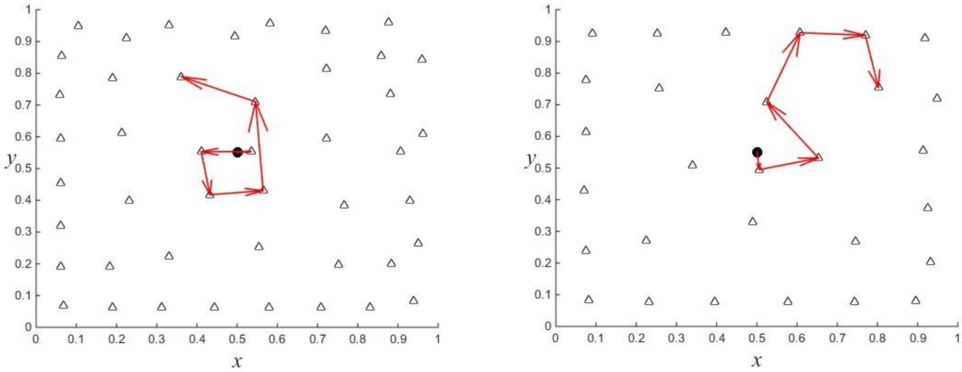
We also compare the performance of the CA approach with and without considering the inbound trip, under various $\bar{f}$ and $\bar{q}$ values. Without the inbound trip, a customer only chooses the optimal facility sequence to obtain the service without considering the return trip to her home location. The optimal service area, optimal system cost and optimal facility number without an inbound trip are denoted by $A_O^*(x)$, C_O^* and N_O^*, respectively. We use C_{IO} to express the actual cost with the inbound trip when $A_O^*(x)$ is implemented. $\varepsilon_O = \left(C^* - C_O^*\right)/C^*$ denotes the difference in total system cost with and without the inbound trip, whereas $\varepsilon_{IO} = \left(C_{IO} - C^*\right)/C^*$ denotes the actual cost deviation after applying the "wrong" facility location design. Table 4 shows the solutions of several problem instances with and without considering the inbound trip with various $\bar{f}$ and $\bar{q}$. The optimal system cost without the inbound trip is lower than that with the inbound trip in all experiments. The difference ε_O is more than 28% and decreases with increasing $\bar{q}$. The optimal facility number without the inbound trip is lower. The actual system cost C_{IO} is less than the optimal system cost C^* under the "wrong" design, with ε_{IO} as high as greater than 4%.

Table 4 Total system cost estimation with and without an inbound trip.

#	$\bar{f}$	$\bar{q}$	N^*	N_O^*	C^*	C_O^*	C_{IO}	ε_O (%)	ε_{IO} (%)
1	0.5	0.05	19.01	12.38	29.13	18.98	30.38	34.83	4.31
2	0.5	0.1	20.38	13.68	31.29	21.03	32.46	32.78	3.73
3	0.5	0.15	21.86	15.07	33.72	23.32	34.81	30.82	3.25
4	0.5	0.2	23.47	16.58	36.52	25.97	37.55	28.90	2.83
5	1	0.05	11.98	7.80	36.70	23.92	38.28	34.83	4.31
6	1	0.1	12.84	8.62	39.42	26.49	40.89	32.79	3.73

7	1	0.15	13.77	9.50	42.44	29.35	43.82	30.85	3.25
8	1	0.2	14.79	10.45	45.90	32.61	47.20	28.96	2.83
9	2	0.05	7.55	4.91	46.24	30.13	48.23	34.83	4.31
10	2	0.1	8.09	5.43	49.66	33.37	51.51	32.80	3.74
11	2	0.15	8.68	5.98	53.44	36.95	55.18	30.87	3.25
12	2	0.2	9.31	6.58	57.73	40.98	59.36	29.02	2.84
13	4	0.05	4.75	3.10	58.25	37.97	60.76	34.83	4.31
14	4	0.1	5.10	3.42	62.56	42.04	64.89	32.80	3.74
15	4	0.15	5.47	3.77	67.30	46.51	69.49	30.88	3.25
16	4	0.2	5.87	4.15	72.62	51.52	74.69	29.06	2.84

Figure 5 shows the visiting sequence with and without an inbound trip in one facility location design problem. Comparing Figure 5(a) and (b), we see that the number of built facilities in Figure 5(a) is less than that in Figure 5(b), which indicates that considering the inbound trip will increase the facility number to guarantee customers the ability to obtain the service near their initial locations. Figure 5(a) shows that the customer visiting sequence is assigned near her initial location. Figure 5(b) shows that the customer visiting sequence is similar to a line and is far from her initial location. The results reflect our realistic situations when customers look for service. If a customer wants to go back her home, she will search the service around her home. On the contrary, if this customer does not need to go back her home, she will visit the most appropriate facility and may far away from her home when she finds the service.



(a) (b)

Figure 5 Visiting sequences (a) with and (b) without an inbound trip.

Finally, we discuss the sensitivity analysis of optimal results for parameters $\bar{q}$, $\bar{f}$, $\bar{\lambda}$ and R.

Table 5 shows the relationship between the cost components and R. We find that providing the backup service can reduce the total system cost when system faces the facility disruptions. However, this reduction gradually diminishes as R increases. When R exceeds 5, all cost components except the penalty cost do not change. This indicates that the benefits by assigning more facilities are almost disappeared. Thus, we can foresee that the penalty cost will be almost equal to 0 when R is large enough. Therefore, we do not set R exceed 5 in most experimental instances of this paper.

Table 5 Analysis of the sensitivity to R

R	Construction cost	Transportation cost	Penalty cost	Total system cost	Facility number
1	19.3498	38.6995	0.3683	58.4176	4.7372
2	19.4129	38.8258	0.024	58.2626	4.7522
3	19.4178	38.8356	0.0016	58.2549	4.7533
4	19.4181	38.8361	0.0001	58.2543	4.7534
5	19.4181	38.8362	7.63E-06	58.2542	4.7534
6	19.4181	38.8362	5.35E-07	58.2542	4.7534
7	19.4181	38.8362	3.78E-08	58.2542	4.7534
8	19.4181	38.8362	2.69E-09	58.2542	4.7534
9	19.4181	38.8362	1.92E-10	58.2542	4.7534
10	19.4181	38.8362	1.37E-11	58.2542	4.7534

Figure 6 shows the results of sensitivity analysis to several key parameters. We set $\bar{q}=0.05$, $\bar{f}=1$, and $\bar{\lambda}=500$ as the default parameter values, and select one parameter to vary at a time. The other parameters are set as $\tau_f=0.1$, $\tau_\lambda=0.9$, $\tau_q=0.5$, $\omega=11.73$, $\varphi=1$, and $R=5$.

Figure 6(a) and Figure 6(b) illustrate how the optimal system cost and optimal facility number change with the average facility disruption probability $\bar{q}$. Both of these values increase as $\bar{q}$ increases. The optimal system cost increases slowly when $\bar{q}$ is less than 0.3. However, as $\bar{q}$ continues to increase, it increases rapidly and becomes very large. The optimal facility number also has a similar tendency, but it is not obvious. Therefore, we should control the probability $\bar{q}$ to a low value to reduce the increase in the optimal system cost. These observations are similar to those in the Part I paper.

Figure 6(c) shows that the optimal system cost increases as the average facility cost $\bar{f}$ increases. Figure 6(d) shows that the optimal facility number decreases as the average facility cost $\bar{f}$ increases. In other words, a higher average facility cost results in fewer facilities, as is commonly found in real-world situations. However, the difference in tendencies in Figure 6(c) and Figure 6(d) indicates that the presence of fewer facilities will result in greater transportation costs, leading to an increase in the optimal system cost.

Figure 6(e) and Figure 6(f) show how the optimal system cost and optimal facility number change with the average demand density $\bar{\lambda}$. Both of these values increase approximately linearly as $\bar{\lambda}$ increases. Therefore, the demand has a constant effect on the optimal location design. To satisfy the increase in customer demand, we must build more facilities to shorten the customers' travel distance and thus reduce the increase in the total system cost.

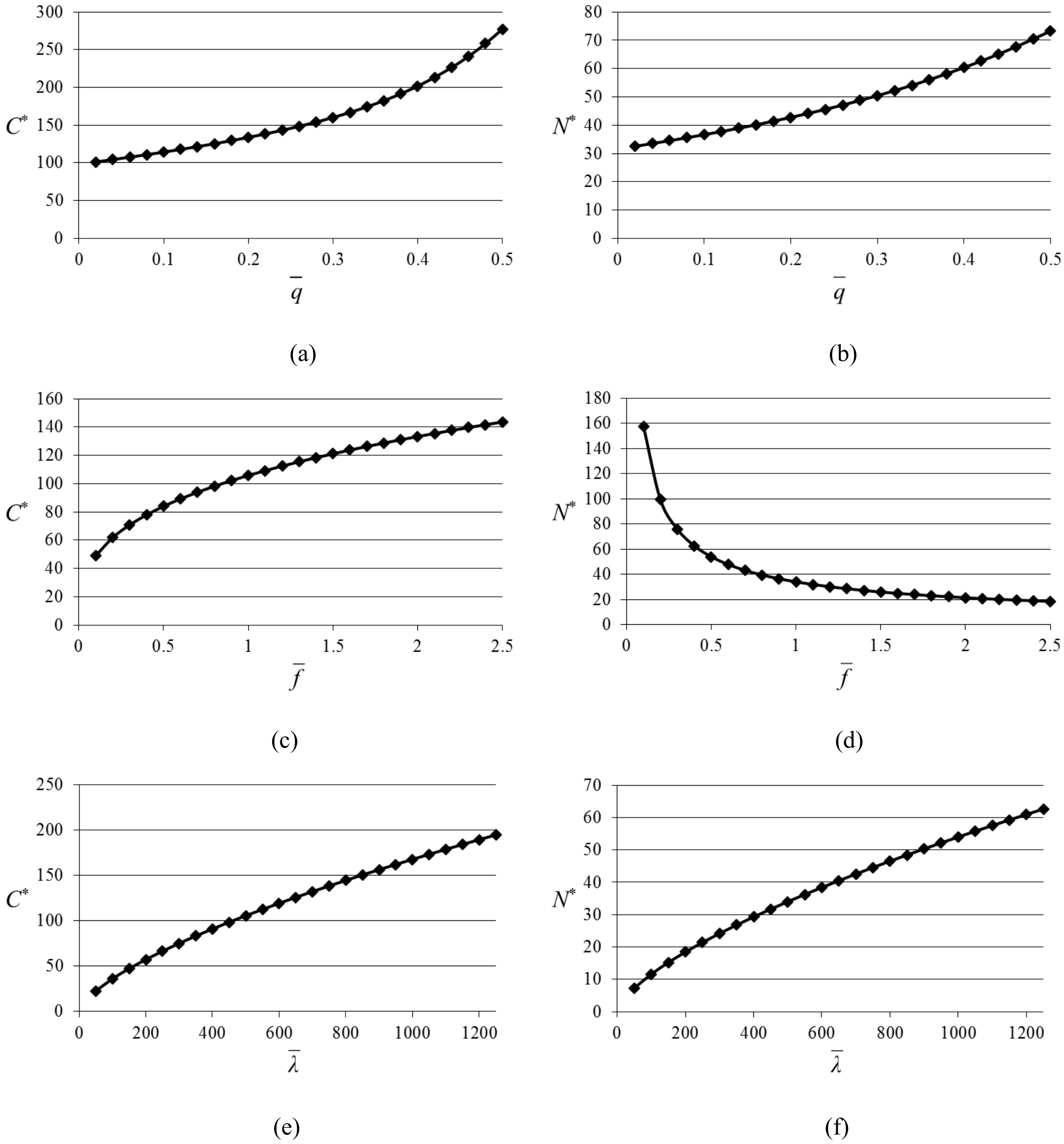


Figure 6 Sensitivity analysis

5. Conclusion

This paper proposes a continuous model for the large-scale RUFL problem considering round-trip transportation under imperfect information, which is a supplement of research on the location problem performed in the Part I paper. In the proposed model, we assume that each facility has a site-dependent disruption probability. In any disruption scenario, a customer has imperfect information regarding facility states and always attempts to visit pre-assigned facilities to obtain the minimum transportation cost. When the customer obtains the service or gives up, she will return to her initial location. The CA formulation starts with an idealized homogeneous plane and is then extended to a general heterogeneous plane for the investigated problem. The simple structure of the CA model allows examination into problem structures for constructing a near-optimum solution (e.g., by constructing a feasible customer visiting sequence and a lower bound cost). Numerical experiments showed that the continuous model adopting the CA technique has superior computational efficiency for solving large-scale instances, whereas the discrete model performs well for small and medium-sized problem instances. Case studies also indicated that the round-trip needs to be considered in reliable facility location problems, particularly with imperfect information. The results of the sensitivity analysis for various parameters indicated that the continuous model can solving the large-scale instances with a good, robust performance.

In the future, we can relax the facility disruption pattern to more general patterns, such as correlated disruption pattern that is investigated by Li et al. (2010) considering perfect information. When relevant data are available (e.g., facility disruption patterns), it is interesting to see how this proposed modeling method can be applied to real-world problems to improve the infrastructure system reliability.

Acknowledgements

This majority of the work was conducted at the University of South Florida under the supervision of Dr. Xiaopeng Li. This research is supported by National Natural Science Foundation of China # 71601014, the U.S. National Science Foundation through Grants CMMI #1558889 and CMMI #1541130, and the Fundamental Research Funds for the Central Universities # 2018RC023.

Reference

Ansari, S., Başdere, M., Li, X., Ouyang, Y., & Smilowitz, K. (2018). Advancements in continuous approximation models for logistics and transportation systems: 1996-2016. Transportation Research Part B: Methodological, 107(C), 229-252.

Cui, T., Ouyang, Y., Shen, Z.M. (2010) Reliable facility location design under the risk of disruptions. Operations Research 58(4-part-1), 998-1011.

Daganzo, C., Newell, G. (1986) Configuration of physical distribution networks. Networks 16(2), 113-132.

Daganzo, C. (2005) Logistics System Analysis. Fourth ed. (Springer, Berlin, Germany).

Daskin, M.S. (1995). Network and Discrete Location: Models, Algorithms, and Applications. (Wiley, New York).

D'Amico, E. (2002) West coast port lockout creates problems for chemical shippers. Chemical Week 164(40), 10.

Drezner, Z. (1995) Facility Location: A Survey of Applications and Methods. Springer, New York.

Godoy, L.A. (2007) Performance of storage tanks in oil facilities damaged by Hurricanes Katrina and Rita. Journal of Performance of Constructed Facilities 21(6), 441-449.

Hirsch, M., Carli, P., Nizard, R., Riou, B., Baroudjian, B., Baubet, T., Chhor, V., Choolet-Xemard, C., Dantchev, N., Fleury, N., Fontaine, J., Yordanov, Y., Raphael, M., Burtz, C.P., Lafont, A. (2015) The medical response to multisite terrorist attacks in Paris. The Lancet 386(10012), 2535-2538.

Fan, H., Yun, L., Li, X. (2018) A linear-time sweeping algorithm for discretization of continuum approximation. Working Paper. University of South Florida.

Langevin, A., Mbaraga, P., Campbell, J. (1996) Continuous approximation models in freight distribution: an overview. Transportation Research Part B 30(3), 163-188.

Li, X., Ouyang, Y. (2010) A continuum approximation approach to reliable facility location design under correlated probabilistic disruptions. Transportation Research Part B 44 (4), 535-548.

Li, X., Ouyang, Y. (2012) Reliable traffic sensor deployment under probabilistic disruptions and generalized surveillance effectiveness measures. Operation Research 60(5), 1183-1198.

Li, X., Ma, J., Cui, J., Ghiasi, A., Zhou. F. (2016) Design framework of large-scale one-way electric vehicle sharing systems: A continuum approximation model. Transportation Research Part B 88, 21-45.

Ouyang, Y., Wang, Z., Yang, H. (2015) Facility location design under continuous traffic equilibrium. Transportation Research Part B 81, 18-33.

Peng, P. Wang, X., Ouyang, Y. (2014) Approximation of discrete spatial data for continuous facility location design. Integrated Computer-Aided Engineering, 21(4), 311-320.

Sharkey, T., Nurre, S., Nguyen, H., Chow, J., Mitchell, J., Wallace, W. (2015) Identification and classification of restoration interdependencies in the wake of hurricane Sandy. Journal of Infrastructure Systems 22(1), 04015007.

Sheppard, PS., Landry, MD. (2016) Lessons from the 2015 earthquake(s) in Nepal: implication for rehabilitation. Disability and Rehabilitation 38(9), 910-913.

Snyder, L.V., Daskin, M.S. (2005) Reliability models for facility location: the expected failure cost case. Transportation Science 39(3), 400-416.

Toth, L.F. (1959) Sur la representation dune population infinie par un nombre fini delements. Acta Mathematica Hungarica 10 (3-4), 299-304.

Wang, X., Ouyang, Y. (2013) A continuum approximation approach to competitive facility location design under facility disruption risks. Transportation Research Part B 50, 90-103.

Wang, X., Lim, M., Ouyang, Y. (2015) Infrastructure deployment under uncertainties and competition: the biofuel industry case. Transportation Research Part B 78, 1-15.

Wang, X., Lim, M.K., Ouyang, Y. (2017) A continuum approximation approach to the dynamic facility location problem in a growing market. Transportation Science 51(1), 343-357.

Yun, L., Fan, H., Li, X. (2018) Reliable Facility Location Design with Round-trip Transportation under Imperfect Information Part I: A Discrete Model. Submitted to Transportation Research Part B.

Appendix A. Notation list

SYMBOL	DESCRIPTION
S	Planning space
$f(x)$	Fixed opening cost for facility x (the facility at location x)
$\lambda(x)$	Demand of customer x (the customer at location x)
$\varphi(x)$	Penalty cost for customer x
$q(x)$	Disruption probability for facility x
$A(x)$	Service area for facility x
$\mathbf{x}$	Set of built facilities
$j_r(x\|\mathbf{x})$	The rth facility that customer x visits
$J(x\,\|\,\mathbf{x})$	Facility visiting sequence for customer x
r	Facility rank for a customer
R	Maximum facility rank for a customer
$d_r(x\|J(x\,\|\,\mathbf{x}))$	Total travel distance for customer x given that she ends at her rank r facility given visiting sequence $J(x\,\|\,\mathbf{x})$
$d_r^{\mathrm{O}}(x\|J(x\,\|\,\mathbf{x}))$	Outbound distance for customer x given that she ends at her rank r facility given visiting sequence $J(x\,\|\,\mathbf{x})$
$d_r^{\mathrm{I}}(x\|J(x\,\|\,\mathbf{x}))$	Inbound distance for customer x given that she ends at her rank r facility given visiting sequence $J(x\,\|\,\mathbf{x})$
$P_r(x\|J(x\,\|\,\mathbf{x}))$	Probability for customer x to be served by facility $j_r(x\|\mathbf{x})$ conditioned on facility visiting sequence $J(x\,\|\,\mathbf{x})$
$\bar{P}(x\|J(x\,\|\,\mathbf{x}))$	Probability that customer x is not served by any facility conditioned on facility visiting sequence $J(x\,\|\,\mathbf{x})$

A	Size of a facility's initial service area
$\mathcal{T}$	Analysis area for the IHP problem
$-^{\text{NO}}$	Superscript tag that the corresponding terms are associated with the near-optimum (NO) solution of the NOVS
C^{F}	Unit-area facility fixed opening cost
C^{P}	Unit-area penalty cost
C^{T}	Unit-area transportation cost
C^{TO}	Unit-area outbound transportation cost
C^{TI}	Unit-area inbound transportation cost
C^{*}	Optimal system cost in the original finite heterogeneous plane
N^{*}	Total number of optimal facilities in the original finite heterogeneous plane
s	Side length of the planning space
$\bar{\lambda}$ ($\bar{f}$, $\bar{q}$)	Average customer density (facility cost, disruption probability)
τ_{λ} (τ_{f}, τ_{q})	Scalar to control the heterogeneity of $\lambda(x)$ ($f(x)$, $q(x)$) over S
ω	Scalar to normalize the average customer density, facility cost and disruption probability
n	Integer parameter for the space granularity
$\mathcal{J}$	Set of all square cells
z_i	Location of customer $i \in \mathcal{J}$
z_j	Location of candidate facility $j \in \mathcal{J}$

Appendix B. IHP transportation cost formulation

To facilitate the derivation of transportation costs, we arrange facilities as illustrated in Figure 7. Again, we investigate a generic central facility (or hexagon) and index the facilities around it in the manner illustrated by Figure 3. Furthermore, the hexagons can be grouped into different layers according to their distance from facility 0. For example, the first layer contains facility 0, the second layer contains facilities 1-7, and so forth. We index the layers with $m \in \mathbb{Z}$. Meanwhile, we divide the space into six sextants, indexed by $s \in \{1,2,3,4,5,6\}$ where $s=1$ indexes the upper right sextant. Note that the m^{th} layer has m facilities in each sextant.

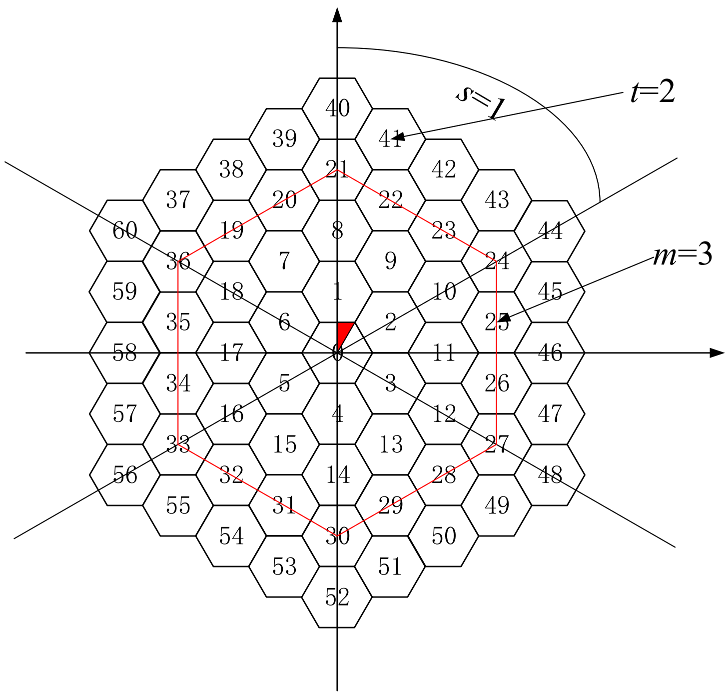


Figure 7 Illustration of the coordinate system

With this system, we can re-index each facility with a triplet (m,s,t) such that $m \in \mathbb{Z}$ indexes the layer of this facility, $s \in \{1,\cdots$ indexes the sextant of this facility, and $t \in \{1,\cdots$ indexes the clockwise position of this facility in this sextant. For example, facility 2 is re-indexed as $(2,1,2)$. In general, the mapping from index j to triplet (m,s,t) is as follows:

$$m = \left\lceil \frac{-3+\sqrt{9+12j}}{6} \right\rceil, \tag{37}$$

$$s=\begin{cases}\left\lceil \dfrac{j-3m^2+2m+1}{m}\right\rceil, & j>3m^2-2m-1\\ \left\lceil \dfrac{j-3m^2+8m+1}{m}\right\rceil, & j\le 3m^2-2m-1\end{cases}, \tag{38}$$

$$t=\begin{cases}j-3m^2+3m+1-sm, & j>3m^2-2m-1\\ j-3m^2+9m+1-sm, & j\le 3m^2-2m-1\end{cases}. \tag{39}$$

Since the area size of a hexagon is A, the side length of a hexagon should be $l:=\left(4/27\right)^{\frac{1}{2}}A^{\frac{1}{2}}$. We can easily obtain each facility's Euclidean coordinates $\left(a,b\right)$ by its index $\left(m,s,t\right)$. The results of these coordinates are shown in Table 6.

Table 6 Coordinates $\left(a,b\right)$ of a facility

$s=1$	$s=2$
$\left(m\left(0,\sqrt{3}\right)+\left(t-1\right)\left(\frac{3}{2},-\frac{\sqrt{3}}{2}\right)\right)l$	$\left(m\left(\frac{3}{2},\frac{\sqrt{3}}{2}\right)+\left(t-1\right)\left(0,-\sqrt{3}\right)\right)l$
$s=3$	$s=4$
$\left(m\left(\frac{3}{2},-\frac{\sqrt{3}}{2}\right)+\left(t-1\right)\left(-\frac{3}{2},-\frac{\sqrt{3}}{2}\right)\right)l$	$\left(m\left(0,-\sqrt{3}\right)+\left(t-1\right)\left(-\frac{3}{2},\frac{\sqrt{3}}{2}\right)\right)l$
$s=5$	$s=6$
$\left(m\left(-\frac{3}{2},-\frac{\sqrt{3}}{2}\right)+\left(t-1\right)\left(0,\sqrt{3}\right)\right)l$	$\left(m\left(-\frac{3}{2},\frac{\sqrt{3}}{2}\right)+\left(t-1\right)\left(\frac{3}{2},\frac{\sqrt{3}}{2}\right)\right)l$

With this system, the average distance between customers in $\mathcal{T}$ and any facility $\left(m,s,t\right)$ can be obtained by the following integral.

$$d_{ave}=\frac{\int_0^{\frac{1}{2}}\int_{\sqrt{3}x}^{\frac{\sqrt{3}}{2}}\sqrt{\left(x-a\right)^2+\left(y-b\right)^2}\,dxdy}{\frac{\sqrt{3}}{8}}\left(\frac{4}{27}\right)^{\frac{1}{4}}A^{\frac{1}{2}}. \tag{40}$$