Axel Schumacher · Thomas Vietor
Sierk Fiebig · Kai-Uwe Bletzinger
Kurt Maute *Editors*

Advances in Structural and Multidisciplinary Optimization

Proceedings of the 12th World Congress of Structural and Multidisciplinary Optimization (WCSMO12)

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Foreword

The present proceeding is a collection of contributions from the Twelfth World Congress of Structural and Multidisciplinary Optimization (WCSMO12) held at the Technische Universität Braunschweig in Germany from June 5 to 9, 2017.

The WCSMO12 was organized by the International Society for Structural and Multidisciplinary Optimization (ISSMO) founded in October 1991 and has held the WCSMO biennially since 1995. One of the goals of ISSMO is to bring together researchers and practitioners in the field of structural and multidisciplinary optimization (SMO), by means of international meetings with high scientific standard. The ISSMO aims at stimulating and promoting research in all aspects of optimal design of structures as well as multidisciplinary design optimization, where the involved disciplines deal with the analysis of solids, fluids, or other field problems.

The organizing staff of the WCSMO12 was composed of members of the Technische Universität Braunschweig, the University of Wuppertal, the Volkswagen AG, the Technical University of Munich, and the University of Colorado Boulder.

We would like to express our gratitude to all the contributing authors who helped to create this comprehensive proceeding. Also, we thank the members of the International Papers Committee of the WCSMO12: Byeng Dong Youn from the Seoul National University in Korea, Qing Li from the University of Sydney in Australia, Ramana Grandhi from the Wright State University in Ohio, Zhan Kang from the Dalian University of Technology in China, and Niels Pedersen from the Technical University of Denmark.

This proceeding provides a detailed overview of the current research activities of methods for structural and multidisciplinary optimization. The content is subdivided into several parts. In part I to VI, we collect all contributions concerning general approaches and strategies for optimization processes, like robust design or surrogate models. Parts VII to VIII deal with optimization algorithms. In parts IX to XV, there are contributions dealing with structural optimization methods like shape and topology optimization. Parts XVI to XXII contain contributions dealing with different physical models, like crash simulation, acoustic simulation, or the consideration of manufacturing aspects. Industrial applications are collected in parts XXIII to XXVII.

November 2017

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Contents

Part VI: General Approaches and Strategies: General Aspects of Single-Objective Optimization

Part VII: Optimization Algorithms: Local Mathematical Methods

Part VIII: Optimization Algorithms: Global Methods (e.g. Evolutionary Algorithms)

Part IX: Structural Optimization: Sizing

Part X: Structural Optimization: Fiber and Composite Optimization

Part XV: Structural Optimization: Topology Optimization with Other Methods

Part XX: Optimization with Emphasis on Particular Physics Model: Considering Piezoelectricity, Magnetic and Electrical Fields

Part XXI: Optimization with Emphasis on Particular Physics Model: Considering Other Specialty Disciplines

Part XXII: Optimization with Emphasis on Particular Physics Model: Considering Manufacturing Aspects

Stochastic Sensitivity Analysis for Robust Topology Optimization

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Abstract. Topology optimization under uncertainty poses extreme difficulty to the already challenging topology optimization problem. This paper presents a new computational method for calculating topological sensitivities of statistical moments of high-dimensional complex systems subject to random inputs. The proposed method, capable of evaluating stochastic sensitivities for large-scale, robust topology optimization (RTO) problems, integrates a polynomial dimensional decomposition (PDD) of multivariate stochastic response functions and deterministic topology derivatives. In addition, the statistical moments and their topology sensitivities are both determined concurrently from a single stochastic analysis. When applied in collaboration with the gradient based optimization algorithm, the proposed method affords the ability of solving industrial-scale RTO design problems. Numerical examples indicate that the new method developed provides computationally efficient solutions.

Keywords: Stochastic sensitivity analysis · Polynomial dimensional decomposition · Robust design topology optimization · Topological derivatives

1 Introduction

Topology optimization is a computational design framework to identify the optimal distribution of materials for complex engineering systems [1,2,10,17,18,20,21]. Uncertainties, unavoidable in the manufacturing process and operating environment, often plague those engineering systems, thus need to be taken into account during the design process. Conventional deterministic design approaches typically lead to inefficient and overly conservative designs that overcompensate for uncertainties, or unknowingly risky designs due to the underestimation of uncertainties. Aimed at minimizing the propagation of input uncertainty, robust topology optimization (RTO) discovers insensitive topology design in the presence of uncertainty. In the past decade, it is increasingly viewed as an enabling

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technology for topology design of aerospace, automotive, and civil structures subject to uncertainty.

The objective or constraint functions in RTO usually consist of first two moment properties, such as means and standard deviations, of certain stochastic responses, describing the objective robustness or feasibility robustness of a given topology. Therefore, solving a practical RTO problem draws in statistical moments and their sensitivity analysis for random responses. The fundamental problem rooted in statistical moment analysis entails calculation of a high-dimensional integral with respect to the probability measure $f_{\mathbf{X}}(\mathbf{x})$ of $\mathbf{X}$ over $\mathbb{R}^N$, where N is the number of random variables. In general, such an integral cannot be evaluated analytically. Direct numerical integration can be performed, but it is not economically feasible for the cases that N exceeds three or four, especially when expensive finite element analyses (FEA) are involved in the evaluation of response functions. Existing approximate methods for statistical moment analysis include the point estimate method (PEM) [8], Taylor series expansion or perturbation method [8], tensor product quadrature (TPQ) [9], Neumann expansion method [24], polynomial chaos expansion (PCE) [19], statistically equivalent solution [5], dimension-reduction method [22,23], and others [6].

Two major concerns are relevant to existing approaches when conducting stochastic moment and their sensitivity analysis. First, the commonly used stochastic methods, such as the perturbation or Taylor series expansions, PEM, PCE, TPQ, and dimension-reduction methods begin to be inapplicable or inadequate when performing uncertainty quantification for many large-scale engineering problems. For example, although the Taylor series expansion and PEM are inexpensive and simple, they may deteriorate when the nonlinearity of response function is high and/or when the input uncertainty is large. PCE, commonly used in stochastic mechanics, is an infinite series involving Hermite polynomials of Gaussian variables (or others). However, for high-dimensional systems, PCE requires astronomically large numbers of terms or coefficients to capture the high nonlinearity of a stochastic response, easily succumbing to the curse of dimensionality. The dimension-reduction methods, to some extent, mitigate the curse of dimensionality, but they are based on the referential dimensional decomposition (RDD), often leading to sub-optimal approximations of a response function. Second, applied to design sensitivity analysis of the statistical moments, many of the aforementioned methods invoke finite-difference techniques, which require repetitive stochastic analyses at both perturbed and nominal design points. Therefore, they are often computationally expensive. Although some methods, such as Taylor series expansions, also provide the design sensitivities economically, its sensitivity estimates are either inaccurate or unreliable due to inherited errors from the associated second-moment analysis.

This paper presents a new approach for evaluating topology design sensitivities of statistical moments of complex engineering structures subject to random inputs. The method proposes a novel integration of polynomial dimensional decomposition (PDD) of a multivariate stochastic response function and deterministic topological derivative. For stochastic moment analysis, it leads to analytical formulations for the first two moments. Furthermore, it is capable of incorporating both moment calculation and their topology design sensitivity evaluation in a single stochastic analysis. Section 2 describes the PDD approximation of a multivariate function, resulting in explicit formulae for the first two moments. Section 3 described the new integration of PDD and deterministic topological derivative as well as numerical procedures for topology sensitivities of moments. The calculation of PDD expansion coefficients, required in sensitivity analyses of moments, is discussed in Sect. 4. In Sect. 5, two numerical examples are presented to probe the accuracy and computational efficiency of the proposed method. Finally, conclusions are drawn in Sect. 6.

2 PDD and Moment Analysis

2.1 Robust Topology Optimization

The mathematical formulation for robust topology optimization involving a single objective function and $1 \leq K < \infty$ constraint functions requires:

$$\begin{aligned} \min_{\Omega} \quad & c_0(\Omega) := w_1 \frac{\mathbb{E}\left[y_0(\Omega, \mathbf{X})\right]}{\mu_0^*} + w_2 \frac{\sqrt{\operatorname{var}\left[y_0(\Omega, \mathbf{X})\right]}}{\sigma_0^*}, \\ \text{subject to} \quad & c_k(\Omega) := \alpha_k \sqrt{\operatorname{var}\left[y_k(\Omega, \mathbf{X})\right]} - \mathbb{E}\left[y_k(\Omega, \mathbf{X})\right]; \quad k = 1, \cdots, K, \\ & \Omega \subseteq D, \end{aligned} \tag{1}$$

where $w_1 \in \mathbb{R}_0^+$ and $w_2 \in \mathbb{R}_0^+$ are two non-negative, real-valued weights, satisfying $w_1 + w_2 = 1$, $\mu_0^* \in \mathbb{R} \setminus \{0\}$ and $\sigma_0^* \in \mathbb{R}_0^+ \setminus \{0\}$ are two non-zero, real-valued scaling factors; $\alpha_k \in \mathbb{R}_0^+$, $k = 0, 1, \cdots, K$, are non-negative, real-valued constants associated with the probabilities of constraint satisfaction; $D \subset \mathbb{R}^3$ is a bounded domain in which all admissible Ω are included; $\mathbf{X} := (X_1, \cdots, X_N)^T \in \mathbb{R}^N$ is an N-dimensional random input vector completely defined by a family of joint probability density functions $\{f_{\mathbf{X}}(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^N\}$ on the probability triple $(\Omega_{\mathbf{X}}, \mathscr{F}, P)$, where $\Omega_{\mathbf{X}}$ is the sample space; $\mathscr{F}$ is the σ-field on $\Omega_{\mathbf{X}}$; and P is the probability measure associated with probability density $f_{\mathbf{X}}(\mathbf{x})$. Equation (1) describes a generic RTO problem involving the first two moments of certain responses.

To Solve Eq. (1) for an optimal topology employing gradient based algorithm, efficient methods for stochastic moments and their topology design sensitivities are required. The following subsections describe the recently developed polynomial dimensional decomposition method for stochastic moments and their sensitivities.

2.2 PDD

Consider a multivariate stochastic response $y(\Omega, \mathbf{X})$, representing any of the performance function y_k in Eq. (1) that depends on the random vector $\mathbf{X} = \{X_1, \cdots, X_N\}^T$. Let $\mathscr{L}_2(\Omega_{\mathbf{X}}, \mathscr{F}, P)$ be a Hilbert space of square-integrable functions y with corresponding probability measure $f_{\mathbf{X}}(\mathbf{x})d\mathbf{x}$ supported on $\mathbb{R}^N$. Assuming independent coordinates, the PDD expansion of function y is hierarchical:

$$y(\Omega, \mathbf{X}) = y_\emptyset(\Omega) + \sum_{\emptyset \neq u \subseteq \{1,\cdots,N\}} \sum_{\mathbf{j}_{|u|} \in \mathbb{N}^{|u|}} C_{u\mathbf{j}_{|u|}}(\Omega) \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u; \Omega), \tag{2}$$

in terms of a set of multivariate orthonormal polynomials [12,13] $\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u; \Omega) := \prod_{p=1}^{|u|} \psi_{i_p j_p}(X_i; \Omega)$ where $\mathbf{j}_{|u|} = (j_1, \cdots, j_{|u|}) \in \mathbb{N}^{|u|}$ is a $|u|$-dimensional multi-index; $y_\phi(\Omega)$ is a constant; for $|u| = 1$, $C_{u\mathbf{j}_{|u|}}(\Omega)\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u; \Omega)$ is a univariate component function representing the individual contribution to $y(\Omega, \mathbf{X})$ by a single input variable; for $|u| = 2$, it is a bivariate component function describing the cooperative influence of two input variables; and for $|u| = S$, it is an S-variate component function quantifying the interactive effects of S input variables. For most performance functions, Eq. (2) can be truncated by retaining, at most, the interactive effects of $S < N$ input variables and m-th order polynomials as follows

$$\tilde{y}_{S,m}(\Omega, \mathbf{X}) := y_\emptyset(\Omega) + \sum_{\substack{\emptyset \neq u \subseteq \{1,\cdots,N\} \\ 1 \leq |u| \leq S}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}^{|u|} \\ \|\mathbf{j}_{|u|}\|_\infty \leq m}} C_{u\mathbf{j}_{|u|}}(\Omega) \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u; \Omega), \tag{3}$$

where

$$y_\emptyset(\Omega) = \int_{\mathbb{R}^N} y(\mathbf{x}, \Omega) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{4}$$

and

$$C_{u\mathbf{j}_{|u|}}(\Omega) := \int_{\mathbb{R}^N} y(\mathbf{x}, \Omega) \psi_{u\mathbf{j}_{|u|}}(\mathbf{x}_u; \Omega) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}, \quad \emptyset \neq u \subseteq \{1, \cdots, N\},\ \mathbf{j}_{|u|} \in \mathbb{N}^{|u|}, \tag{5}$$

are various expansion coefficients. For $S > 0$, Eq. (3) contains interactive effects of at most S input variables $X_{i_1}, \cdots, X_{i_S}$, $1 \leq i_1 < \cdots < i_S \leq N$, on y, thus resulting in the S-variate, mth-order PDD approximation. When $S \to N$ and $m \to \infty$, $\tilde{y}_{S,m}$ converges to y and engenders a sequence of hierarchical and convergent approximations of y. Depending on the dimensional structure and nonlinearity of a stochastic response, the truncation parameters S and m can be chosen accordingly. The higher the values of S and m permit the higher the accuracy, but also endow the computational cost of an Sth-order polynomial computational complexity [12,13]. Henceforth, the S-variate, mth-order PDD approximation will be simply referred to as *truncated PDD approximation* in this paper.

2.3 Moment Analysis

Let $m^{(r)}(\Omega) := \mathbb{E}[y^r(\Omega, \mathbf{X})]$, if it exists, denote the raw moment of y of order r, where $r \in \mathbb{N}$. Provided an S-variate, mth-order PDD approximation $\tilde{y}_{S,m}(\Omega, \mathbf{X})$ of $y(\Omega, \mathbf{X})$, let $\tilde{m}^{(r)}(\Omega) := \mathbb{E}[\tilde{y}^r_{S,m}(\Omega, \mathbf{X})]$ denote the raw moment of $\tilde{y}_{S,m}$ of order

r. The analytical expressions or explicit formulae for calculating the moments using PDD approximations are described as follows.

Applying the expectation operator on $\tilde{y}_{S,m}(\Omega,\mathbf{X})$ and $\tilde{y}^2_{S,m}(\Omega,\mathbf{X})$, the first moment or mean [14]

$$\tilde{m}^{(1)}_{S,m}(\Omega) := \mathbb{E}\left[\tilde{y}_{S,m}(\Omega,\mathbf{X})\right] = y_{\emptyset}(\Omega) = \mathbb{E}\left[y(\Omega,\mathbf{X})\right] =: m^{(1)}(\Omega) \tag{6}$$

of the S-variate, mth-order PDD approximation reproduces the exact mean of y, whereas the second moment [14]

$$\tilde{m}^{(2)}_{S,m}(\Omega) := \mathbb{E}\left[\tilde{y}^2_{S,m}(\Omega,\mathbf{X})\right] = y^2_{\emptyset}(\Omega) + \sum_{\substack{\emptyset\neq u\subseteq\{1,\cdots,N\}\\ 1\le|u|\le S}} \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}^{|u|}\\ \left\|\mathbf{j}_{|u|}\right\|_{\infty}\le m}} C^2_{u\mathbf{j}_{|u|}}(\Omega) \tag{7}$$

is evaluated as the sum of squares of all expansion coefficients of $\tilde{y}_{S,m}(\Omega,\mathbf{X})$. It is elementary to show that the estimation of second moment provided by Eq. (7) approaches the exact second moment

$$m^{(2)}(\Omega) := \mathbb{E}\left[y^2(\Omega,\mathbf{X})\right] = y^2_{\emptyset}(\Omega) + \sum_{\emptyset\neq u\subseteq\{1,\cdots,N\}} \sum_{\mathbf{j}_{|u|}\in\mathbb{N}^{|u|}} C^2_{u\mathbf{j}_{|u|}}(\Omega) \tag{8}$$

of y when $S \to N$ and $m \to \infty$. The mean-square convergence of $\tilde{y}_{S,m}$ is guaranteed as its component functions will include all required members of the associated Hilbert spaces. Furthermore, the variance of $\tilde{y}_{S,m}(\Omega,\mathbf{X})$ is also mean-square convergent.

3 Topology Sensitivity of Stochastic Moments

3.1 Deterministic Topological Derivative

For a given reference domain $\Omega \subset \mathbb{R}^n,\ n = 2$ or 3, consider the following linear elastic system

$$\begin{cases} \boldsymbol{\nabla}\cdot(\mathbb{C}:\boldsymbol{\varepsilon}) = \mathbf{0} & \text{in } \Omega \\ \boldsymbol{u} = \bar{\boldsymbol{u}} & \text{on } \Gamma_D \\ \boldsymbol{n}\cdot(\mathbf{C}:\boldsymbol{\varepsilon}) = \bar{\boldsymbol{t}} & \text{on } \Gamma_N \end{cases}, \tag{9}$$

where $\mathbb{C}$ is the elastic tensor, $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{u}\boldsymbol{\nabla} + \boldsymbol{\nabla}\boldsymbol{u}\right)$ is the Cauchy strain tensor, Γ_D and Γ_N are Dirichlet boundary and Neumann boundary of Ω, respectively. For a point $\boldsymbol{\xi}_0 \in \Omega$ and a spherical hole $\omega \in \mathbb{R}^n$ with a fixed radius and boundary $\partial\omega$, the translated and rescaled hole can be defined by $\omega_\rho = \boldsymbol{\xi}_0 + \rho\omega,\ \forall\rho > 0$ and the perforated domain is $\Omega_\rho = \Omega\backslash\overline{\omega_\rho}$, the new problem with the Neumann boundary on the perforated hole is

$$\begin{cases} \boldsymbol{\nabla}\cdot(\mathbb{C}:\boldsymbol{\varepsilon}_\rho) = \mathbf{0} & \text{in } \Omega_\rho \\ \boldsymbol{u}_\rho = \bar{\boldsymbol{u}} & \text{on } \Gamma_D \\ \boldsymbol{n}\cdot(\mathbb{C}:\boldsymbol{\varepsilon}_\rho) = \bar{\boldsymbol{t}} & \text{on } \Gamma_N \\ \boldsymbol{n}\cdot(\mathbb{C}:\boldsymbol{\varepsilon}_\rho) = \mathbf{0} & \text{on } \partial\omega_\rho \end{cases}. \tag{10}$$

In engineering design, it is often of interest to investigate a certain response function y defined over the domain Ω and its variation when the domain is perforated by a small hole. In general, y is the function of Ω as well as the displacement $\boldsymbol{u}$, and is differentiable with respect to $\boldsymbol{u}$, although it is often denoted as $y(\Omega)$ for simplicity. For a small $\rho > 0$, if $y(\Omega_\rho)$ admits the topological asymptotic expansion $y(\Omega_\rho) = y(\Omega) + \rho^k D_T y(\Omega, \boldsymbol{\xi}_0) + o(\rho^k)$, then $\rho^k D_T y(\Omega, \boldsymbol{\xi}_0)$ represents the variation of the response function when the domain changes from Ω to Ω_ρ, and $D_T y(\Omega, \boldsymbol{\xi}_0)$ is called the topological derivative at point $\boldsymbol{\xi}_0$. The concept of topological derivative is applicable to general boundary value problems including linear elastic systems.

When the total compliance of the structure is selected as the response function y, aided by the adjoint method, the domain truncation technique, and the asymptotic expansion with respect to the small parameter ρ, the deterministic topological derivative $D_T y(\Omega, \boldsymbol{\xi}_0)$ reads [4]

$$D_T y(\Omega, \boldsymbol{\xi}_0) = \boldsymbol{a}\left(\mathbb{C} : \boldsymbol{\varepsilon}\left(\boldsymbol{\xi}_0\right)\right) : \tilde{\boldsymbol{\varepsilon}}\left(\boldsymbol{\xi}_0\right) \tag{11}$$

where $\tilde{\boldsymbol{\varepsilon}}$ is the strain solution of the following adjoint problem

$$\begin{cases} \boldsymbol{\nabla} \cdot (\mathbb{C} : \tilde{\boldsymbol{\varepsilon}}) = 0 & \text{in } \Omega \\ \tilde{\boldsymbol{u}} = -\bar{\boldsymbol{u}} & \text{on } \Gamma_D \\ \boldsymbol{n} \cdot (\mathbb{C} : \tilde{\boldsymbol{\varepsilon}}) = \bar{\boldsymbol{t}} & \text{on } \Gamma_N \end{cases} . \tag{12}$$

and $\boldsymbol{a}\left(\mathbb{C} : \boldsymbol{\varepsilon}\left(\boldsymbol{\xi}_0\right)\right)$ is a second order tensor function implicitly related to $\mathbb{C} : \boldsymbol{\varepsilon}\left(\boldsymbol{\xi}_0\right)$ and its components read

$$a_{ij} = \int_{\partial \omega_\rho} \left(\boldsymbol{n} \cdot \mathbb{C} : \hat{\boldsymbol{\varepsilon}}\right) \cdot \boldsymbol{e}_i \xi_j \mathrm{d}\gamma(\boldsymbol{\xi}), \tag{13}$$

in which $\boldsymbol{e}_i$ is ith orthonormal basis vector of the reference frame, and $\hat{\boldsymbol{\varepsilon}}$ is the strain solution of the following problem

$$\begin{cases} \boldsymbol{\nabla} \cdot (\mathbb{C} : \hat{\boldsymbol{\varepsilon}}) = \boldsymbol{0} & \text{in } \mathbb{R}^n \backslash \overline{\omega_\rho} \\ \boldsymbol{n} \cdot (\mathbb{C} : \hat{\boldsymbol{\varepsilon}}) = \boldsymbol{n} \cdot \mathbb{C} : \boldsymbol{\varepsilon}\left(\boldsymbol{\xi}_0\right) & \text{on } \partial \omega_\rho \end{cases} . \tag{14}$$

For the case that the domain consist of isotropic linear elastic material, Eq. (14) has an analytical asymptotic solution [4,7,11], thus Eq. (11) can be further simplified and the topological derivative $D_T y(\Omega, \boldsymbol{\xi}_0)$ has a concrete form

$$D_T y(\Omega, \boldsymbol{\xi}_0) = [\mathbb{C} : \tilde{\boldsymbol{\varepsilon}}(\boldsymbol{\xi}_0)] : \mathbb{A} : [\mathbb{C} : \boldsymbol{\varepsilon}(\boldsymbol{\xi}_0)] \tag{15}$$

in which $\mathbb{A}$ is a fourth order tensor related to Young's modulus E and Poisson's ratio ν as follows

$$\mathbb{A} = \frac{2\pi}{3E(7 - 5\nu)} \left[2\left(11 + \nu - 10\nu^2\right)\mathbb{I} - 3\left(1 + 4\nu - 5\nu^2\right)\mathbf{I} \otimes \mathbf{I}\right] \tag{16}$$

where $\mathbb{I}$ is the symmetric fourth order identity tensor and $\boldsymbol{I}$ is the second order identity tensor.

For anisotropic linear elastic materials or nonlinear materials, the analytical solution for Eq. (14) is hardly available, so further investigations or alternatives are in demands.

3.2 Topology Sensitivity of Stochastic Moments

Let $y(\Omega, \mathbf{X})$ be a response function of the linear system (9) subject to certain random input $\mathbf{X}$. For a point $\boldsymbol{\xi}_0 \in \Omega$, taking topology derivative of rth moments of the response function $y(\Omega, \mathbf{X})$ and applying the Lebesgue dominated convergence theorem, which permits interchange of the differential and integral operators, yields

$$\begin{aligned} D_T m^{(r)}(\Omega, \boldsymbol{\xi}_0) &:= D_T \mathbb{E}\left[y^r(\Omega, \mathbf{X})\right]\big|_{\boldsymbol{\xi}_0} = \int_{\mathbb{R}^N} r y^{r-1}(\Omega, \mathbf{X}) D_T y(\Omega, \mathbf{X}, \boldsymbol{\xi}_0) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \\ &= \mathbb{E}\left[r y^{r-1}(\Omega, \mathbf{X}) D_T y(\Omega, \mathbf{X}, \boldsymbol{\xi}_0)\right] \end{aligned} \tag{17}$$

that is, the topology derivative is obtained from the expected value of the response function multiplied by its topology derivative.

For simplicity, we denote $D_T y(\Omega, \mathbf{X}, \boldsymbol{\xi}_0)$ by $z(\Omega, \mathbf{X}, \boldsymbol{\xi}_0)$, existing an S-variate, mth-order PDD approximation $\tilde{z}_{S,m}$ as

$$\tilde{z}_{S,m}(\Omega, \mathbf{X}, \boldsymbol{\xi}_0) := z_\emptyset(\Omega, \boldsymbol{\xi}_0) + \sum_{\substack{\emptyset \neq u \subseteq \{1, \cdots, N\} \\ 1 \le |u| \le S}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}^{|u|} \\ \left\|\mathbf{j}_{|u|}\right\|_\infty \le m}} D_{u\mathbf{j}_{|u|}}(\Omega, \boldsymbol{\xi}_0) \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u; \Omega), \tag{18}$$

Replacing y and $D_T y$ of Eq. (17) by their S-variate, mth-order PDD approximations $\tilde{y}_{S,m}$ and $\tilde{z}_{S,m}$, respectively, we have

$$D_T \tilde{m}^{(r)}_{S,m}(\Omega, \boldsymbol{\xi}_0) = \mathbb{E}\left[r \tilde{y}^{r-1}_{S,m}(\Omega, \mathbf{X}) \tilde{z}_{S,m}(\Omega, \mathbf{X}, \boldsymbol{\xi}_0)\right] \tag{19}$$

For $r = 1, 2, 3$, employing the zero mean property and orthonormal property of the PDD basis $\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u; \Omega)$ yields analytical formulation for topology sensitivity of first three moments

$$D_T \tilde{m}^{(1)}_{S,m}(\Omega, \boldsymbol{\xi}_0) = z_\emptyset(\Omega, \boldsymbol{\xi}_0), \tag{20}$$

$$D_T \tilde{m}^{(2)}_{S,m}(\Omega, \boldsymbol{\xi}_0) = 2 \times \left[y_\emptyset(\Omega) z_\emptyset(\Omega, \boldsymbol{\xi}_0) + \sum_{\substack{\emptyset \neq u \subseteq \{1, \cdots, N\} \\ 1 \le |u| \le S}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}^{|u|} \\ ||\mathbf{j}_{|u|}||_\infty \le m}} C_{u\mathbf{j}_{|u|}}(\Omega) D_{u\mathbf{j}_{|u|}}(\Omega, \boldsymbol{\xi}_0) \right], \tag{21}$$

$$D_T\tilde{m}^{(3)}_{S,m}(\Omega,\boldsymbol{\xi}_0) = 3\times\left[z_{\emptyset}(\Omega,\boldsymbol{\xi}_0)\tilde{m}^{(2)}_{S,m}(\Omega) + 2y_{\emptyset}(\Omega)\sum_{\substack{\emptyset\neq u\subseteq\{1,\cdots,N\}\\ 1\leq|u|\leq S}}\sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}^{|u|}\\ ||\mathbf{j}_{|u|}||_\infty\leq m}} C_{u\mathbf{j}_{|u|}}(\Omega)D_{u\mathbf{j}_{|u|}}(\Omega,\boldsymbol{\xi}_0) + T_k\right], \quad (22)$$

$$T_k = \sum_{\substack{\emptyset\neq u,v,w\subseteq\{1,\cdots,N\}\\ 1\leq|u|,|v|,|w|\leq S}}\sum_{\substack{\mathbf{j}_{|u|},\mathbf{j}_{|v|},\mathbf{j}_{|w|}\in\mathbb{N}^{|u|}\\ ||\mathbf{j}_{|u|}||_\infty,||\mathbf{j}_{|v|}||_\infty,||\mathbf{j}_{|w|}||_\infty\leq m}} C_{u\mathbf{j}_{|u|}}(\Omega)C_{v\mathbf{j}_{|v|}}(\Omega)D_{w\mathbf{j}_{|w|}}(\Omega,\boldsymbol{\xi}_0)$$
$$\times\mathbb{E}_{\mathbf{d}}\left[\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\Omega)\psi_{v\mathbf{j}_{|v|}}(\mathbf{X}_v;\Omega)\psi_{wj_{|w|}}(\mathbf{X}_w;\Omega)\right], \quad (23)$$

which requires expectations of various products of three random orthonormal polynomials. However, if $\mathbf{X}$ follows Classical distributions such as Gaussian, Exponential, and Uniform distribution, then the expectations are easily determined from the properties of univariate Hermite, Laguerre, and Legendre polynomials [3,15,16]. For general distribution, numerical integration methods will apply.

4 Calculation of PDD Coefficients

The determination of truncated PDD expansion coefficients $y_{\emptyset}(\Omega)$ and $C_{u\mathbf{j}_{|u|}}(\Omega)$, where $\emptyset\neq u\subseteq\{1,\cdots,N\}$ and $\mathbf{j}_{|u|}\in\mathbb{N}^{|u|}$; $|\mathbf{j}_{|u|}||_\infty\leq m_u$, is vitally important for calculating the statistical moments and probability of failure, as well as associated design sensitivities, of the responses of interest. The PDD coefficients in Eq. (2), require calculations of various N-dimensional integrals over $\mathbb{R}^N$. For large N, it is computationally prohibitive to perform a N-dimensional numerical integration by an N-dimensional tensor product of a univariate quadrature rule. Therefore, the dimension-reduction integration (DRI) scheme, developed by Xu and Rahman [22], is adopted in this work to evaluate the coefficients accurately and effectively [22,23].

Let $\mathbf{c}=(c_1,\cdots,c_N)^T\in\mathbb{R}^N$, which is commonly adopted as the mean of $\mathbf{X}$, be a reference point, and $y(\Omega,\mathbf{x}_v,\mathbf{c}_{-v})$ represent an $|v|$-variate RDD component function of $y(\Omega,\mathbf{x})$, where $v\subseteq\{1,\cdots,N\}$. Given a positive integer $S\leq R\leq N$, when $y(\Omega,\mathbf{x})$ in Eqs. (4) and (5) is replaced with its R-variate RDD approximation, the coefficients $y_{\emptyset}(\Omega)$ and $C_{u\mathbf{j}_{|u|}}(\Omega)$ are estimated from [22]

$$y_{\emptyset}(\Omega)\cong\sum_{i=0}^{R}(-1)^i\binom{N-R+i-1}{i}\sum_{\substack{v\subseteq\{1,\cdots,N\}\\ |v|=R-i}}\int_{\mathbb{R}^{|v|}} y(\Omega,\mathbf{x}_v,\mathbf{c}_{-v})f_{\mathbf{X}_v}(\mathbf{x}_v)d\mathbf{x}_v \quad (24)$$

and

$$C_{u\mathbf{j}_{|u|}}(\Omega) \cong \sum_{i=0}^{R}(-1)^i \binom{N-R+i-1}{i} \sum_{\substack{v\subseteq\{1,\cdots,N\}\\ |v|=R-i, u\subseteq v}} \int_{\mathbb{R}^{|v|}} y(\Omega,\mathbf{x}_v,\mathbf{c}_{-v})\psi_{u\mathbf{j}_{|u|}}(\mathbf{x}_u,\Omega) f_{\mathbf{X}_v}(\mathbf{x}_v)d\mathbf{x}_v, \tag{25}$$

respectively, requiring evaluation of at most R-dimensional integrals. The reduced integration facilitates calculation of the coefficients approaching their exact values as $R \to N$ and is significantly more efficient than performing one N-dimensional integration, particularly when $R \ll N$. Hence, the computational effort is significantly lowered using the dimension-reduction integration. For instance, when $R = 1$ or 2, Eqs. (24) and (25) involve one-, or at most, two-dimensional integrations, respectively. Nonetheless, numerical integrations are still required for performing various $|v|$-dimensional integrals over $\mathbb{R}^{|v|}$, where $0 \le |v| \le R$.

5 Numerical Examples

Two examples, comprising a three-point bending beam and a three-hole bracket, are illustrated to examine the efficiency of the PDD methods developed for calculating the first-order topology sensitivities of statistical moments. The PDD expansion coefficients were estimated by dimension-reduction integration with the mean input as the reference point, $R = S$, and the number of Gauss points $n_g = m + 1$, where $S = 1$ and $m = 1$ for all two problems. In both examples, orthonormal polynomials and associated Gauss quadrature rules consistent with the probability distributions of input variables, including classical forms, if they exist, were employed. No unit for length, force, and Young's modulus is specified in both examples for simplicity, while permitting any consistent unit system for the results.

5.1 A Three Point Bending Beam

The three point bend test is a classical experiment in engineering mechanics, often employed to measure the Young's modulus of a material. In this example, we employ it to illustrate topology design sensitivities. Consider a beam, of length 300 and height 100, resting on two roller supports and is subject to a concentrated load F at its top centre as shown in Fig. 1, where $F \sim \mathcal{N}(10, 1)$ is a Gaussian random variable with mean value of 10 and standard deviation of 1. The beam consists of isotropic linear elastic material, of random Young's modulus E and deterministic Poisson's ratio 0.3, where $E \sim \mathcal{N}(100, 1)$ is another Gaussian random variable with mean value of 100 and standard deviation of 1. The response function of interest is the total compliance of the beam, that is

$$y(\Omega) = \int_{\Omega} \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon} d\Omega. \tag{26}$$

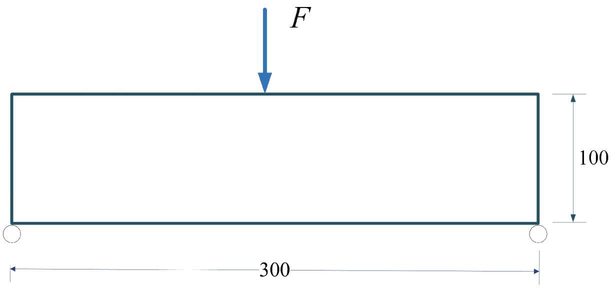

Fig. 1. A three point bending beam

Fig. 2. A three point bending beam: topology sensitivity of the first moment of the total compliance

The objective of this example is to evaluate the topology sensitivity of the first two moments for the compliance response, i.e. $D_T m^{(1)}(\Omega, \boldsymbol{\xi}_0)$ and $D_T m^{(2)}(\Omega, \boldsymbol{\xi}_0)$, for any point $\boldsymbol{\xi}_0$ in the beam. Figures 2 and 3 present the contours of approximate topology sensitivities, $D_T \tilde{m}^{(1)}_{S,m}(\Omega, \boldsymbol{\xi}_0)$ and $D_T \tilde{m}^{(2)}_{S,m}(\Omega, \boldsymbol{\xi}_0)$ by the proposed truncated PDD method. Based on the Eqs. (20) and (21), it is reasonable that two contour plots demonstrated a similar pattern by which the topology sensitivity varies with the location. For the particular response function, the total compliance, investigated in this example, the significant values of topology sensitivities of moments occurred in the location with high stress value, for instance, the top centre where the concentrated force is applied and the two lower corners where two roller supports are defined. Whereas two upper corners incurs negligible topology sensitivities, which can be inferred from the inappreciable stress of those locations.

5.2 A Three-Hole Bracket

This example involves topology sensitivity analysis of a two-dimensional, three-hole bracket. The bottom two holes are fixed, and a deterministic horizontal force F is applied at the center of the top hole as shown in Fig. 4, where $F \sim \mathscr{N}(15000, 1500)$ is a Gaussian random variable with mean value of 15000 and

Fig. 3. A three point bending beam: topology sensitivity of the second moment of the total compliance

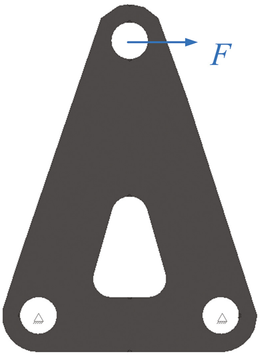

Fig. 4. A three-hole bracket

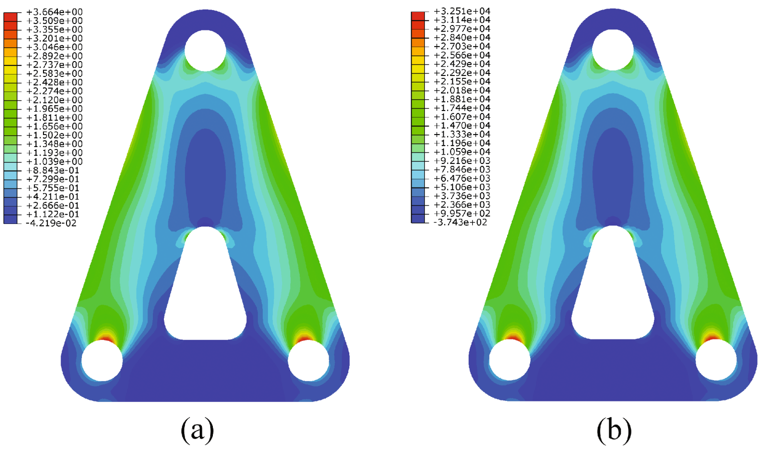

Fig. 5. A three-hole bracket: topology sensitivity for (a) the first moment and (b) the second moment of the total compliance

standard deviation of 1500. The bracket material has random elastic modulus E and deterministic Poisson's ratio $\nu = 0.3$, where E follows the Gaussian distribution with mean value of 207400 and standard deviation of 20740. The total compliance $y(\Omega) = \int_\Omega \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon} d\Omega$ is also selected as the response function.

The topology sensitivity of the first two moments for the compliance response, i.e. $D_T m^{(1)}(\Omega, \boldsymbol{\xi}_0)$ and $D_T m^{(2)}(\Omega, \boldsymbol{\xi}_0)$, are calculated by the proposed method. Their contours are presented in Fig. 5(a) and (b). In the same manner with the beam example, two contour plots demonstrated a similar pattern by which the topology sensitivity varies with the location. The location with high stress value likewise has the significant value of topology sensitivities. To obtain the contour of topology sensitivity, the total number of FEA required in either beam example or bracket example is 5, owing to the truncated polynomial dimensional decomposition of the response function and its deterministic topology derivatives.

6 Conclusions

The novel computational method grounded in PDD was developed for topology sensitivity analysis of high-dimensional complex systems subject to random inputs. The proposed method, capitalizing on a novel integration of PDD and deterministic topological derivatives, provides analytical expressions of approximate topology sensitivities of the first three moments that are mean-square convergent. Both the statistical moments and their topology sensitivities are determined concurrently from a single stochastic analysis. Numerical results indicate that the new methods developed provide computationally efficient solutions. The future work can be envisioned at least from the following three aspects: (1) convergence study for different S and m values, (2) subtly designed examples, for which the analytical topology sensitivity exists, to benchmark the accuracy of the proposed method, and (3) cases involving nonlinear materials.

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