STABLE STRATA OF GEODESICS IN OUTER SPACE

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ABSTRACT. In this paper we propose an Outer space analogue for the principal stratum of the unit tangent bundle to the Teichmüller space $\mathcal{T}(S)$ of a closed hyperbolic surface S. More specifically, we focus on properties of the geodesics in Teichmüller space determined by the principal stratum. We show that the analogous Outer space "principal" periodic geodesics share certain stability properties with the principal stratum geodesics of Teichmüller space. We also show that the stratification of periodic geodesics in Outer space exhibits some new pathological phenomena not present in the Teichmüller space context.

1. INTRODUCTION

Let S be a closed oriented surface of genus ≥ 2 and let $\mathcal{T}(S)$ be the Teichmüller space of S. Recall that the unit (co)tangent bundle to $\mathcal{T}(S)$ is canonically identified with the space $\mathcal{Q}^1(S)$ of unit area holomorphic quadratic differentials on S. The space $\mathcal{Q}^1(S)$ has a natural stratification, invariant under the Teichmüller geodesic flow, according to the orders of zeros of a quadratic differential, with the *principal stratum* $\mathcal{Q}^1_{princ}(S)$ consisting of quadratic differentials where all zeros are simple. This stratification of $\mathcal{Q}^1(S)$ defines the corresponding stratification of the space $\mathbb{G}\mathcal{T}(S)$ of all bi-infinite directed Teichmüller geodesics in $\mathcal{T}(S)$, by looking at the unit tangent vector to $L \in \mathbb{G}\mathcal{T}(S)$ at some (equivalently, any) point $X \in L$. Thus we also get a subset $\mathbb{G}_{princ}\mathcal{T}(S) \subseteq \mathbb{G}\mathcal{T}(S)$ consisting of all Teichmüller geodesics $L \in \mathbb{G}\mathcal{T}(S)$ with defining tangent vectors in $\mathcal{Q}^1_{princ}(S)$.

By the classic work of Thurston, the space $PMF(S)$ of projective measured foliations on S can be viewed as the Thurston boundary $\partial\mathcal{T}(S)$ of $\mathcal{T}(S)$. Note that there is no canonical stratification of $\partial\mathcal{T}(S)$ corresponding to the stratification of $\mathcal{Q}^1(S)$ discussed above. Indeed, one can show that for every point $[\mu] \in \partial\mathcal{T}(S)$ there exists some Teichmüller geodesic ray $\rho \subseteq \mathcal{T}(S)$ such that the initial tangent vector of ρ belongs to $\mathcal{Q}^1_{princ}(S)$. However, for a pseudo-Anosov $g \in MCG(S)$ there is a well-defined notion of g being *principal*, which corresponds to the bi-infinite g-periodic geodesic in $\mathcal{T}(S)$ being defined by a tangent vector from $\mathcal{Q}^1_{princ}(S)$, or equivalently, to the stable foliation $[\mu_+(g)]$ coming from a quadratic differential in the principal stratum $\mathcal{Q}^1_{princ}(S)$.

An important result of Kaimanovich and Masur [KM96] concerns the boundary behavior of a random walk on the mapping class group $MCG(S)$, satisfying some mild restrictions. They proved that for every basepoint $X \in \mathcal{T}(S)$, for almost every trajectory ω of this random walk, projecting ω to $\mathcal{T}(S)$ from the basepoint X via the orbit map gives a sequence in $\mathcal{T}(S)$ that converges to a uniquely ergodic point of $\partial\mathcal{T}(S) = PMF(S)$. Thus we get an

exit measure ν_X on $\partial \mathcal{T}(S)$ for the projected random walk in $\mathcal{T}(S)$ starting at X. This exit measure is supported on the set $\mathcal{UE} \subseteq PMF(S)$ of uniquely ergodic projective measured foliations. Maher [Mah11] later showed that, again under some mild assumptions on the random walk, the element $g_n \in MCG(S)$, obtained after n steps of the walk, is pseudo-Anosov with probability tending to 1 and $n \to \infty$. (Rivin [Riv08] had earlier proved the same conclusion about g_n for the simple random walk on $MCG(S)$ corresponding to a finite generating set of $MCG(S)$.) However, until recently, little else has been known about the properties of a "random" point of $\partial \mathcal{T}(S)$ corresponding to the exit measure ν_X, or about the properties of the stable foliation $[\mu_+(g_n)]$ of the pseudo-Anosov g_n as above.

In [GM17], Gadre and Maher shed light on these questions. They proved that if the support of a random walk on $MCG(S)$ is "sufficiently large" and contains a principal pseudo-Anosov g, then for every $X \in \mathcal{T}(X)$ and for ν_X-a.e. point $\xi \in \mathcal{UE} \subseteq \partial \mathcal{T}(S)$, the Teichmüller geodesic from X to ξ has its initial tangent vector in $\mathcal{Q}^1_{princ}(S)$. They also proved that in this setting, with probability tending to 1 as $n \to \infty$, after n steps the random walk produces a principal pseudo-Anosov $g_n \in MCG(S)$. Gadre and Maher also obtained the following stability result for principal axes. For X, Y in the axis $L_g \subseteq \mathcal{T}(S)$ of g and for $R \geq 0$, denote by $\Gamma_R(X, Y)$ the collection of all oriented bi-infinite Teichmüller geodesics $L \subseteq \mathcal{T}(S)$ with uniquely ergodic vertical and horizontal foliations such that $B(X, R) \cap L \neq \emptyset$, $B(Y, R) \cap L \neq \emptyset$ and such that the first point in $L \cap (B(X, R) \cup B(Y, R))$ belongs to $B(X, R)$. Here the balls $B(X, R)$ and $B(Y, R)$ are taken with respect to the Teichmüller metric on $\mathcal{T}(S)$. A crucial ingredient in the proof of the main results of [GM17] is the following "stability" property of L_g for a principal pseudo-Anosov g; see [GM17, Proposition 2.7]:

Theorem 1.1 (Gadre-Maher). *Let S be a closed oriented surface of genus ≥ 2 and let $g \in MCG(S)$ be a principal pseudo-Anosov. Then for any $R \geq 0$ there exists $D > 0$ such that if $X, Y \in L_g$ have $d(X, Y) \geq D$, then every $L \in \Gamma_R(X, Y)$ is principal.*

We are interested in investigating the corresponding questions in the $Out(F_r)$ setting, where $r \geq 2$. Similar to the mapping class group setting, it is by now well-known [MT14, TT16] that, under some mild assumption on the support, for a random walk on $Out(F_r)$, an element $\varphi_n \in Out(F_r)$ obtained after n steps of the walk is atoroidal fully irreducible with probability tending to 1 as $n \to \infty$. It is also known that projecting a random orbit of this walk to the Culler-Vogtmann Outer space CV_r (starting at some basepoint $X \in CV_r$) gives a sequence in CV_r that with probability 1 converges to some point in $[T] \in \partial CV_r$ and, moreover, that the $\mathbb{R}$-tree T is uniquely ergodic [NPR14]. The proofs that φ_n is generically fully irreducible and atoroidal involve projecting a random walk on $Out(F_r)$ to the free factor complex in the first case and to the co-surface graph in the second case. These are both Gromov hyperbolic and one argues that φ_n acts loxodromically on the hyperbolic graph in question. Addressing the index properties, Kapovich and Pfaff [KP15] proved that for a "train-track directed" random walk on $Out(F_r)$, the element φ_n is, with an asymptotically positive probability, an ageometric fully irreducible outer automorphism with a 1-element index list $\{\frac{3}{2} - r\}$ and that the corresponding ideal Whitehead graph is complete (the relevant definitions are discussed next below). We wish to understand how this statement generalizes to the case of a more general random walk on $Out(F_r)$.

One of the difficulties in the $Out(F_r)$ setting is finding a suitable notion of a "principal stratum." In the original context of a closed hyperbolic surface X, if $[\mu] \in PMF(S)$ is uniquely ergodic and with the dual $\mathbb{R}$-tree having all branch-points being trivalent, then

$[\mu] \in \mathcal{Q}_{princ}$. This fact motivates us to use the index properties of a geodesic in Outer space when defining strata in the space of such geodesics.

Given a nongeometric fully irreducible $\varphi \in Out(F_r)$ (where "nongeometric" means that φ is not induced by a homeomorphism of a surface S with $\pi_1(S) \cong F_r$) one can define a conjugacy class invariant called the *ideal Whitehead graph* $IW(\varphi)$ of φ (see §2.6 for details). The graph $IW(\varphi)$ captures essential information about the structure of the attracting lamination of φ and therefore of branch-points of the stable $\mathbb{R}$-tree T_φ of φ (as well as about the interaction of "directions" in T_φ at those branch-points). The graph $IW(\varphi)$ can be read-off, via an explicit procedure, from any train track representative of φ. In addition, one can also define the *index list* for φ (recording the sizes of components of $IW(\varphi)$), and the *index sum* $i(\varphi)$, obtained by summing up the numbers in the index list of φ. Unlike in the case of pseudo-Anosovs (we remind the reader that the invariant foliations for a pseudo-Anosov cannot have saddle connections), there may be many types of Ideal Whitehead graphs with the same index list and we shall see that stability properties are related to the graph types rather than the index lists (or sums). We say that a finite graph $\mathcal{G}$ is *r-dominant* if $\mathcal{G}$ is a union of complete graphs, each with ≥ 3 vertices, and if the index sum of $\mathcal{G}$ is $\frac{3}{2} - r$. Of special interest is the r-dominant graph all of whose components are triangles, we denote it Δ_r.

Let $r \geq 3$ and let CV_r denote the (projectivized) Culler-Vogtmann Outer space for the free group F_r. We denote by $\mathcal{F}_r$ the set of all bi-infinite fold lines in CV_r, where folds are performed one at a time (see Definition 2.16 below for the precise formulation). Note that all elements of $\mathcal{F}_r$ are bi-infinite geodesics for the asymmetric Lipschitz metric on CV_r. We denote by $\mathcal{A}_r$ the set of all "axes," i.e. the set of all $L \in \mathcal{F}_r$ such that L is a periodic fold line defined by an expanding irreducible train track representative of some $\varphi \in Out(F_r)$. We endow $\mathcal{F}_r$ with a natural topology, where for $L, L' \in \mathcal{F}_r$, the line L' is "close" to L if there exist a "large" $R \geq 1$ and a "small" $\varepsilon > 0$ such that some subsegment J of L' of length R is contained in the ε-neighborhood of L (with respect to the symmetrized Lipschitz metric on CV_r). See Definition 3.2 below for details. All of the various subsets of $\mathcal{F}_r$ discussed below are then given the subspace topology.

Definition 1.2 (Dominant and principal strata, and their basins)**.** Let $r \geq 3$ and let $\mathcal{G}$ be a graph. We define the $\mathcal{G}$-*basin* $B\mathcal{S}_r(\mathcal{G}) \subseteq \mathcal{A}_r$ as the set of all $L \in \mathcal{A}_r$ such that L is φ-periodic for $\varphi \in Out(F_r)$ ageometric fully irreducible and satisfying that $IW(\varphi)$ is a union of components of $\mathcal{G}$. We define the $\mathcal{G}$-*stratum* $\mathcal{S}_r(\mathcal{G}) \subseteq B\mathcal{S}_r(\mathcal{G})$ as those lines for which the corresponding φ satisfies $IW(\varphi) \cong \mathcal{G}$. Thus $\mathcal{S}_r(\mathcal{G}) \subseteq B\mathcal{P}_r(\mathcal{G}) \subseteq \mathcal{A}_r$.

If $\mathcal{G}$ is an r-dominant (resp. r-principal) graph, we say $\mathcal{S}_r(\mathcal{G})$ is a *dominant stratum* (resp. $\mathcal{P}_r$ is the r-principal stratum) and $B\mathcal{S}_r(\mathcal{G})$ is dominant (resp. $B\mathcal{P}_r$ is the principal basin).

The results of Mosher-Pfaff [MP16] imply that if $\varphi \in Out(F_r)$ is $\mathcal{G}$-dominant for some r-dominant graph $\mathcal{G}$, then φ is a "lone axis" fully irreducible outer automorphism, i.e. φ has a unique axis in CV_r. In particular, this fact applies to all principal $\varphi \in Out(F_r)$.

Recall that CV_r is a simplicial complex of dimension $3r - 4$ (with some faces missing). For an integer $k \geq 0$, we will denote by $CV_r^{(k)}$ the k-skeleton of CV_r. Our main result is the following attracting/stability property for dominant strata:

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Theorem A. *Let $r \geq 3$ and let $\mathcal{G}$ be an r-dominant graph. Let $L \in \mathcal{S}_r(\mathcal{G})$ and k the minimal integer so that $L \subset CV_r^{(k)}$. Then there exists a neighborhood $U \subseteq \mathcal{A}_r$ of L with the following properties:*

(a) For each $L' \in U$ with $L' \subseteq CV_r^{(k)}$, we have $L' \in B\mathcal{S}_r(\mathcal{G})$.

(b) For each $L' \in U$ with $L' \subseteq CV_r^{(k)}$ and with L' containing no full folds, we have $L' \in \mathcal{S}_r(\mathcal{G})$.

See Definition 2.18 for terminology regarding full folds. Note, the conclusion of Theorem A implies that each $L' \in U$ is an axis of an ageometric fully irreducible element of $Out(F_r)$. Moreover, in the case of (b), L' is the unique axis of that fully irreducible in CV_r ([MP16]).

Our results suggest that for a reasonable random walk on $Out(F_r)$, for a random fully irreducible $\varphi_n \in Out(F_r)$ obtained after n steps of the walk, there are several possibilities for $IW(\varphi_n)$ that each occur with an asymptotically positive probability as $n \to \infty$.

Question 1.3. *Is the conclusion of Theorem A true only for an r-dominant $\mathcal{G}$?*

It turns out that it is, in general, not possible to replace $B\mathcal{S}_r(\mathcal{G})$ by $\mathcal{S}_r(\mathcal{G})$ in the conclusion of Theorem A(a) above. We show that certain kinds of pathologies exist that can force $L' \in U$ to fall out of the dominant $\mathcal{G}$-stratum and that the best one can conclude is that $L' \in B\mathcal{S}_r(\mathcal{G})$:

Theorem B. *There exists a principal fully irreducible outer automorphism $\varphi \in Out(F_3)$ with a train track representative $f \colon \Gamma \to \Gamma$ with a Stallings fold decomposition $\mathfrak{f}$, such that for every $n \geq 1$ there exists a nonprincipal fully irreducible outer automorphism $\psi_n \in Out(F_3)$ with a train track representative $g_n \colon \Gamma \to \Gamma$ with a Stallings fold decomposition $\mathfrak{g}_n$ such that $\mathfrak{g}_n$ starts with $\mathfrak{f}^n$.*

Theorem B immediately implies:

Corollary C. *For $r = 3$, there exist a principal periodic geodesic $L \in \mathcal{P}_r$ in CV_r and a sequence of nonprincipal periodic geodesics $\{L_n\}_{n=1}^\infty \subseteq B\mathcal{P}_r - \mathcal{P}_r$ such that $\lim_{n \to \infty} L_n = L$.*

The cause of the pathologies exhibited in Theorem B is that the folding process may identify vertices. Hence, some f-periodic vertices of Γ may become nonperiodic for g_n. Such vertices contribute to $IW(\varphi)$ but not to $IW(\psi_n)$.

2. Background & Definitions

Given a free group F_r of rank $r \geq 2$, we choose once and for all a free basis $A = \{X_1, \ldots, X_r\}$. Let $R_r = \vee_{i=1}^r \mathbb{S}^1$ denote the graph with one vertex and r edges. We choose also once and for all an orientation on R_r and identify each positive edge of R_r with an element of the chosen free basis. Thus, a cyclically reduced word in the basis corresponds to an immersed loop in R_r.

2.1. Outer space CV_r.

Definition 2.1 (Marked metric F_r-graph)**.** Let $r \geq 2$ be an integer. A *marked metric graph* for F_r is a triple (Γ, m, ℓ) which satisfies:

- Γ is a finite 1-dimensional CW complex, with the 0-cells *vertices* and the 1-cells *edges*.
- For each vertex v, $deg(v) \geq 3$.
- Each open 1-cell e of Γ is given a positive *length* $L(e) > 0$, and Γ is endowed with a metric ℓ such that for each open 1-cell e of Γ there is a locally isometric bijection between e and the interval $(0, L(e)) \subseteq \mathbb{R}$.
- m is a homotopy equivalence $m \colon R_r \to \Gamma$, which we call a *marking*.

Notation 2.2. We use the following notation.

(1) We sometimes write (Γ, m) for (Γ, m, ℓ) if the metric is otherwise clear or irrelevant.
(2) Given a graph Γ (metric or topological), we let $E(\Gamma)$ denote the set of oriented edges of Γ and let $V(\Gamma)$ denote the vertex set of Γ.

Definition 2.3 (Change of marking & marked graph equivalence)**.** Let (Γ, m) and (Γ', m') be marked graphs. Then a *change of marking* is a continuous map $f \colon \Gamma \to \Gamma'$ so that m' is homotopic to $f \circ m$. Two F_r-marked (metric) graphs (Γ, m) and (Γ', m') are *equivalent* if there exists an isometric change of marking $\varphi \colon \Gamma \to \Gamma'$.

Definition 2.4 (Unprojectivized Outer space)**.** The *(rank-r) unprojectivized Outer space* $\widehat{CV}_r$ is the space of equivalence classes of F_r-marked metric graphs. By abuse of notation, we usually still denote the equivalence class of (Γ, m) by (Γ, m), or of (Γ, m, ℓ) by (Γ, m, ℓ).

For a marked metric graph (Γ, m) denote by $vol(\Gamma, m, \ell)$, or just $vol(\Gamma)$, the sum of the ℓ-lengths of the 1-cells in Γ. Note that $vol(\Gamma, m, \ell)$ is preserved by the above equivalence relation, so that $vol(\Gamma, m)$ is well-defined for points of $\widehat{CV}_r$.

Definition 2.5 ((Projectivized) Outer space)**.** Let $r \geq 2$ be an integer. For each $r \geq 2$ the *(rank-r) Outer space* CV_r is the set of $(\Gamma, m, \ell) \in \widehat{CV}_r$ with $vol(\Gamma, m, \ell) = 1$. There is a map from $q \colon \widehat{CV}_r \to CV_r$ normalizing the graph volume, i.e. if (Γ, m, ℓ) is a marked metric graph, then $q(\Gamma, m, \ell) = (\Gamma, m, \frac{1}{vol(\Gamma, m, \ell)}\ell)$.

Note that $\mathbb{R}_{>0}$ has a natural action on $\widehat{CV}_r$ by multiplying the metric on Γ by a positive real number. There is a canonical identification between CV_r and the quotient set $\widehat{CV}_r / \mathbb{R}_{>0}$ and we will usually not distinguish between these two sets.

Definition 2.6 (Simplicial structure on CV_r)**.** Let Γ be a topological graph and $m \colon R_r \to \Gamma$ a homotopy equivalence, so that (Γ, m) is a marked graph. We denote the *simplex* σ in CV_r corresponding to (Γ, m) by

$$\sigma_{(\Gamma, m)} := \{(\Gamma, m, \ell) \in CV_r\}.$$

By enumerating $E(\Gamma)$, we can identify $\sigma_{(G, \mu)}$ with the open simplex

$$\Sigma_{|E|} = \left\{ \vec{v} \in \mathbb{R}_+^{|E|} \ \middle| \ \sum_{i=1}^{|E|} v_i = 1 \right\}.$$

Definition 2.7 ($CV_r^{(k)}$)**.** We let $CV_r^{(k)}$ denote the k-skeleton of CV_r.

Definition 2.8 (Simplicial metric)**.** Given an open simplex $\sigma_{(\Gamma,m)}$ in CV_r, the *simplicial metric* on $\sigma_{(\Gamma,m)}$ is the Euclidean metric on $\Sigma_{|E|}$. We also denote by d_{simp} the extension of this metric to a path metric on CV_r. (There is another (asymmetric) metric on Outer space see Definition 2.21).

Definition 2.9 (Topology on $\widehat{CV}_r$)**.** We call the full preimage under q (see Definition 2.4) of a simplex in CV_r an *unprojectivized simplex* in $\widehat{CV}_r$. The unprojectivized Outer space $\widehat{CV}_r$ is topologized by giving it the the structure of an ideal simplicial complex built from (unprojectivized) open simplices (see [Vog02] for details). Faces of $\sigma_{(\Gamma,m)}$ arise by letting the edges of a tree in Γ have length 0. The projectivized outer space $CV_r \subseteq \widehat{CV}_r$ is given the subspace topology, and can also be thought of as an ideal simplicial complex built from open simplices. The subspace topology on CV_r coincides with the quotient topology on $\widehat{CV}_r / \mathbb{R}_{>0}$.

2.2. Train track maps & gate structures.

Definition 2.10 (Graph maps & train track maps)**.** We call a continuous map of graphs $g\colon \Gamma \to \Gamma'$ a *graph map* if it takes vertices to vertices and is locally injective on the interior of each edge. A self graph map $g\colon \Gamma \to \Gamma$ is a *train track map* if g is a homotopy equivalence and if for each $k \geq 1$ the map g^k is locally injective on edge interiors.

We call the train track map g *expanding* if for each edge $e \in E(\Gamma)$ we have that $|g^n(e)| \to \infty$ as $n \to \infty$, where for a path γ we use $|\gamma|$ to denote the number of edges γ traverses (with multiplicity).

Definition 2.11 (Directions)**.** For each $x \in \Gamma$ we let $\mathcal{D}(x)$ denote the set of *directions* at x, i.e. germs of initial segments of edges emanating from x. For each edge $e \in E(\Gamma)$, we let $D(e)$ denote the initial direction of e. For an edge-path $\gamma = e_1 \ldots e_k$, we let $D\gamma = D(e_1)$. Let $g\colon \Gamma \to \Gamma'$ be a graph map. We denote by Dg the map of directions induced by g, i.e. $Dg(d) = D(g(e))$ for $d = D(e)$. For a self-map, i.e. one where $\Gamma = \Gamma'$, a direction d is *periodic* if $Dg^k(d) = d$ for some $k > 0$ and *fixed* when $k = 1$.

Definition 2.12 (Turns & gates)**.** Let $g\colon \Gamma \to \Gamma'$ be a graph map. We call an unordered pair of directions $\{d_i, d_j\}$ a *turn*, and a *degenerate turn* if $d_i = d_j$. We denote by Tg the map induced by Dg on the turns of Γ. A turn τ is called g-*prenull* if $Tg(\tau)$ is degenerate. When $g\colon \Gamma \to \Gamma$ is a self-map, the turn τ is called an *illegal turn* for g if $Tg^k(\tau)$ is degenerate for some k and a *legal turn* otherwise. We call a g *transparent* if each illegal turn is prenull. Notice that every graph self-map has a transparent power.

Considering the directions of an illegal turn equivalent, one can define an equivalence relation on the set of directions at a vertex. We call the equivalence classes *gates* and call the partitioning of the directions at each vertex into gates the *induced gate structure*.

For a path $\gamma = e_1 e_2 \ldots e_{k-1} e_k$ in Γ where e_1 and e_k may be partial edges, we say γ *takes* $\{\overline{e_i}, e_{i+1}\}$ for each $1 \leq i < k$. For both edges and paths we more generally use an "overline" to denote a reversal of orientation. Given a graph map $g\colon \Gamma \to \Gamma'$, we say that a turn T in Γ' is *g-taken* if there exists an edge e so that $g(e)$ takes T. A path γ is *legal* with respect to a train track structure on Γ if γ only takes turns that are legal in this train track structure.

Definition 2.13 (Irreducible & fully irreducible)**.** We call a train track map *irreducible* if it has no proper invariant subgraph with a noncontractible component.

An outer automorphism $\varphi \in \mathrm{Out}(F_r)$ is *fully irreducible* if no positive power preserves the conjugacy class of a proper free factor of F_r. Bestvina and Handel [BH92] proved that every (fully) irreducible outer automorphism admits an irreducible train track representatives.

Definition 2.14 (Transition matrix, Perron-Frobenius matrix, Perron-Frobenius eigenvalue)**.** The *transition matrix* of a train track map $g \colon \Gamma \to \Gamma$ is the square $|E(\Gamma)| \times |E(\Gamma)|$ matrix (a_{ij}) such that a_{ij}, for each i and j, is the number of times $g(e_i)$ passes over e_j in either direction. A transition matrix $A = [a_{ij}]$ is *Perron-Frobenius (PF)* if there exists an N such that, for all $k \geq N$, A^k is strictly positive. By Perron-Frobenius theory, we know that each such matrix has a unique eigenvalue of maximal modulus and that this eigenvalue is real. This eigenvalue is called the *Perron-Frobenius (PF) eigenvalue* of A.

2.3. Fold lines.

Definition 2.15 (Fold lines)**.** A *fold line* in $\widehat{CV}_r$ is a continuous, injective, proper function $\mathbb{R} \to \widehat{CV}_r$ defined by a continuous 1-parameter family of marked graphs $t \to \Gamma_t$ and a family of differences of markings $\hat{h}_{st} \colon \hat{\Gamma}_t \to \hat{\Gamma}_s$ defined for $t \leq s \in \mathbb{R}$, satisfying:

    (1) $\hat{h}_{ts}$ is a local isometry on each edge for all $s \leq t \in \mathbb{R}$.
    (2) $\hat{h}_{us} \circ \hat{h}_{st} = \hat{h}_{ut}$ for all $t \leq s \leq u \in \mathbb{R}$ and $\hat{h}_{ss} \colon \Gamma_s \to \Gamma_s$ is the identity for all $s \in \mathbb{R}$.
A fold line in CV_r is the q-image (where q is the normalizing map, see Definition 2.5) of a fold line. We shall denote $q(\hat{\Gamma}_t)$ and $q \circ \hat{h}_{s,t}$ by Γ_t and $h_{s,t}$ respectively.

Definition 2.16 (Simple fold lines)**.** A fold line in Outer space $\mathbb{R} \to CV_r$ is said to be *simple* if there exists a subdivision of $\mathbb{R}$ by points $(t_i)_{i \in \mathbb{Z}}$

$$\ldots t_{i-1} < t_i < t_{i+1} \ldots$$

such that $\lim_{i \to \infty} t_i = \infty$, $\lim_{i \to -\infty} t_i = -\infty$ and such that the following holds:

    For each $i \in \mathbb{Z}$ there exist distinct edges e, e' in Γ_{t_i}, with a common initial vertex, such that: For each $s \in (t_i, t_{i+1}]$ the map $h_{st_i} \colon \Gamma_{t_i} \to \Gamma_s$ identifies an initial segment of e with an initial segment of e', with no other identifications (that is, h_{st_i} is injective on the complement of those two initial segments in Γ_{t_i}).

Remark 2.17 (Simple fold lines)**.** All fold lines that we consider in this paper will be simple.

Definition 2.18 (Stallings folds)**.** Stallings introduced "folds" in [Sta83]. Let $g \colon \Gamma \to \Gamma'$ be a homotopy equivalence of marked graphs. Let $e_1' \subset e_1$ and $e_2' \subset e_2$ be maximal, initial, nontrivial subsegments of edges e_1 and e_2 emanating from a common vertex and satisfying that $g(e_1') = g(e_2')$ as edge paths and that the terminal endpoints of e_1' and e_2' are distinct points in $g^{-1}(\mathcal{V}(\Gamma))$. Redefine Γ to have vertices at the endpoints of e_1' and e_2' if necessary. One can obtain a graph Γ_1 by identifying the points of e_1' and e_2' that have the same image under g, a process we will call *folding*.

    Let $\mathcal{F}$ be a fold of e_1 and e_2. We call $\mathcal{F}$ a *full fold* if the entirety of e_1 and e_2 are identified. We call $\mathcal{F}$ a *proper full fold* if only an initial subsegment of one of e_1 or e_2 is folded with the entirety of the other. We call $\mathcal{F}$ a *partial fold* if neither e_1 nor e_2 is entirely folded.

Definition 2.19 (Stallings fold decomposition)**.** Stallings [Sta83] also showed that if $g \colon \Gamma \to \Gamma'$ is a homotopy equivalence graph map, then g factors as a composition of folds and a final homeomorphism. We call such a decomposition a *Stallings fold decomposition*. It can be obtained as follows: At an illegal turn for $g \colon \Gamma \to \Gamma'$, one can fold two maximal initial

segments having the same image in Γ' to obtain a map $\mathfrak{g}_1 \colon \Gamma_1 \to \Gamma'$ of the quotient graph Γ_1. The process can be repeated for $\mathfrak{g}_1$ and recursively. If some $\mathfrak{g}_k \colon \Gamma_{k-1} \to \Gamma$ has no illegal turn, then $\mathfrak{g}_k$ will be a homeomorphism and the fold sequence is complete.
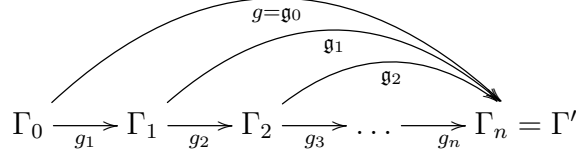

FIGURE 1. Constructing a Stallings folds decomposition

Notice that choices of illegal turns are made in this process and that different choices lead to different Stallings fold decompositions of the same homotopy equivalence.

When Γ is a marked metric graph (of volume 1), we obtain an induced metric on each Γ_k, which we may renormalize to be again of volume 1.

In [Sko89], Skora interpreted a Stallings fold decomposition for a graph map homotopy equivalence $g \colon \Gamma \to \Gamma'$ as a sequence of folds performed continuously. Let $g \colon \Gamma \to \Gamma$ be an irreducible train track map representing an outer automorphism $\varphi \in Out(F_r)$ and let λ be its Perron-Frobenius eigenvalue. Repeating a Stallings fold decomposition of g defines a periodic fold line in Outer space. The discretization of this fold line is depicted in Equation 1 below, where it should be noted that $\Gamma_{nK} = \frac{1}{\lambda^n} \Gamma_0 \cdot \varphi^n$, for each integer n.

$$(1) \qquad \dots \to \Gamma_0 \xrightarrow{g_1} \Gamma_1 \xrightarrow{g_2} \cdots \xrightarrow{g_K} \Gamma_K \xrightarrow{g_{K+1}} \Gamma_{K+1} \xrightarrow{g_{K+2}} \cdots \xrightarrow{g_{2K}} \Gamma_{2K} \xrightarrow{g_{2K+1}} \dots$$

Definition 2.20 (Periodic fold lines). Let $g \colon \Gamma \to \Gamma$ be an expanding irreducible train track map representing an outer automorphism $\varphi \in Out(F_r)$ and let $\lambda > 1$ be its Perron-Frobenius eigenvalue. If $g_1, \dots, g_k$ is a Stallings fold sequence for g, the process of Skora defines a path $\mathcal{L}_0 \colon [0, \log \lambda] \to CV_r$ so that the union of φ^k-translates of $\mathcal{L}_0$ for all k gives the entire fold line $\mathcal{L}$ determined by g, see Definition 2.19. That is, $\mathcal{L} \colon \mathbb{R} \to CV_r$ is defined by $\mathcal{L}(t) = \mathcal{L}_0(t - \lfloor \frac{t}{\log \lambda} \rfloor) \varphi^{\lfloor \frac{t}{\log \lambda} \rfloor}$. $\mathcal{L}$ is called a *periodic fold line* for φ or, if φ is fully irreducible, an *axis* for φ.

2.4. **Geodesics in Outer space.**

Definition 2.21 (Lipschitz metric). Given an ordered pair of points (X, Y) in the Outer space CV_r, the *Lipschitz distance* $d(X, Y)$ from $X = (\Gamma_X, m_X, \ell_X)$ to $Y = (\Gamma_Y, m_Y, \ell_Y)$ is defined as the logarithm of the minimal Lipschitz constant of a Lipschitz difference of markings. (It is known [FM11] this minimum is in fact realized and that $d(X, Y) \geq 0$, with $d(X, Y) = 0$ if and only if $X = Y$ in CV_r.) We sometimes also denote $d(X, Y)$ by $d_L(X, Y)$.

Let α be an element of F_r. We also denote by α the corresponding loop in the base rose R_r. Let $X = (\Gamma, m)$ be a point in Outer space. Denote by α_X the immersed loop (unique up to cyclic reparametrization) in X that is freely homotopic to $m(\alpha)$.

Definition 2.22 (Witness). It is proved in [FM11] that for each ordered pair of points (X, Y) in Outer space there exists an element α of F_r so that $\log \frac{len(\alpha_Y)}{len(\alpha_X)} = d(X, Y)$. We call each such α a *witness* of $d(X, Y)$ or of the change of marking from X to Y.

Definition 2.23 (Candidate). Let $X \in CV_r$. A loop in X whose image is an embedded circle, an embedded figure-8, or an embedded barbell is called a *candidate* of X.

Lemma 2.24. [FM11] *For each ordered pair of points (X, Y) in Outer space there exists a candidate loop in X that is a witness of $d(X, Y)$.*

Definition 2.25 (Geodesic). A map $\mathcal{L}\colon [0, \ell] \to CV_r$ is a *Lipshitz geodesic* if

(1) for all $s, t, r \in [0, \ell]$ so that $s \leq t \leq r$ we have

$$d(\mathcal{L}(s), \mathcal{L}(r)) = d(\mathcal{L}(s), \mathcal{L}(t)) + d(\mathcal{L}(t), \mathcal{L}(r)) \text{ and}$$

(2) there exists no $X_0 \in CV_r$ and no nontrivial subinterval $[a, b]$ of $[0, \ell]$ so that $\mathcal{L}(t) = X_0$ for all $t \in [a, b]$.

Lemma 2.26. *Let $\mathcal{L} = \{\Gamma_t\}_{t=0}^{\infty}$ be a Lipschitz geodesic ray. Then there exists an element $\alpha \in F_r$ so that, for each $t \geq 0$, if α_t denotes the immersed loop representing the conjugacy class of α in Γ_t, then α_t is a witness to $d(\Gamma_t, \Gamma_s)$ for each $t \leq s$.*

Proof. For each $0 \leq t \leq s$ we have $d(\Gamma_0, \Gamma_s) = d(\Gamma_0, \Gamma_t) + d(\Gamma_t, \Gamma_s)$. Let $f_{s,0}, f_{t,0}, f_{s,t}$ be optimal maps, i.e. maps that realize the equality in Definition 2.21, and let α be a witness loop for $d(\Gamma_0, \Gamma_s)$. Then $\frac{len(\alpha_s)}{len(\alpha_0)} = Lip f_{s,0} = Lip f_{s,t} Lip f_{t,0} \geq \frac{len(\alpha_s)}{len(\alpha_t)} \frac{len(\alpha_t)}{len(\alpha_0)}$. Thus, all of the inequalities are in fact equalities i.e. $Lip f_{s,t} = \frac{len(\alpha_s)}{len(\alpha_t)}$ and $Lip f_{t,0} = \frac{len(\alpha_t)}{len(\alpha_0)}$. Hence, α_0 is a witness for $f_{t,0}$ for all $t \leq s$ and α_t is a witness for $f_{s,t}$. Moreover, by Lemma 2.24, α_0 can be chosen to be a candidate in Γ_0. Notice that there are only finitely many candidate loops in Γ_0. Let Θ_s be the set of candidate loops that are $f_{s,0}$ witnesses. Then the sequence $\{\Theta_s\}_{s=0}^{\infty}$ consists of a non-empty decreasing finite sets of loops, hence stabilizes as $s \to \infty$. We let

$$\Theta_\infty := \bigcap_{s=0}^{\infty} \Theta_s.$$

Any $\alpha_0 \in \Theta_\infty$ is a witness for each $f_{s,0}$ with $0 \leq s$ and, by the discussion above, α_t is a witness of $f_{s,t}$ for each $0 \leq t \leq s$. $\qquad\square$

Lemma 2.27. *Let g be an expanding train track representative of $\varphi \in \mathrm{Out}(F_r)$ and $\mathcal{L}$ the periodic fold line determined by g as in Definition 2.20. Then $\mathcal{L}$ is a Lipschitz geodesic.*

Proof. For each interval $[t, s] \subset \mathbb{R}$, let $\mathcal{L}_{t,s}$ denote the restriction of $\mathcal{L}$ to $[t, s]$. It suffices to show $\mathcal{L}_{t,s}$ is a geodesic segment for each pair t, s of positive integer multiples of $\log \lambda$. For each such t, s the change of marking map from $\mathcal{L}(t)$ to $\mathcal{L}(s)$ is g^k, where $k = \frac{s-t}{\log \lambda}$. Given $\alpha \in F_r$, we denote by α_t the immersed loop in $\mathcal{L}(t)$ representing α. Let $\beta \in F_r$ satisfy that β_t is g-legal. Then $h_{u,t}(\beta_t) = \beta_u$ is immersed for each $t \leq u \leq s$, thus it is a witness for $d(\mathcal{L}(t), \mathcal{L}(u))$. Moreover, β_u is still not $h_{s,u}$-prenull for each $t \leq u \leq s$. Therefore, β_u is a witness for $d(\mathcal{L}(u), \mathcal{L}(s))$ for all $u \leq s$. Taking the logarithm of both sides of $\frac{\ell_s(\beta_s)}{\ell_t(\beta_t)} = \frac{\ell_u(\beta_u)}{\ell_t(\beta_t)} \frac{\ell_s(\beta_s)}{\ell_u(\beta_u)}$, we see that $\mathcal{L}$ is a geodesic. $\qquad\square$

2.5. Nielsen paths & principal points.

Definition 2.28 (Nielsen paths). Let $g\colon \Gamma \to \Gamma$ be an expanding irreducible train track map. Bestvina and Handel [BH92] define a nontrivial immersed path ρ in Γ to be a *periodic Nielsen path (PNP)* if, for some power $R \geq 1$, we have $g^R(\rho) \cong \rho$ rel endpoints (and just a *Nielsen path (NP)* if $R = 1$). An NP ρ is called *indivisible* (hence is an "iNP") if it cannot be written as $\rho = \gamma_1 \gamma_2$, where γ_1 and γ_2 are themselves NPs.

Definition 2.29 (Ageometric)**.** A fully irreducible outer automorphism is called *ageometric* if it has a train track representative with no PNPs.

Bestvina and Handel describe in [BH92, Lemma 3.4] the structure of iNPs:

Lemma 2.30 ([BH92])**.** *Let $g\colon \Gamma \to \Gamma$ be an expanding irreducible train track map and ρ an iNP for g. Then $\rho = \bar{\rho}_1\rho_2$, where ρ_1 and ρ_2 are nontrivial legal paths originating at a common vertex v and such that the turn at v between ρ_1 and ρ_2 is a nondegenerate illegal turn for g.*

Definition 2.31 (Principal points)**.** Given a train track map $g\colon \Gamma \to \Gamma$, following [HM11] we call a point *principal* that is either the endpoint of a PNP or is a periodic vertex with ≥ 3 periodic directions. Thus, in the absence of PNPs, a point is principal if and only if it is a periodic vertex with ≥ 3 periodic directions

Definition 2.32 (Rotationless)**.** An expanding irreducible train track map is called *rotationless* if each principal point and periodic direction is fixed and each PNP is of period one. By [FH11, Proposition 3.24], one then defines a fully irreducible $\varphi \in Out(F_r)$ to be *rotationless* if some (equivalently, all) of its train track representatives is rotationless.

2.6. **Whitehead graphs.** The following definitions are in [HM11] and [MP16].

Definition 2.33 (Whitehead graphs & indices)**.** Let $g\colon \Gamma \to \Gamma$ be a train track map. The *local Whitehead graph* $LW(v; \Gamma)$ at a point $v \in \Gamma$ has a vertex for each direction at v and an edge connecting the vertices corresponding to a pair of directions $\{d_1, d_2\}$ if the turn $\{d_1, d_2\}$ is g^k-taken for some $k \geq 0$. The *stable Whitehead graph* $SW(v; \Gamma)$ at a principal point v is the subgraph of $LW(v; \Gamma)$ obtained by restricting to the periodic direction vertices.

Let $g\colon \Gamma \to \Gamma$ be a PNP-free train track representative of a fully irreducible $\varphi \in Out(F_r)$. Then the *ideal Whitehead graph* $IW(\varphi)$ of φ is isomorphic to the disjoint union $\bigsqcup \mathcal{SW}(g; v)$ taken over all principal vertices. Justification of this being an outer automorphism invariant can be found in [HM11, Pfa12].

Let $\varphi \in Out(F_r)$ be fully irreducible. For each component C_i of $IW(\varphi)$, let k_i denote the number of vertices of C_i. Then the *index sum* is defined as $i(\varphi) := \sum 1 - \frac{k_i}{2}$. Since the index sum can be computed as such from the ideal Whitehead graph, we can define an index sum for an ideal Whitehead graph, or in fact any graph. For a graph $\mathcal{G}$, we write the index sum as $i(\mathcal{G})$. Writing the terms $1 - \frac{k_i}{2}$ as a list, we obtain the *index list* for φ.

Remark 2.34. By [GJLL98], we know that all fully irreducible $\varphi \in Out(F_r)$ satisfy $0 > i(\varphi) \geq 1 - r$. An ageometric fully irreducible $\varphi \in Out(F_r)$ can be characterized by satisfying $0 > i(\varphi) > 1 - r$. The definition we have given for an ideal Whitehead graph only works for ageometric fully irreducibles. However the index sum is always defined from the ideal Whitehead graph as in Definition 2.33 and general definitions of the ideal Whitehead graph can be found in [Pfa12] or [HM11].

A train track map g induces a simplicial (hence continuous) map $Dg\colon LW(g, v) \to LW(g, g(v))$ extending the map of vertices defined by the direction map Dg. When g is rotationless and v a principal vertex, the map $Dg\colon LW(g, v) \to LW(g, v)$ has image in $SW(g, v)$. Since Dg acts as the identity on $SW(g, v)$, when viewed as a subgraph of $LW(g, v)$, this map is in fact a surjection $Dg\colon LW(g, v) \to SW(g, v)$.

2.7. **Full irreducibility criterion.** The following lemma is essentially [Pfa13, Proposition 4.1], with the added observation that a fully irreducible outer automorphism with a PNP-free train track representative is in fact ageometric (by definition). [Kap14] has a related result.

Proposition 2.35 ([Pfa13]). *(The Ageometric Full Irreducibility Criterion (FIC)) Let $g \colon \Gamma \to \Gamma$ be a PNP-free, irreducible train track representative of $\varphi \in Out(F_r)$. Suppose that the transition matrix for g is Perron-Frobenius and that all the local Whitehead graphs are connected. Then φ is an ageometric fully irreducible outer automorphism.*

2.8. **Lone Axis Fully Irreducible Outer Automorphisms.** In [MP16] Mosher and Pfaff defined the property of being a lone axis fully irreducible outer automorphism. In lay terms this means that there is only one fold line in CV_r that is invariant under φ.

Theorem 2.36 ([MP16]). *Let $\varphi \in Out(F_r)$ be an ageometric fully irreducible outer automorphism. Then φ is a lone axis fully irreducible if and only if*
  (1) the rotationless index satisfies $i(\varphi) = \frac{3}{2} - r$ and
  (2) no component of the ideal Whitehead graph $\mathcal{IW}(\varphi)$ has a cut vertex.

Remark 2.37. It will be important for our purposes that each train track representative of an ageometric lone axis fully irreducible φ is PNP-free ([MP16, Lemma 4.4]).

The unique axis is a periodic fold line and one may choose a particularly nice train track representative to generate it (Definition 2.19).

Proposition 2.38 ([MP16]). *Let φ be an ageometric lone axis fully irreducible outer automorphism. Then there exists a train track representative $g \colon \Gamma \to \Gamma$ of some power φ^R of φ so that all vertices of Γ are principal, and fixed, and all but one direction is fixed.*

Definition 2.39 (A_φ). Given a lone axis fully irreducible outer automorphism φ, we denote its axis by A_φ. In particular, A_φ will be the periodic fold line determined by any (and every) train track representative of any positive power of φ.

3. Stratification of the space of fold lines

3.1. **The space of fold lines.** We fix a rank $r \geq 3$ throughout this section. Notice that the Outer space CV_r has dimension $0 \leq k \leq 3r - 4$.

Definition 3.1 ($\mathcal{F}_r$ & $\mathcal{A}_r$). $\mathcal{F}_r$ will denote the set of all simple fold lines in CV_r (see Definition 2.16). $\mathcal{A}_r \subset \mathcal{F}_r$ will denote the set of all periodic fold lines in CV_r (see Definition 2.20).

Definition 3.2 (Topology on $\mathcal{F}_r$). Let $\mathcal{L}$ be a geodesic in CV_r. We let $N(\mathcal{L}, \varepsilon)$ denote the ε-neighborhood of $\mathcal{L}$ in CV_r with respect to the symmetrized Lipschitz metric on CV_r. We let $B(\mathcal{L}, R, \varepsilon) \subset \mathcal{F}_r$ denote the set of all $\mathcal{L}' \in \mathcal{F}_r$ such that $\mathcal{L}'$ has a length-R subsegment β with $\beta \subset N(\mathcal{L}, \varepsilon)$. For each integer k with $0 \leq k \leq 3r - 4$, we denote by $B^k(\mathcal{L}, R, \varepsilon)$ the set of all $\mathcal{L}' \in B(\mathcal{L}, R, \varepsilon)$ such that the line $\mathcal{L}'$ is contained in the k-skeleton $CV_r^{(k)}$.

We topologize $\mathcal{F}_r$ by using, for each $\mathcal{L} \in \mathcal{F}_r$, the family of sets $\{B(\mathcal{L}, R, \varepsilon)\}_{\varepsilon > 0, R \geq 1}$ as the basis of neighborhoods of $\mathcal{L}$ in $\mathcal{F}_r$.

Remark 3.3. It is a subtle but rather minor point to decide which metric to use in Definition 3.2. The Lipschitz metric d_L is not symmetric. Define the symmetrized Lipchitz metric as $d_s(X, Y) = d_L(X, Y) + d_L(Y, X)$. Consider the 4 possible topologies arising from the following

generating sets: balls in the symmetrized metric, balls in the simplicial metric, "incoming balls" in the Lipschitz metric $B_{in}(X, r) = \{Y \in CV_r \mid d_L(Y, X) < r\}$, and "outgoing balls" in the Lipschitz metric $B_{out}(X, r) = \{Y \in CV_r \mid d_L(X, Y) < r\}$. The four topologies on CV_r coincide [AK12]. However the same is not true for neighborhoods of geodesics. Let $\mathcal{L}$ be a geodesic and consider $N_{simp}(\mathcal{L}, \varepsilon) = \{Y \mid d_{simp}(Y, \mathcal{L}) < \varepsilon\}$, similarly define $N_{sym}(\mathcal{L}, \varepsilon)$. Define $N_{in}(\mathcal{L}, \varepsilon) = \{Y \mid d(Y, \mathcal{L}) < \varepsilon\}$ and $N_{out}(\mathcal{L}, \varepsilon) = \{Y \mid d(\mathcal{L}, Y) < \varepsilon\}$. Then sets of the first three types are equivalent, in that for each of the two types and for all ε, one may find an ε' so that $N(\mathcal{L}, \varepsilon)$ of one of the types contains $N(\mathcal{L}, \varepsilon')$ of the other type. The same is not true for $N_{out}(\mathcal{L}, \varepsilon)$. There exists a geodesic $\mathcal{L}$ and $\varepsilon > 0$ so that for all ε', $N_{sym}(\mathcal{L}, \varepsilon)$ does not contain $N_{out}(\mathcal{L}, \varepsilon')$. The outgoing neighborhoods are "too big," hence we use the others (these geodesics $\mathcal{L}$ necessarily don't stay in any "thick part" of CV_r).

Remark 3.4. Note, with the topology defined above, the space $\mathcal{F}_r$ is non-Hausdorff: If two distinct fold lines $\mathcal{L}, \mathcal{L}' \in \mathcal{F}_r$ overlap along a common subray, then each neighborhood of $\mathcal{L}$ in $\mathcal{F}_r$ contains $\mathcal{L}'$ and each neighborhood of $\mathcal{L}'$ contains $\mathcal{L}$. Nevertheless, the topology on $\mathcal{F}_r$ given in Definition 3.2 is natural for our purposes. Moreover, the topology is better behaved when restricted to the subspace $\mathcal{A}_r \subseteq \mathcal{F}_r$, the main object of interest in this paper.

We remark that the analogous topology on the space of periodic Te-ichmüller geodesics in Teichmüller space is not Hausdorff either. We thank Chris Leininger for the following example. Teichmüller spaces have Teichmüller disks which are isometrically embedded copies of the hyperbolic plane. We find in such a Teichmüller disk two periodic fold lines so that any pair of open sets (in our topology) respectively containing these fold lines will contain a third fold line, i.e. point, in its intersection. Consider a Teichmüller disk whose Veech group is a lattice so that the quotient X is a hyperbolic surface (see [HS07] for the relevant background info). Let γ_1, γ_2 be a disjoint pair of essential simple closed curves in X and let γ_3 be a third essential simple closed curve in X that intersects each of γ_1 and γ_2 precisely once. We construct a sequence of simple closed geodesic curves $\{\alpha_i\}_{i=1}^{\infty}$ in X so that any pair of open neighborhoods of the lifts $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, respectively, contain some lift $\tilde{\alpha}_i$ in its intersection. For each i, the curve α_i, as a fundamental group element, wraps i times around γ_1, follows γ_3 to γ_2, wraps i times around γ_2, and then returns via γ_3 to γ_1.

Let $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, $\tilde{\gamma}_2$, and $\tilde{\alpha}_i$ be lifts to the disk (hence Teichmüller space) of respectively $\gamma_1, \gamma_2, \gamma_3$, and the α_i satisfying that all of the intersection points are in the same fundamental domain. For $j = 1, 2$, the endpoints of the $\tilde{\alpha}_i$ in the direction in which they fellow travel $\tilde{\gamma}_j$ for increasing distance, will converge to the endpoint of $\tilde{\gamma}_j$ in that same direction. Since the Teichmüller disk is hyperbolic, it follows that, for any open set U_1 containing $\tilde{\gamma}_1$ and for any open set U_2 containing $\tilde{\gamma}_2$, some $\tilde{\alpha}_i$ is contained in the intersection of U_1 and U_2, so that $U_1 \cap U_2 \neq \varnothing$. Thus the space of periodic Teichmüller geodesics in Teichmüller space, equipped with the topology defined above and similar to the space $\mathcal{F}_r$, is not Hausdorff.
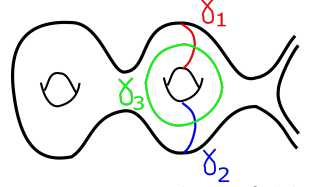
3.2. Strata of geodesics.

Definition 3.5 (*r-Dominant graph*)**.** Let $r \geq 3$ be fixed. A finite graph $\mathcal{G}$ is *r-dominant* if it is a disjoint union of complete graphs and has index sum $\frac{3}{2} - r$, see Definition 2.33.

Definition 3.6 (*r-Dominant outer automorphism*)**.** Let $\varphi \in \text{Out}(F_r)$ be fully irreducible and suppose $IW(\varphi)$ is r-dominant. Then we say that φ is an *(r-)dominant* outer automorphism.

Remark 3.7. Notice that if φ is dominant then it is ageometric and by Theorem 2.36 φ is also a lone axis outer automorphism.

Definition 3.8 (Stratum $\mathcal{S}_r(\mathcal{G})$). For a finite graph $\mathcal{G}$, we define the *stratum* $\mathcal{S}_r(\mathcal{G}) \subset \mathcal{A}_r$ *for* $\mathcal{G}$:

$$\mathcal{S}_r(\mathcal{G}) := \{\mathcal{L} \mid \mathcal{L} \text{ is an axis for a fully irreducible } \varphi \in Out(F_r) \text{ with } IW(\varphi) \cong \mathcal{G}\}.$$

If $\mathcal{G}$ is an r-dominant graph, we call $\mathcal{S}_r(\mathcal{G})$ a *dominant stratum.*

Definition 3.9 ($\mathcal{G}$-basin $B\mathcal{S}_r(\mathcal{G})$). For an r-dominant graph $\mathcal{G}$, we define the $\mathcal{G}$-*basin*:

$$B\mathcal{S}_r(\mathcal{G}) := \{\mathcal{L} \mid \mathcal{L} \text{ is an axis for an ageometric fully irreducible } \varphi \in Out(F_r)$$

$$\text{where } IW(\varphi) \text{ is the union of some subset of the components of } \mathcal{G}\}.$$

Thus $B\mathcal{S}_r(\mathcal{G}) \subseteq \mathcal{S}_r(\mathcal{G}) \subset \mathcal{A}_r$. Notice that each element in $B\mathcal{S}_r - \mathcal{S}_r$ is not a lone axis automorphism since its index sum is strictly larger than $\frac{3}{2} - r$.

We give special names (and attention) to the following dominant strata.

Definition 3.10 (Principal strata $\mathcal{P}_r$). Let Δ_r denote the graph that is a disjoint union of $2r - 3$ triangles. Notice that, in particular, Δ_r has index sum $\frac{3}{2} - r$. We define the *(rank-r) principal stratum* of $\mathcal{A}_r$ as $\mathcal{P}_r = \mathcal{S}_r(\Delta_r)$.

In light of the above, we call a fully irreducible outer automorphism $\varphi \in Out(F_r)$ with $IW(\varphi) \cong \Delta_r$ a *principal* outer automorphism.

We define the *(rank-r) principal stratum basin* in $\mathcal{A}_r$ as $B\mathcal{P}_r := B\mathcal{S}_r(\Delta_r)$. Outer automorphisms with axes in $B\mathcal{P}_r$ will be called *principal basin* outer automorphisms.

Note that

$$B\mathcal{P}_r := \{\mathcal{L} \mid \mathcal{L} \text{ is an axis for a fully irreducible } \varphi \in Out(F_r) \text{ having } IW(\varphi) \cong \Delta_{r'} \text{ with } r' \leq r\}.$$

Remark 3.11. We have noted that every dominant outer automorphism is a lone axis outer automorphism. If an outer automorphism is principal then its axis intersects the interior of a maximum dimensional simplex in CV_r. This can be seen to follow from Proposition 2.38. Moreover, if φ is dominant and not principal, then it will not pass through the interior of a maximum dimensional simplex, as one of its Whitehead graphs comes from a stable whitehead graph of a vertex with more than three stable directions.

Definition 3.12 (Principal index list). Since the index list for a principal outer automorphism is comprised of terms $-\frac{1}{2}$ summing to $\frac{3}{2} - r$, we call this the *principal* index list.

4. Nielsen path prevention

Definition 4.1 is more or less in the spirit of [CL$^+$15]. We shall prove that under certain conditions a map g is *legalizing* (see Definition 4.9). Our goal will be to show that if a train track map h factors through g then it cannot admit a PNP.

Definition 4.1 (Long turns). Suppose that we have a train track structure on Γ induced by a train track map g on Γ (see Definition 2.12). By a *long turn* at a vertex v we will mean a pair of legal paths $\{\alpha, \beta\}$ emanating from v. If $\{D(\alpha), D(\beta)\}$ is legal, then we call $\{\alpha, \beta\}$ *legal*. If $\{D(\alpha), D(\beta)\}$ is illegal, then we call $\{\alpha, \beta\}$ *illegal*.

If either $g(\alpha)$ is an initial subpath of $g(\beta)$ or vice versa, then we call $\{\alpha, \beta\}$ *extendable*. Those long turns that are not extendable can be characterized as either *safe* or *dangerous*

depending on whether, respectively, $g_\#(\overline{\alpha}\beta)$ is a legal path or not (whether the cancellation of $g_\#(\alpha)$ and $g_\#(\beta)$ ends with a legal turn or an illegal turn).

The following is a relatively direct consequence of Lemma 2.30.

Lemma 4.2. *Let $g\colon \Gamma \to \Gamma$ be an expanding irreducible train track map and ρ an iNP for g. Then $\rho = \bar{\rho}_1 \rho_2$, where $\{\rho_1, \rho_2\}$ is a dangerous long turn for each positive power g^k of g. More generally, if $g\colon \Gamma \to \Gamma$ has a PNP, then Γ contains dangerous long turns for each positive power g^k of g. Thus, an expanding irreducible train track map with no dangerous long turns has no PNPs.*

Definition 4.3 (k-Protected path)**.** Let $g\colon \Gamma \to \Gamma$ be an expanding irreducible train track map. Let γ be a path in Γ and α a subpath of γ whose endpoints are at vertices. We say that α is *k-protected* if

- γ contains $\geq k$ edges to the right of α and $\geq k$ edges to the left of α and
- the length-k subpath of γ directly to the right of α and the length-k subpath of γ directly to the left of α are each legal.

Definition 4.4 (Splitting)**.** Let $g\colon \Gamma \to \Gamma$ be an expanding irreducible train track map. Let γ be a path in Γ. We say that $\gamma = \ldots \gamma_{l-1}\gamma_l \ldots$ is a *k-splitting* if $g_\#^k(\gamma) = \ldots g_\#^k(\gamma_{l-1})g_\#^k(\gamma_l) \ldots$ is a decomposition into subpaths. γ is a *splitting* if it is a k-splitting for all $k > 0$.

The following is a special case of the definition of P_r on pg. 558 of [BFH00].

Definition 4.5 (Pth_g)**.** Let $g\colon \Gamma \to \Gamma$ be an expanding irreducible train track map. We let Pth_g denote the paths in γ so that:

(1) Each $g_\#^k(\gamma)$ contains exactly one illegal turn.
(2) The number of edges in $g_\#^k(\gamma)$ is bounded independently of k.

The following is [BFH00, Lemma 4.2.5].

Lemma 4.6 ([BFH00])**.** *Let $g\colon \Gamma \to \Gamma$ be an expanding irreducible train track map. Then Pth_g is finite.*

The next lemma states that there is a uniform k so that for each long turn $\{\alpha, \beta\}$ the iterate $g^k(\bar{\alpha}\beta)$ splits into at most three well understood parts.

Lemma 4.7. *Suppose that g is an expanding irreducible train track map representing a fully irreducible outer automorphism $\varphi \in Out(F_r)$. Then there exists some power g^k of g so that for each long turn $\{\alpha, \beta\}$ for g we have that $g_\#^k(\bar{\alpha}\beta)$ splits with respect to g into paths that are legal paths except for at most a single iNP.*

Proof. First notice that, since φ is fully irreducible, $g\colon \Gamma \to \Gamma$ has only a single EG stratum. Thus, by [BFH00, Lemma 4.2.2], there exists a constant K so that, if τ is a path in Γ and if σ is a K-protected subpath of τ, then τ can be split at the endpoints of σ.

There are only finitely many paths of length $\leq 4K$. Let $\mathcal{S}$ denote the set of all paths of length $\leq 4K$ with only a single illegal turn. Let $k \in \mathbb{Z}$ be the power so that, if $\rho \in \mathcal{S}$, then either $g_\#^k(\rho)$ is legal or for no $n \in \mathbb{Z}$ is $g_\#^{nk}(\rho)$ legal. By [BFH00, Lemma 4.2.6] we can then replace k with a higher power, if necessary, so that for each $\rho \in \mathcal{S}$, we have that $g_\#^k(\rho)$ splits into subpaths that are either legal or an element of Pth_g (a uniform power is possible since $\mathcal{S}$

is finite). The subpaths that are elements of Pth_g are permuted. Thus, by replacing k with a higher power yet, we can assume that they are iNPs (hence have only one illegal turn).

Let $\{\alpha, \beta\}$ be a long turn. Then using trivial paths as K-protected subpaths, $\bar{\alpha}\beta$ can be split into legal paths and a path ρ of length $\leq 4K$ containing the single illegal turn. Since $\rho \in \mathcal{S}$, we can use the power k of the previous paragraph and obtain that $g_{\#}^k(\bar{\alpha}\beta)$ splits into subpaths that are either legal or iNPs. But $\bar{\alpha}\beta$ had only one illegal turn, and the number of illegal turns cannot increase under $g_{\#}^k$. So there can only be one iNP in the splitting. $\qquad\square$

Lemma 4.8. *Suppose that g is an expanding irreducible train track map with no PNPs. Then there exists some power g^k of g with no dangerous long turns.*

Proof. Let k be as in Lemma 4.7 and suppose, for the sake of contradiction, that g^k had a dangerous long turn $\tau = \{\alpha, \beta\}$. By Lemma 4.7, $g_{\#}^k(\bar{\alpha}\beta)$ splits into legal paths and iNPs. Since g^k admits no iNPs, $g_{\#}^k(\bar{\alpha}\beta)$ is legal, contradicting that τ is dangerous. $\qquad\square$

Definition 4.9 (Legalizing train track maps)**.** We call a train track map $g\colon \Gamma \to \Gamma$ *legalizing* if it has no dangerous long turns.

Proposition 4.10. *Suppose that g is a PNP-free expanding irreducible train track map. Then there exists some $p > 0$ so that g^p is legalizing.*

Proof. This follows from Lemmas 4.2 and 4.8. $\qquad\square$

Proposition 4.11. *Suppose φ is a lone axis fully irreducible outer automorphism. Then there is a fully stable transparent legalizing train track representative $g\colon \Gamma \to \Gamma$ of a power φ^R of φ so that all vertices of Γ are principal and fixed, and all but one direction is fixed.*

Proof. This follows from Remark 2.37, Proposition 2.38, and Proposition 4.10. $\qquad\square$

Definition 4.12 (Convenient train track maps)**.** For a lone axis fully irreducible outer automorphism φ we call a train track representative of a power φ^R of φ satisfying the properties of Proposition 4.11 *convenient*.

5. Proof of the main result

Lemma 5.1. *Let $r \geq 3$ be an integer, so that $n = 3r - 4$ is the dimension of CV_r. For each lone axis fully irreducible $\varphi \in Out(F_r)$ there exists an integer $k \leq n$ so that $A_\varphi \subset CV_r^{(k)} \setminus CV_r^{(k-2)}$. In particular, all folds in A_φ are proper full folds.*

Proof. Let $g\colon \Gamma \to \Gamma$ be a convenient train track map representing φ, guaranteed by Proposition 4.11. Let $k = |E(\Gamma)|$, i.e. $1+$ the dimension of the open simplex containing Γ. The fold line A_φ is a periodic fold line for a Stallings fold decomposition of g. It suffices to show that each fold of A_φ is a proper full fold. But, if one of the folds were full, then some vertex of Γ would not be g-periodic, hence not principal. This contradicts that g is convenient. $\qquad\square$

If $\mathfrak{g}$ is a Stallings fold decomposition of g we denote by $\mathfrak{g}^p$ the Stallings fold sequence obtained by juxtaposing p copies of $\mathfrak{g}$. Note that $\mathfrak{g}^p$ is a decomposition of g^p. In the next lemma we need not assume the outer automorphism represented by h is fully irreducible.

Lemma 5.2. *Let $r \geq 3$ be an integer, so that $n = 3r - 4$ is the dimension of CV_r, and let $2 \leq k \leq n$. Suppose that $\mathcal{L} \subset CV_r^{(k)} \setminus CV_r^{(k-2)}$ is the periodic fold line for a Stallings fold decomposition $\mathfrak{g}$ of a train track map g. Then there exist constants $R, \varepsilon > 0$ so that: For*

each fold line $\mathcal{L}' \in B^k(\mathcal{L}, R, \varepsilon)$ *that is the periodic fold line corresponding to a Stallings fold decomposition* $\mathfrak{h}$ *of some train track map* h, *there exists a power* p *so that* $\mathfrak{h}^p$ *contains* $\mathfrak{g}$.

In particular, h *and* g *are self-maps of the same topological graph* Γ.

Proof. Let R be three times the length of a $\mathfrak{g}$-segment of $\mathcal{L}$. Since $\mathcal{L}$ is periodic and contained in $CV_r^{(k)} \setminus CV_r^{(k-2)}$, there exists some $\varepsilon_0 > 0$ such that $N(\mathcal{L}, \varepsilon_0) \cap CV_r^{(k)} \subset CV_r^{(k)} \setminus CV_r^{(k-2)}$. Therefore, for any $0 < \varepsilon \le \varepsilon_0$, any geodesic segment γ of length R contained in $N(\mathcal{L}, \varepsilon)$ passes through the same sequence of simplices as a subsegment of $\mathcal{L}$ of length $\ge R - 2\varepsilon$, and hence shares a fold sequence with this subsegment of $\mathcal{L}$. Choose $\varepsilon = min\{\varepsilon_0, \frac{R}{6}\}$. Then any subsegment of $\mathcal{L}$ of length $\ge R - 2\varepsilon \ge \frac{2}{3}R$ contains twice the length of a $\mathfrak{g}$-segment of $\mathcal{L}$ so must contain a $\mathfrak{g}$-segment of $\mathcal{L}$. Hence, any periodic fold line $\mathcal{L}' \in B^k(\mathcal{L}, R, \varepsilon)$ will in fact contain the full fold sequences $\mathfrak{g}$. We can now take the power p of $\mathfrak{h}$ high enough so that $\mathfrak{h}^p$ contains any length-R subsegment of $\mathcal{L}'$ and the conclusion of the theorem will hold. $\quad\square$

Lemma 5.3. *Let* $r \ge 3$ *be an integer, so that* $n = 3r - 4$ *is the dimension of* CV_r, *and let* $2 \le k \le n$. *Suppose* φ *is a dominant lone axis fully irreducible with axis* $A_\varphi \subset CV_r^{(k)} \setminus CV_r^{(k-2)}$. *Then there exist constants* $R, \varepsilon > 0$ *and a convenient train track representative* $g \colon \Gamma \to \Gamma$ *of a power* φ^p *of* φ *so that for each periodic fold line* $\mathcal{L} \in B^k(A_\varphi, R, \varepsilon)$ *there exist and a self-map* h *on* Γ *with a Stallings fold decomposition yielding* $\mathcal{L}$ *and such that:*

(a) $h \colon \Gamma \to \Gamma$ *is a train track map.*
(b) h *does not admit a PNP.*
(c) *The transition matrix for* h *is Perron-Frobenius.*
(d) $\displaystyle\bigcup_{v \in V(\Gamma)} LW(g, v) = \bigcup_{v \in V(\Gamma)} LW(h, v)$.
(e) *If the vertex* w *of* Γ *is* h-*periodic then* $SW(h, w) = SW(g, w)$.
(f) *If* $\mathcal{L}$ *contains no proper full fold, then all vertices of* Γ *are principal with respect to both* g *and* h *and* $\displaystyle\bigcup_{v \in V(\Gamma)} SW(g, v) = \bigcup_{v \in V(\Gamma)} SW(h, v)$.

Proof. Since φ is a lone axis fully irreducible, by Proposition 2.38, there exists a rotationless power φ^p of φ with a convenient train track representative $g \colon \Gamma \to \Gamma$. We call the nonfixed direction d. Since φ, hence φ^p, is a lone axis fully irreducible, A_φ is the unique periodic fold line for g and is formed by iterating the fold sequence $\mathfrak{g}$ for g. Replace g with a power so that each turn in $LW(g)$ is taken by $g(e)$ for each edge e.

Notice that A_φ is also the periodic fold line for g^3 and that $\mathfrak{g}^3$ is the fold sequence for g^3. Applying Lemma 5.2, there exist $R, \varepsilon > 0$ so that for any periodic fold line $\mathcal{L} \in B^k(A_\varphi, R, \varepsilon)$ corresponding to a train track map h' and fold sequence $\mathfrak{h}'$ of h', there exists a power p such that $(\mathfrak{h}')^p$ contains $\mathfrak{g}^3$. Thus, replacing h with this power and possibly applying a cyclic permutation, h' factors as $h' = f \circ g^3$, see (2).

The trickiest aspect of this proof, and the reason to use g^3 instead of g, is to prove item (b). We will show all items for the cyclic permutation $h = g \circ f \circ g^2$ of h'.

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$$(2) \qquad \Gamma \xrightarrow{g} \Gamma \xrightarrow{g} \Gamma \xrightarrow{f} \Gamma \xrightarrow{g} \Gamma \xrightarrow{g} \Gamma \xrightarrow{g} \Gamma \xrightarrow{f} \Gamma$$

with h' arcing over the right portion and h arcing under the left portion.

We first show (a). Suppose h is not a train track map, i.e. $h^p(e)$ contains a backtracking segment for some $e \in E(\Gamma)$ and power p. We parametrize $\mathcal{L} \colon \mathbb{R} \to CV_r$ so that the graphs appearing in (2) are $\mathcal{L}(0), \mathcal{L}(1), \mathcal{L}(2), \dots$ respectively. Let $\gamma \in F_r$ be the witness guaranteed by Lemma 2.26, i.e. $\gamma_t \in \mathcal{L}(t)$ is legal for all $t \geq 0$. Note that $\gamma_4 = g(\gamma_3)$, and since g maps each edge onto the entire graph, γ_4 contains e. Thus, e cannot be h-legal, a contradiction.

We now show (b). Recall that g is convenient, hence if $\{\alpha, \beta\}$ is a long turn then either $g(\alpha)$ is an initial subsegment of $g(\beta)$ or $g_{\#}(\bar{\alpha}\beta) = \bar{\alpha}'\beta'$ is legal, where α', β' are nontrivial terminal subsegments of α, β. In the second case all turns of $g(\bar{\alpha}'\beta')$ are g-taken (since $\{Dg\alpha', Dg\beta'\}$ cannot contain d so it, too, is g-taken). Notice that each g-taken turn is $h^p \circ g \circ f$-legal since for any witness loop γ, $g^2(\gamma)$ maps over all g-taken turns and $h^p \circ g \circ f(g^2(\gamma))$ is legal. Concluding, we get that the path $g(\bar{\alpha}'\beta')$ is legal with respect to $h^p \circ g \circ f$ for each p. Now if $\rho = \bar{\alpha}\beta$ is an iPNP, then for some p, $h^p_{\#}(\rho) = \rho$, which is illegal. But $h^p(\bar{\alpha}\beta) = h^{p-1} \circ g \circ f \circ g(g_{\#}(\bar{\alpha}\beta)) = h^{p-1} \circ g \circ f(g(\bar{\alpha}'\beta'))$, which is legal. We get a contradiction to the fact that ρ is an iPNP.

To prove (c) recall that each edge of Γ maps onto Γ under the map g. Since $h = g \circ (f \circ g^2)$ is a train track map, the same is true for h. Thus the transition matrix of h is PF.

To prove (d), recall that $g(e)$ contains all turns in each local Whitehead graph. Note also that for any witness loop γ for $\mathcal{L}$, $h(\gamma)$ contains all g-taken turns. Thus $LW(h, w) \supset LW(g, w)$. Let d be the unique g-nonperiodic direction, and let v be its initial vertex. Note that $\cup_{w \in \Gamma} LW(g, w)$ contains all turns not involving d. Thus, if $\cup LW(h, w) \setminus \cup LW(g, w) \neq \emptyset$ then $\{d, d'\} \in LW(h, v) \setminus LW(g, v)$ for some $d' \neq d$. Therefore, there exists an edge e and a natural p so that $h^p(e)$ crosses $\tau = \{d, d'\}$. Denoting $\alpha = f \circ g^2 \circ h^{p-1}(e)$ (which is an immersed g-legal path) we have $h^p(e) = g(\alpha)$ contains τ. But τ is not g-taken and not in $Im(Tg)$, since $d \notin Im(Dg)$, a contradiction. So $LW(h, w) = LW(g, w)$ for each $w \in \Gamma$.

To prove (e) we again denote by v the initial vertex of the direction d that is g-nonperiodic. First let $w \neq v$ be h-periodic. Since all turns at w are h-taken, Dh is injective on the directions at w, hence all directions at w are periodic and $SW(h, w) = LW(h, w) = LW(g, w) = SW(g, w)$. For v, since $LW(v, g) = LW(v, h)$, we have that $LW(v, h)$ contains a complete graph C on $deg(v) - 1$ vertices (all directions except d). Dh is injective on C, since otherwise a taken turn would be illegal. If v is not h-periodic then there is nothing to prove, so we assume that v is h-periodic. Let h^p be a rotationless power of h. Then $Dh^p \colon LW(h, v) \to SW(h, v)$ sends C to an isomorphic graph. Moreover, we know that $deg(v) = V(C) + 1$ and that d is in an h-illegal turn, so $SW(h, v)$ cannot contain more than $V(C)$ vertices. Hence, $SW(h, v) \cong C \cong SW(g, v)$.

To prove (f) note that the containment in (e) is proper if and only if not all vertices are principal. Since the local Whitehead graphs contain all edges without d, this happens only when for some $w \neq u \in V(\Gamma)$ we have $h^p(w) = h^p(u)$. When this happens, some fold in the fold sequence does not restrict to an injective map on the vertices. This implies that the

fold is full. Therefore, if no fold in the fold sequence of $\mathcal{L}$ is full, then all vertices of Γ are h-principal, and the stable Whitehead graphs of h and g are identical. $\qquad\square$

The basin of any dominant stratum has the following "rigidity" properties:

Theorem A. *Let $r \geq 3$ and let $\mathcal{G}$ be an r-dominant graph. Let $L \in \mathcal{S}_r(\mathcal{G})$ and k the minimal integer so that $L \subset CV_r^{(k)}$. Then there exists a neighborhood $U \subseteq \mathcal{A}_r$ of L with the following properties:*

(a) *For each $\mathcal{L}' \in U$ with $\mathcal{L}' \subseteq CV_r^{(k)}$, we have $\mathcal{L}' \in B\mathcal{S}^r(\mathcal{G})$.*
(b) *For each $\mathcal{L}' \in U$ with $\mathcal{L}' \subseteq CV_r^{(k)}$ and with $\mathcal{L}'$ containing no full folds, we have $\mathcal{L}' \in \mathcal{S}^r(\mathcal{G})$.*

Proof. Using Lemma 5.3, we have that $\mathcal{L}$ is a periodic fold line for a Stallings decomposition of the train track map h with no PNPs and with connected local Whitehead graphs. By the FIC (Proposition 2.35) we get that the outer automorphisms φ' represented by h is ageometric fully irreducible. Let $W \subset V(\Gamma)$ be the set of h-principal vertices of Γ. Then by Lemma 5.3(e),

$$IW(\varphi') = \bigcup_{v \in W} SW(h, v) \subset \bigcup_{v \in V(\Gamma)} SW(g, v) = IW(\varphi)$$

and the unions are disjoint. Thus, $IW(\varphi')$ is a union of components of $IW(\varphi)$.

We prove (b). By Proposition 5.3(f), all vertices are h-principal, hence

$$IW(\varphi') = \bigcup_{v \in V(\Gamma)} SW(h, v) = \bigcup_{v \in V(\Gamma)} SW(g, v) = IW(\varphi).$$

$\qquad\square$

Corollary 5.4. *Suppose $\mathcal{L} \in \mathcal{P}_r$. Then there exists an open neighborhood U of $\mathcal{L}$ so that:*

(a) *$U \cap \mathcal{A}_r \subset B\mathcal{P}_r$ and*
(b) *each periodic fold line in U containing no proper folds is contained in $\mathcal{P}_r$.*

6. EXAMPLES

Example 6.1 (Principal outer automorphisms exist)**.** We claim that, for each rank $r \geq 3$, the examples constructed in [CL$^+$15] to have the principal index list are in fact principal outer automorphisms. For each $r \geq 3$, we denote this outer automorphism in $Out(F_r)$ by φ_r. By [CL$^+$15, Theorem 6.2] we know that each φ_r is an ageometric fully irreducible outer automorphism. To show that φ_r is principal we must prove that $IW(\varphi_r) = \Delta_r$.

The proof of [CL$^+$15, Proposition 4.3] indicates that the stable Whitehead graph at each vertex is a complete graph. Since there are no periodic Nielsen paths (again by [CL$^+$15, Theorem 6.2]), this indicates that each component of the ideal Whitehead graph is a complete graph. Since the map is constructed to have the principal index list, this implies that the ideal Whitehead graph is in fact Δ_r, as desired.
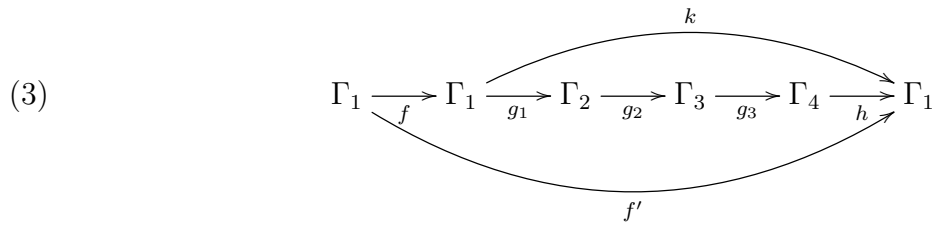
Example 6.2. We give an example of an ageometric fully irreducible outer automorphism φ' that is a composition $\psi \circ \varphi$ where:

- φ is a principal outer automorphism and
- φ' is not principal, but is only a principal basin outer automorphism.

This example reveals the necessity on our restricting in Theorem A to fold lines consisting of proper full folds and, in so doing, indicates that, unlike in the Teichmüller space setting, being in the principal stratum is not quite an "open" condition.

The outer automorphism φ is φ_3 of Example 6.1 (from [CL$^+$15]). Let $f: \Gamma_1 \to \Gamma_1$ be the train track representative used by Coulbois-Lustig to define it (see Figure 6.2 below). We use the notation e_1, e_2, e_3, e_4, e_5 to respectively denote the edges $\bar{c}_3, \bar{c}_2, \bar{c}_1, \bar{a}_1, \bar{b}_1$ in [CL$^+$15, §3](see "maximal odd" case on pages 1117-1118). We replace f with a high enough power to be both transparent and legalizing in the sense of Definition 4.9 (see Proposition 4.10). The only illegal turn of f is $\{\bar{e}_3, \bar{e}_4\}$ and all other gates are singleton directions.

We shall define a map $k: \Gamma_1 \to \Gamma_1$ as a composition of folds, namely g_1, g_2, g_3 and a homeomorphism h, and we define $f' := k \circ f$. The map f' will be shown to be a train track map representing an ageoemtric fully irreducible $\varphi' \in \mathrm{Out}(F_r)$.

(3)
$$\Gamma_1 \xrightarrow{f} \Gamma_1 \xrightarrow{g_1} \Gamma_2 \xrightarrow{g_2} \Gamma_3 \xrightarrow{g_3} \Gamma_4 \xrightarrow{h} \Gamma_1$$
with k over the top (from Γ_1 after f to Γ_1) and f' under the bottom.

The maps g_1, g_2, g_3, h are described in Figure 6.2. Composing the maps in the diagram yields:

$$k(e_1) = \bar{e}_1\bar{e}_4, \quad k(e_2) = e_1, \quad k(e_3) = e_2\bar{e}_5, \quad k(e_4) = e_2\bar{e}_5, \quad k(e_5) = \bar{e}_3\bar{e}_5$$

Notice the following facts:

(1) The only k-prenull turn is $\{\bar{e}_3, \bar{e}_4\}$.
(2) $\{\bar{e}_3, \bar{e}_4\}$ is the only f-illegal turn, and it is not in $Im(Tf)$.
(3) $Im(Dk)$ does not contain $\bar{e}_4$.
(4) $\{\bar{e}_3, \bar{e}_4\}$ is not a k-taken turn. The k-taken turns are: $\{\bar{e}_2, \bar{e}_5\}, \{e_3, \bar{e}_5\}, \{e_1, \bar{e}_4\}$.
(5) $k(v_1) = k(v_3) = v_2$ and $k(v_2) = v_1$ (see the bottom of the label of Figure 6.2).
(6) The Tk images of all of the f-legal turns at v_1 and v_2 are: $\{e_2, e_5\}, \{\bar{e}_1, e_5\}, \{\bar{e}_1, e_2\}$ at the vertex v_2 and $\{e_1, \bar{e}_3\}, \{e_1, e_4\}, \{e_4, \bar{e}_3\}$ at the vertex v_1.
(7) f takes all turns not involving $\bar{e}_4$.

Lemma 6.3. $f' = k \circ f$ *is an irreducible train track map.*

Proof. We first show that f' is a train track map. Suppose not - then for some $p \in \mathbb{N}$ and edge e, $(f')^p(e)$ would contain backtracking. Note that then $f \circ (f')^p(e)$ would contain backtracking. We show by induction on p that $f \circ (f')^p(e)$ does not contain backtracking. Since f is a train track map, $f(e)$ can contain no backtracking. Inductively assume $\beta = f \circ (f')^{p-1}(e)$ has no backtracking. All of the turns in β are either f-taken or in $Im(Tf)$, so also f-taken. No f-taken turn is k-prenull, so $k(\beta)$ has no backtracking. All turns in $k(\beta)$ are either k-taken or in $Im(Tk)$ so by properties (3) and (4) above, $k(\beta)$ does not contain $\{\bar{e}_3, \bar{e}_4\}$. So by (2) it is f-legal. This completes the induction step. The fact that f' is irreducible follows from the fact that $f(e)$ contains all edges and k is onto. $\square$

Lemma 6.4. $f' = k \circ f$ *has no PNPs.*

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Proof. Recall that f is legalizing and transparent, which implies that if α and β are legal paths initiating at the same vertex then, without loss of generality, either $f(\alpha)$ is an initial subpath of $f(\beta)$ (then $\{\alpha, \beta\}$ is an f-extendable long turn) or $f_\#(\bar{\alpha}\beta) = \bar{\alpha}'\beta'$ is f-legal, where α', β' are terminal subsegments of α, β respectively.

Now suppose $\rho = \bar{\alpha}\beta$ is an iPNP for f'. Since there exists a $p > 0$ so that $(f')_\#^p(\rho) = \rho$, we would then have that ρ is not f'-extendable, which would imply that it is not f-extendable. By the first paragraph we have $f_\#(\rho) = \bar{\alpha}'\beta'$, where α', β' are f-taken paths and the turn $\{D\alpha', D\beta'\}$ is f-legal, so it is not equal to $\{\bar{e}_3, \bar{e}_4\}$, the only k-prenull turn. Thus $k(\bar{\alpha}'\beta')$ contains no backtracking. Therefore, the turns in $f'(\rho) = k(\bar{\alpha}'\beta')$ are in the image of Tk or are k-taken turns. By properties (3) and (4) above, $f'(\rho)$ does not contain $\{\bar{e}_3, \bar{e}_4\}$, so is g-legal. Continuing this way, applying k and g alternately to $k(\alpha'\beta')$, we see that the turn $\{\bar{e}_3, \bar{e}_4\}$ never appears, so $f'(\rho)$ is f' legal, contradicting the assumption that ρ is a PNP. $\square$

Lemma 6.5. *Let $f' = k \circ f$ represents an ageometric fully irreducible outer automorphism.*

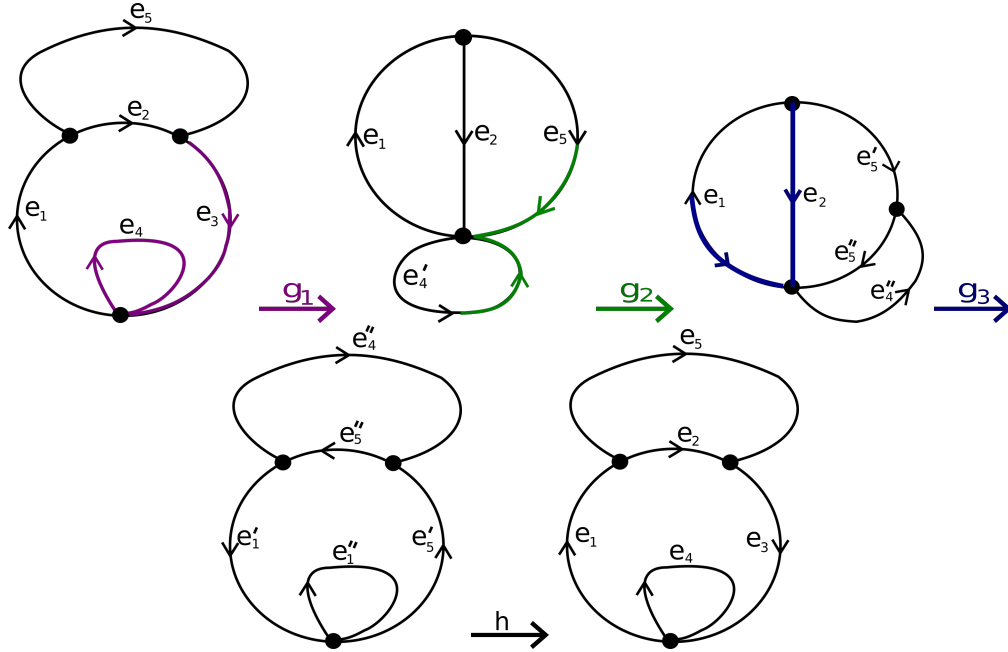*Proof.* By the FIC, it suffices to show that each local Whitehead graph is connected.

FIGURE 2. From left to right the graphs are $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_1$. Colors indicate the (partial) edges folded by the subsequent maps. g_1 is the full fold of $\overline{e_3}$ and $\overline{e_4}$ (and e_4' is the edge formed by the identification of $\overline{e_3}$ and $\overline{e_4}$). g_2 is a partial fold of $\overline{e_4'}$ and $\overline{e_5}$ (and e_5'' is the edge formed from the identification of the initial portions of $\overline{e_4'}$ and $\overline{e_5}$, and e_5' is the portion of e_5 not folded, and e_4'' is the portion of e_4 not folded). g_3 is a proper full fold of e_1 over $\overline{e_2}$ (and e_1'' is the portion of e_1 remaining after the fold), i.e. $g_3(e_1) = e_1'e_1''$. h is a homeomorphism sending e_1'' to $\overline{e_4}$, and e_1' to $\overline{e_1}$, and e_4'' to e_2, and e_5'' to $\overline{e_5}$, and e_5' to $\overline{e_3}$. The bottom vertex in Γ_1 is v_1, the upper left vertex is v_2, and the upper right vertex is v_3.

By item (6), we have that all turns at v_2 are f'-taken and all turns at v_1 not involving $d = \bar{e}_4$ are f'-taken. It follows that $LW(f', v_2)$ is a triangle and $LW(f', v_1)$ contains a triangle. Since f' is an irreducible train track map and for no such map does $LW(v_1, f')$ have an isolated vertex, we have that $\bar{e}_4$ in $LW(v_1, f')$ is also connected via an edge to another vertex, hence $LW(v_1, f')$ is connected. By (4), $LW(f', v_3)$ is also connected. $\qquad\square$

Lemma 6.6. *Let $f' = k \circ f$ and suppose that f' represents the automorphism $\varphi' \in Out(F_3)$. Then the ideal Whitehead graph $IW(\varphi')$ is a union of two triangles.*

Proof. By the proof of Lemma 6.5, $LW(f', v_2)$ is a triangle and $LW(f', v_1)$ contains a triangle. Now v_1 and v_2 are permuted by f and k, hence are permuted by f'. Thus they are fixed by a rotationless power f'^p of f'. Moreover, Df'^p cannot identify any of the directions at v_2, since that would collapse a taken turn. Therefore, the triangle in $LW(f', v_2)$ is taken by f'^p to a triangle in $SW(f'^p, v_2)$. Similarly, no two of the directions $\{d_1, d_2, d_3\}$ at v_1, distinct from d, can be identified by Df'^p. Thus their images span a triangle in $SW(f'^p, v_1)$. Thus $SW(f', v_1)$ contains a triangle. But since d forms an illegal turn with some other direction, there are only 3 gates, and the graph is in fact a triangle. Lastly note that $k(v_3) = v_2$, so v_3 is not a principal vertex (this is in fact the key for dropping from a union of 3 triangles in $IW(f)$ to two triangles in $IW(f')$). The ideal Whitehead graph is the union of the stable Whitehead graphs of principal vertices glued along PNPs but, since f' admits none, $IW(\varphi')$ is a union of two disjoint triangles. $\qquad\square$

We can now prove one of the main results stated in the introduction:

Theorem B. *There exists a principal fully irreducible outer automorphism $\varphi \in Out(F_3)$ with a train track representative $f \colon \Gamma \to \Gamma$ with a Stallings fold decomposition $\mathfrak{f}$ such that, for each $n \geq 1$, there exists a nonprincipal fully irreducible outer automorphism $\psi_n \in Out(F_3)$ with a train track representative $g_n \colon \Gamma \to \Gamma$ with a Stallings fold decomposition $\mathfrak{g}_n$ such that $\mathfrak{g}_n$ starts with $\mathfrak{f}^n$.*

Proof. The lemmas and proofs above apply verbatim when f is replaced with f^n. By the FIC (Proposition 2.35), both f and $g_n = k \circ f^n$ represent fully irreducible outer automorphisms φ, ψ_n of F_3. The ideal Whitehead graph of φ is a union of three triangles, while $IW(\psi_n)$ is a union of two triangles. Hence φ is principal, while ψ_n is not. $\qquad\square$

Theorem B immediately implies:

Corollary C. *For $r = 3$, there exist a principal periodic geodesic $\mathcal{L} \in \mathcal{P}_r$ in CV_r and a sequence of nonprincipal periodic geodesics $\{\mathcal{L}_n\}_{n=1}^{\infty} \subseteq B\mathcal{P}_r - \mathcal{P}_r$ such that $\lim\limits_{n \to \infty} \mathcal{L}_n = \mathcal{L}$.*

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