Received: 2 August 2018 | Revised: 30 October 2018 | Accepted: 9 December 2018
DOI: 10.1002/rnc.4453

RESEARCH ARTICLE

WILEY

# Transverse function control with prescribed performance guarantees for underactuated marine surface vehicles

Shi-Lu Dai[1] | Shude He[1] | Hai Lin[2]

[1]School of Automation Science and Engineering, South China University of Technology, Guangzhou, China

[2]Department of Electrical Engineering, University of Notre Dame, Notre Dame, Indiana

**Correspondence**
Hai Lin, Department of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556.
Email: hlin1@nd.edu

**Funding information**
National Natural Science Foundation of China, Grant/Award Number: 61473121, 61773169, and 61527811; Guangdong Natural Science Foundation, Grant/Award Number: 2017A030313381 and 2017A030313369; Science and Technology Program of Guangzhou, Grant/Award Number: 201604016082; Fundamental Research Funds for the Central Universities

**Summary**

This paper studies the problem of stabilizing reference trajectories (also called as the trajectory tracking problem) for underactuated marine vehicles under predefined tracking error constraints. The boundary functions of the predefined constraints are asymmetric and time-varying. The time-varying boundary functions allow us to quantify *prescribed* performance of tracking errors on both transient and steady-state stages. To overcome difficulties raised by underactuation and nonzero off-diagonal terms in the system matrices, we develop a novel transverse function control approach to introduce an additional control input in backstepping procedure. This approach provides practical stabilization of *any* smooth reference trajectory, whether this trajectory is *feasible or not*. By practical stabilization, we mean that the tracking errors of vehicle position and orientation converge to a small neighborhood of zero. With the introduction of an error transformation function, we construct an inverse-hyperbolic-tangent-like *barrier* Lyapunov function to show practical stability of the closed-loop systems with prescribed transient and steady-state performances. To deal with unmodeled dynamic uncertainties and external disturbances, we employ neural network (NN) approximators to estimate uncertain dynamics and present disturbance observers to estimate unknown disturbances. Subsequently, we develop adaptive control, based on NN approximators and disturbance estimates, that guarantees the prescribed performance of tracking errors during the transient stage of on-line NN weight adaptations and disturbance estimates. Simulation results show the performance of the proposed tracking control.

**KEYWORDS**

neural networks, prescribed performance guarantees, robust control, transverse function control, uncertain underactuated systems

## 1 | INTRODUCTION

A typical motion control of an autonomous vehicle is *trajectory tracking*, which is concerned with designing a feedback control law such that the vehicle could track a time parameterized reference trajectory.[1,2] The degree of difficulty on solving the trajectory tracking control problem depends on the vehicle configuration. For a fully actuated system, this problem has been reasonably well understood. For an underactuated vehicle that has fewer independent actuators than the freedom degrees, trajectory tracking control still receives considerable attention due to its theoretical challenges and

 © 2019 John Wiley & Sons, Ltd. 

important applications. In practical applications, for example, most of marine surface vehicles are equipped only with two propellers for surge and yaw motions, but without any actuator for direct control of sway motion.[3,4] The technical difficulty of controlling underactuated vehicle is that a smooth time-invariant feedback controller cannot asymptotically stabilize the vehicle at an equilibrium point.[5] Additionally, an underactuated vehicle typically suffers from a nonintegrable nonholonomic constraint.[6] The nonholonomic constraint makes the tracking control properties strictly depend on the reference trajectories. Most of the studies on trajectory tracking control of underactuated vehicles traditionally concentrate on kinematically *feasible* (or admissible) trajectories. A reference trajectory is called to be kinematically *feasible* or *admissible* if it could satisfy the nonholonomic constraints of the vehicle kinematics.[3,7] In the real-time implementation, however, a desired trajectory might be not feasible.[7-9] A typical example of an *infeasible* trajectory (that is, the trajectory does not satisfy the vehicle kinematics equation) for a marine vehicle[9] is a fixed position and orientation, where a desired orientation is not aligned with the ocean current direction. Hence, we are motivated to study the problem of stabilizing *any* smooth reference trajectory, no matter whether the trajectory is feasible or not, for underactuated marine vehicles. In addition to considering *infeasible* trajectory tracking control design, we also consider the problem of tracking error constraints, in which the vehicle outputs are not allowed to exceed a predefined allowable distance from the reference trajectory. Such constraints are critical for vehicle's safety and system's performance, especially in narrow waterways or in riverine applications, since any violation of the constraints during the vehicle motions may deteriorate system's performance or even cause potential collisions and vehicle damages. It is worth pointing out our main motivation is not to obtain a stronger result of *asymptotic* tracking but to develop tracking controllers that are theoretically capable of stabilizing in a *practical* fashion for any smooth reference trajectory (either feasible or infeasible trajectory), with guaranteeing prescribed transient performance. This is reasonable because of the infeasible trajectory, ie, trajectory that is not a solution to system's motion equations and thus cannot be stabilized asymptotically.[8,9]

## 1.1 | Related work

The main motivation of this paper arises from the recent development of feedback control with output/state constraints and trajectory tracking control of underactuated vehicles.

### 1.1.1 | Feedback control with output/state constraints

Many practical systems are often subject to the constraints on system outputs, inputs, or states, which might be presented in the forms of safety, saturation, or performance specifications. Consequently, considering output/state constraints in feedback control design is receiving increasing attention. Without depending on the trial-and-error method, several elegant design techniques of enforcing the constraints in the feedback control synthesis have been proposed, including funnel control,[10] barrier Lyapunov function approach,[11,12] and prescribed performance control (PPC) methodology.[13] A funnel control[10] could ensure the evolution of the tracking error being within the predefined region by adjusting its time-varying control gain, where the control gain is a smooth function having the following properties: if tracking error approaches the funnel boundary, then the gain attains values large sufficiently to prevent boundary contact. In the work of Tee et al,[11] a *barrier* function with a remarkable property of finite escape whenever its argument approaches the boundary of constraints is introduced in constructing a control Lyapunov function for stability analysis for nonlinear systems with output/state constrains. By guaranteeing the barrier Lyapunov function along the system trajectory bounded, the stability of the closed-loop system is ensured and these constraints are never violated. Time-invariant output/state constraints were greatly extended to the time-varying constraint case,[12] where time-varying output constraints could specify the transient performance bounds as the functions of time and the constraints are not violated by the use of time-varying barrier Lyapunov functions. Barrier Lyapunov function approach provides a flexible and powerful tool in dealing with state/output constraints and has been applied to practical systems such as robotic manipulators,[14] gantry crane system,[15] and fully actuated surface vessels.[16,17] The PPC methodology[13] was originally presented to design feedback control systems, in which explicitly *predefined* transient and steady-state performances of output tracking errors were satisfied. Prescribed performance guarantees means that the tracking errors evolve always within predefined allowable regions that are bounded by decaying functions of time, usually chosen as exponential functions. To achieve a prescribed performance, tracking errors are transformed by a smooth and strictly increasing/decreasing function,[13,18] similar to a barrier function, which goes to infinity while its argument approaching the predefined bounds. By guaranteeing boundedness of the transformed errors, the tracking errors evolve always within the predefined regions.

Although the constraints on system outputs/states are of practical importance for the safety and performance of control systems, enforcing these constraints in the motion control design has not been fully addressed, especially for underactuated vehicles. In the work of He et al,[16] a barrier Lyapunov function was applied for control of a fully actuated vessel with time-invariant output constraints. An asymmetric time-varying barrier Lyapunov function was developed to address the output constraints for a fully actuated marine surface vehicle.[17] Using the PPC methodology, adaptive tracking control with prescribed performance was proposed for a group of fully actuated marine vehicles.[19,20] In the work of Bechlioulis et al,[21] the PPC methodology was applied to solve the trajectory tracking control problem for torpedo-like and unicycle-like vehicles with prescribed performance. For a group of underactuated surface vehicles,[22] considering symmetric constraints on tracking errors of the line-of-sight ranges and angles, adaptive formation control laws were designed by employing novel barrier Lyapunov functions. Different from the studies[21-23] where the matrices of vehicle mass and damping were assumed to be diagonal, we consider the case of nonzero off-diagonal terms in the system matrices. For detailed comparisons and technical analyses, please see Remark 4. Additionally, we design tracking controllers that stabilize *any* reference trajectory, either feasible or infeasible.

### 1.1.2 | Trajectory tracking control of underactuated vehicles

A good number of design techniques and interesting solutions for *asymptotically* stabilizing *feasible* reference trajectories have been presented for underactuated vehicles (eg, other works[24-26]). For the problem of stabilizing *infeasible* reference trajectories, however, the number of available results is very limited in the literature due to the difficulty of the problem. A *transverse function* approach was proposed to construct smooth feedback control laws that guarantee *practical* stabilization of *any* (feasible or infeasible) reference trajectory (ie, stabilization of system output in a small neighborhood of the reference trajectory) for a controllable driftless nonlinear system.[7] Recently, the transverse function control approach has been effectively applied to *practically* stabilize any smooth reference trajectory, whether this trajectory is feasible or not, for underactuated mobile robots[8] and marine surface vehicles,[9,27-29] where the calculations of transverse functions or/and stability analysis usually require an accurate vehicle model (see Remarks 4 and 5 presented in this paper for detailed discussions and technical analyses). In an uncertain maritime environment, however, an accurate model may not be obtained a priori, eg, the hydrodynamic damping effects,[30-33] and the vehicle model usually suffers from external disturbances induced by ocean currents, winds, and waves (winds generated).[34-36] The presences of unmodeled dynamic uncertainties and external disturbances may result in violating tracking error constraints or in deteriorating system's performance.[37,38] In this paper, we consider unmodeled dynamics and unknown external disturbances and develop transverse function control of underactuated vehicles under tracking error constraints. In comparison with the existing transverse function control approaches,[7-9,27-29] the proposed controllers could guarantee the prescribed transient performance of the tracking errors, ie, the convergence speeds of the tracking errors are faster than the preselected values and the maximum overshoots are less than the given constants.

## 1.2 | Our contributions

The present control design is based on transverse function control approaches,[9,27-29] disturbance observers,[39-41] backstepping procedure, barrier functions, and control Lyapunov synthesis. It is not straightforward, even for an accurately known vehicle model, to combine the transverse function approaches developed in related works[9,27-29] and the references therein with the prescribed output-constrained control methodologies in other works[11-13,21,22] to design a tracking controller for underactuated vehicle. This is due to the fact that the approaches in related works[9,27-29] and the references therein introduce three transverse functions in the vehicle kinematics design, appearing in the first step of backstepping procedure, and thus any large transverse function may violate the tracking error constraints and make the problem of prescribed performance guarantees for the tracking errors difficult or impossible. Different from the existing transverse function approaches, we introduce two transverse functions in the vehicle kinetics design (ie, in the second step of backstepping procedure), and thus the additional transverse functions do not present in the tracking error equations of the vehicle position and orientation. It is worth noticing that transverse function controllers highly depend on the selected transverse functions. So far, the calculations of such functions in the existing literature are only applicable for controlling vehicle kinematics, and there is no result available on the construction of transverse functions for vehicle kinetics design. The technical contributions of the present paper are summarized as follows.

i. To overcome difficulties raised by underactuation and nonzero off-diagonal terms in the system matrices, we develop the transverse function control approach to introduce an additional control input in the sway dynamics. One of the noticeable features is that it is not required to prove the stability of the sway dynamics separately. This feature is of great importance to robust stabilization of underactuated systems with unmodeled dynamics. Furthermore, the present control design gives *continuous* feedback control laws such that practical stabilization of any smooth trajectory, no matter whether the trajectory is feasible or not, is achieved.

ii. Different from the existing transverse function approaches where three transverse functions were introduced in the vehicle kinematics design, we design two novel transverse functions in the vehicle kinetics design and duly give the determination of these two functions.
iii. With the introduction of an error transformation function, we construct a novel *inverse-hyperbolic-tangent-like* barrier Lyapunov function to show the stability of the closed-loop systems and to guarantee that tracking errors always stay within predefined constraint bounds.
iv. To provide robust stability and performance with respect to model uncertainties and external disturbances, we develop disturbance-observer–based adaptive NN control laws such that the prescribed performance of tracking errors is guaranteed during the transient stage of on-line NN weight adaptations and disturbance estimates.

### 1.3 | Outline of the paper

The rest of this paper is organized as follows. Section 2 gives the problem formulation and provides some preliminary knowledge on tracking error constraints. Section 3 proposes a constructive design of model-based tracking control laws that ensure the stability of the closed-loop systems and guarantee the position and orientation tracking errors being always within certain predefined bounds. To deal with unmodeled dynamics, in Section 4, approximation-based tracking control is developed to provide robust stability and performance with respect to model uncertainties and unknown disturbances from the maritime environments. In Section 5, simulation studies are performed to show the transient and steady-state performances of the control systems. Finally, the conclusion is included in Section 6.

### 1.4 | Notation

Throughout this paper, $\|\bullet\|$ is the Euclidean norm of vectors; $\lambda_{\min}(\bullet)$ and $\lambda_{\max}(\bullet)$ are the minimum and maximum eigenvalues of a symmetric matrix, respectively; det($\mathbf{Q}$) is the determinant of matrix $\mathbf{Q}$; $I_n$ is the $n \times n$ identity matrix; $\mathcal{L}_\infty$ denotes the space of all essentially bounded functions; and $(\hat{\cdot})$ and $(\tilde{\cdot})$ with $\tilde{(\cdot)} = \hat{(\cdot)} - (\cdot)$, respectively, denote the estimates of unknown disturbances/parameters and estimate errors.

## 2 | PROBLEM FORMULATION AND PRELIMINARIES

### 2.1 | Underactuated marine vehicle

Consider an underactuated marine vehicle equipped only with two independent propellers that provide the force in surge and the control torque in yaw. The kinematics and dynamics of the underactuated vehicle moving in a horizontal plane are described by[41,42] the following:

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \mathbf{J}(\psi)\boldsymbol{\nu} \\ \mathbf{M}\dot{\boldsymbol{\nu}} &= -\mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{\tau} + \boldsymbol{\tau}_w(t), \end{aligned} \tag{1}$$

where $\boldsymbol{\eta} = [x, y, \psi]^T$ are the vehicle position $(x, y)$ and orientation $(\psi)$ in the earth-fixed frame; $\boldsymbol{\nu} = [u, v, r]^T$ are the linear velocities in surge $(u)$ and sway $(v)$, and the angular velocity in yaw $(r)$ in the body-fixed frame; $\boldsymbol{\tau} = [\tau_u, 0, \tau_r]^T$ are the control inputs with the surge force $\tau_u$ and yaw moment $\tau_r$; $\boldsymbol{\tau}_w = [\tau_{wu}, \tau_{wv}, \tau_{wr}]^T$ denote the unknown external disturbances induced by ocean currents, winds, and waves (winds generated); $\mathbf{J}(\psi)$ is the rotation matrix; and $\mathbf{M} = \mathbf{M}^T > 0$, $\mathbf{C}(\boldsymbol{\nu})$, and $\mathbf{D}(\boldsymbol{\nu})$ are the vehicle inertia matrix, the Coriolis and centripetal matrix, and hydrodynamic damping matrix, respectively, all of which are nonzero off-diagonal. The coefficients of system (1) are given by

$$\mathbf{J}(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & m_{23} \\ 0 & m_{32} & m_{33} \end{bmatrix}$$

$$\mathbf{C}(\boldsymbol{\nu}) = \begin{bmatrix} 0 & 0 & -m_{22}v - m_{23}r \\ 0 & 0 & m_{11}u \\ m_{22}v + m_{23}r & -m_{11}u & 0 \end{bmatrix}$$

$$\mathbf{D}(\boldsymbol{\nu}) = \begin{bmatrix} d_{11}(u) & 0 & 0 \\ 0 & d_{22}(v,r) & d_{23}(v,r) \\ 0 & d_{32}(v,r) & d_{33}(v,r) \end{bmatrix},$$

where $m_{11} = m - X_{\dot{u}}$, $m_{22} = m - Y_{\dot{v}}$, $m_{23} = m_{32} = mx_g - Y_{\dot{r}}$, $m_{33} = I_z - N_{\dot{r}}$, $d_{11}(u) = -(X_u + X_{|u|u}|u| + X_{uuu}u^2)$, $d_{22}(v,r) = -(Y_v + Y_{|v|v}|v| + Y_{|r|v}|r|)$, $d_{23}(v,r) = -(Y_r + Y_{|v|r}|v| + Y_{|r|r}|r|)$, $d_{32}(v,r) = -(N_v + N_{|v|v}|v| + N_{|r|v}|r|)$, $d_{33}(v,r) = -(N_r + N_{|v|r}|v| + N_{|r|r}|r|)$. Here, $m$ is the vehicle mass; $X_{\dot{u}}$, $Y_{\dot{v}}$, $Y_{\dot{r}}$, and $N_{\dot{r}}$ are the added masses; $x_g$ is the $x_b$-coordination of the vehicle center of gravity; $I_z$ is the moment of inertia in yaw; and $X_{(\cdot)}$, $Y_{(\cdot)}$, and $N_{(\cdot)}$ are the linear and quadratic hydrodynamic damping coefficients in surge, sway, and yaw. Noticing that the surface vehicle (1) has three freedom degrees (surge, sway, and yaw) to be controlled and only has two independent control inputs (surge force and yaw moment), and thus system (1) is underactuated.

**Assumption 1.** The external disturbance $\boldsymbol{\tau}_w$ is time-varying, continuously differentiable, and directly unmeasurable (or too expensive to measure). The disturbance $\boldsymbol{\tau}_w$ and its first time derivative are bounded.

*Remark* 1. The vessel mass parameters are determined quite accurately using semi-empirical methods or software packages.[41,43] The damping effects are the results of several hydrodynamic phenomena, including potential damping, skin friction, wave drift damping, and the damping due to vortex shedding, which might make accurate modeling for hydrodynamic damping effects difficult.[41,43] Many practical disturbance terms, including wind disturbance, wave disturbance, ocean currents, are bounded and satisfy $\|\dot{\boldsymbol{\tau}}_w\| < \bar{\tau}_w$.[34,41,44]

## 2.2 | Control objectives

Let $\boldsymbol{\eta}_d = [x_d, y_d, \psi_d]^T$ denote the desired trajectory. Define the output tracking errors as follows:

$$\begin{aligned} e_1 &= x - x_d \\ e_2 &= y - y_d \\ e_3 &= \psi - \psi_d, \end{aligned} \tag{2}$$

where $e_1, e_2$ are the position tracking errors, and $e_3$ is the orientation tracking error. We consider the following time-varying constraint problem: the system output $\boldsymbol{\eta}$ is required to keep within a time-varying constrained distance from the reference trajectory $\boldsymbol{\eta}_d$. More specifically, the output tracking error $e_i$ needs to satisfy the following constrained condition:

$$-\underline{e}_i(t) < e_i(t) < \bar{e}_i(t), \quad \forall t > 0, \quad i = 1,\ 2,\ 3, \tag{3}$$

where $-\underline{e}_i(t)$ and $\bar{e}_i(t)$, respectively, are the predefined lower and upper bounds of the tracking error $e_i(t)$ with $\underline{e}_i(t) > 0$, $\bar{e}_i(t) > 0$, and $-\underline{e}_i(0) < e_i(0) < \bar{e}_i(0)$, and thus constraint (3) includes the equilibrium point at the origin. Constraint (3) is imposed to bind the distance of the vehicle outputs from the desired positions and orientation. This constraint problem is of great importance in practice because it is often desired to steer the vehicle to keep within an allowable distance from a reference trajectory, especially in narrow waterways or in riverine applications.

It is important to notice that the time-varying boundary functions of the tracking error constraint (3), which include time-invariant ones as a special class, allow us to quantify a prescribed performance of tracking errors on both transient and steady-state stages. For example, if the constraint boundary functions are taken as the prescribed performance function presented in the works of Bechlioulis et al,[13,21]

$$\begin{cases} \underline{e}_i(t) = \bar{e}_i(t) = \delta_i \rho_i(t), \ i = 1,\ 2,\ 3 \\ \rho_i(t) = (\rho_{i0} - \rho_{i\infty}) \exp(-\kappa_i t) + \rho_{i\infty}, \end{cases} \tag{4}$$

where $\rho_i(t)$ is called as a *performance function*,[13] and $\delta_i$, $\rho_{i0}$, $\rho_{i\infty}$, and $\kappa_i$ are positive design constants with $\rho_{i0} > \rho_{i\infty} \neq 0$ and $-\delta_i\rho_i(0) < e_i(0) < \delta_i\rho_i(0)$, then the output tracking error constraint (3) with Equation (4) could characterize prescribed transient and steady-state performances of tracking errors as follows. The prescribed performances are that (i) the decrease of tracking error $e_i$ is faster than the exponentially decaying function $\exp(-\kappa_i t)$; (ii) the maximum overshoot of the tracking error $e_i$ is $\delta_i\rho_{i0}$; and (iii) the steady-state tracking error is within the range $(-\delta_i\rho_{i\infty}, \delta_i\rho_{i\infty})$. Additionally, the transient performance with respect to the convergence rate and the maximum overshoot, and the steady-state performance with respect to the maximum allowable size of tracking error $e_i$ could be improved by adjusting the preselected parameters $\kappa_i$, $\delta_i$, $\rho_{i0}$, and $\rho_{i\infty}$.

**Assumption 2.** The desired trajectories $\boldsymbol{\eta}_d^{(j)}$ and the constraint boundary functions $\underline{e}_i^{(j)}$, $\bar{e}_i^{(j)}$ given in (3) exist and are known functions for all $j = 0, 1, 2$, where $\boldsymbol{\eta}_d^{(j)}$ denotes the $j$th time derivative of $\boldsymbol{\eta}_d$. At the initial time $t = 0$, the tracking errors $e_i(t)$ satisfy $-\underline{e}_i(0) < e_i(0) < \bar{e}_i(0)$, $i = 1, 2, 3$.

*Control objectives:* Under Assumptions 1 and 2, the objective of this paper is to design feedback control laws $\tau_u$, $\tau_r$ for underactuated marine vehicle (1) with nonzero off-diagonal terms in the system matrices such that

- the vehicle outputs $\boldsymbol{\eta} = [x, y, \psi]^T$ track the desired trajectory $\boldsymbol{\eta}_d$, and all the signals in the closed-loop system are uniformly ultimately bounded; and
- the tracking error constraint (3) is *never* violated.

## 2.3 | Inverse-error-transformation-functions and barrier Lyapunov functions

Define $\gamma_{ei}(t) = \underline{e}_i(t)/\overline{e}_i(t)$. To simplify the notation, the time $t$ is dropped sometimes without any confusion. To establish the relationship between the tracking error $e_i$ with its boundary functions $\underline{e}_i$, $\bar{e}_i$, motivated by the PPC methodology,[13] we introduce the following tracking error *transformation function*:

$$e_i = \overline{e}_i T_i(z_{1i}, \gamma_{ei}),\ i = 1,\ 2,\ 3, \tag{5}$$

where $z_{1i}$ is called as the *transformed error*,[13] and the transformation function $T_i(\cdot)$ is a *smooth and strictly increasing* function with respect to $z_{1i}$ and has the following properties:

$$\begin{cases} -\gamma_{ei} < T_i(z_{1i}, \gamma_{ei}) < 1, \quad \forall z_{1i} \in \mathcal{L}_\infty \\ \lim_{z_{1i}\to-\infty} T_i(z_{1i}, \gamma_{ei}) = -\gamma_{ei} \\ \lim_{z_{1i}\to+\infty} T_i(z_{1i}, \gamma_{ei}) = 1 \\ T_i(z_{1i}, \gamma_{ei}) = 0, \qquad \text{iff } z_{1i} = 0, \end{cases} \tag{6}$$

which implies that $\frac{\partial T_i(\cdot)}{\partial z_{1i}} > 0$, and thus, the inverse function of $T_i(\cdot)$ with respect to $z_{1i}$ exists. From (5) and (6), we have

$$z_{1i} = T_i^{-1}\left(\frac{e_i}{\overline{e}_i}, \gamma_{ei}\right),\ i = 1,\ 2,\ 3, \tag{7}$$

where $T_i^{-1}(\cdot)$ is the inverse function of $T_i(\cdot)$.

*Remark* 2. The definition of the transformation function $T_i(\cdot)$ was introduced originally in the work of Bechlioulis and Rovithakis.[13] In condition (6), we further extend the properties of the transformation function $T_i(\cdot)$ in the works of Bechlioulis et al[13,21] by imposing the requirement: $T_i(z_{1i}, \gamma_{ei}) = 0$, if and only if (iff) $z_{1i} = 0$. This property is of great importance, which guarantees that asymptotic convergence of the tracking error $e_i$ to the origin if $z_{1i}$ converges to an equilibrium point at the origin (please see Remark 3 for a more detailed discussion). Additionally, the constraint boundary functions $\underline{e}_i$ and $\overline{e}_i$ in the aforementioned works[13,21] are highly correlated and are required to stratify $\gamma_{ei} = \frac{\underline{e}_i}{\overline{e}_i} = 1$, eg, condition (4), or $\gamma_{ei} =$ constant, whereas the lower and upper bounds in constraint (3) could be any smooth time-varying functions and thus the variable $\gamma_{ei}$ is also time-varying.

*Remark* 3. It is interesting to notice that the inverse function $T_i^{-1}(\cdot)$ given in (7) is a *barrier* function, which possesses the property of finite escape whenever the argument $\frac{e_i}{\overline{e}_i}$ approaches the boundary of the open set $(-\gamma_{ei}, 1)$ containing the origin, ie, $z_{1i} = T_i^{-1}\left(\frac{e_i}{\overline{e}_i}, \gamma_{ei}\right) \to \infty$ as $\frac{e_i}{\overline{e}_i} \to 1$, and $z_{1i} = T_i^{-1}\left(\frac{e_i}{\overline{e}_i}, \gamma_{ei}\right) \to -\infty$ as $\frac{e_i}{\overline{e}_i} \to -\gamma_{ei}$. Therefore, the *barrier* function $T_i^{-1}(\cdot)$ could be used to guarantee the constraint (3) is not violated.

In order to ensure that, if the barrier function $T_i^{-1}(\cdot)$ converges, then it converges to an equilibrium point at the origin, we impose the transformation function $T_i(\cdot)$ in (6) to satisfy $T_i(0, \gamma_{ei}) = 0$, ie, $T_i^{-1}(0, \gamma_{ei}) = 0$. It should be noticed that the barrier function $z_{1i} = T_i^{-1}(\cdot)$ is not positive definite. In order to obtain an everywhere nonnegative function that allows Lyapunov-based analysis and control synthesis to be developed, we define the quadratic function of barrier function $T_i^{-1}(\cdot)$ in the following form:

$$V_0 = \frac{1}{2} z_{1i}^2 = \frac{1}{2}\left[T_i^{-1}\left(\frac{e_i}{\overline{e}_i}, \gamma_{ei}\right)\right]^2. \tag{8}$$

It is clear that the function $V_0$ is continuously differentiable, positive semidefinite on the open set $(-\gamma_{ei}, 1)$ containing the origin, and thus it could be considered as a Lyapunov function candidate. By guaranteeing boundedness of barrier Lyapunov function $V_0$ in (8) along system trajectories, we could show that the constraint (3) is *never* violated, and this is a key basis of our control design methodology.

Inspiring in part by the *hyperbolic-tangent-type* function,[13,21] in this paper, we could take the transformation function $T_i(\cdot)$ as

$$T_i(z_{1i}, \gamma_{ei}) = \frac{e^{z_{1i}} - e^{-z_{1i}}}{e^{z_{1i}} + \gamma_{ei}^{-1} e^{-z_{1i}}}. \tag{9}$$

It is easily verified that $T_i(\cdot)$ in (9) satisfies condition (6), whose inverse function is given by

$$z_{1i} = T_i^{-1}\left(\frac{e_i}{\bar{e}_i}, \gamma_{ei}\right) = \frac{1}{2}\ln\left(\frac{\underline{e}_i + e_i}{\gamma_{ei}(\bar{e}_i - e_i)}\right), \tag{10}$$

where $\ln(\cdot)$ denotes the natural logarithm. For the symmetric constraint case, eg, prescribed performance condition (4), it follows $\gamma_{ei} = \frac{\underline{e}_i}{\bar{e}_i} = 1$, and thus the transformation function $T_i(\cdot)$ could be taken as a hyperbolic tangent function,[13,21] which has the form

$$T(z_{1i}) = \tanh(z_{1i}) = \frac{e^{z_{1i}} - e^{-z_{1i}}}{e^{z_{1i}} + e^{-z_{1i}}}.$$

## 3 | CONTROL DESIGN FOR KNOWN VEHICLE MODEL

In this section, we consider the case where the hydrodynamic damping terms $d_{11}$, $d_{22}$, $d_{23}$, $d_{32}$, $d_{33}$ are known. Under tracking error constraint (3), we propose a constructive design technique of model-based tracking controllers that force system (1) to follow the desired trajectory $\boldsymbol{\eta}_d$. The control design is based on transverse function control approach, disturbance observers, backstepping procedure, barrier functions, and Lyapunov synthesis.

Expanding system (1) yields

$$\begin{aligned}
\dot{x} &= u\cos(\psi) - \upsilon\sin(\psi) \\
\dot{y} &= u\sin(\psi) + \upsilon\cos(\psi) \\
\dot{\psi} &= r \\
\dot{u} &= \phi_u - f_u + \frac{1}{m_{11}}\tau_u + \bar{\tau}_{wu} \\
\dot{\upsilon} &= \phi_\upsilon - f_\upsilon - \frac{m_{23}}{\bar{m}_{33}}\tau_r + \bar{\tau}_{w\upsilon} \\
\dot{r} &= \phi_r - f_r + \frac{m_{22}}{\bar{m}_{33}}\tau_r + \bar{\tau}_{wr},
\end{aligned} \tag{11}$$

where $\phi_u = \frac{m_{22}}{m_{11}}\upsilon r + \frac{m_{23}}{m_{11}}r^2$, $\phi_\upsilon = \frac{\bar{m}_{12}}{\bar{m}_{33}}ur + \frac{\bar{m}_{13}}{\bar{m}_{33}}u\upsilon$, $\phi_r = \frac{\bar{m}_{22}}{\bar{m}_{33}}u\upsilon + \frac{\bar{m}_{23}}{\bar{m}_{33}}ur$, $\bar{m}_{12} = -m_{11}m_{33} + m_{23}^2$, $\bar{m}_{13} = m_{22}m_{23} - m_{11}m_{23}$, $\bar{m}_{22} = m_{11}m_{22} - m_{22}^2$, $\bar{m}_{23} = m_{11}m_{23} - m_{22}m_{23}$, $\bar{m}_{33} = m_{22}m_{33} - m_{23}^2$, $\bar{\tau}_{wu} = \tau_{wu}/m_{11}$, $\bar{\tau}_{w\upsilon} = (m_{33}\tau_{w\upsilon} - m_{23}\tau_{wr})/\bar{m}_{33}$, $\bar{\tau}_{wr} = (m_{22}\tau_{wr} - m_{23}\tau_{w\upsilon})/\bar{m}_{33}$, and

$$\begin{aligned}
f_u &= \frac{1}{m_{11}}d_{11}(u)u \\
f_\upsilon &= -\frac{1}{\bar{m}_{33}}(m_{23}d_{32}\upsilon - m_{33}d_{22}\upsilon + m_{23}d_{33}r - m_{33}d_{23}r) \\
f_r &= -\frac{1}{\bar{m}_{33}}(m_{23}d_{22}\upsilon - m_{22}d_{32}\upsilon - m_{22}d_{33}r + m_{23}d_{23}r),
\end{aligned} \tag{12}$$

in which $f_u$, $f_\upsilon$, and $f_r$ are hydrodynamic damping effects.

For system (11), we will employ control Lyapunov synthesis and backstepping procedure to design control laws $\tau_u$ and $\tau_r$ to achieve our control objective. The control design procedure includes two steps. In the first step, the first three equations of system (11) are designed by viewing $u$, $\upsilon$ as virtual control inputs for the $(x, y)$-subsystem, and by viewing $r$ as a virtual control input for the $\psi$-subsystem. In the second step, the actual control inputs $\tau_u$ and $\tau_r$ are designed to stabilize the $(u, \upsilon)$-subsystem and $r$-subsystem at the origin. Using such design procedure, the control objective would be achieved if system (1) was fully actuated meaning that there are three independent control inputs in $(u, \upsilon, r)$-subsystem. However, system (1) does not have an independent actuator in the sway motion. Additionally, the vehicle inertia matrix $\mathbf{M}$ is not assumed to be diagonal and thus the yaw moment control $\tau_r$ acts directly on the sway and yaw motions, which makes the stabilization of both $\upsilon$-subsystem and $r$-subsystem difficult. To tackle this difficulty, we develop the transverse function approaches presented in the works of Morin and Samson[7,8] to introduce an additional control input in the second step of the backstepping design. Define the following error coordinate transformations:

$$\begin{aligned} z_{21} &= u - \alpha_1 \\ z_{22} &= v - \alpha_2 - h_1(\beta) \\ z_{23} &= r - \alpha_3 - h_2(\beta), \end{aligned} \tag{13}$$

where $\alpha_i$, $i = 1, 2, 3$ are virtual control inputs, $h_1(\beta)$ and $h_2(\beta)$ are bounded differentiable functions with respect to $\beta$ for all $\beta \in \mathbb{R}$, and $h_i(\beta)$ $(i = 1, 2)$ are called as *transverse functions*,[7-9] and $\dot{\beta}$ is called as the *additional control input*.[7-9] The transverse function $h_i(\beta)$ will be further defined and computed in (30) and (31). The additional control input $\dot{\beta}$ will be specified in (39).

*Remark* 4. For a vehicle equipped with a rudder, the yaw moment control $\tau_r$ directly enters the sway dynamics due to nonzero off-diagonal terms in the system matrices. To avoid that the yaw moment control $\tau_r$ enters the sway dynamics, two methods are usually applied in the literature of motion control of underactuated vehicles. The first is to assume that the off-diagonal terms of system matrices is zero, eg,[6,21-24,35,46,47] so that the coupling terms between the sway dynamics and the yaw dynamics are avoided. Another commonly used method is the use of coordinate transformations (changing the vehicle positions), eg, other works,[27,28,48] to change the vehicle dynamic model into a form without off-diagonal terms. Consequently, the yaw moment control $\tau_r$ acts only on the yaw dynamics and there is no sway force acting on the sway dynamics. The vehicle's center of oscillation is controlled, through changing the vehicle positions, rather than controlling the vehicle's center of gravity. Since there is no control input evolved in the sway dynamic equation, in general, either the sway velocity is assumed to be passive-bounded,[22,49,50] or the stability analysis for the sway dynamics is implemented separately,[27-29,51] where the stability analysis usually requires exact information on the hydrodynamic damping term. Different from the existing methods, in this paper, we expand directly system (1) without changing the vehicle positions and obtain system (11), where we allow the yaw moment control $\tau_r$ to act directly on the sway dynamics and the yaw dynamics as well. We present the transverse function approach[7,8] to introduce an additional control input $\dot{\beta}$ in the sway and yaw error system (27). By designing the control input $[\tau_r, \dot{\beta}]^T$ for stabilizing $\mathbf{z}_3$-system (27) at the origin, the stability analysis for the sway dynamics is not required to prove separately. It is worth pointing out that the present feedback controllers do not require another propeller to realize the additional control input $\dot{\beta}^*$ because the actual control input $\tau_r^*$ in (38) is affected greatly by designing the matrix $Q^{-1}$ given in (32). That is, it is clear that $\tau_r^*$ in (38) is influenced by $\beta^*$ due to $\bar{h}_0^* = \varepsilon_2 \arctan(\beta^*)$.

*Remark* 5. The transverse function control approaches have been effectively applied in solving the trajectory tracking control problem for underactuated vehicles,[9,27-29] where three transverse functions were introduced in the tracking errors $e_i$, $i = 1, 2, 3$, in the first step of backstepping procedure, and thus they causes the difficulty of guaranteeing prescribed performance of tracking errors because any large transverse function might break the tracking error constraints. Additionally, the stability of the sway dynamics is proved separately in other works[9,27-29] and the stability analysis usually requires exact information on the hydrodynamic damping term (please see Remark 4 for a detailed discussion). Different from the existing transverse function approaches,[9,27-29] we introduce two transverse functions $h_1(\beta)$ and $h_2(\beta)$ in the second step of backstepping procedure, and thus the two additional functions do not present in the tracking error equations $e_i$, see the error coordinate transformation (13). Accordingly, we give a specific calculation for the two additional functions, see equation (30). Furthermore, we develop the transverse function approach for designing adaptive control that provides robust performance with respect to model uncertainties and disturbances.

*Step 1:* Under Assumption 2, the derivatives of $e_i, i = 1, 2, 3$ in (2) along system (11) are

$$\dot{e}_1 = u\cos(\psi) - v\sin(\psi) - \dot{x}_d \tag{14}$$

$$\dot{e}_2 = u\sin(\psi) + v\cos(\psi) - \dot{y}_d \tag{15}$$

$$\dot{e}_3 = r - \dot{\psi}_d. \tag{16}$$

To design virtual control inputs $\alpha_i, i = 1, 2, 3$ that guarantee constraint (3) is not violated, we choose the barrier Lyapunov function candidate in the form (8) with Equation (10). Consider the following *logarithm* Lyapunov function candidate:

$$V_1 = \frac{1}{2}\sum_{i=1}^{3} z_{1i}^2 = \frac{1}{2}\sum_{i=1}^{3}\left[\frac{1}{2}\ln\left(\frac{\underline{e}_i + e_i}{\gamma_{ei}(\overline{e}_i - e_i)}\right)\right]^2, \tag{17}$$

whose derivative along systems (13)-(16) yields

$$\begin{aligned}\dot{V}_1 = \sum_{i=1}^{3} z_{1i}(p_i\dot{e}_i + l_i) = {} & z_{11}[p_1(\alpha_1\cos(\psi) - \alpha_2\sin(\psi) - \dot{x}_d) + l_1] \\ & + z_{12}[p_2(\alpha_1\sin(\psi) + \alpha_2\cos(\psi) - \dot{y}_d) + l_2] \\ & + z_{21}(z_{11}p_1\cos(\psi) + z_{12}p_2\sin(\psi)) - z_{11}p_1h_1\sin(\psi) \\ & + z_{22}(-z_{11}p_1\sin(\psi) + z_{12}p_2\cos(\psi)) + z_{12}p_2h_1\cos(\psi) \\ & + z_{13}[p_3(\alpha_3 - \dot{\psi}_d) + l_3] + z_{23}z_{13}p_3 + z_{13}p_3h_2, \end{aligned} \tag{18}$$

where $p_i$ and $l_i$, $i = 1, 2, 3$ are given by

$$p_i = \frac{1}{2}\left[\frac{1}{\underline{e}_i + e_i} + \frac{1}{\bar{e}_i - e_i}\right] \tag{19}$$

$$l_i = \frac{1}{2}\left[\frac{\dot{\underline{e}}_i}{\underline{e}_i + e_i} - \frac{\dot{\bar{e}}_i}{\bar{e}_i - e_i} - \frac{\dot{\gamma}_{ei}}{\gamma_{ei}}\right], \tag{20}$$

in which $p_i$ and $l_i$ are available for feedback control design. Substituting (5) into (19) yields

$$p_i = \frac{1}{2\bar{e}_i}\left[\frac{1}{\gamma_{ei} + T_i(z_{1i}, \gamma_{ei})} + \frac{1}{1 - T_i(z_{1i}, \gamma_{ei})}\right]. \tag{21}$$

From the definition of $T_i(\cdot)$ in (6), it follows that $p_i$ is bounded and $p_i > 0$. Consider the following virtual control:

$$\alpha_1 = \cos(\psi)\Psi_1 + \sin(\psi)\Psi_2 \tag{22}$$

$$\alpha_2 = -\sin(\psi)\Psi_1 + \cos(\psi)\Psi_2 \tag{23}$$

$$\alpha_3 = p_3^{-1}(-k_{13}z_{13} - l_3) + \dot{\psi}_d, \tag{24}$$

where $\Psi_1 = p_1^{-1}(-k_{11}z_{11} - l_1) + \dot{x}_d$, $\Psi_2 = p_2^{-1}(-k_{12}z_{12} - l_2) + \dot{y}_d$ with $k_{1i}$, $i = 1, 2, 3$ being positive design parameters to be specified later. Then, substituting virtual control laws (22)-(24) into Equation (18) gives

$$\begin{aligned}\dot{V}_1 = {} & -k_{11}z_{11}^2 - k_{12}z_{12}^2 - k_{13}z_{13}^2 + z_{21}(z_{11}p_1\cos(\psi) + z_{12}p_2\sin(\psi)) \\ & - z_{11}p_1h_1\sin(\psi) + z_{12}p_2h_1\cos(\psi) + z_{22}(-z_{11}p_1\sin(\psi) + z_{12}p_2\cos(\psi)) \\ & + z_{23}z_{13}p_3 + z_{13}p_3h_2. \end{aligned} \tag{25}$$

*Step 2:* In this step, we will design the actual control inputs $\tau_u$, $\tau_r$, and additional control $\dot{\beta}$. For system (11), differentiating both sides of (13) with Assumption 2 gives

$$\dot{z}_{21} = \phi_u - f_u + \frac{1}{m_{11}}\tau_u - \dot{\alpha}_1 + \bar{\tau}_{wu} \tag{26}$$

$$\dot{\mathbf{z}}_3 = \mathbf{\Phi}_{vr} - \mathbf{F}_{vr} + \mathbf{Q}\begin{bmatrix}\tau_r \\ \dot{\beta}\end{bmatrix} - \dot{\boldsymbol{\alpha}}_{23} + \boldsymbol{\tau}_{wvr}, \tag{27}$$

where $\mathbf{z}_3 = [z_{22}, z_{23}]^T$, $\Phi_{vr} = [\phi_v, \phi_r]^T$, $\mathbf{F}_{vr} = [f_v, f_r]^T$, $\dot{\boldsymbol{\alpha}}_{23} = [\dot{\alpha}_2, \dot{\alpha}_3]^T$, $\boldsymbol{\tau}_{wvr} = [\bar{\tau}_{wv}, \bar{\tau}_{wr}]^T$,

$$\mathbf{Q} = \begin{bmatrix} -\frac{m_{23}}{\bar{m}_{33}} & -\frac{\partial h_1}{\partial \beta} \\ \frac{m_{22}}{\bar{m}_{33}} & -\frac{\partial h_2}{\partial \beta} \end{bmatrix}, \tag{28}$$

and $\dot{\alpha}_1$, $\dot{\boldsymbol{\alpha}}_{23}$ are available for controller implementation because every term in the first time derivative of $\alpha_i$ in (22)-(24) are computable with $\dot{p}_i^{-1} = -\dot{p}_i/p_i^2$, $i = 1, 2, 3$ and $p_i$ defined in (19). It is clear for system (27) that the designed control laws are closely related to the selection of $\mathbf{Q}$ according to feedback linearization technique. Consequently, we will give systematic method to construct the invertible matrix $\mathbf{Q}$ and the transverse function $h_i(\beta)$ $(i = 1, 2)$. It follows from (28) the determinant of matrix $\mathbf{Q}$ is

$$\det(\mathbf{Q}) = \frac{1}{\bar{m}_{33}}\left(m_{23}\frac{\partial h_2}{\partial \beta} + m_{22}\frac{\partial h_1}{\partial \beta}\right), \tag{29}$$

where the functions $\frac{\partial h_1}{\partial \beta}$ and $\frac{\partial h_2}{\partial \beta}$ are required to make the matrix $\mathbf{Q}$ invertible. To satisfy the nonzero of $\det(\mathbf{Q})$ for all $\beta \in \mathbb{R}$, we could consider $\frac{\partial h_1}{\partial \beta} = c\frac{\cos^2(\beta)}{m_{22}}$, $\frac{\partial h_2}{\partial \beta} = c\frac{\sin^2(\beta)}{m_{23}}$ with constant $c > 0$, which yields $h_1 = \frac{c}{2m_{22}}(\beta + \sin(\beta)\cos(\beta))$,

$h_2 = \frac{c}{2m_{23}}(\beta - \sin(\beta)\cos(\beta))$, $\det(\mathbf{Q}) = \frac{c}{\bar{m}_{33}} > 0$. However, the boundedness of $h_1$ and $h_2$ can not be guaranteed as $\beta$ tends to infinity. To ensure the boundedness of $h_1$ and $h_2$ for all $\beta \in \mathbb{R}$, we would employ a differentiable bounded function $\bar{h}_0(\beta)$, for example, to replace $\beta$ in the functions $h_1$ and $h_2$, whose derivatives give $\frac{\partial h_1}{\partial \beta} = c\frac{\cos^2(\bar{h}_0)}{m_{22}}\frac{\partial \bar{h}_0}{\partial \beta}$, $\frac{\partial h_2}{\partial \beta} = c\frac{\sin^2(\bar{h}_0)}{m_{23}}\frac{\partial \bar{h}_0}{\partial \beta}$, and then we have $\det(\mathbf{Q}) = \frac{c}{\bar{m}_{33}}\frac{\partial \bar{h}_0}{\partial \beta}$. Thus, the matrix $\mathbf{Q}$ is invertible if $\frac{\partial \bar{h}_0}{\partial \beta} \neq 0$ holds for all $\beta \in \mathbb{R}$. For instance, $\bar{h}_0(\beta)$ could be taken as $\arctan(\beta)$, $\tanh(\beta)$, etc. Therefore, an example of the calculations of transverse functions $h_i$ $(i = 1, 2)$ are given by

$$\begin{aligned} h_1 &= \frac{\varepsilon_1}{2m_{22}}(\bar{h}_0 + \sin(\bar{h}_0)\cos(\bar{h}_0)) \\ h_2 &= \frac{\varepsilon_1}{2m_{23}}(\bar{h}_0 - \sin(\bar{h}_0)\cos(\bar{h}_0)), \end{aligned} \tag{30}$$

where $\bar{h}_0 = \varepsilon_2 \arctan(\beta)$. Then, we have

$$\begin{aligned} \frac{\partial h_1}{\partial \beta} &= \frac{\varepsilon_1\varepsilon_2\cos^2(\bar{h}_0)}{m_{22}(1+\beta^2)} \\ \frac{\partial h_2}{\partial \beta} &= \frac{\varepsilon_1\varepsilon_2\sin^2(\bar{h}_0)}{m_{23}(1+\beta^2)} \\ \det(\mathbf{Q}) &= \frac{\varepsilon_1\varepsilon_2}{\bar{m}_{33}(1+\beta^2)} > 0, \end{aligned} \tag{31}$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are design constants. Note that $\det(\mathbf{Q}) \neq 0$, $\forall \beta \in \mathbb{R}$, $\mathbf{Q} = \begin{bmatrix}\mathbf{X} & \frac{\partial \mathbf{h}}{\partial \beta}\end{bmatrix}$, $\mathbf{X} = \begin{bmatrix}-\frac{m_{23}}{\bar{m}_{33}} & \frac{m_{22}}{\bar{m}_{33}}\end{bmatrix}^T$, $\frac{\partial \mathbf{h}}{\partial \beta} = \begin{bmatrix}-\frac{\partial h_1}{\partial \beta} & -\frac{\partial h_2}{\partial \beta}\end{bmatrix}^T$, and then the vector $\mathbf{X}$ and $\frac{\partial \mathbf{h}}{\partial \beta}$ form a basis of $\mathbb{R}^2$. That is, the gradient of $\mathbf{h}(\beta) = \begin{bmatrix}-h_1(\beta) - h_2(\beta)\end{bmatrix}^T$ is *transversal* to the direction given by vector $\mathbf{X}$. Therefore, the bounded function $\mathbf{h}(\beta)$ is said to be a *transverse function*.[7] It is worth pointing out that the calculation of such transverse function is not unique.

From (28) and (31), we have

$$\mathbf{Q}^{-1} = \begin{bmatrix} -\frac{\bar{m}_{33}\sin^2(\bar{h}_0)}{m_{23}} & \frac{\bar{m}_{33}\cos^2(\bar{h}_0)}{m_{22}} \\ -\frac{m_{22}(1+\beta^2)}{\varepsilon_1\varepsilon_2} & -\frac{m_{23}(1+\beta^2)}{\varepsilon_1\varepsilon_2} \end{bmatrix}. \tag{32}$$

*Remark* 6. From Equations (30) and (13), it is clear that decreasing the design parameters $\varepsilon_1$ and $\varepsilon_2$ could reduce the velocity errors $z_{22}$ and $z_{23}$ significantly. However, we do not suggest the use of very small parameters $\varepsilon_1$ and $\varepsilon_2$ in practical applications because this might result in a very small $\det(\mathbf{Q})$ in Equation (31) and high-gain control inputs with powerful actuators may require duly. Therefore, the design parameters $\varepsilon_1$ and $\varepsilon_2$ should be adjusted carefully in practical applications for achieving suitable transient performance and control inputs.

To estimate the unknown disturbance $\bar{\tau}_{wu}$ in system (26), we could design the disturbance observer as follows:

$$\begin{cases} \dot{\xi}_1 &= z_{21} - k_{d1}\left(\phi_u - f_u + \frac{1}{m_{11}}\tau_u - \dot{\alpha}_1 + \hat{\tau}_{wu}\right) \\ \hat{\tau}_{wu} &= \xi_1 + k_{d1}z_{21}, \end{cases} \tag{33}$$

where $\xi_1$ is the observer state, $\hat{\tau}_{wu}$ is the estimate of $\bar{\tau}_{wu}$, and $k_{d1} > 0$ is a design parameter. From systems (26) and (33), we have the following observer error dynamics:

$$\dot{\tilde{\tau}}_{wu} = z_{21} - k_{d1}\tilde{\tau}_{wu} - \dot{\bar{\tau}}_{wu}, \tag{34}$$

where the observer error $\tilde{\tau}_{wu} = \hat{\tau}_{wu} - \bar{\tau}_{wu}$. For system (27), the disturbance observer is taken as

$$\begin{cases} \dot{\boldsymbol{\xi}}_2 &= \mathbf{z}_3 - \mathbf{K}_{d2}\left\{\Phi_{vr} - \mathbf{F}_{vr} + \mathbf{Q}\begin{bmatrix}\tau_r \\ \dot{\beta}\end{bmatrix} - \dot{\boldsymbol{\alpha}}_{23}\right\} - \mathbf{K}_{d2}\hat{\boldsymbol{\tau}}_{wvr} \\ \hat{\boldsymbol{\tau}}_{wvr} &= \boldsymbol{\xi}_2 + \mathbf{K}_{d2}\mathbf{z}_3, \end{cases} \tag{35}$$

which yields the observer error dynamics

$$\dot{\tilde{\boldsymbol{\tau}}}_{wvr} = \mathbf{z}_3 - \mathbf{K}_{d2}\tilde{\boldsymbol{\tau}}_{wvr} - \dot{\boldsymbol{\tau}}_{wvr}, \tag{36}$$

where $\mathbf{K}_{d2}^T = \mathbf{K}_{d2} > 0$ is a design parameter and $\tilde{\boldsymbol{\tau}}_{wvr} = \hat{\boldsymbol{\tau}}_{wvr} - \boldsymbol{\tau}_{wvr}$. Based on disturbance observers (33) and (35), feedback control laws $\tau_u^*$, $\tau_r^*$, and additional control $\dot{\beta}^*$ are given by

$$\tau_u^* = m_{11}(-k_{31}z_{21} - \phi_u + f_u - \hat{\tau}_{wu} + \dot{\alpha}_1 - z_{11}p_1\cos\psi - z_{12}p_2\sin\psi) \tag{37}$$

$$\tau_r^* = -\frac{\bar{m}_{33}\sin^2\left(\bar{h}_0^*\right)}{m_{23}}\bar{z}_{22}^* + \frac{\bar{m}_{33}\cos^2\left(\bar{h}_0^*\right)}{m_{22}}\bar{z}_{31}^* \tag{38}$$

$$\dot{\beta}^* = -\frac{m_{22}(1+\beta^{*2})}{\varepsilon_1\varepsilon_2}\bar{z}_{22}^* - \frac{m_{23}(1+\beta^{*2})}{\varepsilon_1\varepsilon_2}\bar{z}_{31}^*, \tag{39}$$

where $\bar{h}_0^* = \varepsilon_2 \arctan(\beta^*)$, $\bar{z}_{22}^* = -k_{21}z_{22} - \phi_v + f_v + \dot{\alpha}_2 - \hat{\tau}_{wv} + z_{11}p_1 \sin\psi - z_{12}p_2 \cos\psi$, $\bar{z}_{31}^* = -k_{22}z_{23} - \phi_r + f_r + \dot{\alpha}_3 - \hat{\tau}_{wr} - z_{13}p_3$ with $k_{31} > 0$, $k_{21} > 0$, and $k_{22} > 0$ being design parameters to be specified later. Let $\mathbf{K}_2 = \text{diag}[k_{21}, k_{22}]$. Substituting control laws (37), (38), additional control $\dot{\beta}^*$ (39) into systems (26) and (27) yields the following closed-loop error system:

$$\dot{z}_{21} = -k_{31}z_{21} - \tilde{\tau}_{wu} - z_{11}p_1\cos\psi - z_{12}p_2\sin\psi \tag{40}$$

$$\dot{\mathbf{z}}_3 = -\mathbf{K}_2\mathbf{z}_3 - \tilde{\boldsymbol{\tau}}_{wvr} + \begin{bmatrix} z_{11}p_1\sin\psi - z_{12}p_2\cos\psi \\ -z_{13}p_3 \end{bmatrix}. \tag{41}$$

Consider the following Lyapunov function candidate

$$V_2^* = V_1 + \frac{1}{2}z_{21}^2 + \frac{1}{2}\mathbf{z}_3^T\mathbf{z}_3 + \frac{1}{2}\tilde{\tau}_{wu}^2 + \frac{1}{2}\tilde{\boldsymbol{\tau}}_{wvr}^T\tilde{\boldsymbol{\tau}}_{wvr}, \tag{42}$$

whose derivative along systems (40), (41), (34), (36), (25) is

$$\begin{aligned}\dot{V}_2^* = &-k_{11}z_{11}^2 - k_{12}z_{12}^2 - k_{13}z_{13}^2 - k_{31}z_{21}^2 - \mathbf{z}_3^T\mathbf{K}_2\mathbf{z}_3 \\ &- k_{d1}\tilde{\tau}_{wu}^2 - z_{11}p_1h_1\sin(\psi) + z_{12}p_2h_1\cos(\psi) \\ &+ z_{13}p_3h_2 - \tilde{\boldsymbol{\tau}}_{wvr}^T\mathbf{K}_{d2}\tilde{\boldsymbol{\tau}}_{wvr} - \tilde{\tau}_{wu}\dot{\tau}_{wu} - \tilde{\boldsymbol{\tau}}_{wvr}^T\dot{\boldsymbol{\tau}}_{wvr}.\end{aligned} \tag{43}$$

By completion of squares, we have

$$\begin{aligned}-z_{11}p_1h_1\sin(\psi) &\leq \frac{\kappa_1}{2}z_{11}^2 + \frac{\bar{p}_1^2}{2\kappa_1}\left(\frac{\varepsilon_1\varepsilon_2\pi}{4m_{22}}\right)^2 \\ z_{12}p_2h_1\cos(\psi) &\leq \frac{\kappa_1}{2}z_{12}^2 + \frac{\bar{p}_2^2}{2\kappa_1}\left(\frac{\varepsilon_1\varepsilon_2\pi}{4m_{22}}\right)^2 \\ z_{13}p_3h_2(\beta) &\leq \frac{\kappa_1}{2}z_{13}^2 + \frac{\bar{p}_3^2}{2\kappa_1}\left(\frac{\varepsilon_1\varepsilon_2\pi}{4m_{23}}\right)^2 \\ -\tilde{\tau}_{wu}\dot{\tau}_{wu} &\leq \frac{\kappa_1\tilde{\tau}_{wu}^2}{2} + \frac{\bar{\bar{\tau}}_{wu}^2}{2\kappa_1} \\ -\tilde{\boldsymbol{\tau}}_{wvr}^T\dot{\boldsymbol{\tau}}_{wvr} &\leq \frac{\kappa_1\tilde{\boldsymbol{\tau}}_{wvr}^T\tilde{\boldsymbol{\tau}}_{wvr}}{2} + \frac{\bar{\tau}_{wvr}^2}{2\kappa_1},\end{aligned}$$

with constants $\kappa_1 > 0$, where $\bar{\bar{\tau}}_{wu}, \bar{\tau}_{wvr}, \bar{p}_i\ (i = 1,2,3)$ are positive constants denoting the bounds of $\dot{\tau}_{wu}, \dot{\boldsymbol{\tau}}_{wvr}, p_i$ according to Assumption 1, Equation (21). Notice that we only require the existence of the bounds of $\dot{\tau}_{wu}$, $\dot{\boldsymbol{\tau}}_{wvr}$, and $p_i$, without the need for explicit knowledge of these bounds. Thus, we have the following inequality:

$$\dot{V}_2^* \leq -\rho_1^*V_2^* + \delta_1^*, \tag{44}$$

where

$$\delta_1^* = \frac{(\varepsilon_1\varepsilon_2\pi)^2}{32\kappa_1}\left(\frac{\bar{p}_1^2 + \bar{p}_2^2}{m_{22}^2} + \frac{\bar{p}_3^2}{m_{23}^2}\right) + \frac{\bar{\bar{\tau}}_{wu}^2 + \bar{\tau}_{wvr}^2}{2\kappa_1} \tag{45}$$

$$\rho_1^* = \min\{2k_{11} - \kappa_1, 2k_{12} - \kappa_1, 2k_{13} - \kappa_1, 2\lambda_{\min}(\mathbf{K}_2),\ 2k_{31}, 2k_{d1} - \kappa_1, \lambda_{\min}(2\mathbf{K}_{d2} - \kappa_1 I_2)\}. \tag{46}$$

Next, we present the proposed model-based control such that the stability and the transient and steady-state performances of the closed-loop system can be guaranteed.

**Theorem 1.** *Under Assumptions 1 and 2, consider underarcuated marine vehicle (1) whose model dynamics are available for control design, tracking control laws $\tau_u^*$, $\tau_r^*$ given in (37), (38), and disturbance observers (33), (35). If the design parameters are chosen appropriately such that*

$$\begin{aligned}&2k_{11} - \kappa_1,\ \ 2k_{12} - \kappa_1,\ \ 2k_{13} - \kappa_1 > 0,\ \ \mathbf{K}_2 > 0,\ \ k_{31} > 0, \\ &2k_{d1} - \kappa_1 > 0,\ \ 2\mathbf{K}_{d2} - \kappa_1 I_2 > 0\end{aligned} \tag{47}$$

*with constant $\kappa_1 > 0$, then, for any initial condition satisfying condition (3), we have*

i. *all the signals in the closed-loop system remain bounded;*
ii. *the tracking errors always evolve within the predefined time-varying asymmetric bounds, ie, $-\underline{e}_i(t) < e_i(t) < \overline{e}_i(t)$, $\forall t > 0$, $i = 1, 2, 3$, and especially, the prescribed transient and steady-state performances of the tracking errors in the sense of (3) and (4) are guaranteed; and*
iii. *the tracking errors $e_i$, $i = 1, 2, 3$ and disturbance observer errors $\tilde{\tau}_{wu}$, $\tilde{\tau}_{wv}$, $\tilde{\tau}_{wr}$ converge exponentially to a small neighborhood of zero, whose size is adjustable by tuning the design parameters $\varepsilon_1$, $\varepsilon_2$, and $\kappa_1$.*

*Proof.* See Appendix. □

## 4 | CONTROL DESIGN FOR MODEL UNCERTAINTIES

In this section, we develop approximation-based control design technique for system (1) in presences of unmodeled dynamic uncertainties and time-varying disturbances. In Section 3, we assume that hydrodynamic damping effects are accurately known a priori. In maritime environments, however, the exact knowledge of the hydrodynamic damping might be not available for feedback control design. Subsequently, we make the following assumption on system (1).

**Assumption 3.** Assume that the hydrodynamic damping effects $f_u$, $f_v$, and $f_r$ given in (12) are unmodeled dynamic uncertainties.

Thanks to the universal approximation abilities of radial basis function (RBF) NNs,[52,53] the continuous functions $f_u$, $f_v$, and $f_r$ could be approximated to any accuracy and could be represented as follows:

$$\begin{aligned} f_u(Z_1) &= W_1^{*T} S_1(Z_1) + \epsilon_1(Z_1) \\ f_v(Z_2) &= W_2^{*T} S_2(Z_2) + \epsilon_2(Z_2) \\ f_r(Z_2) &= W_3^{*T} S_3(Z_2) + \epsilon_3(Z_2), \end{aligned} \tag{48}$$

where $Z_1 = u \in \Omega_{Z_1} \subset \mathbb{R}^1$ and $Z_2 = [v, r]^T \in \Omega_{Z_2} \subset \mathbb{R}^2$ are the NN input vectors, and $\Omega_{Z_1}$ and $\Omega_{Z_2}$ denote compact sets; $W_i^*$, $i = 1, 2, 3$, are the true/optimal constant weight vectors; $\epsilon_i(\cdot)$ are approximation errors, and $|\epsilon_i| < \epsilon_i^*$ with constant $\epsilon_i^* > 0$; and $S_i(\cdot)$ are RBF vectors. The RBFs are typically taken as Gaussian functions and Gaussian RBF vector $S_i(\cdot)$ are bounded, that is, $\|S_i(\cdot)\| \leq s^*$ with constant $s^* > 0$. Considering Equation (48) and system (26), we have

$$\dot{z}_{21} = \phi_u - W_1^{*T} S_1(Z_1) + \frac{1}{m_{11}} \tau_u + k_{wu} d_{wu} - \dot{\alpha}_1, \tag{49}$$

where

$$d_{wu} = k_{wu}^{-1} \left[ \frac{1}{m_{11}} \tau_{wu} - \epsilon_1(Z_1) \right] \tag{50}$$

is a lumped disturbance and $k_{wu} > 0$ is a design parameter. To estimate the unknown term $d_{wu}$ for system (49), we could design the following disturbance observer:

$$\begin{cases} \dot{\xi}_3 &= k_{wu} z_{21} - k_{d3} \left[ \phi_u - \hat{W}_1^T S_1(Z_1) + \frac{1}{m_{11}} \tau_u - \dot{\alpha}_1 \right] - k_{d3} k_{wu} \hat{d}_{wu} \\ \hat{d}_{wu} &= \xi_3 + k_{d3} z_{21}, \end{cases} \tag{51}$$

where $\xi_3$ is the observer state, $\hat{W}_1^T S_1(Z_1)$ is NN approximator of the unknown function $f_u(Z_1)$ with $Z_1 \in \Omega_{Z_1}$, and $k_{d3} > 0$ is a design parameter. From systems (49), (51), and Equation (48), the observer error dynamics is

$$\dot{\tilde{d}}_{wu} = k_{wu} z_{21} - k_{d3} k_{wu} \tilde{d}_{wu} + k_{d3} \tilde{W}_1^T S_1(Z_1) - \dot{d}_{wu} \tag{52}$$

with $\tilde{d}_{wu} = \hat{d}_{wu} - d_{wu}$ and $\tilde{W}_1 = \hat{W}_1 - W_1^*$. For system (27), we design the following disturbance observer:

$$\begin{cases} \dot{\boldsymbol{\xi}}_4 = \mathbf{K}_{wvr} \mathbf{z}_3 - \mathbf{K}_{d4} \left[ \Phi_{vr} - \dot{\boldsymbol{\alpha}}_{23} + \mathbf{K}_{wvr} \hat{\boldsymbol{d}}_{wvr} \right] \\ \qquad -\mathbf{K}_{d4} \left\{ - \begin{bmatrix} \hat{W}_2^T S_2(Z_2) \\ \hat{W}_3^T S_3(Z_2) \end{bmatrix} + \mathbf{Q} \begin{bmatrix} \tau_r \\ \dot{\beta} \end{bmatrix} \right\} \\ \hat{\boldsymbol{d}}_{wvr} = \boldsymbol{\xi}_4 + \mathbf{K}_{d4} \mathbf{z}_3, \end{cases} \tag{53}$$

which gives the observer error dynamics

$$\dot{\tilde{\boldsymbol{d}}}_{wvr} = \mathbf{K}_{wvr}\mathbf{z}_3 - \mathbf{K}_{d4}\mathbf{K}_{wvr}\tilde{\boldsymbol{d}}_{wvr} + \mathbf{K}_{d4}\begin{bmatrix}\tilde{W}_2^T S_2(Z_2)\\ \tilde{W}_3^T S_3(Z_2)\end{bmatrix} - \dot{\boldsymbol{d}}_{wvr}, \tag{54}$$

where $\boldsymbol{\xi}_4 = [\xi_{41}, \xi_{42}]^T$ is the observer states, $\mathbf{K}_{wvr} = \mathrm{diag}[k_{wv}, k_{wr}] > 0$ and $\mathbf{K}_{d4}^T = \mathbf{K}_{d4} > 0$ are design parameters, $\boldsymbol{d}_{wvr} = \mathbf{K}_{wvr}^{-1}\boldsymbol{\tau}_{wvr} - \mathbf{K}_{wvr}^{-1}\begin{bmatrix}\epsilon_2(Z_2)\\ \epsilon_3(Z_2)\end{bmatrix}$ and $\tilde{W}_j = \hat{W}_j - W_j^*, j = 2, 3$.

*Remark* 7. In system (50), the external disturbance $\tau_{uw}$ and NN approximation error $\epsilon_1(Z_1)$ are lumped together, and then disturbance observer (51) is employed to estimate the lumped disturbance $d_{uw}$. By estimating the lumped disturbance, we incorporate the NN approximator $\hat{W}_1^T S_1(Z_1)$ into the disturbance observer (51) to compensate for the unknown vessel dynamics such that both tracking errors and observer errors could converge to a small neighborhood of zero by appropriately choosing design parameters. The motivation for using disturbance observers (51) and (53) mainly lies in the following: (i) both the external disturbances and NN approximation errors are estimated by the disturbance observers, and then compensated for in a feedforward control loop; and (ii) the disturbance feedforward compensation term can be considered as a "patch" to the baseline feedback control that is designed for the nominal systems (ie, without the external disturbances and NN approximation errors). This disturbance-observer–based compensation is added to improve the robustness against the disturbances.

The disturbance-observer–based adaptive control laws $\tau_u, \tau_r$ could be given by

$$\tau_u = m_{11}\left(-k_{41}z_{21} - \phi_u + \hat{W}_1^T S_1(Z_1) - k_{wu}\hat{d}_{wu} + \dot{\alpha}_1 - z_{11}p_1\cos\psi - z_{12}p_2\sin\psi\right) \tag{55}$$

$$\tau_r = -\frac{\bar{m}_{33}\sin^2(\bar{h}_0)}{m_{23}}\bar{z}_{22} + \frac{\bar{m}_{33}\cos^2(\bar{h}_0)}{m_{22}}\bar{z}_{31}, \tag{56}$$

where $\bar{h}_0 = \varepsilon_2\arctan(\beta)$, $\bar{z}_{22} = -k_{51}z_{22} - \phi_v + \hat{W}_2^T S_2(Z_2) + \dot{\alpha}_2 - k_{wv}\hat{d}_{wv} + z_{11}p_1\sin\psi - z_{12}p_2\cos\psi$, $\bar{z}_{31} = -k_{52}z_{23} - \phi_r + \hat{W}_3^T S_3(Z_2) + \dot{\alpha}_3 - k_{wr}\hat{d}_{wr} - z_{13}p_3$ with design parameters $k_{41} > 0, k_{51} > 0, k_{52} > 0$, and

$$\dot{\beta} = -\frac{m_{22}(1+\beta^2)}{\varepsilon_1\varepsilon_2}\bar{z}_{22} - \frac{m_{23}(1+\beta^2)}{\varepsilon_1\varepsilon_2}\bar{z}_{31}. \tag{57}$$

Consider the following adaptive laws:

$$\dot{\hat{W}}_i = -\Gamma_i[S_i(Z_i)z_{2i} + \sigma_i\hat{W}_i], \quad i = 1, 2, 3, \tag{58}$$

where $\Gamma_i = \Gamma_i^T > 0$ are adaptation matrices and $\sigma_i > 0$ are the $\sigma$-modification parameters. Then, we obtain the following closed-loop error systems

$$\dot{z}_{21} = -k_{41}z_{21} + \tilde{W}_1^T S_1(Z_1) - k_{wu}\tilde{d}_{wu} - z_{11}p_1\cos\psi - z_{12}p_2\sin\psi \tag{59}$$

$$\dot{\mathbf{z}}_3 = -\mathbf{K}_5\mathbf{z}_3 - \mathbf{K}_{wvr}\tilde{\boldsymbol{d}}_{wvr} + \begin{bmatrix}\tilde{W}_2^T S_2(Z_2)\\ \tilde{W}_3^T S_3(Z_2)\end{bmatrix} + \begin{bmatrix}z_{11}p_1\sin\psi - z_{12}p_2\cos\psi\\ -z_{13}p_3\end{bmatrix}, \tag{60}$$

where $\mathbf{K}_5 = \mathrm{diag}[k_{51}, k_{52}]$. Consider the following Lyapunov function candidate:

$$V_2 = V_1 + \frac{1}{2}z_{21}^2 + \frac{1}{2}\mathbf{z}_3^T\mathbf{z}_3 + \frac{1}{2}\tilde{d}_{wu}^2 + \frac{1}{2}\tilde{\boldsymbol{d}}_{wvr}^T\tilde{\boldsymbol{d}}_{wvr} + \frac{1}{2}\sum_{i=1}^{3}\tilde{W}_i{}^T\Gamma_i^{-1}\tilde{W}_i, \tag{61}$$

whose derivative along systems (59), (60), (52), (54), (58), and (25) yields

$$\dot{V}_2 \leq -\rho_1 V_2 + \delta_1. \tag{62}$$

Accordingly, we have the following theorem that summarizes the stability and transient behaviors of the closed-loop adaptive systems.

**Theorem 2.** *Consider underarcuated marine vehicle (1) satisfying Assumptions 1 to 3, adaptive control laws $\tau_u, \tau_r$ given in (55), (56), NN weight updating law (58), and disturbance observers (51), (53). If there exists sufficiently large compact sets $\Omega_{Z_1}, \Omega_{Z_2}$ such that $Z_1 \in \Omega_{Z_1}, Z_2 \in \Omega_{Z_2}$ for all $t \geq 0$, then, for the initial conditions satisfying condition (3), we have the following.*

*i. All the signals in the closed-loop system remain bounded.*

*ii. The tracking errors always evolve within the predefined time-varying asymmetric bounds, ie,* $-\underline{e}_i(t) < e_i(t) < \bar{e}_i(t)$*,* $\forall t > 0, i = 1, 2, 3$*, and especially, the prescribed transient and steady-state performances of the tracking errors in the sense of (3) and (4) are guaranteed.*

*iii. The tracking errors* $e_i$*,* $i = 1, 2, 3$ *and observer errors* $\tilde{d}_{wu}$*,* $\tilde{\mathbf{d}}_{wvr}$ *converge to a small neighborhood of zero by appropriately choosing design parameters.*

*Proof.* The proof is similar to that of Theorem 1 and is therefore omitted. □

## 5 | SIMULATION STUDIES

To show the performance of the proposed tracking controllers, we perform numerical simulation on a Cybership-II model with vehicle mass $m = 23.8$ kg and length $L = 1.255$ m.[43] The Cybership-II dynamics is described by system (1) and the system parameters are taken from the work of Skjetne et al[43] and also presented in Table 1. All quantities in Table 1 are expressed in the international system of units (SI). Without loss of generality, the unmeasurable external disturbances are given by $\boldsymbol{\tau}_w = 0.5\mathbf{M}[1.5 + \sin(t), 1.5 + \cos(t), 1.5 + \sin(t)]^T$, which is borrowed from section 7.5 in the work of Do and Pan.[3] The choice of such disturbances indicates there exist both constant bias and time-varying external disturbances that act on surge, sway, and yaw dynamics of the vehicle. In practical maritime environments, the external disturbances may be different. We take the aforementioned disturbances for an illustration of the robustness properties of our proposed controllers. The desired trajectory in the phase space is a straight line and an ellipse. When $t \leq t_c$, the desired trajectory is a straight line that is given by $x_d = 3t, y_d = \psi_d = 0$; and when $t > t_c$, the reference trajectory is an ellipse that is described by $x_d = 3t_c + 30\sin(0.1(t - t_c))$, $y_d = 20 - 20\cos(0.1(t - t_c))$, $\psi_d = 0.1(t - t_c)$, where $t_c \geq 0$ is a time constant. The tracking error $\boldsymbol{e}$ is subject to the following asymmetric constraints:

$$-\underline{e}_i(t) < e_i(t) < \bar{e}_i(t),\ \forall t > 0,\ i = 1,\ 2,\ 3, \tag{63}$$

where $\bar{e}_1 = (1-0.1)\exp(-0.5t)+0.1$, $\underline{e}_1 = (1-0.1)\exp(-0.5t)+0.1$, $\bar{e}_2 = (5-0.1)\exp(-0.5t)+0.1$, $\underline{e}_2 = (1-0.1)\exp(-0.5t)+0.1$, $\bar{e}_3 = (3 - 0.05)\exp(-0.5t) + 0.05$, $\underline{e}_3 = (6 - 3.2)\exp(-0.5t) + 3.2$, which mean the prescribed performance of tracking errors for vehicle (1) are that (i) the decrease of tracking error $e_i$ is faster than $\exp(-0.5t)$, and (ii) the steady-state errors are smaller than 0.1, 0.1, and 3.2, respectively. Let $t_c = 10$ seconds, the initial states of the vehicle $\boldsymbol{\eta}(0) = [0, 3, -2]^T$, $\boldsymbol{v}(0) = [0.5, 2, -0.5]^T$, and the initial condition $\dot{\beta}(0) = 0$.

*Model-based control:* Suppose that all hydrodynamic damping terms in system (1) are accurately known a priori, but the disturbance $\boldsymbol{\tau}_w$ is not available for feedback control inputs $\tau_u$, $\tau_r$. Thus, we apply model-based tracking controllers (37) and (38) with additional control $\dot{\beta}^*$ in (39), and disturbance observers (33), (35) to achieve trajectory tracking control of system (1) with guaranteeing the prescribed performance. The controller parameters are $k_{d1} = 2$, $\mathbf{K}_{d2} = \text{diag}[2, 2]$, $k_{11} = 0.1$, $k_{12} = 0.4$, $k_{13} = 0.8$, $\mathbf{K}_2 = \text{diag}[2, 5]$, $k_{31} = 10$, $\varepsilon_1 = 6$, and $\varepsilon_2 = 4$. The initial states of the disturbance observers are $\xi_1(0) = 0$ and $\boldsymbol{\xi}_2(0) = [9, 2]^T$.

*Adaptive NN control:* Assume that both hydrodynamic damping terms $d_{11}$, $d_{22}$, $d_{23}$, $d_{32}$, $d_{33}$, and the disturbance $\boldsymbol{\tau}_w$ are unknown. Disturbance-observer–based adaptive controllers (55), (56) with additional control $\dot{\beta}$ in (57), NN weight updating law (58), and disturbance observers (51), (53) are applied to obtain the control objectives. We use the Gaussian RBF NN $W_1^T S_1(Z_1)$ with nine nodes, where the centers of the receptive field are evenly spaced on $[-2.4, 0]$, the widths of

**TABLE 1** Vehicle parameters

| Parameter | Value | Parameter | Value | Parameter | Value |
|---|---|---|---|---|---|
| $m$ | 23.8 | $X_u$ | -0.7225 | $Y_{\lvert r\rvert r}$ | -3.450 |
| $L$ | 1.255 | $X_{\lvert u\rvert u}$ | -1.3274 | $N_v$ | 0.1052 |
| $I_z$ | 1.7600 | $X_{uuu}$ | -5.8664 | $N_{\lvert v\rvert v}$ | 5.0437 |
| $x_g$ | 0.0460 | $Y_v$ | -0.8612 | $N_{\lvert r\rvert v}$ | 0.130 |
| $X_{\dot{u}}$ | -2.0 | $Y_{\lvert v\rvert v}$ | -36.2823 | $N_r$ | -1.900 |
| $Y_{\dot{v}}$ | -10.0 | $Y_{\lvert r\rvert v}$ | -0.805 | $N_{\lvert v\rvert r}$ | 0.080 |
| $Y_{\dot{r}}$ | -0.0 | $Y_r$ | 0.1079 | $N_{\lvert r\rvert r}$ | -0.750 |
| $N_{\dot{r}}$ | -1.0 | $Y_{\lvert v\rvert r}$ | -0.845 | | |

the receptive field are 0.4. Both Gaussian RBF NNs $W_j^T S_j(Z_2), j = 2, 3$ contain 60 nodes, in which the centers are evenly spaced on $[0, 1.5] \times [-3, 4]$ and the widths are 0.6. The design parameters of the adaptive controllers are $k_{11} = 0.1$, $k_{12} = 0.1$, $k_{13} = 0.4$, $k_{41} = 2$, $\mathbf{K}_5 = \text{diag}[3, 3]$, $\Gamma_1 = 2$, $\Gamma_2 = 10$, $\Gamma_3 = 4$, $\sigma_1 = \sigma_2 = 0.2$, $\sigma_3 = 0.4$, $k_{d3} = 4$, $\mathbf{K}_{d4} = \text{diag}[3, 1]$, $k_{wu} = 2$, $\mathbf{K}_{wvr} = \text{diag}[6, 10]$, $\epsilon_1 = 6$, and $\epsilon_2 = 4$. The initial NN weight estimates are $\hat{W}_1(0) = \hat{W}_2(0) = \hat{W}_3(0) = 0$, the initial states of the disturbance observers are $\xi_1(0) = -2$ and $\xi_4(0) = [6, 9]^T$.

Simulation results for both model-based control (MBC) and adaptive NN control (ANNC) are presented in Figures 1 to 6. The vehicle position outputs shown in Figure 1 follow the desired reference trajectory $(x_d, y_d)$ successfully for both controllers. It can be seen from Figures 2 to 4 that the tracking errors of vehicle position and orientation are always within the predefined bound (63). The control inputs $\tau_u, \tau_r$ are given in Figures 5 and 6, respectively. It follows from Figures 2 to 4 that the prescribed transient and steady-state tracking performances of vehicle (1) are achieved using ANNC, even though no accurate vehicle model of the vehicle is available. Hence, the proposed ANNC is robust stability and performance with respect to model uncertainties and external disturbances. Compared with the tracking errors in Figures 2 to 4, it is clear that adaptive control yields slower exhibit damped oscillations and steady-state errors and requires larger control input signals shown in Figures 5 and 6. This is because adaptive control has to adapt to unmodeled dynamic uncertainties through online adjustment of NN weights. It should be noticed that the prescribed performance of tracking errors is still guaranteed, as shown in Figures 2 to 4, during the transient stage of the parameter adaptations and disturbance estimates.

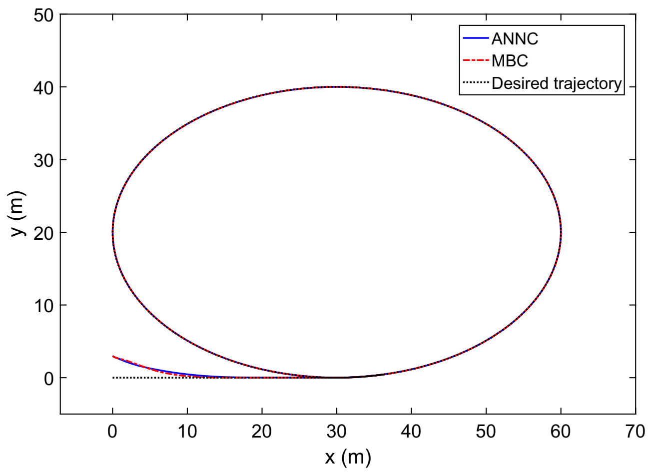


**FIGURE 1** Position output $(x, y)$ follows the reference trajectory $(x_d, y_d)$ ("- -") in the phase space: MBC ("-.-"), ANNC ("–"). ANNC, adaptive neural network control; MBC, model-based control [Colour figure can be viewed at wileyonlinelibrary.com]

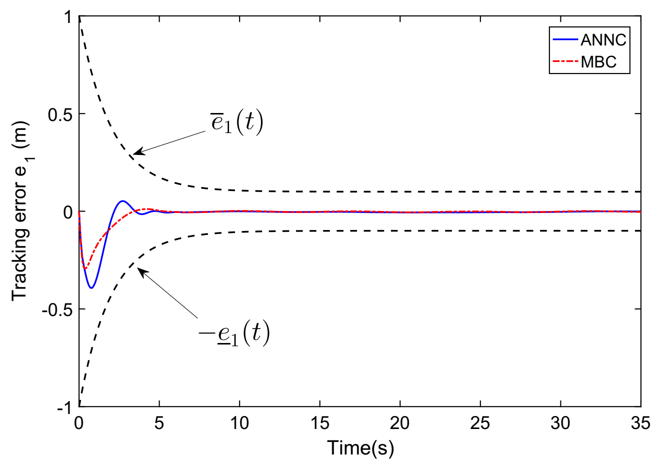


**FIGURE 2** Tracking error $e_1$: MBC ("-.-") and ANNC ("–"). ANNC, adaptive neural network control; MBC, model-based control [Colour figure can be viewed at wileyonlinelibrary.com]

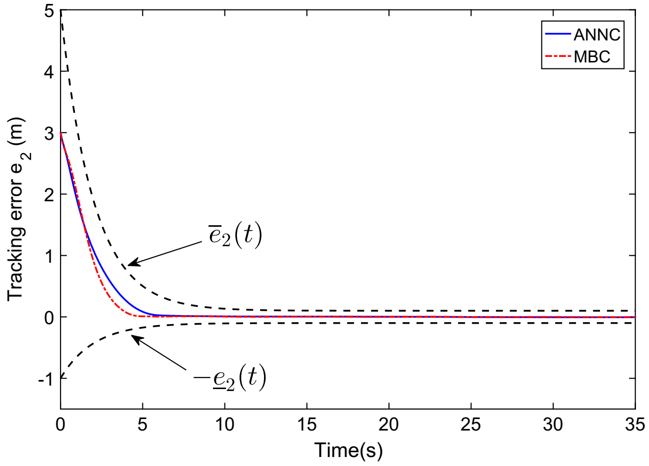


**FIGURE 3** Tracking error $e_2$: MBC ("-.-") and ANNC ("–"). ANNC, adaptive neural network control; MBC, model-based control [Colour figure can be viewed at wileyonlinelibrary.com]

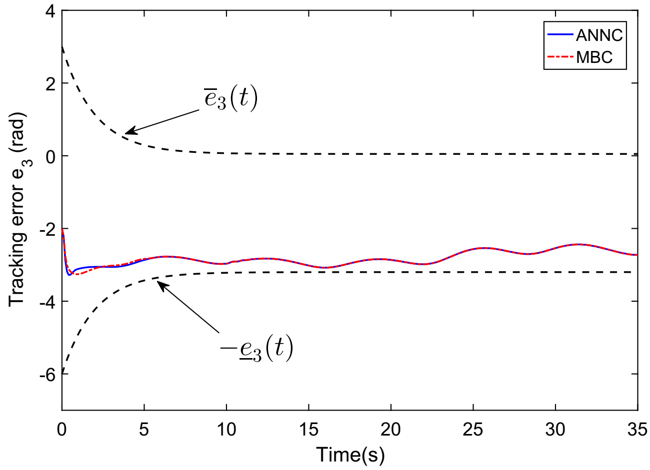


**FIGURE 4** Tracking error $e_3$: MBC ("-.-") and ANNC ("–"). ANNC, adaptive neural network control; MBC, model-based control [Colour figure can be viewed at wileyonlinelibrary.com]

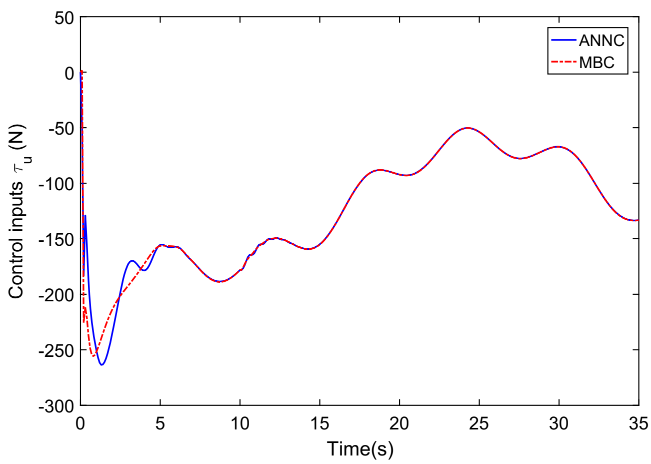


**FIGURE 5** Control input $\tau_u$: MBC ("-.-") and ANNC ("–"). ANNC, adaptive neural network control; MBC, model-based control [Colour figure can be viewed at wileyonlinelibrary.com]

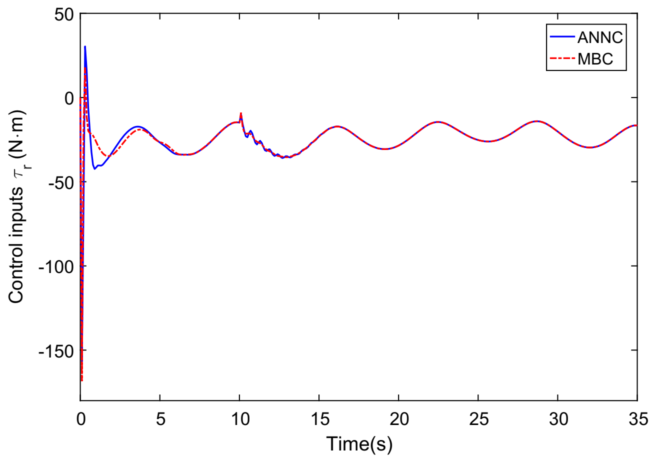


**FIGURE 6** Control input $\tau_r$: MBC ("-.-") and ANNC ("–"). ANNC, adaptive neural network control; MBC, model-based control [Colour figure can be viewed at wileyonlinelibrary.com]

## 6 | CONCLUSION

This paper has developed a constructive design technique of trajectory tracking control, under predefined tracking error constraints, for off-diagonal underactuated marine vehicles with uncertain damping effects and unknown external disturbances. The keys to the design technique include (i) the use of transverse function approach that introduces an additional control input to overcome the difficulties raised by underactuation and nonzero off-diagonal system matrices; (ii) the integration of backstepping procedure, barrier function, and Lyapunov synthesis for constructing stable feedback control with prescribed performance guarantees; and (iii) the introduction of NN approximation and disturbance estimates techniques for the compensation of uncertain hydrodynamic damping and external disturbances. Two novel transverse functions have been designed for controlling vehicle kinetics and the calculations of the transverse functions have been given duly. The design technique yields continuous control laws that guarantee practical stabilization of any smooth reference trajectory, whether this trajectory is feasible or not. An important feature of the design technique is that it is not required to prove the stability of the sway dynamics separately. This feature is of great significance to robust stabilization of uncertain underactuated systems, and then adaptive control has been developed to ensure the prescribed performance of tracking errors during the transient stage of on-line NN weight adaptations and disturbance estimates.

The present control design is based on control Lyapunov synthesis, backstepping technique, and disturbance observers, which provides a *smooth/continuous* tracking controller with *smooth* disturbance observer that achieves practical stability of closed-loop systems with guaranteed prescribed performance, where the external disturbances are assumed to be time-varying, continuously differentiable, and unmeasurable. Stochastic disturbances or the disturbances with abrupt changes might occur in marine practical applications. Opportunities for future work include performance-guaranteed control design of underactuated marine surface vehicles with nondifferentiable disturbances.

## ACKNOWLEDGEMENTS

The authors would like to thank the Associate Editor and the anonymous reviewers for their helpful and insightful comments for further improving the quality and presentation of this paper. This work was supported in part by the National Natural Science Foundation of China under grants 61473121, 61773169, and 61527811; in part by the Guangdong Natural Science Foundation under grants 2017A030313381 and 2017A030313369; in part by the Science and Technology Program of Guangzhou under grant 201604016082; and in part by the Fundamental Research Funds for the Central Universities.

## ORCID

*Shi-Lu Dai* https://orcid.org/0000-0002-8115-8787
*Shude He* https://orcid.org/0000-0002-3511-0600
*Hai Lin* https://orcid.org/0000-0002-5242-2366

## REFERENCES


1. Aguiar AP, Hespanha JP. Trajectory-tracking and path-following of underactuated autonomous vehicles with parametric modeling uncertainty. *IEEE Trans Autom Control*. 2007;52(8):1362-1379.
2. Paliotta C, Lefeber E, Pettersen KY, Pinto J, Costa M, Sousa JTFB. Trajectory tracking and path following for underactuated marine vehicles [early access online]. *IEEE Trans Control Syst Technol*. 2018. https://doi.org/10.1109/TCST.2018.2834518
3. Do KD, Pan J. *Control of Ships and Underwater Vehicles: Design for Underactuated and Nonlinear Marine Systems*. London, UK: Springer-Verlag; 2009.
4. Lefeber E, Pettersen KY, Nijmeijer H. Tracking control of an underactuated ship. *IEEE Trans Control Syst Technol*. 2003;11(1):52-61.
5. Brockett RW. Asymptotic stability and feedback stabilization. *Differential Geometric Control Theory*. 1983;27(1):181-191.
6. Behal A, Dawson DM, Dixon WE, Fang Y. Tracking and regulation control of an underactuated surface vessel with nonintegrable dynamics. *IEEE Trans Autom Control*. 2002;47(3):495-500.
7. Morin P, Samson C. Practical stabilization of driftless systems on Lie groups: the transverse function approach. *IEEE Trans Autom Control*. 2003;48(9):1496-1508.
8. Morin P, Samson C. Control of nonholonomic mobile robots based on the transverse function approach. *IEEE Trans Robot*. 2009;25(5):1058-1073.
9. Hamel T, Samson C. Transverse function control of a motorboat. *Automatica*. 2016;65:132-139.
10. Ilchmann A, Ryan EP, Trenn S. Tracking control: performance funnels and prescribed transient behaviour. *Syst Control Lett*. 2005;54(7):655-670.
11. Tee KP, Ge SS, Tay EH. Barrier Lyapunov functions for the control of output-constrained nonlinear systems. *Automatica*. 2009;45(4):918-927.
12. Tee KP, Ren B, Ge SS. Control of nonlinear systems with time-varying output constraints. *Automatica*. 2011;47(11):2511-2516.
13. Bechlioulis CP, Rovithakis GA. Robust adaptive control of feedback linearizable MIMO nonlinear systems with prescribed performance. *IEEE Trans Autom Control*. 2008;53(9):2090-2099.
14. Jin X, Xu J-X. A barrier composite energy function approach for robot manipulators under alignment condition with position constraints. *Int J Robust Nonlinear Control*. 2014;24(17):2840-2851.
15. He W, Ge SS. Cooperative control of a nonuniform gantry crane with constrained tension. *Automatica*. 2016;66:146-154.
16. He W, Yin Z, Sun C. Adaptive neural network control of a marine vessel with constraints using the asymmetric barrier Lyapunov function. *IEEE Trans Cybern*. 2017;47(7):1641-1651.
17. Zheng Z, Huang Y, Xie L, Zhu B. Adaptive trajectory tracking control of a fully actuated surface vessel with asymmetrically constrained input and output. *IEEE Trans Control Syst Technol*. 2018;26(5):1851-1859.
18. Zhang J-X, Yang G-H. Fault-tolerant leader-follower formation control of marine surface vessels with unknown dynamics and actuator faults. *Int J Robust Nonlinear Control*. 2018;28(14):4188-4208.
19. Dai S-L, He S, Lin H, Wang C. Platoon formation control with prescribed performance guarantees for USVs. *IEEE Trans Ind Electron*. 2018;65(5):4237-4246.
20. He S, Wang M, Dai S-L, Luo F. Leader-follower formation control of USVs with prescribed performance and collision avoidance [early access online]. *IEEE Trans Ind Inform*. 2018. https://doi.org/10.1109/TII.2018.2839739
21. Bechlioulis CP, Karras GC, Heshmati-Alamdari S, Kyriakopoulos KJ. Trajectory tracking with prescribed performance for underactuated underwater vehicles under model uncertainties and external disturbances. *IEEE Trans Control Syst Technol*. 2017;25(2):429-440.
22. Jin X. Fault tolerant finite-time leader–follower formation control for autonomous surface vessels with LOS range and angle constraints. *Automatica*. 2016;68(6):228-236.
23. Dai S-L, He S, Wang M, Yuan C. Adaptive neural control of underactuated surface vessels with prescribed performance guarantees [early access online]. *IEEE Trans Neural Netw Learn Syst*. 2018. https://doi.org/10.1109/TNNLS.2018.2876685
24. Jiang Z-P. Global tracking control of underactuated ships by Lyapunov's direct method. *Automatica*. 2002;38(2):301-309.
25. Do KD, Jiang Z-P, Pan J. Underactuated ship global tracking under relaxed conditions. *IEEE Trans Autom Control*. 2002;47(9):1529-1536.
26. Wondergem M, Lefeber E, Pettersen KY, Nijmeijer H. Output feedback tracking of ships. *IEEE Trans Control Syst Technol*. 2011;19(2):442-448.
27. Do KD. Practical control of underactuated ships. *Ocean Engineering*. 2010;37(13):1111-1119.
28. Do KD. Formation control of multiple elliptical agents with limited sensing ranges. *Automatica*. 2012;48(7):1330-1338.
29. Dai S-L, He S, Xu Y. Transverse function approach to practical stabilisation of underactuated surface vessels with modelling uncertainties and unknown disturbances. *IET Control Theory Appl*. 2017;11(15):2573-2584.
30. Almeida J, Silvestre C, Pascoal A. Cooperative control of multiple surface vessels in the presence of ocean currents and parametric model uncertainty. *Int J Robust Nonlinear Control*. 2010;20(14):1549-1565.
31. Peng Z, Wang J, Wang D. Distributed containment maneuvering of multiple marine vessels via neurodynamics-based output feedback. *IEEE Trans Ind Electron*. 2017;64(5):3831-3839.
32. Enayat M, Khorasani K. Semi-decentralized nonlinear cooperative control strategies for a network of heterogeneous autonomous underwater vehicles. *Int J Robust Nonlinear Control*. 2017;27(16):2688-2707.
33. Shojaei K. Observer-based neural adaptive formation control of autonomous surface vessels with limited torque. *Robot Auton Syst*. 2016;78:83-96.

34. Chen M, Jiang B, Cui R. Actuator fault-tolerant control of ocean surface vessels with input saturation. *Int J Robust Nonlinear Control*. 2016;26(3):542-564.
35. Do KD, Jiang Z-P, Pan J. Robust adaptive path following of underactuated ships. *Automatica*. 2004;40(6):929-944.
36. Gao T, Huang J, Zhou Y, Song Y-D. Robust adaptive tracking control of an underactuated ship with guaranteed transient performance. *Int J Syst Sci*. 2017;48(2):272-279.
37. Dai S-L, Yang C, Ge SS, Lee TH. Robust adaptive output feedback control of a class of discrete-time nonlinear systems with nonlinear uncertainties and unknown control directions. *Int J Robust Nonlinear Control*. 2013;23(13):1472-1495.
38. Shojaei K. Neural network formation control of underactuated autonomous underwater vehicles with saturating actuators. *Neurocomputing*. 2016;194:372-384.
39. Chen W-H, Yang J, Guo L, Li S. Disturbance observer-based control and related methods: an overview. *IEEE Trans Ind Electron*. 2016;63(2):1083-1095.
40. Yang J, Li S, Yu X. Sliding-mode control for systems with mismatched uncertainties via a disturbance observer. *IEEE Trans Ind Electron*. 2013;60(1):160-169.
41. Wang X, Li S, Lam J. Distributed active anti-disturbance output consensus algorithms for higher-order multi-agent systems with mismatched disturbances. *Automatica*. 2016;74:30-37.
42. Fossen TI, Gravdahl JT. *Marine Control Systems: Guidance, Navigation and Control of Ships, Rigs and Underwater Vehicles*. Trondheim, Norway: Marine Cybernetics; 2002.
43. Skjetne R, Fossen TI, Kokotovic PV. Adaptive maneuvering, with experiments, for a model ship in a marine control laboratory. *Automatica*. 2005;41(2):289-298.
44. Do KD, Pan J. Global robust adaptive path following of underactuated ships. *Automatica*. 2006;42(10):1713-1722.
45. Fischer N, Hughes D, Walters P, Schwartz EM, Dixon WE. Nonlinear RISE-based control of an autonomous underwater vehicle. *IEEE Trans Robot*. 2014;30(4):845-852.
46. Huang J, Wen C, Wang W, Song Y-D. Global stable tracking control of underactuated ships with input saturation. *Syst Control Lett*. 2015;85:1-7.
47. Abdelaal M, Fränzle M, Hahn A. Nonlinear model predictive control for trajectory tracking and collision avoidance of underactuated vessels with disturbances. *Ocean Engineering*. 2018;160:168-180.
48. Fossen TI, Pettersen KY. On uniform semiglobal exponential stability (USGES) of proportional line-of-sight guidance laws. *Automatica*. 2014;50(11):2912-2917.
49. Peng Z, Wang D, Chen Z, Hu X, Lan W. Adaptive dynamic surface control for formations of autonomous surface vehicles with uncertain dynamics. *IEEE Trans Control Syst Technol*. 2013;21(2):513-520.
50. Li J-H, Lee P-M, Jun B-H, Lim Y-K. Point-to-point navigation of underactuated ships. *Automatica*. 2008;44(12):3201-3205.
51. Sun Z, Zhang G, Yi B, Zhang W. Practical proportional integral sliding mode control for underactuated surface ships in the fields of marine practice. *Ocean Eng*. 2017;142:217-223.
52. Sanner RM, Slotine JJE. Gaussian networks for direct adaptive control. *IEEE Trans Neural Netw*. 1992;3(6):837-863.
53. Dai S-L, Wang C, Wang M. Dynamic learning from adaptive neural network control of a class of nonaffine nonlinear systems. *IEEE Trans Neural Netw Learn Syst*. 2014;25(1):111-123.


**How to cite this article:** Dai S-L, He S, Lin H. Transverse function control with prescribed performance guarantees for underactuated marine surface vehicles. *Int J Robust Nonlinear Control*. 2019;29:1577–1596. https://doi.org/10.1002/rnc.4453

## APPENDIX

### PROOF OF THEOREM 1

i. Solving inequality (44), we obtain

$$V_2^* \leq \varrho_1^* + c_0^* e^{-\rho_1^* t}, \ \forall t \geq 0, \tag{A1}$$

where constants $\varrho_1^* = \delta_1^* / \rho_1^* > 0$ and $c_0^* = V_2^*(0) - \varrho_1^*$. It follows from (17), (42), and (A1) that

$$\frac{1}{2}\left(z_{11}^2 + z_{12}^2 + z_{13}^2 + z_{21}^2 + \|\mathbf{z}_3\|^2 + \tilde{\tau}_{wu}^2 + \|\tilde{\boldsymbol{\tau}}_{wvr}\|^2\right) \leq V_2^* \leq c_0^* e^{-\rho_1^* t} + \varrho_1^*, \ \forall t \geq 0,$$

which means that

$$\frac{1}{2} z_{1i}^2 \leq c_0^* e^{-\rho_1^* t} + \varrho_1^* \leq c_1^*, \ \forall t \geq 0, \ i = 1, \ 2, \ 3, \tag{A2}$$

where $c_1^* = c_0^* + \varrho_1^* > 0$. From (A2), if $c_0^* = V_2^*(0) - \varrho_1^* = 0$, then the following inequality $|z_{1i}| \le \sqrt{2\varrho_1^*}$ holds. If $c_0^* = V_2^*(0) - \varrho_1^* \neq 0$, from (A2), we can conclude that, given any $\zeta_1 > \sqrt{2\varrho_1^*}$, there exists $t_{z1}$ such that, for any $t > t_{z1}$, the inequality $|z_{1i}| \le \zeta_1$ holds, where $t_{z1} = -\frac{1}{\rho_1} \ln\left(\frac{\zeta_1^2 - 2\varrho_1^*}{2c_0^*}\right)$, $V_2^*(0) \neq \varrho_1^*$. Therefore, when $t$ tends to infinity, it follows

$$|z_{1i}| \le \sqrt{2\varrho_1^*}, \tag{A3}$$

which means that $z_{1i}$ is uniformly ultimately bounded. A similar conclusion can be made on $z_{21}$, $\mathbf{z}_3$, $\tilde{\tau}_{wu}$, and $\tilde{\boldsymbol{\tau}}_{wvr}$. From Equations (5) and (6), it is clear that the boundedness of $z_{1i}$ guarantees that the tracking error $e_i$ is uniformly ultimately bounded. Since $\boldsymbol{\eta}_d$ is bounded using Assumption 2, we have that system state $\boldsymbol{\eta}$ is bounded. According to Equation (13), it follows that $u, v$, and $r$ are bounded due to the boundedness of $\alpha_i$, $i = 1, 2, 3$ in (22)-(24), and $h_j(\beta), j = 1, 2$ in (30). Under Assumption 2, $\dot{\alpha}_i$, $i = 1, 2, 3$ are also bounded because every term in the first time derivative of $\alpha_i$ (22)-(24) is bounded. Considering $\tilde{\tau}_{wj} = \hat{\tau}_{wj} - \bar{\tau}_{wj}, j = u, v, r$ and the bounded external disturbances $\bar{\tau}_{wj}$ according to Assumption 1, it is clear that the disturbance estimates $\hat{\tau}_{wj}$ are bounded. Subsequently, the observer states $\xi_1$ in (33) and $\boldsymbol{\xi}_2$ in (35) are also bounded because the boundedness of $\hat{\tau}_{wu}$, $\hat{\boldsymbol{\tau}}_{wvr}$, $z_{21}$, and $\mathbf{z}_3$, and then it can be concluded that the control inputs $\tau_u^*$ in (37) and $\tau_r^*$ in (38) are also bounded. Therefore, all the signals in the closed-loop system are uniformly ultimately bounded.

ii. From Equations (5) and (9), we have

$$\frac{e_i}{\bar{e}_i} = T_i(z_{1i}, \gamma_{ei}) = \frac{e^{z_{1i}} - e^{-z_{1i}}}{e^{z_{1i}} + \gamma_{ei}^{-1} e^{-z_{1i}}}, \quad i = 1, \ 2, \ 3. \tag{A4}$$

Note that $T_i(\cdot)$ in (A4) is a strictly increasing function with respect to $z_{1i}$, and then it is clear from (A2) that

$$-\gamma_{ei} < \frac{1 - e^{2\sqrt{2c_1^*}}}{1 + \gamma_{ei}^{-1} e^{2\sqrt{2c_1^*}}} \le \frac{e_i}{\bar{e}_i} \le \frac{1 - e^{-2\sqrt{2c_1^*}}}{1 + \gamma_{ei}^{-1} e^{-2\sqrt{2c_1^*}}} < 1$$

for all $t > 0$, which yields $-\underline{e}_i(t) < e_i(t) < \bar{e}_i(t), \forall t > 0, i = 1, 2, 3$ because of $\gamma_{ei} = \underline{e}_i / \bar{e}_i$, and $\bar{e}_i > 0$. If $\underline{e}_i(t) = \bar{e}_i(t) = \delta_i \rho_i(t)$, $i = 1, 2, 3$, then we have

$$|e_i| < \delta_i \rho_i(t) = \delta_i(\rho_{i0} - \rho_{i\infty}) \exp(-\kappa_i t) + \delta_i \rho_{i\infty}, \forall t > 0,$$

which means that the prescribed transient and steady-state performances in the sense of (3) and (4) are guaranteed.

iii. From inequalities (A3) and (A2), it is clear that when $t$ tends to infinity, we obtain

$$|z_{1i}| \le \sqrt{2\varrho_1^*}, \quad |\tilde{\tau}_{wj}| \le \sqrt{2\varrho_1^*}, \ i = 1, \ 2, \ 3, \ j = u, \ v, \ r, \tag{A5}$$

which means that the errors $z_{1i}$ and $\tilde{\tau}_{wj}$ converge exponentially to a small residual set $\sqrt{2\varrho_1^*}$. Consequently, we can conclude from (A4) and (A5) that the tracking errors $e_i$, $i = 1, 2, 3$, converge exponentially to a small neighborhood of zero. It follows from (45) and (A1) that (i) the size of $\sqrt{2\varrho_1^*}$ depends on the bounds of $\dot{\tau}_{wj}$, and design parameters $\varepsilon_1$, $\varepsilon_2$, $\kappa_1$; (ii) when the external disturbances $\tau_{wj}$ are constant or slowly varying, ie, $\dot{\tau}_{wj} = 0$, the disturbance observer error $\tilde{\tau}_{wj}$ could converge exponentially to zero; (iii) increasing the design parameter $\kappa_1$ may result in smaller $\delta_1^*$; and (iv) decreasing $\varepsilon_1, \varepsilon_2$ will help to reduce $\delta_1^*$. Thus, decreasing $\varepsilon_1, \varepsilon_2$ and increasing $\kappa_1$ could significantly improve tracking accuracy on the steady-state stage. This completes the proof.