# Chance Constraint based design of Input Shapers

Souransu Nandi[1] and Tarunraj Singh[2]

*Abstract*— **The focus of this paper is on the design of input shapers for systems with uncertainties in the parameters of the vibratory modes which need to be attenuated. A probabilistic framework is proposed for the design of the robust input shaper, when the uncertain modal parameters are characterized by probability density functions. A convex chance constrained optimization problem is posed to determine the parameters of input shapers (time-delay filter) which can accommodate the users acceptable risk levels for a prescribed residual energy threshold. Robust input shapers are developed for various compact support distributions to illustrate the ability of the proposed formulation to synthesize input shapers which can satisfy a residual energy threshold with a given risk level. This problem formulation can conceivably reduce the conservative nature of worst case controllers which have to ensure that all realizations of the uncertain system have to satisfy a prescribed performance index. The chance constrained input shaper is designed for a spring-mass-dashpot system with three different distributions for the uncertain spring stiffness. Results provide encouragement for the extension of the proposed approach to multi-dimensional and multi-model uncertainties.**

## I. INTRODUCTION

Precise regulation of a lightly damped system has been a topic of interest to the control community for over five decades. Increasing the damping characteristic via feedback control comes at the cost of increasing the settling time of the system response for any rest-to-rest maneuver. A finite-time rest-to-rest (also called a dead-beat) response has appeal and was addressed by Tallman and Smith [1]. Their approach exploited the linear superposition principle in suggesting a two step inputs where the second step is delayed to generate a system response which is 180 degrees out of phase with the response generated by the first step. If the amplitudes of the two steps are appropriately designed such that the amplitudes of the responses of the first and second (delayed) step are out of phase and of the same magnitude, then one can generate a dead-beat response. They remark that the uncertainty in the estimated location of the second-order under-damped poles, characterized by the s-plane distance between the estimated and true location of the poles is proportional to the residual oscillations.

There have been numerous subsequent publications which addressed the control problem assuming a nominal model. The issue of developing control profiles which are insensitive to uncertainties in estimated location of the under-damped poles, was brought to the fore by Singer and Seering [2] in their article on *Input Shaping*, where the local sensitivity of system response to variation in damping ratio or natural frequency of the under-damped poles was forced to zero. The tradeoff between the settling time and the degree of local robustness was characterized by the number of impulses in the Input Shaper design. The idea of using local sensitivity was extended to multi-mode systems, for time-optimal, fuel-time optimal and for controllers which optimized other cost functions [3], [4], [5].

The concept of using the variation of the cost function over a compact support of the uncertain variables was another approach used to develop controllers which are insensitive to uncertainties in the model parameters. This resulted in the minimax time-delay filter [6] design. A slightly different approach where the acceptable level of residual vibration was prescribed and the domain over which the cost function was below the specified threshold was identified; resulting in the Extra-Insensitive Input Shaper [7]. Using knowledge of the probability density functions (pdf), Chang et al. [8] formulated a cost function which is the expected value of the residual energy to determine the impulse sequence of the input shaper. They considered a uniform and Gaussian distribution to illustrate their technique. Since minimizing the expected value alone does not correspond to reducing the variance, the resulting input shaper can lead to a residual energy distribution with a large variance. This implies that many realization of the uncertain system can have large residual vibrations. A polynomial chaos based approach was proposed by Singh et al. [9], which permits selection of any number of moments in the design of the input shaper. As the number of moments included in the cost functions increased, the solution tended to a minimax solution. Since the minimax problem formulation requires raster scan sampling over the uncertain space, it is afflicted by the curse of dimensionality. The polynomial chaos approach on the other hand provides an approach to alleviate the computational cost in the design of input shapers.

The aforementioned approaches for the design of input shapers have either considered the nominal model alone and used local sensitivity for the design, or have considered the support of the uncertainty which in conjunction with their probability distribution functions resulted in a worst case design. One can imagine many scenarios where the worst case design caters to model realizations which have very small probability. This would then result in a very conservative design. There is clearly a need for a problem formulation where the user can specify a level of risk, that is a bound on the probability of violating a prescribed threshold for residual energy. This paper presents such a problem

[1]Souransu Nandi is a Ph.D. student of Mechanical Engineering, University at Buffalo, Buffalo, NY 14260, USA. Email: souransu@buffalo.edu

[2]Tarunraj Singh is with Faculty of Department of Mechanical and Aerospace Engineering, University at Buffalo, Buffalo, NY 14260, USA. Email: tsingh@buffalo.edu

formulation which results in providing the user with a suite of solutions as a function of the acceptable risk.
The paper has been organised as follows. Section I introduces the problem statement, provides a background on the existing literature and motivates the need for chance constraint based input shaping. Section II explains the concept of probabilistic constraints and introduces the robust version of it. This is followed by a review of Polynomial Chaos in Section III. Section IV then formulates the input shaper (using developments from previous sections) by posing it as an optimization problem. Section V is finally used to present the results from a numerical simulation before finishing with concluding remarks in Section VI.

## II. CHANCE CONSTRAINT

This section first introduces the generic nature of a chance constraint and its exact representation as an inequality. It then focuses on linear probabilistic constraints emphasizing its exact as well as its robust implementation.
Chance constraints are probabilistic constraints of the form:

$$P\left(h(x,\xi) \leq 0\right) \geq \eta \tag{1}$$

where $\eta \in [0,1]$ is the probability level, $x$ corresponds to the decision variable(s) and $\xi$ represents the random variable(s). Equation (1) can also be written as:

$$P\left(h(x,\xi) \leq 0\right) \geq 1-\epsilon \tag{2}$$

where $\epsilon \in [0,1]$ represents the acceptable risk level [10].
For a linear chance constraints of the form:

$$P\left(\xi^T x \leq b\right) \geq \eta \tag{3}$$

where $\xi \sim \mathcal{N}\left(\bar{\xi},\Sigma\right)$, $\bar{\xi}$ and $\Sigma$ are the mean and the covariance of the Gaussian random variable $\xi$ respectively, we can represent:

$$P\left(\xi^T x - b \leq 0\right) = \Phi\left(\frac{b-\bar{\xi}^T x}{x^T \Sigma x}\right) \tag{4}$$

where $\Phi$ represents the cumulative distribution function (cdf) of a normal distribution wth 0 mean and unit variance. This permits rewriting the linear chance constraint as:

$$P\left(\xi^T x - b \leq 0\right) \geq \eta \Longleftrightarrow b - \bar{\xi}^T x \geq \Phi^{-1}\left(\eta\right)\|\Sigma^{1/2}x\|. \tag{5}$$

Equation (5) is a cone constraint and is convex for $\eta > 0.5$ [11]. For an illustrative example, consider the linear constraint:

$$\xi_1 x_1 + \xi_2 x_2 \geq 400 \tag{6}$$

where the random variables $\xi_1$ and $\xi_2$ are given by the distributions $\mathcal{N}\left(40,10^2\right)$ and $\mathcal{N}\left(200,40^2\right)$ respectively. The corresponding chance constraint is given by

$$P\left(\xi_1 x_1 + \xi_2 x_2 - 400 > 0\right) \geq 0.9. \tag{7}$$

for a risk level of $\epsilon = 0.1$. The analytical expression given by Equation 7 is the exact form for a linear chance constraint with Gaussian random coefficients. However, for linear chance constraints where the pdf of the random coefficients are time varying and might not be characterized by a well known pdf, then the problem of imposing the exact chance constraint is challenging. This is the scenario we encounter when we study imposing chance constraints on states of a dynamic system with uncertain model parameters. The issue, however, can be dealt with a robust version of the chance constraint as detailed below.
Calafiore and El Ghaoui in [10] provides an approach to rewrite the linear probabilistic inequality:

$$P\left(\boldsymbol{\xi}^T \boldsymbol{x} + b \leq 0\right) \geq 1-\epsilon \tag{8}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{x}$ are the vectors of random variables and decision variables respectively, as a convex non-probabilistic constraint. In their work, they prove that if $\boldsymbol{\xi}$ and $b$ are random variables with known means and variances, then the constraint in equation (8) is equivalent to the convex constraint

$$\sqrt{\frac{1-\epsilon}{\epsilon}}\{var[\boldsymbol{\xi}^T\boldsymbol{x}+b]\}^{1/2} + E[\boldsymbol{\xi}^T\boldsymbol{x}+b] \leq 0 \tag{9}$$

where $\epsilon$ represents the risk level i.e. the probability with which the constraint is permitted to be violated. It should be noted that the constraint is conservative since it subsumes all distributions with the same mean and variance. Therefore, if only the first 2 moments of the random variables $(\boldsymbol{\xi}, b)$ are known, equation (9) allows one to enforce equation (8) no matter what the true distribution of $(\boldsymbol{\xi}, b)$ is. However, since this constraint is robust to all distributions, it yields conservative solutions.
Figure 1 visually illustrates the exact as well as the robust versions of the constraints. The solid red line is the deterministic constraint with the random variables $\xi_1$ and $\xi_2$ taking their mean values. The robust chance constraint is shown as the darker region bounded by a dashed black line (given by Equation (9)). The permissible region with the exact chance constraint (given by Equation (7)) is bounded by the solid blue line and denoted by the lighter area. We can see that the exact constraint shows a larger feasibile region; which means it is less conservative relative to the constraint defined by Equation (9) (the darker region).
Therefore, although the robust constraint is more conserva-

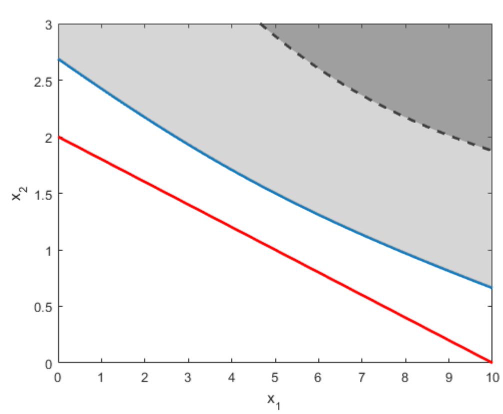



Fig. 1. Comparison of Linear Chance Constraints

tive, its use is still warranted as it poses convex inequalities for distributions of any nature. The only requirement for its implementation is the knowledge of mean and variance of the random variables. The next section now presents a tool which can be used to determine these desired means and variances.

## III. POLYNOMIAL CHAOS

Polynomial Chaos (PC) is a popular approach for characterizing the uncertainty of evolving states of dynamical systems, which are functions of probabilistically represented uncertain system parameters. Polynomial chaos is build on the pioneering *Homogeneous Chaos* approach of Wiener [12] which deals with Gaussian random variables. In contrast to the Monte Carlo approach for estimating the evolving pdf of the stochastic states of a dynamical system, the intrusive form of PC implementation is a non-sampling based approach. The non-intrusive implementation of PC (on the other hand) uses samples that are quadrature based [13]. Kim et al. [13] illustrate via a simple example, the precise estimate of the mean and variance of a low-order PC expansion, providing encouragement for its use in uncertainty quantification. Since the chance constraint problem formulation posed in this paper for the design of robust input shapers only require information of the mean and variance of the uncertain states, polynomial chaos is a germane approach to pose a convex optimization problem.

Homogeneous chaos, which is specific to Gaussian random variables has been generalized by Xiu and Karniadakis [14] where they showed that any stochastic process can be approximated by an infinite series expansion where the basis functions are given by the Wiener-Askley scheme. This generalized Polynomial Chaos (gPC) approach is used in this paper to develop the robust input shapers.

The simplest implementation of the input shaper is one which targets one mode. If a system has multiple modes which contribute to the output, the input shapers for each mode are convolved together to generate an input shaper which targets all the modes of interest. A concurrent design of input shapers which accounts for all the modes of interest can result in shorter maneuver time input shaper. In this paper, we present the design of input shapers which target one mode at a time.

To illustrate the proposed technique of using PC for the determination of a robust input shaper, we consider the second order system:

$$\ddot{x} + c\dot{x} + kx = ku \tag{10}$$

where $k$ is an uncertain parameter of the system which is known to lie in the interval $[a\ b]$. We assume it to be a function of random variable $\xi$ with known probability density function $f(\xi)$. Thus, the uncertain parameter $k$ can be represented as:

$$k(\xi) = \sum_{i=0}^{N} k_i \phi_i(\xi). \tag{11}$$

Furthermore, if $\xi \in [-1\ 1]$, only two terms are necessary to represent $k(\xi)$, i.e.

$$k(\xi) = k_0 + k_1\xi,\ k_0 = \frac{a+b}{2},\ k_1 = \frac{b-a}{2}. \tag{12}$$

which results from the fact that $\phi_0 = 1$ and $\phi_0 = \xi$. This does not preclude Normal distributions, since $k_0$ and $k_1$ can represent the mean and standard deviation of $k(\xi)$ when $\xi \in (-\infty\ \infty)$.

Now, the displacement $x$ can be approximated by the finite series as:

$$x = \sum_{i=0}^{N} x_i(t)\phi_i(\xi) \tag{13}$$

where $\phi_i(\xi)$ represents the orthogonal polynomial set which is orthogonal with respect to the pdf $f(\xi)$, i.e.

$$\langle \phi_i(\xi), \phi_j(\xi)\rangle = \int_\Omega \phi_i(\xi)\phi_j(\xi)f(\xi)d\xi = c_i^2\delta_{ij}. \tag{14}$$

$c_i^2$ are positive numbers which depend on the orthogonal polynomials and $\delta_{ij}$ is Kronecker delta product.

For example, the Legendre and Hermite polynomials constitute the orthogonal polynomial sets for uniform and normal distributions, respectively. In general, these polynomials can be constructed by making use of *Gram-Schmidt Orthogonalization process*. Now, substituting for $x$ and $k$ from Eqs. (13) and (11) in Eq. (10) leads to

$$\sum_{i=0}^{N} \phi_i(\xi)\left(\ddot{x}_i + c\dot{x}_i\right) + \left(k_0\phi_0(\xi) + k_1\phi_1(\xi)\right)\sum_{i=0}^{N}\phi_i(\xi)x_i = \left(k_0\phi_0(\xi) + k_1\phi_1(\xi)\right)u. \tag{15}$$

Using the Galerkin projection method, the dynamics of $x_i$ can be determined. Making use of the fact that system equation error due to polynomial chaos approximation (Eq. (15)) should be orthogonal to basis function set $\phi_j(\xi)$, we arrive at the equation:

$$\mathcal{M}\underbrace{\begin{bmatrix}\ddot{x}_0\\ \ddot{x}_1\\ \vdots\\ \ddot{x}_N\end{bmatrix}}_{\ddot{\mathbf{x}}} + c\mathcal{M}\underbrace{\begin{bmatrix}\dot{x}_0\\ \dot{x}_1\\ \vdots\\ \dot{x}_N\end{bmatrix}}_{\dot{\mathbf{x}}} + \mathcal{K}\underbrace{\begin{bmatrix}x_0\\ x_1\\ \vdots\\ x_N\end{bmatrix}}_{\dot{\mathbf{x}}} = \mathcal{D}u. \tag{16}$$

The elements of the $\mathcal{M}$ matrix are given by

$$\mathcal{M}_{ij} = \langle \phi_i(\xi), \phi_j(\xi)\rangle = \int_\Omega \phi_i(\xi)\phi_j(\xi)f(\xi)d\xi = c_i^2\delta_{ij} \tag{17}$$

where $i, j$ varies from 0 to $N$. The elements of the $\mathcal{K}$ matrix are given by

$$\mathcal{K}_{ij} = k_0\langle \phi_i(\xi), \phi_j(\xi)\rangle + k_1\langle \xi\phi_i(\xi), \phi_j(\xi)\rangle \tag{18}$$

It is already know that every orthogonal polynomial set satisfies a three-term recurrence relation[15]:

$$\xi\phi_n(\xi) = \frac{a_n}{a_{n+1}}\phi_{n+1}(\xi) + \frac{c_n^2}{c_{n-1}^2}\frac{a_{n-1}}{a_n}\phi_{n-1}(\xi) \tag{19}$$

where $a_n$ and $a_{n-1}$ are the leading coefficients of $\phi_n(\xi)$ and $\phi_{n-1}(\xi)$, respectively. Exploiting this recurrence relationship, the elements of the $\mathcal{K}$ matrix can be written as

$$\mathcal{K}_{ii} = k_0 \langle \phi_i(\xi), \phi_j(\xi) \rangle = k_0 c_i^2 \tag{20}$$

$$\mathcal{K}_{i,i+1} = k_1 \langle \phi_{i+1}(\xi), \phi_j(\xi) \rangle = k_1 c_{i+1}^2 \frac{a_i}{a_{i+1}} \tag{21}$$

$$\mathcal{K}_{i,i-1} = k_1 \frac{c_i^2}{c_{i-1}^2} \langle \phi_{i-1}(\xi), \phi_j(\xi) \rangle = k_1 c_i^2 \frac{a_{i-1}}{a_i} \tag{22}$$

and the $\mathcal{D}$ matrix as

$$\mathcal{D} = \begin{bmatrix} c_0^2 k_0 & c_1^2 k_1 & 0 & 0 & \dots \end{bmatrix}^T. \tag{23}$$

A state space representation of the system is given by

$$\underbrace{\begin{Bmatrix} \dot{\boldsymbol{X}} \\ \ddot{\boldsymbol{X}} \end{Bmatrix}}_{\dot{\boldsymbol{Z}}} = \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\mathcal{M}^{-1}\mathcal{K} & -\mathcal{M}^{-1}c\mathcal{M} \end{bmatrix}}_{A'} \underbrace{\begin{Bmatrix} \boldsymbol{X} \\ \dot{\boldsymbol{X}} \end{Bmatrix}}_{\boldsymbol{Z}} + \underbrace{\begin{bmatrix} \boldsymbol{0} \\ \mathcal{M}^{-1}D \end{bmatrix}}_{B'} \boldsymbol{u}. \tag{24}$$

Assuming $\boldsymbol{X} \in \mathbb{R}^{N+1}$ (i.e. $\boldsymbol{X} = [x_0, \dots, x_N]^T$), $\boldsymbol{Z} \in \mathbb{R}^{2N+2}$. The system in equation (24) can be discretized to obtain

$$\boldsymbol{Z}(\tilde{k}+1) = A\boldsymbol{Z}(\tilde{k}) + B\boldsymbol{u}(\tilde{k}) \tag{25}$$

where $\tilde{k}$ represents the $\tilde{k}^{th}$ time step under a zero order hold assumption. This final development where the coefficients of PC are now described by a discrete linear system (equation (25)) concludes the section on PC; as the mean and the variance of the $x$ and $\dot{x}$ can be easily derived from these coefficients.

## IV. INPUT SHAPER DESIGN

This section presents the terminal development allowing the input shaper to be posed as a convex optimization problem.

The control objective is to determine $\boldsymbol{u}(\tilde{k})$ which can be used to drive the system from an initial state ($\boldsymbol{Z}(0)$ at time $t = 0$) to a final desired state ($\boldsymbol{Z_d}(T_f)$ at time $t = T_f$). Parameterizing the input-shaper/time-delay filter as:

$$G(s) = \sum_{i=0}^{P} A_i e^{-siT_s} \tag{26}$$

where $T_s$ is the sampling interval and $P$ are the total number of delays in the time-delay filter, results in a total of $P+1$ parameters to be solved for.

Equation (25) can be easily solved for a parameterized shaped input $u$ which is the output of Equation (26) subject to a unit step. Since the residual energy at the final time $T_P$ can be represented as:

$$V(T_P, \xi) = \frac{1}{2}\left(\sum_{i=0}^{P} \dot{x}_i \phi_i(\xi)\right)^T \left(\sum_{i=0}^{P} \dot{x}_i \phi_i(\xi)\right) + \frac{1}{2}\left(\sum_{i=0}^{P} x_i \phi_i(\xi) - x_f\right)^T k(\xi) \left(\sum_{i=0}^{P} x_i \phi_i(\xi) - x_f\right) \tag{27}$$

is a quadratic it does permit posing a convex chance constraint problem. Consequently, a $l_1$ norm approximation of the $l_2$ norm will be used to design the robust input-shapers. Figure 2 illustrate two polygon approximations of a two norm, given by a circle. The polygon approximation permits the use of straight line constraints to approximate the quadratic constraint. For the second order system considered

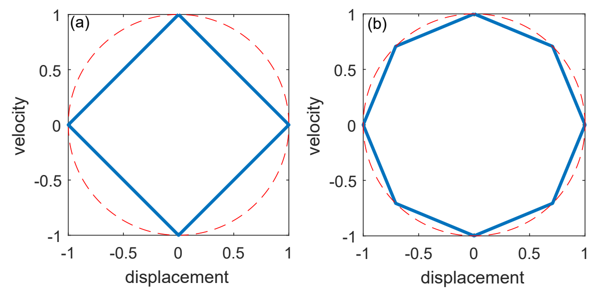



Fig. 2. Approximation of $l_2$ norm with polygons

here, the abscissa corresponds to the displacement state and the ordinate corresponds to the velocity state. As an illustrative example, consider a second order constraint of the form:

$$x^2(T_P) + \dot{x}^2(T_P) \leq 1 \tag{28}$$

where $x(T_P)$ and $\dot{x}(T_P)$ represent the displacement and velocity states at the terminal time $T_P$. It can be approximated by the linear constraints:

$$+x(T_P) + \dot{x}(T_P) < 1 \tag{29}$$

$$+x(T_P) - \dot{x}(T_P) < 1 \tag{30}$$

$$-x(T_P) + \dot{x}(T_P) < 1 \tag{31}$$

$$-x(T_P) - \dot{x}(T_P) < 1 \tag{32}$$

as shown in Figure 2(a). A better approximation using linear constraints as shown in Figure 2(b), will be given by the set of linear constraints:

$$\pm x(T_P) + (\sqrt{2}-1)\dot{x}(T_P) < 1 \tag{33}$$

$$\pm(\sqrt{2}-1)x(T_P) + \dot{x}(T_P) < 1 \tag{34}$$

$$\pm(\sqrt{2}-1)x(T_P) - \dot{x}(T_P) < 1 \tag{35}$$

$$\pm x(T_P) - (\sqrt{2}-1)\dot{x}(T_P) < 1. \tag{36}$$

Since the quadratic cost function has been approximated by a series of linear constraints, the convex representation of a linear chance constraint can be used to solve for the robust input shaper.

We will use the linear constraint given by Equation (29) as an exemplar to derive the convex probabilistic chance constraint which is given as:

$$P\left(x(T_P) + \dot{x}(T_P) - f \leq 0\right) \geq 1 - \epsilon \tag{37}$$

where $f$ permits changing the point of intersection of the line with the ordinate. Equation (37) can be rewritten as:

$$\psi\{var[x(T_P) + \dot{x}(T_P) - f]\}^{1/2} + E[x(T_P) + \dot{x}(T_P) - f] \leq 0 \tag{38}$$

where $\psi = \sqrt{\frac{1-\epsilon}{\epsilon}}$.
The linear chance constraint requires knowledge of the mean ($\overline{x}$) and variance ($\sigma$) of the terminal states. It can be shown that they are given by

$$\overline{x} = E[x(T_P,\xi)] = x_0(T_P)\phi_0(\xi) \quad (39)$$

$$\sigma = E[(x(T_P,\xi)-\overline{x})^2] = \sum_{i=1}^{N} x_i^2 \langle \phi_i(\xi), \phi_i(\xi)\rangle \quad (40)$$

where $\langle \phi_i(\xi), \phi_i(\xi)\rangle$ is the inner product. Substituting Equations (39) and (40) into Equation (38), we arrive at a convex optimization problem to determine the input shaped profile. The convex optimization problem can now be stated as:

$$\min \quad f = \sum_{i=0}^{P} (i+1)^\lambda |A_i| \quad (41a)$$

subject to

$$\psi\sqrt{\Sigma\left(x(T_P)+\dot{x}(T_P)-f\right)} + \mu\left(x(T_P)+\dot{x}(T_P)-f\right) \le 0 \quad (41b)$$

$$\psi\sqrt{\Sigma\left(x(T_P)-\dot{x}(T_P)-f\right)} + \mu\left(x(T_P)-\dot{x}(T_P)-f\right) \le 0 \quad (41c)$$

$$\psi\sqrt{\Sigma\left(-x(T_P)+\dot{x}(T_P)-f\right)} + \mu\left(-x(T_P)+\dot{x}(T_P)-f\right) \le 0 \quad (41d)$$

$$\psi\sqrt{\Sigma\left(-x(T_P)-\dot{x}(T_P)-f\right)} + \mu\left(-x(T_P)-\dot{x}(T_P)-f\right) \le 0 \quad (41e)$$

$$A_i > 0 \quad \forall i \quad (41f)$$

where we define $\Sigma\,(.)$ as the variance of the argument and $\mu\,(.)$ as the mean of the argument.

## V. NUMERICAL RESULTS

The proposed approach was used to design robust input shapers for the spring-mass-dashpot system:

$$\ddot{x} + c\dot{x} + kx = ku \quad (42)$$

where the damping constant $c = 0.1$ and the stiffness $k$ is an uncertain variable. A chance constrained based optimization problem is posed where the maneuver time is selected to be one period of the damped natural frequency of the system. The constraint that has to be satisfied is:

$$P\left(V(T_P,k) \le 0.02^2\right) \ge 0.7 \quad (43)$$

which states that the residual energy (Equation (27)) should be less than 0.02 for more than 70% of the realizations of the uncertain stiffness $k$. The maneuver time of one period of the damped natural frequency permits comparing the performance of the proposed robust input shaper to the three-impulse or Zero-Vibration-Derivative (ZVD) input shaper [2].

We will consider three different probability density functions with the same mean and variance for $k$ which are 1 and $0.018$ respectively. The three probability distributions functions are: The first distribution is a uniform one and is defined in terms of the r.v. $\xi_1 \in U[-1,1]$. Therefore, we have $k = 1 + 0.2324\xi_1$.
The second distribution is defined via a beta distributed r.v. $\xi_2 \in [-1,1]$ with parameters $a = 1$ and $b = 1$ making $k = 1 + 0.3\xi_2$.
The final distribution is chosen from the article [9]. The r.v. $\xi_3 \in [-1,1]$ and has a pdf given by

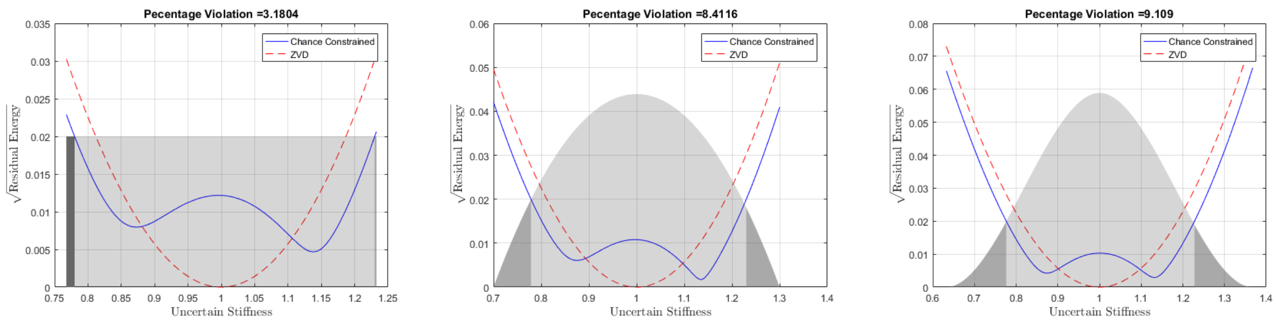
$$p(\xi_3) = 1 - W\sum_{i=0}^{1} Q_i|\xi_3|^{2-i+1} \quad (44)$$

where $W = -(3)!\,;\quad Q_i = \frac{(-1)^{i\,1}R_i}{2-i+1}$; and ${}^1R_i = \frac{1!}{i!(1-i)!}$.

A discrete time model is used to parameterize the terminal states in terms of the control profile and a cost function which is the sum of a time weighted control increment is minimized. A maneuver time of $T_P = 14.05$ and $P = 750$ is used which results in a sampling interval of $T_s = 0.0187$. The chance constraint is imposed to ensure that the control perfomance ensures that the residual energy is below a threshold of $0.02^2$ for at least 70% of the realizations.

Figures 3(a), 3(b) and 3(c) present the variation of residual energy for the chance constrained based robust input shaper design for a uniform distribution, a beta distribution and a compact support polynomial distribution for the uncertain spring stiffness, respectively. It should be noted that the design requires more than 70% of the realization of the uncertain system should have a residual energy below $0.02^2$. The regions of the pdf that violate the prescribed threshold are shown by the darker regions of the pdf. It can be seen that the violations for all the three cases are significantly smaller than the permitted 30%. The residual energy distribution for the ZVD input shaper designed with the same maneuver time is shown by the dashed red lines on all three figure and it is clear from the figures that a greater fraction of the uncertain system realization violate the prescribed threshold relative to the robust chance constrained based design. It should be noted that the acceptable risk level for a prescribed threshold of residual energy can result in an infeasible optimization problem if the risk levels are slected to very small for the prescribed residual energy threshold or a very small residual energy threshold is prescribed for a given risk level. A binary search in one dimensional on the risk level threshold is carried out for each presrcribed residual energy threshold to identify the risk level which corresponds to the boundary between the feasible and infeasible region.

Figure 4 presents the feasibility region in the Energy - Risk level space when the spring constant is assumed to have an uniform distribution. The black line marks the boundary of the feasibility region. A control solution exists for any desired point in the grey space, i.e. if any point (residual energy,risk level) is chosen in the grey space, a control solution can be found such that realizations of the stochastic system violate the residual energy level y at most x fraction of times. It is interesting to note that at lower risk levels the residual energy levels needed for a feasible solution are higher. This is consistent with the intuition that when the




(a) Uniform (b) Beta (c) Polynomial

Fig. 3. Residual Energy distribution and Violations of Constraint

probability of constraint violation is required to be low, the residual energy level naturally needs to be higher perfectly capturing the performance vs robustness trade off. Similar charts can be use to characterize the tradeoff between the residual-energy and the acceptable risk-level.

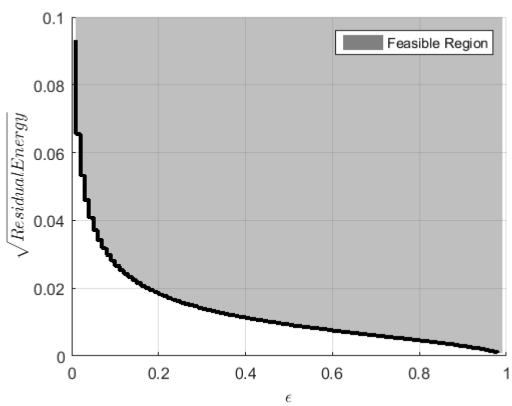



Fig. 4. Feasibility Region when the spring constant is uniformly distributed

## VI. CONCLUSIONS

The paper presents a convex optimization problem formulation for the design of input shapers that are robust to model parameter uncertainties. A chance constraint formulation is used to prescribe an acceptable risk level for a residual-energy threshold. Approximating a $l_2$ norm with a set of linear constraints, a convex optimiation problem is formulated to determine the parameters of an input shaper. The proposed approach is illustrated on a second order spring-mass-dashpot system for three different distributions of the uncertain spring stiffness. Results illustrate that the % violation if always smaller than the prescribed risk level. This is attributed to the fact that the chance constraint which is used in the problem is conservative. The proposed technique can be easily extended to system with multiple modes and with multiple uncertain paramters which are defined probabilistically.

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