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TRAVELING WAVES FOR A CLASS OF DIFFUSIVE DISEASE-TRANSMISSION MODELS WITH NETWORK STRUCTURES*

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Abstract. In this paper, the necessary and sufficient conditions for the existence of traveling wave solutions are derived for a class of diffusive disease-transmission models with network structures. The existence of traveling semifronts is obtained by Schauder's fixed-point theorem, and these traveling semifronts are shown to be bounded by transforming the boundedness problem into the classification problem of nonnegative solutions to a linear elliptic system on $\mathbb{R}$. To overcome the reducibility problem arising in the proofs, Harnack's inequality for positive supersolutions on $\mathbb{R}$ is proved.

Key words. disease-transmission models, traveling wave, classification problem, rescaling method, Harnack's inequality

AMS subject classifications. 35K57, 35C07, 92D30

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1. Introduction.

Compared with simple compartmental epidemic models, those with complex network structures can better describe the disease-transmission behaviors [22]. In this paper, we aim to show the existence of traveling waves for a class of diffusive disease-transmission models with network structures, which are formulated by a noncooperative reaction-diffusion system and usually consist of more than three equations. To that end, methods for traveling waves of noncooperative reaction-diffusion systems will be developed.

In our model, hosts are assumed to be divided into $n+1$ subclasses, in which each individual is either susceptible or infected. If a host is infected, we call it a carrier, who may be infectious or noninfectious (e.g., exposed or infective class; see Britton [4, Chapter 3]). Let $u(x,t)$ and $v_i(x,t)$ denote the densities of susceptible and carrier hosts with infection character i, respectively. Here x is the space variable and t is the time. Then our diffusive model is given by

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_0\Delta u + f(u) - g_0(u,v), \\ \dfrac{\partial v_i}{\partial t} = d_i\Delta v_i + g_i(u,v), \quad i=1,\dots,n, \end{cases} \tag{1.1}$$

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where

$$v = (v_1, \dots, v_n),\ x \in \mathbb{R}^d,\ \Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2},\ \ f(u) = \delta(K - u),$$

$$g_0(u, v) = \sum_{j=1}^{n} g_j^0(u, v),\quad g_i^0(u, v) = u \sum_{j=1}^{n} \frac{\beta_{ij} v_j}{1 + \gamma_{ij} v_j},\ \ g_i(u, v) = g_i^0(u, v) + \sum_{j=1}^{n} m_{ij} v_j$$

with $m_{jk} \geq 0$ if $j \neq k$ and $m_{jj} \leq 0$. Here d_j ($j = 0, 1, \dots, n$), δ and K are positive constants, and β_{ij} and γ_{ij} are nonnegative constants such that $\beta_{i_0 j_0} > 0$ for some index i_0 and j_0. The function $\frac{\beta_{ij} u v_j}{1+\gamma_{ij} v_j}$ stands for the disease incidence due to the jth carrier class v_j, which results in the susceptible u becoming carrier v_i. It is the famous bilinear incidence if $\gamma_{ij} = 0$ and the saturation incidence if $\gamma_{ij} > 0$ [27].

For system (1.1), we introduce the notations for the two matrices $\mathbb{M}$, $G^0 \in \mathbb{R}^{n\times n}$, given by $(\mathbb{M})_{ij} = D_{v_j} g_i(0, 0) = m_{ij}$ and $(G^0)_{ij} = D_{v_j} g_i(K, 0) = K\beta_{ij} + m_{ij}$, which satisfy

(1.2)
$$g_i(0, v) = \sum_{k=1}^{n} m_{ik} v_k = (\mathbb{M}v)_i \quad \text{and} \quad g_i(K, v) = (G^0 v)_i + o(\|v\|) \ \text{ for } 1 \leq i \leq n.$$

Both matrices are essentially nonnegative and constant (i.e., off-diagonal entries are nonnegative). In fact, $(G^0)_{ij} \geq (\mathbb{M})_{ij}$ for all i, j. These two matrices play important roles in determining the critical wave speed and other properties of traveling wave solutions.

To illustrate the range of disease models to which our methods for (1.1) apply, we consider a multistage epidemiological model. Guo, Li, and Shuai [17] proposed a general class of multistage epidemiological models that allow possible deterioration and amelioration between any two infected stages. That model can describe disease progression through multiple latent or infectious stages, as in the cases of HIV and tuberculosis. The host population is partitioned into the following compartments: a susceptible compartment S, a succession of infectious compartments I_i, $i = 1, \dots, n$, whose members are in the ith stage of the disease progression, and a removed compartment R. Generally speaking, hosts can diffuse freely, and thus we consider a special case of Guo's model with diffusion and bilinear incidence, which is as follows:

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_0 \Delta S + \delta(K - S) - S \displaystyle\sum_{j=1}^{n} \beta_j I_j, \\ \dfrac{\partial I_1}{\partial t} = d_1 \Delta I_1 + S \displaystyle\sum_{j=1}^{n} \beta_j I_j + \sum_{j=1}^{n} \phi_{1j} I_j - \psi_1 I_1, \\ \dfrac{\partial I_i}{\partial t} = d_i \Delta I_i + \displaystyle\sum_{j=1}^{n} \phi_{ij} I_j - \psi_i I_i, \quad i = 2, \dots, n, \end{cases} \tag{1.3}$$

where δ, K, d_i, $i = 0, 1, \dots, n$, are positive, and $\psi_i = \sum_{j=1}^{n} \phi_{ji} + \zeta_i$ for all i. Moreover, $\phi_{ii} = 0, \phi_{ij} \geq 0$ for all i, j, and $\sum_{j=1}^{n} \phi_{ji} > 0$ for all i; $\beta_{i_0} > 0$ for some index i_0, and $\zeta_i > 0$ for all i. Obviously, model (1.3) is a special case of (1.1). There are two network structures in (1.3): the network between S and $I := (I_1, \dots, I_n)$ and that among different progression stages I_i, $i = 1, \dots, n$. In the first network, $\beta_j S I_j$ stands for the disease incidence due to I_j. In the second network, ϕ_{ij} measures the transfer (deterioration or amelioration) rate from I_j to I_i. Similarly, for model (1.1), the matrix $\left(\frac{\beta_{ij}uv_j}{1+\gamma_{ij}v_j}\right)_{n\times n}$ stands for the transfer network from u to v and the matrix $\mathbb{M}$ measures the transfer rates among v_i, $i = 1, \ldots, n$.

Besides the multistage epidemiological model (1.3), system (1.1) can also model the spatial virulence-mutation behaviors [16, 15, 31, 14]. If $m_{ij} = 0 = \beta_{ij}$ for all $i \neq j$, and $\gamma_{ij} = 0$ for all i, j, then our model (1.1) becomes the one in [31]. If the total host size is constant (i.e., $u(x,t) + \sum_{i=1}^{n} v_i(x,t)$ is constant over $\mathbb{R}^d \times [0,\infty)$) and the matrix $\mathbb{M}$ is irreducible, then our model results in the system of [14], the one in [16] when $n = 2$, or the model in [15] when $n = 2$ and $d_1 = d_2$. Note that the existence of traveling waves for our models in special cases [15, 14] has been studied completely. However, unlike those in [16, 15, 31, 14], our model is more general and allows general mutation matrix $\mathbb{M}$ and varying total host size, which would better describe virulence evolution among different pathogen strains.

Apart from model (1.3) and those in [16, 15, 31, 14], system (1.1) also contains, as special cases, the models in [40, 36] and those in [21, 30, 29, 23, 13, 11] with diffusion. Clearly, in our model (1.1), susceptible hosts u have positive effects on carrier hosts v, whereas the carrier hosts have negative effects on the susceptible. This means that system (1.1) is noncooperative. The goal of this paper is to develop a novel method for the existence of traveling waves of the noncooperative system (1.1) and to apply this method to model (1.3) and the models in [16, 15, 31, 14, 21, 30, 29, 23, 13, 11].

Note that the disease incidence in (1.1) has the specific form $\frac{\beta_{ij}uv_j}{1+\gamma_{ij}v_j}$, which is unsaturated (bilinear) if $\gamma_{ij} = 0$ and saturated if $\gamma_{ij} > 0$. Like [17, 36], we could certainly make this incidence be a general nonlinear function with some tedious assumptions. However, the paper organized in this manner may seem complex in writing, and lots of efforts have to be paid for tedious assumptions. In this paper, in order to avoid this situation and let the readers easily grasp the main ideas, we thus take the disease incidence to be the specific function $\frac{\beta_{ij}uv_j}{1+\gamma_{ij}v_j}$ including saturated and unsaturated cases. We hope to make the main ideas of the proofs more transparent in this way.

It is easy to verify that the following properties hold for system (1.1):

(C1) $g_0(u,v)$ is nondecreasing with respect to $u \geq 0$ and $v \geq 0$, and $g_j(u,v)$, $j = 1, \ldots, n$, are nondecreasing with respect to $u \geq 0$ and $v_i \geq 0, i \neq j$.

(C2) For $u > 0$ and $v > 0$, and $i = 0, 1, \ldots, n$, the Hessian matrices $D_v^2 g_i(u,v)$ are negative semidefinite. These two properties will be frequently used in the proofs of this paper.

For simplicity, we introduce some notations that will be used throughout this paper and then give some basic definitions.

1.1. Notations.

1. $[n] := \{1, 2, \ldots, n\}$.
2. $0_n :=$ zero vector with n entries.
3. $\hat{i}$ denotes the imaginary unit, i.e., $\hat{i}^2 = -1$.
4. $A^T :=$ transpose of the matrix A.
5. $\Lambda_1(M) :=$ Perron–Frobenius dominant (or principal) eigenvalue of essentially nonnegative matrix M.
6. $(M)_{ij} :=$ the (i,j) entry of matrix M; $(v)_i :=$ the ith entry of vector v; $(m_{ij})_{n\times n}$ denotes the $n \times n$ matrix with entries m_{ij}.
7. $\|v\| := \sum_{j=1}^{n} |(v)_j|$, where v is a vector with n entries.

8. For vectors $v, \hat{v} \in \mathbb{R}^n$, define

$$\begin{aligned} &v \geq \hat{v} \text{ if } (v)_j \geq (\hat{v})_j \quad \text{for all } j \in [n]; \\ &v > \hat{v} \text{ if } v \geq \hat{v} \text{ and } v \neq \hat{v}; \\ &v \gg \hat{v} \text{ if } (v)_j > (\hat{v})_j \quad \text{for all } j \in [n]. \end{aligned}$$

9. $s \succ 1$: s is sufficiently large. $s \prec -1$: $-s$ is sufficiently large.
10. For $\tilde{v} = (\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_n)$ and a function $g_i(\tilde{v})$, set

$$g_{i,j}(\tilde{v}) := \frac{\partial g_i}{\partial \tilde{v}_j}(\tilde{v}), \quad g_{i,jk}(\tilde{v}) := \frac{\partial^2 g_i}{\partial \tilde{v}_j \partial \tilde{v}_k}(\tilde{v}), \quad i, j, k = 0, 1, \ldots, n.$$

1.2. Definitions.

Definition 1.1.

1. *A solution* $(u, v) = (u, v_1, v_2, \ldots, v_n)(x, t)$ *of* (1.1) *is said to be a traveling wave solution (TWS) if*

$$(1.4) \qquad (u, v)(x, t) = (U, V_1, V_2, \ldots, V_n)(s), \quad s = x^T \nu + ct,$$

for which c *is referred to as the wave speed and* $\nu \in \mathbb{R}^d$ *is the unit vector of the traveling direction.*

2. *A positive TWS* (1.4) *is called a traveling semifront if*

$$(1.5) \qquad (U, V)(-\infty) = E_0(K, 0_n),$$

where E_0 *denotes the invasion-free equilibrium.*

3. *A traveling semifront* $(U, V)(s)$ *is called persistent if it is bounded and satisfies*

$$(1.6) \qquad \liminf_{s \to +\infty} U(s) > 0, \quad \liminf_{s \to +\infty} V_j(s) > 0, \quad j \in [n].$$

4. *We say that a square matrix* $M = (M_{ij})_{n \times n}$ *is essentially nonnegative if* $M_{ij} \geq 0$ *whenever* $i \neq j$. *And we say that* M *is irreducible if for some* $k \in \mathbb{N}$, *all entries of* $\tilde{M}^k$ *are positive, where*

$$\tilde{M}_{ij} := \begin{cases} 0 & \text{if } i = j, \\ M_{ij} & \text{if } i \neq j. \end{cases}$$

We recall the classical Perron–Frobenius theorem for nonnegative matrices. See, e.g., [3, pp. 26–27].

Theorem 1.2. *Every essentially nonnegative matrix* M *has a Perron–Frobenius dominant eigenvalue* $\Lambda_1(M) \in \mathbb{R}$, *which is the eigenvalue with the greatest real part. Moreover, if* M *is irreducible, there exists a strictly positive right (resp., left) eigenvector* $\vec{v}_R$ *(resp.,l* $\vec{v}_L$*), such that*

$$M\vec{v}_R = \Lambda_1(M)\vec{v}_R \qquad (\textit{resp.}, \quad (\vec{v}_L)^T M = \Lambda_1(M)(\vec{v}_L)^T).$$

There may be other real eigenvalues besides $\Lambda_1(M)$, *but* $\Lambda_1(M)$ *is the only one with nonnegative eigenvectors.*

1.3. Methods and related results in the literature. We now review the methods frequently used for the existence of traveling wave solutions for reaction-diffusion systems.

Wu and Zou [35], Li, Weinberger, and Lewis [24], and Liang and Zhao [26] set up the general theory on the existence of TWSs for cooperative (or monotonic) systems by monotonic theories. For noncooperative systems that can be controlled from above and from below by cooperative systems, Wang [33] obtained results similar to that of [24] by comparison arguments. Recently, by using Schauder's fixed-point theorem and rescaling method, Girardin [14] studied a noncooperative system, the linearization of which at invasion-free equilibrium results in an irreducible (and essentially nonnegative) matrix.

Unfortunately, a large number of models, such as the famous predator-prey model and SI disease-transmission model, cannot be controlled by cooperative systems, and the linearization of these models at invasion-free equilibrium (e.g., prey-only equilibrium or disease-free equilibrium) is not cooperative. In this case, we say that the model is essentially noncooperative. There are two methods commonly used for essentially noncooperative reaction-diffusion systems, i.e., the geometric approach (or shooting method) and Schauder's fixed-point theorem approach. The shooting method was proposed by Dunbar [9, 10] for predator-prey models and has been adopted by many researchers for more than 30 years. This method was developed further by Huang [20] for a class of general noncooperative systems. Though it is powerful, the geometric method is mainly used for noncooperative systems consisting of two equations. It is usually challenging to analyze the geometric behaviors of noncooperative systems consisting of more than three equations. The approach via Schauder's fixed-point theorem is also widely used for essentially noncooperative systems; it was proposed by Ma [28] and developed by Huang and Zou [18, 19] and Li, Lin, and Ruan [25]. Typically, to apply Schauder's fixed-point theorem, one needs to construct a pair of appropriate super- and subsolutions connecting two equilibria, which is generally challenging. To overcome this difficulty, Schauder's fixed-point theorem method was developed further by Ducrot, Langlais, and Magal [8], Fu and Tsai [12], and Zhang, Wang, and Wang [39] by constructing a pair of super- and subsolutions connecting only invasion-free equilibrium at $-\infty$ and by using LaSalle's invariance principle to conclude convergence to a positive equilibrium at $+\infty$. Zhang, Wang, and Wang [39] also developed Schauder's fixed-point theorem by introducing persistence theory (see Thieme [32]) into the study of traveling waves where Lyapunov function is not available, whereby LaSalle's invariance principle cannot be applied. By Schauder's fixed-point theorem, Zhang [36] studied the existence of traveling waves with the minimal wave speed for a general noncooperative system (with or without recruitment) consisting of three equations.

We say that system (1.1) is unsaturated if there exist i and j such that $\beta_{ij} > 0, \gamma_{ij} = 0$, i.e., $g_i^0(u,v)$ is unbounded with respect to $v > 0$ for fixed $u > 0$. Then system (1.1) may be unsaturated and essentially noncooperative with recruitment (i.e., $f(u) \not\equiv 0$). The existence of traveling waves for an unsaturated and essentially noncooperative system without recruitment can be studied by the methods proposed by Wang and Wu [34] or Zhang and Wang [38]. However, to the best of our knowledge, there is not much literature on unsaturated and essentially noncooperative systems consisting of more than two equations with recruitment. Zhao and Wang [40] studied such a diffusive model, but there are some restrictions on the diffusive coefficients. Therefore, all the aforementioned methods cannot be directly applied to system (1.1) since they mainly deal with low-dimensional noncooperative systems (such as the geometric method or the methods in [8, 12]) or saturated noncooperative systems (such as the methods in [39, 36])

In this paper, we study the existence and nonexistence of traveling waves for system (1.1) by the rescaling method, which was used by Ducrot, Langlais, and Magal [8] for the nonexistence of traveling waves and by Berestycki et al. [2] and Girardin [14] for the existence of traveling waves. However, the methods utilized in these three papers cannot be directly applied to our model, and the reasons are as follows: (i) The irreducibility of the linearization matrix plays a key role in [14] since Harnack's inequality for elliptic systems works well in that case. In this paper, we assume that G^0 is irreducible but do not require $\mathbb{M}$ to be irreducible (see (1.2) for the definition of $\mathbb{M}$ and G^0). (ii) In [14], the system is, for instance, of Lotka–Volterra type and the boundedness of traveling waves can be guaranteed by the growth of the competition terms (assumption (H4) in [14]). This cannot be done for (1.1) in general since our system (1.1) may be unsaturated. It is a challenge to show the boundedness of traveling waves, especially the traveling wave with minimal wave speed. (iii) The hyperbolic property was used in [36] when Zhang studied the traveling wave with minimal wave speed in [36]. However, this property cannot be easily obtained for higher-dimensional systems such as (1.1). We overcome the obstacles (i)–(iii) by developing Harnack's inequality for positive supersolution in entire space (see Lemma 2.1) and by transforming the boundedness problem of traveling waves into the classification problem of nonnegative solutions to a linear elliptic system (Proposition 2.4). These two results are of independent interest in linear theory.

1.4. Main results. Recall that $\mathbb{M} = (m_{ij})_{n\times n}$ and G^0 are given in (1.2). The following theorem summarizes the main results of this work.

THEOREM 1.3. *Assume that G^0 is irreducible:*

(a) *Suppose $\Lambda_1(G^0) < 0$; then for any $c \in \mathbb{R}$, system* (1.1) *has no bounded traveling semifronts with wave speed c.*

(b) *Suppose $\Lambda_1(G^0) > 0$; then there exists $c_0^* > 0$ such that the following hold:*

(i) *For any $c \in (-\infty, c_0^*)$, system* (1.1) *has no traveling semifronts with wave speed c.*

(ii) *For any $c \in [c_0^*, +\infty)$, system* (1.1) *has a persistent traveling semifront with wave speed c if, in addition, $\Lambda_1(\mathbb{M}) < 0$ holds.*

We discuss briefly the assumption $\Lambda_1(\mathbb{M}) < 0 < \Lambda_1(G^0)$ for the existence of traveling semifronts. The first condition $\Lambda_1(\mathbb{M}) < 0$ is natural, as it means that the disease will become extinct in the absence of susceptibles ($u \equiv 0$). On the other hand, the second condition $\Lambda_1(G^0) > 0$ is, in most cases, equivalent to saying that the disease can establish when susceptibles are at carrying capacity ($u \equiv K$) (or that the basic reproduction number is greater than one). Hence it is necessary for the spread of the disease. This theorem will be divided into two theorems (Theorems 3.2 and 6.1) to facilitate the organization of this paper.

The remainder of this paper is organized as follows. In section 2, some linear problems are prepared for the main proofs. Specifically, Harnack's inequality is developed for positive supersolution in entire space, and nonnegative solutions for a linear elliptic system are completely classified. In section 3, we give the definition of the minimal wave speed c_0^* and show the nonexistence of traveling semifronts of system (1.1) when $c < c_0^*$. Section 4 is devoted to the existence and boundedness of traveling semifronts of system (1.1) with wave speed $c > c_0^*$, and section 5 deals with the existence of traveling semifronts in the case $c = c_0^*$. In section 6, the traveling semifronts of (1.1) with wave speed $c \geq c_0^*$ are shown to be persistent. In section 7, we apply our theorems to the multistage epidemiological model (1.3), and the TWSs for this model are shown to connect two equilibria by LaSalle's invariance principle.

2. Preliminary on a linear elliptic system. In this section, three important results about some linear problems are established, that is, Lemma 2.1, Proposition 2.4, and Lemma 2.9, which play key roles in the proofs following section 2. The results of this section are independent of other sections.

We first consider the following Harnack's inequality (see also Arapostathis, Ghosh, and Marcus [1, Theorem 2.2]).

LEMMA 2.1. *Assume that $\psi(\cdot) \in C^2(\mathbb{R})$ is nonnegative in $\mathbb{R}$ and $\psi(s)$ satisfies*

$$\psi'' + p_1(s)\psi' + p_2(s)\psi \leq 0 \quad \text{for} \quad s \in \mathbb{R}, \tag{2.1}$$

where $p_1(\cdot), p_2(\cdot) \in C(\mathbb{R})$, $|p_1(s)| + |p_2(s)| \leq M_1$ for some positive constant M_1 and all $s \in \mathbb{R}$:

(i) *If $\psi(s) > 0$ for all $s \in \mathbb{R}$, then there exists a positive constant M_2 depending only on M_1 such that*

$$\left|\frac{\psi'(s)}{\psi(s)}\right| \leq M_2 \quad \text{for all } s \in \mathbb{R}.$$

(ii) *There exists some positive constant M_3 such that*

$$\sup_{[a,b]} \psi \leq M_3 \inf_{[a,b]} \psi,$$

where M_3 depends only on M_1 and $b - a$.

Proof. If $\psi \equiv 0$, there is nothing to prove. If $\psi \not\equiv 0$, then the strong maximum principle implies that $\psi > 0$ in $\mathbb{R}$, which we henceforth assume.

First consider the proof of (i). Set $\varphi = \ln \psi$ and $\psi = e^{\varphi}$. Substituting this transform into (2.1) yields

$$\varphi'' + (\varphi')^2 + p_1(s)\varphi' + p_2(s) \leq 0.$$

By setting $\varphi' = w$, it follows that

$$w'(s) \leq -w^2(s) - p_1(s)w(s) - p_2(s).$$

Note that $w(s) = \psi'(s)/\psi(s)$. By the boundedness of p_1 and p_2, there exists $M_2 > 0$ depending only on M_1 such that

$$w'(s) \leq -\frac{1}{2}|w(s)|^2 \quad \text{whenever } |w(s)| \geq M_2. \tag{2.2}$$

Suppose $w(s_1) \leq -M_2$ for some $s_1 \in \mathbb{R}$. Then it follows from (2.2) that $w'(s) < 0$ for all $s > s_1$ and thus $w(s)$ is strictly decreasing in $[s_1, +\infty)$. In particular, $w(s) \leq w(s_1) \leq -M_2$ for all $s \geq s_1$, which means that (2.2) holds in $[s_1, +\infty)$. By the comparison principle of ordinary differential equations (ODEs), we have

$$w(s) \leq \frac{2w(s_1)}{2 + (s - s_1)w(s_1)} \quad \text{for all } s > s_1.$$

But then $w(s) \to -\infty$ as $s \nearrow s_1 - 2/w(s_1)$, contradicting $w(s) = \psi'(s)/\psi(s) \in C(\mathbb{R})$. Hence we conclude that $w(s) \geq -M_2$ for all $s \in \mathbb{R}$.

Similarly, suppose $w(s_1) \geq M_2$ for some $s_1 \in \mathbb{R}$; then $w'(s) < 0$ and $w(s) \geq M_2$ for all $s \leq s_1$. Hence (2.2) holds for all $s \leq s_1$, and by the comparison principle,

$$w(s) \geq \frac{2w(s_1)}{2 + (s - s_1)w(s_1)} \quad \text{for all } s < s_1.$$

Then it follows that $w(s) \to +\infty$ as $s \searrow s_1 - 2/w(s_1)$, which is a contradiction. Hence we conclude that $w(s) \leq M_2$ for all $s \in \mathbb{R}$. This proves part (i).

Now consider (ii). Let $s_1, s_2 \in [a, b]$; then it follows from (i) that

$$\varphi(s_2) - \varphi(s_1) \leq \sup_{[a,b]} |w(s)|(b - a) \leq M_2(b - a),$$

implying

$$\psi(s_2) \leq \psi(s_1)e^{M_2(b-a)}.$$

(ii) follows from the arbitrariness of $s_1, s_2 \in [a, b]$. ◻

Remark 2.2. Note that the result (i) of Lemma 2.1 has been established by Lemma 3.7 in Zhang and Jin [37]. The proof of (i) in this paper is more direct than that in [37]. Obviously, (ii) of Lemma 2.1 generalizes Harnack's inequality in Arapostathis, Ghosh, and Marcus [1, Theorems 2.1 and 2.2] to positive supersolution in entire space. We use Lemma 2.1 to deal with the homogeneous linear elliptic system with an essentially nonnegative (not necessarily irreducible) coefficient matrix.

Throughout this section, let $P = (P_{ij})_{n\times n}$ be a given essentially nonnegative matrix and let

$$H_{\lambda,c} := \text{diag}(d_i\lambda^2 - c\lambda)$$

denote the diagonal matrix with diagonal entries $d_i\lambda^2 - c\lambda$, $i \in [n]$. The following lemma is needed to describe Proposition 2.4.

LEMMA 2.3. *For each $c \in \mathbb{R}$, let*

$$\Lambda(c) := \{\lambda \in \mathbb{R} : \Lambda_1(H_{\lambda,c} + P) = 0\}. \tag{2.3}$$

(i) *If $\Lambda_1(P) < 0$, then for any $c \in \mathbb{R}$ we have $\Lambda(c) = \{\underline{\lambda}, \overline{\lambda}\}$ for some $\underline{\lambda} < 0 < \overline{\lambda}$.*
(ii) *If $\Lambda_1(P) > 0$, then there exists $c^* > 0$ such that*

$$\Lambda(c) = \begin{cases} \{\underline{\lambda}, \overline{\lambda}\} & \text{for some } \underline{\lambda} < \overline{\lambda} < 0 \text{ when } c < -c^*, \\ \{\underline{\lambda} = \overline{\lambda}\} & \text{for some } \underline{\lambda} = \overline{\lambda} < 0 \text{ when } c = -c^*, \\ \emptyset & \text{when } -c^* < c < c^*, \\ \{\underline{\lambda} = \overline{\lambda}\} & \text{for some } \underline{\lambda} = \overline{\lambda} > 0 \text{ when } c = c^*, \\ \{\underline{\lambda}, \overline{\lambda}\} & \text{for some } 0 < \underline{\lambda} < \overline{\lambda} \text{ when } c > c^*. \end{cases}$$

(iii) *If $\underline{\lambda} < \overline{\lambda}$, then $\Lambda_1(H_{\lambda,c} + P) < 0$ for all $\lambda \in (\underline{\lambda}, \overline{\lambda})$.*
(iv) *If $c \geq c^*$, then $\overline{\lambda}$ is nonincreasing with respect to P_{ij}, $i, j \in [n]$. Moreover, $\overline{\lambda}$ is strictly decreasing with respect to $P_{ij}, i, j \in [n]$, if P is irreducible.*

Proof. Denote $\mu(\lambda) := \Lambda_1(H_{\lambda,0} + P)$. Then it is obvious that $\Lambda_1(H_{\lambda,c} + P) = \mu(\lambda) - c\lambda$. It is easy to verify that

$$\mu(\lambda) = \tilde{d}\lambda^2 + \tilde{\Lambda}(\lambda),$$

where

$$\tilde{d} = \frac{1}{2}\min\{d_1, \dots, d_n\}, \quad \tilde{\Lambda}(\lambda) = \Lambda_1(\text{diag}((d_i - \tilde{d})\lambda^2) + P).$$

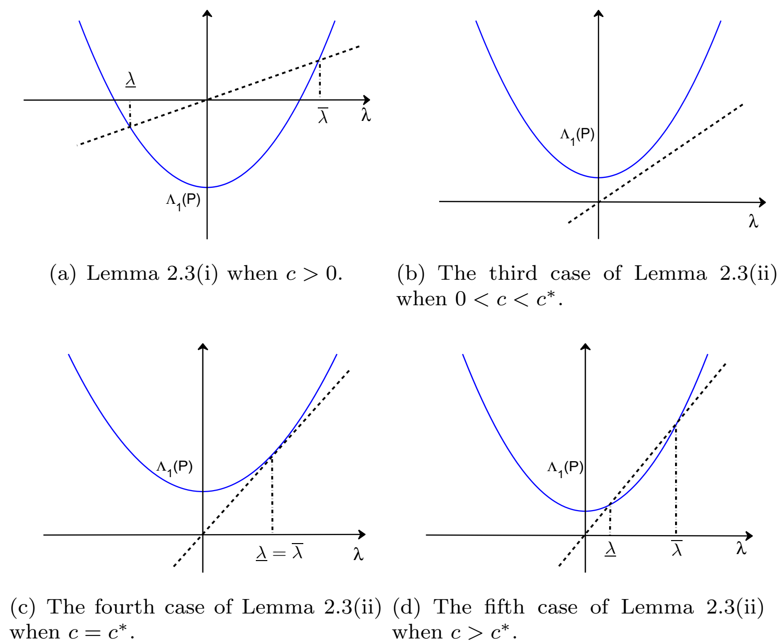

FIG. 1. *Diagram illustrating Lemma 2.3. The solid curve and dashed line are* $\mu(\lambda) = \Lambda_1(H_{\lambda,0}+P)$ *and* $c\lambda$*, respectively.*

Obviously, we have

$$\mu''(\lambda) = 2\tilde{d} + \tilde{\Lambda}''(\lambda) \geq 2\tilde{d} > 0,$$

where we used the fact that $\tilde{\Lambda}(\lambda)$ is convex in λ (see [5]). This means that $\mu(\lambda)$ is strictly convex in $\lambda \in \mathbb{R}$. It is obvious that $\mu(\lambda)$ is an even function and thus symmetric with respect to the vertical axis. Since $\Lambda_1(H_{\lambda,c} + P) = 0$ if and only if $\mu(\lambda) = c\lambda$, then (i), (ii), and (iii) can be given by the convexity and symmetry of $\mu(\lambda)$ (see Figure 1). It follows from [3, (1.5) Corollary, p. 27] that $\mu(\lambda)$ is nondecreasing in P_{ij} and strictly increasing in P_{ij}, provided that P is irreducible. Then (iv) follows from the convexity and symmetry of $\mu(\lambda)$. ◻

For each $c \in \mathbb{R}$, we define

$$\Gamma(c) = \{\lambda \in \mathbb{C} : 0 \text{ is an eigenvalue of } H_{\lambda,c} + P\}. \tag{2.4}$$

It is obvious that $\Lambda(c) \subset \Gamma(c)$. We have the following classification result.

PROPOSITION 2.4. *Assume* $\tilde{V}(s)$ *is a nonnegative solution to*

$$d_i\tilde{V}_i''(s) - c\tilde{V}_i'(s) + \sum_{j=1}^{n} P_{ij}\tilde{V}_j(s) = 0, \quad s \in \mathbb{R}, \quad i \in [n], \tag{2.5}$$

and for all $\lambda \in \Gamma(c)$ *let* ζ_λ *be the unit eigenvector of* $H_{\lambda,c} + P$ *corresponding to the eigenvalue* 0*. Then the following three conclusions hold:*

(i)

$$\tilde{V}(s) = \sum_{\lambda \in \Gamma(c) \cap \mathbb{R}} c_\lambda e^{\lambda s} \zeta_\lambda \tag{2.6}$$

with the restriction $c_\lambda \zeta_\lambda \geq 0$.

(ii) *If* P *is irreducible, then* (2.6) *can be strengthened to be*

$$\tilde{V}(s) = \sum_{\lambda \in \Lambda(c)} c_\lambda e^{\lambda s} \zeta_\lambda,$$

where the set $\Lambda(c)$ *is defined in Lemma* 2.3 *and it contains at most two real numbers. Moreover, either* $c_\lambda \zeta_\lambda = 0$ *or* $c_\lambda \zeta_\lambda \gg 0$.

(iii) *If* P *is irreducible and* $\Lambda(c) = \emptyset$, *then* $\tilde{V}(s) \equiv 0$.

Let $I_{n\times n}$ be the identity matrix of size n. By writing (2.5) as a system of $2n$ first-order ODEs

$$\begin{pmatrix} V \\ W \end{pmatrix}' = A_c \begin{pmatrix} V \\ W \end{pmatrix}, \quad \text{where } A_c = \begin{pmatrix} 0 & I_{n\times n} \\ -\operatorname{diag}(1/d_i)P & \operatorname{diag}(c/d_i) \end{pmatrix} \in \mathbb{R}^{2n\times 2n},$$

we can write any solution $\tilde{V}$ of system (2.5) in the form

$$\tilde{V}(s) = \sum_{\lambda \in \Gamma(c)} c_\lambda \sum_{j=0}^{k_\lambda} \operatorname{Re}\left(e^{\lambda s} s^{k_\lambda - j} \zeta_\lambda^j\right), \tag{2.7}$$

where $k_\lambda \geq 0$ is an integer smaller than the dimension $m_\lambda = \dim \cup_{j\geq 1} \ker(\lambda I_{2n\times 2n} - A_c)^j$ (so that $\sum_{\lambda\in\Gamma(c)} m_\lambda = 2n$), ζ_λ^j, $j \in [k_\lambda]$, are constant vectors, and ζ_λ^0 is an eigenvector corresponding to the zero eigenvalue of $H_{\lambda,c} + P$. By possibly replacing ζ_λ^j by $-\zeta_\lambda^j$, we may assume without loss of generality that $c_\lambda \geq 0$ for all λ.

Remark 2.5. In the above, we used the elementary fact that 0 is an eigenvalue of $H_{\lambda,c} + P$ with eigenvector $\zeta_\lambda^0 \in \mathbb{R}^n$ if and only if λ is an eigenvalue of A_c with eigenvector $(\zeta_\lambda^0, \lambda\zeta_\lambda^0) \in \mathbb{R}^{2n}$.

The following three lemmas are needed to complete the proof of Proposition 2.4.

Lemma 2.6. *Let*

$$\varphi(s) = \operatorname{Re}\left(\sum_{j=1}^{j_0} a_j e^{\hat{i}\beta_j s}\right),$$

where $a_j \in \mathbb{C}$, $\beta_j \in \mathbb{R}$, $\beta_j \neq 0$ *for all* $j \in [j_0]$, *and there exists some* $a_j \neq 0$. *Then*

(i)

$$\liminf_{s\to+\infty} \varphi(s) < 0, \quad \liminf_{s\to-\infty} \varphi(s) < 0, \quad \limsup_{s\to+\infty} \varphi(s) > 0, \quad \limsup_{s\to-\infty} \varphi(s) > 0.$$

(ii)

$$\lim_{L\to+\infty}\left[\sup_{s_0\in\mathbb{R}} \frac{1}{2L}\int_{s_0-L}^{s_0+L} \varphi(s)\,ds\right] = 0.$$

Proof. It is obvious that $\varphi(s)$ and $\varphi'(s)$ are bounded on $\mathbb{R}$ and that

$$\lim_{s\to+\infty} \frac{1}{s-s_0}\int_{s_0}^{s} \varphi(t)dt = 0$$

for any $s_0 \in \mathbb{R}$. By Corduneanu [6, Proposition 3.7], for any $\epsilon > 0$ there exists $l = l(\epsilon) > 0$ such that there exist $\tau_k \in (kl, (k+1)l)$, $k \in \mathbb{Z}$, with the property

$$|\varphi(s_0 + \tau_k) - \varphi(s_0)| < \epsilon \quad \text{for all } s_0 \in \mathbb{R}. \tag{2.8}$$

If $\varphi(s) \geq 0$ for all $s \in \mathbb{R}$, then (2.8), the fact that $\varphi(s) \not\equiv 0$, and the boundedness of $\varphi'(s)$ imply $\limsup_{s\to+\infty} \frac{1}{s-s^*}\int_{s^*}^{s} \varphi(t)dt > 0$, a contradiction. Hence there exists s_1 such that $\varphi(s_1) < 0$. Therefore, (2.8) shows that $\liminf_{s\to+\infty} \varphi(s) < 0$. Other inequalities in (i) can be similarly proved.

Assertion (ii) is a direct consequence of $\varphi(s)$ being a finite linear combination of sine and cosine functions. ∎

LEMMA 2.7. *Suppose for some $\lambda, \lambda' \in \Gamma(c)$ we have*

$$(H_{\lambda,c} + P)\zeta_\lambda = 0 \quad \textit{and} \quad (H_{\lambda',c} + P)\zeta_{\lambda'} = 0$$

for some $\zeta_\lambda \in [0, +\infty)^n \setminus \{0\}$ and $\zeta_{\lambda'} \in \mathbb{C}^n \setminus \{0\}$. If $\lambda \in \mathbb{R}$ and $\lambda' = \lambda + \hat{i}\beta$ for some $\beta \in \mathbb{R} \setminus \{0\}$, then there exists a component i such that $(\zeta_{\lambda'})_i \neq 0$ and $(\zeta_\lambda)_i = 0$.

Proof. Assume to the contrary that there exists $\lambda' = \lambda + \hat{i}b \in \Gamma(c) \setminus \mathbb{R}$ with corresponding eigenvector $\zeta_{\lambda'}$ such that

$$\{i : (\zeta_{\lambda'})_i \neq 0\} \subset \{i : (\zeta_\lambda)_i > 0\}.$$

Choose

$$\bar{k} := \inf\left\{k \in \mathbb{R} : e^{\lambda s}\left[\mathrm{Re}(e^{\hat{i}bs}\zeta_{\lambda'}) + k\zeta_\lambda\right] \geq 0 \text{ for all } s \in \mathbb{R}\right\}.$$

Since at least one entry of $\mathrm{Re}(e^{\hat{i}bs}\zeta_{\lambda'})$ changes sign and is periodic (with period $2\pi/b$) on $\mathbb{R}$ by Lemma 2.6, we can deduce that $0 < \bar{k} < \infty$ and that

$$V_0(s) := e^{\lambda s}\left[\mathrm{Re}(e^{\hat{i}bs}\zeta_{\lambda'}) + \bar{k}\zeta_\lambda\right]$$

is a nonnegative, nontrivial solution of (2.5) such that for some index j, $(V_0(s))_j$ has zero as a strict minimum. This contradicts Harnack's inequality (Lemma 2.1(ii)). ∎

LEMMA 2.8. *Let $W(s) = \sum_{k=1}^{k_1} e^{\lambda_k s} s^{l_k}(\zeta_k + \varphi_k(s) + \epsilon_k(s))$ satisfy*

$$W'' + aW' + bW \leq 0 \quad \textit{and} \quad W \geq 0 \quad \textit{for } s \in \mathbb{R},$$

where $\lambda_1 < \lambda_2 < \cdots < \lambda_{k_1}$, $l_k \in \mathbb{N} \cup \{0\}$, $\zeta_k, a, b \in \mathbb{R}$, and ζ_1 and $\varphi_1(s)$ are not both identically zero. Furthermore, assume that $\lim_{|s|\to\infty} |\epsilon_k(s)| = 0$ and that $\varphi_k(s)$ is a finite linear combination of sine and cosine functions, as in Lemma 2.6. Then

$$\liminf_{s\to-\infty} e^{-\lambda_1 s} W(s) > 0.$$

Proof. Let $\tilde{W}(s) = e^{-\lambda_1 s}W(s)$; then $\tilde{W}'' + (2\lambda_1 + a)\tilde{W}' + (\lambda_1^2 + a\lambda_1 + b)\tilde{W} \leq 0$ on $s \in \mathbb{R}$. Hence we may assume without loss of generality that $0 = \lambda_1 < \lambda_2 < \cdots$.

If $\varphi_1(s) \equiv 0$, then $\zeta_1 \neq 0$. Hence $\liminf_{s\to-\infty} |s|^{-l_1}W(s) = |\zeta_1| > 0$. This proves $\liminf_{s\to-\infty} W(s) > 0$ in the case of $\varphi_1(s) \equiv 0$.

It remains to prove the case when $\varphi_1(s) \not\equiv 0$. We prove only the case for l_1 being even, as the proof for the other case is similar. In this case,

$$\zeta_1 + \liminf_{s\to-\infty} \varphi_1(s) = \liminf_{s\to-\infty} s^{-l_1}W(s) \geq 0.$$

By Lemma 2.6(i), $\liminf_{s\to-\infty} \varphi_1(s) < 0$. Hence $\zeta_1 > 0$. Assume to the contrary that there exists $s_j \to -\infty$ such that $W(s_j) \to 0$ as $j \to \infty$. By Harnack's inequality (Lemma 2.1(ii)), we deduce that, for each $L > 0$,

$$\lim_{j\to\infty} \frac{1}{2L}\int_{s_j - L}^{s_j + L} W(s)\, ds = 0.$$

Hence, for each $L > 0$,

$$\lim_{j\to\infty} \frac{1}{2L}\int_{s_j-L}^{s_j+L} (\zeta_1 + \varphi_1(s))\,ds = \lim_{j\to\infty} \frac{1}{2L}\int_{s_j-L}^{s_j+L} s^{-l_1} W(s)\,ds = 0. \tag{2.9}$$

Next, choose a constant $L_0 > 0$ such that

$$\sup_{s_0\in\mathbb{R}} \left| \frac{1}{2L_0}\int_{s_0-L_0}^{s_0+L_0} \varphi_1(s)\,ds \right| < \frac{1}{2}\zeta_1, \tag{2.10}$$

which is possible due to Lemma 2.6(ii) and the fact that $\zeta_1 > 0$. Finally, by (2.9) and (2.10),

$$0 = \lim_{j\to\infty} \frac{1}{2L_0}\int_{s_j-L_0}^{s_j+L_0} (\zeta_1 + \varphi_1(s))\,ds \geq \zeta_1 - \frac{1}{2}\zeta_1 > 0.$$

This leads to a contradiction, and the assertion is approved as $s \to -\infty$. ∎

Proof of Proposition 2.4. Let $\tilde{V}(s)$ be a nonnegative solution of (2.5). Then $\tilde{V}(s)$ can be written in the form (2.7): $\tilde{V}(s) = \tilde{V}_1(s) + \tilde{V}_2(s)$, where

$$\tilde{V}_1(s) = \sum_{\lambda\in\Gamma(c)\cap\mathbb{R}} c_\lambda e^{\lambda s} s^{l_\lambda}(\zeta_\lambda^0 + o_1(1)), \quad \tilde{V}_2(s) = \sum_{\alpha+\hat{i}\beta\in\Gamma(c)\setminus\mathbb{R}} c'_\alpha e^{\alpha s} s^{k_\alpha}(\varphi_\alpha(s) + o_2(1)),$$

with $c_\lambda \geq 0$, $c'_\alpha \in \mathbb{R}$, $\zeta_\lambda^0 \neq 0$ $(\in \mathbb{R}^n)$; each entry of $\varphi_\alpha(s)$ has the form of $\varphi(s)$ in Lemma 2.6; l_λ and k_α are nonnegative integers; $o_i(1) \to 0$, $i \in [2]$, when $|s| \to \infty$, $o_1(1) \equiv 0$ if $l_\lambda = 0$, and $o_2(1) \equiv 0$ if $k_\alpha = 0$. Define

$$\bar{\mu}_1 = \max\{\lambda : \lambda \in \Gamma(c)\cap\mathbb{R}, c_\lambda \neq 0\},$$

and define, when $\tilde{V}_2 \not\equiv 0$ (i.e., $c'_{\alpha_0} \neq 0$ for some $\alpha_0 \in \{\alpha \in \mathbb{R} : \alpha + \hat{i}\beta \in \Gamma(c)\setminus\mathbb{R}\}$),

$$\bar{\mu}_2 = \max\{\alpha : \alpha + \hat{i}\beta \in \Gamma(c)\setminus\mathbb{R}, c'_\alpha \neq 0\}.$$

For each i, $d_i(\tilde{V})''_i - c(\tilde{V})'_i + P_{ii}(\tilde{V})_i \leq 0$ on $\mathbb{R}$. The strong maximum principle implies that, for each i, either $(\tilde{V})_i \equiv 0$ or $(\tilde{V})_i > 0$ on $\mathbb{R}$. By considering only the nontrivial components of $\tilde{V}(s)$, we may assume without loss of generality that $(\tilde{V})_i(s) > 0$ for all i and for all $s \in \mathbb{R}$.

Step 1. $\bar{\mu}_1$ is well-defined, and $c_{\bar{\mu}_1} > 0$, $\zeta^0_{\bar{\mu}_1} > 0$. Furthermore, if $\tilde{V}_2 \not\equiv 0$, then $\bar{\mu}_1 \geq \bar{\mu}_2$.

If $\tilde{V}_2(s) \equiv 0$, then it follows from the positivity of $\tilde{V}(s)$ that $\bar{\mu}_1$ is well-defined and $c_{\bar{\mu}_1} > 0$. For each i such that $(\zeta^0_{\bar{\mu}_1})_i \neq 0$, we have (recall that "$s \succ 1$" means "$s$ is sufficiently large")

$$0 < \operatorname{sgn}\, (\tilde{V})_i(s) = \operatorname{sgn}\, (c_{\bar{\mu}_1}(\zeta^0_{\bar{\mu}_1})_i) \quad \text{for all } s \succ 1.$$

Since we have chosen $c_{\bar{\mu}_1}$ to be positive, we conclude that $\zeta^0_{\bar{\mu}_1} > 0$.

Next, suppose $\tilde{V}_2(s) \not\equiv 0$, so that $\bar{\mu}_2$ is well-defined and, by Lemma 2.6(i), $\bar{\mu}_1$ is also well-defined. Since $\varphi_{\bar{\mu}_2}(s) \not\equiv 0$ is almost periodic, Lemma 2.6(i) implies that $\bar{\mu}_1 \geq \bar{\mu}_2$ and that $\zeta^0_{\bar{\mu}_1} > 0$.

Step 2. If $\tilde{V}_2(s) \not\equiv 0$, then either (a) $\bar{\mu}_1 > \bar{\mu}_2$; or (b) $\bar{\mu}_1 = \bar{\mu}_2$ and $l_{\bar{\mu}_1} > k_{\bar{\mu}_2}$.

Suppose to the contrary that the above result does not hold. Then it follows from Step 1 that $\zeta^0_{\bar{\mu}_1} > 0$, $\bar{\mu}_1 = \bar{\mu}_2$, and $l_{\bar{\mu}_1} \leq k_{\bar{\mu}_2}$. Moreover, by the fact that $\varphi_{\bar{\mu}_2} \not\equiv 0$ is almost periodic, Lemma 2.6(i) and $\tilde{V} \geq 0$ imply that $l_{\bar{\mu}_1} = k_{\bar{\mu}_2}$. Considering the facts that (i) $\tilde{V}(s) \gg 0$, and that (ii) $(\tilde{V}_1)_i(s) = o(e^{\bar{\mu}_1 s})$ as $s \to +\infty$ for those components i such that $(\zeta^0_{\bar{\mu}_1})_i = 0$, we deduce that $\{i : (\varphi_{\bar{\mu}_2}(s))_i \not\equiv 0\} \subset \{i : (\zeta^0_{\bar{\mu}_1})_i > 0\}$. But this is in contradiction to Lemma 2.7.

Step 3. For each component $i \in [n]$ such that $(\zeta^0_{\bar{\mu}_1})_i > 0$, we have

$$\liminf_{|s|\to+\infty} e^{-\bar{\mu}_1 s}|s|^{-l_{\bar{\mu}_1}}(\tilde{V})_i(s) \geq c_{\bar{\mu}_1}(\zeta^0_{\bar{\mu}_1})_i > 0. \tag{2.11}$$

It follows directly from Step 2 that if $(\zeta^0_{\bar{\mu}_1})_i > 0$, then

$$\lim_{s\to+\infty} e^{-\bar{\mu}_1 s}|s|^{-l_{\bar{\mu}_1}}(\tilde{V})_i(s) = c_{\bar{\mu}_1}(\zeta^0_{\bar{\mu}_1})_i > 0.$$

To prove (2.11), it suffices to consider the case where $s \to -\infty$. Suppose $\bar{\mu}_1$ is the only exponent appearing in $(\tilde{V})_i(s)$; then clearly (2.11) holds by Step 2. Otherwise, by Lemma 2.8, there exists $\mu' < \bar{\mu}_1$ such that $\liminf_{s\to-\infty} e^{-\mu' s}(\tilde{V})_i(s) > 0$, and hence

$$\liminf_{s\to-\infty} e^{-\bar{\mu}_1 s}|s|^{-l_{\bar{\mu}_1}}(\tilde{V})_i(s) = \liminf_{s\to-\infty}[e^{-(\bar{\mu}_1-\mu')s}|s|^{-l_{\bar{\mu}_1}}][e^{-\mu' s}(\tilde{V})_i(s)] = +\infty,$$

where we used the fact that $\liminf_{s\to-\infty} e^{-(\bar{\mu}_1-\mu')s}|s|^{-l_{\bar{\mu}_1}} = +\infty$. Thus (2.11) holds.

Step 4. $l_{\bar{\mu}_1} = 0$. (Particularly, part (b) of Step 2 is impossible.)

Suppose to the contrary that $l_{\bar{\mu}_1} \geq 1$. By Step 3, this implies that there exist γ_0 and s_0 such that $\tilde{V}(s) - \gamma_0 e^{\bar{\mu}_1 s}\zeta^0_{\bar{\mu}_1}$ is a nonnegative solution to (2.5) where one of the components achieves minimum value zero at some $s_0 \in \mathbb{R}$. This is impossible in view of the strong maximum principle for cooperative systems. This proves $l_{\bar{\mu}_1} = 0$. By Step 2, we must have $\bar{\mu}_2 < \bar{\mu}_1$.

From Steps 3 and 4, we deduce that, for each component i such that $(\zeta^0_{\bar{\mu}_1})_i > 0$,

$$\liminf_{|s|\to+\infty} e^{-\bar{\mu}_1 s}(\tilde{V})_i(s) \geq c_{\bar{\mu}_1}(\zeta^0_{\bar{\mu}_1})_i > 0. \tag{2.12}$$

It follows from Steps 2 and 4 that the term in $\tilde{V}(s)$ including $e^{\bar{\mu}_1 s}$ is exactly $c_{\bar{\mu}_1}e^{\bar{\mu}_1 s}\zeta^0_{\bar{\mu}_1}$.

Step 5. $\tilde{V}(s) - c_{\bar{\mu}_1}e^{\bar{\mu}_1 s}\zeta^0_{\bar{\mu}_1} \geq 0$ in $\mathbb{R}$.

Let $\tilde{V}_\gamma(s) := \tilde{V}(s) - \gamma c_{\bar{\mu}_1}e^{\bar{\mu}_1 s}\zeta^0_{\bar{\mu}_1}$.

CLAIM 1. $\tilde{V}_\gamma(s) > 0$ *for all* $s \in \mathbb{R}$ *and* $0 < \gamma < 1$.

If not, then by (2.12), there exists $0 < \gamma_0 < 1$ such that the minimum value zero of $\tilde{V}_{\gamma_0}(s)$ (i.e., a nonnegative solution of (2.5) associated with γ_0) is attained at some component i at some $s_0 \in \mathbb{R}$. By this contradiction with the strong maximum principle, the claim is established.

By continuity and the above claim, we let $\gamma \nearrow 1$ and establish Step 5.

Finally, by applying Steps 1 to 5 to the nonnegative solution $\tilde{V}_{new} := \tilde{V} - c_{\bar{\mu}_1}e^{\bar{\mu}_1 s}\zeta^0_{\bar{\mu}_1}$ of (2.5) and by repeating this procedure finitely many times, we conclude that $\tilde{V}_2 \equiv 0$, and thus $\tilde{V}$ satisfies (2.6) and $c_\lambda\zeta_\lambda \geq 0$. This completes the proof of Proposition 2.4(i).

If P is irreducible, then "0 is an eigenvalue of $H_{\lambda,c} + P$ with a nonnegative eigenvector" if and only if "$0 = \Lambda_1(H_{\lambda,c} + P)$" if and only if $\lambda \in \Lambda(c)$. Using Proposition 2.4(i) and Lemma 2.3, this proves (ii) and (iii). ∎

The following lemma will be used in section 4.3. This lemma is presented here, as its proof is independent of other sections.

Lemma 2.9. *Let $\Lambda(c)$ and $\Gamma(c)$ be given by* (2.3) *and* (2.4)*, respectively. If*

$$\min \Lambda(c) < \max \Lambda(c),$$

then

$$\Gamma(c) \cap \{\lambda \in \mathbb{C} : \min \Lambda(c) < \operatorname{Re} \lambda < \max \Lambda(c)\} = \emptyset. \tag{2.13}$$

Proof. To prove this lemma, we first assume that P is irreducible. Let $\min \Lambda(c) = \underline{\lambda}$ and $\max \Lambda(c) = \overline{\lambda}$ with corresponding unit eigenvectors $\underline{\zeta} \gg 0$, $\overline{\zeta} \gg 0$, so that $(H_{\underline{\lambda},c} + P)\underline{\zeta} = 0$ and $(H_{\overline{\lambda},c} + P)\overline{\zeta} = 0$.

Let $\lambda = a + b\hat{i} \in \Gamma(c)$ such that $\underline{\lambda} < a < \overline{\lambda}$, yielding that zero is an eigenvalue of $H_{\lambda,c} + P$. Suppose $b = 0$; then the matrix $H_{a,c} + P$ is real and essentially nonnegative. Hence, by the Perron–Frobenius theorem, Theorem 1.2, for each eigenvalue μ of $H_{a,c} + P$,

$$\operatorname{Re} \mu \leq \Lambda_1(H_{a,c} + P) < 0;$$

i.e., zero is not an eigenvalue of $H_{a,c} + P$, where the second inequality follows from Lemma 2.3(iii). Therefore, we must have $\lambda = a + b\hat{i}$ for some $b \neq 0$. Let ζ_λ be a corresponding eigenvector, and choose

$$\bar{k} := \inf\{k \in \mathbb{R} : k[e^{\underline{\lambda} s}\underline{\zeta} + e^{\overline{\lambda} s}\overline{\zeta}] + \operatorname{Re}(e^{\lambda s}\zeta_\lambda) \geq 0 \text{ for all } s \in \mathbb{R}\}.$$

Since at least one entry of $\operatorname{Re}(e^{\lambda s}\zeta_\lambda)$ changes sign on $\mathbb{R}$, we deduce that $0 < \bar{k} < \infty$, and that

$$V_0(s) = \bar{k}[e^{\underline{\lambda} s}\underline{\zeta} + e^{\overline{\lambda} s}\overline{\zeta}] + \operatorname{Re}(e^{\lambda s}\zeta_\lambda)$$

is a nontrivial, nonnegative solution of (2.5) such that for some component j and $s_0 \in \mathbb{R}$, $(V_0)_j(s_0) = 0$ is a strict minimum of $(V_0)_j(s)$. This is in contradiction to the strong maximum principle, and thus (2.13) holds if P is irreducible.

Now suppose P is reducible and denote $P_\epsilon = P + \epsilon\mathcal{I}$, where $\mathcal{I}$ is an $n \times n$ matrix with entries being one. Then, for each $\epsilon > 0$, P_ϵ is irreducible and (2.13) holds. By continuous dependence of the roots of $\det(H_{\lambda,c} + P_\epsilon) = 0$ on ϵ, we may let $\epsilon \to 0$ and deduce that (2.13) holds for P as well. □

3. Nonexistence of traveling semifronts of (1.1).

It is easy to show that the traveling profile $(U, V)(s)$ of system (1.1) defined by Definition 1.1 satisfies the following system:

$$\begin{cases} cU' = d_0 U'' + f(U) - g_0(U, V), \\ cV_i' = d_i V_i'' + g_i(U, V), \quad i \in [n], \end{cases} \tag{3.1}$$

where $'$ refers to the derivative with respect to s.

First, we linearize the equations for V_i of system (3.1) at $E_0 = (K, 0)$. Precisely, if $e^{\lambda s}\zeta$ is a solution of the associated linear system, then necessarily $(H_{\lambda,c} + G^0)\zeta = 0$.

Definition 3.1. *Whenever $\Lambda_1(G^0) > 0$, define $c_0^* > 0$ to be the quantity c^* given by Lemma* 2.3(ii) *with $P = G^0$.*

The following theorem establishes the nonexistence of traveling semifronts.

Theorem 3.2. *Assume that G^0 is irreducible. If $\Lambda_1(G^0) < 0$, then for any $c \in \mathbb{R}$ system* (1.1) *has no bounded traveling semifronts with wave speed c. If $\Lambda_1(G^0) > 0$, then for any $c \in (-\infty, c_0^*)$ system* (1.1) *has no traveling semifronts with wave speed c.*

Proof. We will adopt the idea of Girardin [14], but Lemma 6.1 of [14] cannot be directly used in this proof.

Assume system (1.1) has a bounded traveling semifront $(u, v)(x, t) = (U, V)(s)$, $s = x + ct$. We claim that $u^*_\infty := \limsup_{s\to+\infty} U(s) \le K$. Suppose to the contrary that $u^*_\infty > K$. If $U(s)$ is fluctuating for $s \succ 1$, there exists $s_k \to +\infty$ such that

$$U(s_k) \to u^*_\infty, \quad U'(s_k) \to 0, \quad U''(s_k) \to U''_* \le 0 \tag{3.2}$$

for some constant U''_*. If $U(s)$ is monotonic for $s \succ 1$, (3.2) still obviously holds for some $s_k \to +\infty$. It follows by passing to a further subsequence that $\lim_{s\to+\infty} V(s_k)$ exists. By the first equation of (3.1),

$$0 = d_0 U''_* + \delta(K - u^*_\infty) - g_0(u^*_\infty, \lim_{s\to+\infty} V(s_k)) < 0,$$

which leads to a contradiction. We therefore have $u^*_\infty \le K$. Assume that there exists s_0 such that $U(s_0) > K$. It follows from (1.5) and $u^*_\infty \le K$ that there exists s_1 such that $U(s_1) > K, U'(s_1) = 0, U''(s_1) \le 0$, contradicting the first equality of (3.1). We thus have $U(s) \le K$ for all $s \in \mathbb{R}$. It can be similarly shown that $U(s) < K$ for all $s \in \mathbb{R}$.

Now let $\Lambda_1(G^0) < 0$ and let $\zeta \gg 0$ be the corresponding principal eigenvector. Obviously, for any $\tau > 0$, $\hat{v}(t) = \tau e^{\Lambda_1(G^0)t}\zeta$ is a positive supersolution of the second equation of (1.1) such that $\hat{v}(t) \to 0$. Then we have

$$\frac{\partial}{\partial t}(\hat{v} - v) - \mathrm{diag}(d_i)\Delta(\hat{v} - v) = G^0\hat{v} - g(u, v) \ge G^0(\hat{v} - v),$$

where $g(u, v)(x, t) \le g(K, v) \le G^0 v$ is used (see property (C2) in section 1). Let τ be sufficiently large such that $\tau\zeta > v(x, 0)$ for all $x \in \mathbb{R}$. It follows from the comparison principle that $0 \le v(x, t) \le \hat{v}(t) \to 0$. Therefore, it is impossible for (1.1) to admit bounded traveling semifronts.

Suppose now that $\Lambda_1(G^0) > 0$ and system (1.1) has a traveling semifront $(U, V)(s)$ with wave speed $c < c^*_0$, which is the positive solution of (3.1) satisfying (1.5). Obviously, it is impossible that $(V(s))_1$ is nonincreasing for $s \prec -1$. Thus there exists a sequence $s_i \to -\infty$ such that $(V'(s_i))_1 \ge 0$. Define

$$\tilde{V}^{(i)}(s) := \frac{V(s + s_i)}{\|V(s_i)\|},$$

and thus $\|\tilde{V}^{(i)}(0)\| = 1, (\tilde{V}^{(i)})'_1(0) \ge 0$. Lemma 2.1 shows that $\tilde{V}^{(i)}(\cdot)$ converges to some $\tilde{V}_*(\cdot)$ in $C^2_{loc}(\mathbb{R})$, where $\tilde{V}_*(\cdot)$ is a nonnegative solution of (2.5) with $P = G^0$. If $-c^*_0 < c < c^*_0$, Lemma 2.3(ii) says that $\Lambda(c) = \emptyset$, and Proposition 2.4(iii) says that $\tilde{V}_*(\cdot) \equiv 0$, contradicting $\|\tilde{V}_*(0)\| = 1$. If $c \le -c^*_0$, Proposition 2.4(ii) and Lemma 2.3(ii) yield that

$$\tilde{V}_*(s) = c_{\underline{\lambda}} e^{\underline{\lambda} s}\zeta_{\underline{\lambda}} + c_{\overline{\lambda}} e^{\overline{\lambda} s}\zeta_{\overline{\lambda}},$$

where

$$c_{\underline{\lambda}} \ge 0, \quad c_{\overline{\lambda}} \ge 0, \quad c_{\underline{\lambda}} + c_{\overline{\lambda}} > 0, \quad \underline{\lambda} \le \overline{\lambda} < 0, \quad \zeta_{\underline{\lambda}} \gg 0, \quad \zeta_{\overline{\lambda}} \gg 0.$$

However,

$$\tilde{V}'_*(0) = c_{\underline{\lambda}}\underline{\lambda}\zeta_{\underline{\lambda}} + c_{\overline{\lambda}}\overline{\lambda}\zeta_{\overline{\lambda}} \ll 0,$$

contradicting $(\tilde{V}'_*(0))_1 \ge 0$. ∎

In what follows (i.e., sections 4, 5, and 6), to study the existence of traveling semifronts and by Theorem 3.2, we assume the following assumption (A1) holds.

(A1) $\Lambda_1(\mathbb{M}) < 0 < \Lambda_1(G^0)$, and G^0 is irreducible,
where $\mathbb{M} = (m_{ij})_{n\times n}$ and G^0 are given in (1.2).

4. Existence of traveling semifronts of (1.1) with $c > c_0^*$. Noting that assumption (A1) holds, in this section we assume that $c > c_0^*$. Since $\Lambda_1(G^0) > 0$, where G^0 is given by (1.2), $c_0^* > 0$ is well-defined by Definition 3.1. We will show the existence of traveling semifronts of (1.1) with wave speed $c > c_0^*$. This is accomplished by using Schauder's fixed-point theorem with the aid of a pair of super- and subsolutions. In addition, we show the boundedness of these traveling semifronts in $L^\infty(\mathbb{R})$ by a rescaling argument.

4.1. The super- and subsolutions. Now we construct a pair of super- and subsolutions. Denote

$$A_{\lambda,c} := H_{\lambda,c} + G^0 \quad \text{and} \quad \lambda_1 := \underline{\lambda},$$

where $\underline{\lambda} > 0$ is determined by Lemma 2.3(ii) with $P = G^0$. Define

$$\begin{aligned} &\overline{U}(s) := K, && \underline{U}(s) := \max\{K - \sigma_0 e^{\alpha s}, 0\}, \\ &\overline{V}_i(s) := \kappa_i e^{\lambda_1 s}, && \underline{V}_i(s) := \max\{\kappa_i e^{\lambda_1 s}(1 - \sigma_i e^{\epsilon s}), 0\} \end{aligned} \tag{4.1}$$

for $i \in [n]$, where $\kappa = (\kappa_1, \ldots, \kappa_n)^T$ is the unit positive eigenvector associated with $\Lambda_1(A_{\lambda_1,c})$, i.e., $A_{\lambda_1,c}\kappa = 0$, and $\epsilon, \alpha, \sigma_i\ (i = 0, 1, \ldots, n)$ are positive constants to be determined later. Note that the vector $\kappa \gg 0$, as G^0 is irreducible. The following results establish the inequities that this pair of super- and subsolutions satisfy.

LEMMA 4.1. *The function $\overline{V}_i$, $i \in [n]$, satisfies*

$$c\overline{V}_i' \ge d_i \overline{V}_i'' + g_i(K, \overline{V}), \quad \overline{V} = (\overline{V}_1, \ldots, \overline{V}_n).$$

Proof. By definition (4.1) and Taylor's theorem, we obtain

$$\begin{aligned} &d_i\overline{V}_i'' - c\overline{V}_i' + g_i(K, \overline{V}) \\ &= d_i\overline{V}_i'' - c\overline{V}_i' + \sum_{j=1}^n g_{i,j}(E_0)\overline{V}_j + \frac{1}{2}\sum_{j,k=1}^n g_{i,jk}(E_0^*)\overline{V}_j\overline{V}_k \\ &= (A_{\lambda_1,c}\kappa)_i e^{\lambda_1 s} + \frac{1}{2}\sum_{j,k=1}^n g_{i,jk}(E_0^*)\overline{V}_j\overline{V}_k \\ &= \frac{1}{2}\sum_{j,k=1}^n g_{i,jk}(E_0^*)\overline{V}_j\overline{V}_k \le 0, \end{aligned}$$

where $E_0^* = (1 - t_0)E_0 + t_0(K, \kappa e^{\lambda_1 s}) = (K, t_0\kappa e^{\lambda_1 s})$ for some $t_0 = t_0(s) \in [0, 1]$, and property (C2) in section 1 is used for the last inequality. ∎

LEMMA 4.2. *Choose α, σ_0 such that*

$$0 < \alpha < \frac{1}{2}\min\left\{\frac{c}{d_0}, \lambda_1\right\}, \quad \sigma_0 > \max\left\{K, \frac{\sum_{j=1}^n g_{0,j}(E_0)\kappa_j}{\alpha(c - d_0\alpha)}\right\}. \tag{4.2}$$

Then the function $\underline{U}(s)$ satisfies the following inequality:

$$c\underline{U}' \le d_0\underline{U}'' + f(\underline{U}) - g_0(\underline{U}, \overline{V}) \quad \textit{for } s \ne \underline{s}_0 := \frac{1}{\alpha}\ln\frac{K}{\sigma_0}. \tag{4.3}$$

Proof. Since $\sigma_0 > K$, it is clear that $\underline{s}_0 = \frac{1}{\alpha}\ln(K/\sigma_0) < 0$. If $s > \underline{s}_0$, then $\underline{U} = 0$ and (4.3) is clearly satisfied. If $s < \underline{s}_0 < 0$, then we have

$$\begin{aligned}
&d_0\underline{U}'' - c\underline{U}' + f(\underline{U}) - g_0(\underline{U}, \overline{V})\\
\geq&\ -\sigma_0(d_0\alpha^2 - c\alpha)e^{\alpha s} - g_0(K, \overline{V})\\
=&\ -\sigma_0(d_0\alpha^2 - c\alpha)e^{\alpha s} - \sum_{j=1}^{n} g_{0,j}(E_0)\overline{V}_j - \frac{1}{2}\sum_{j,k=1}^{n} g_{0,jk}(E_0^*)\overline{V}_j\overline{V}_k\\
\geq&\ -\sigma_0(d_0\alpha^2 - c\alpha)e^{\alpha s} - \sum_{j=1}^{n} g_{0,j}(E_0)\kappa_j e^{\lambda_1 s}\\
=&\ e^{\alpha s}\left(\sigma_0\alpha(c - d_0\alpha) - \sum_{j=1}^{n} g_{0,j}(E_0)\kappa_j e^{(\lambda_1-\alpha)s}\right)\\
\geq&\ e^{\alpha s}\left(\sigma_0\alpha(c - d_0\alpha) - \sum_{j=1}^{n} g_{0,j}(E_0)\kappa_j\right)\\
\geq&\ 0,
\end{aligned}$$

where property (C2) in section 1 is used for the second inequality, and (4.2) is used for the last inequality. ∎

LEMMA 4.3. *Let α and σ_0 be chosen such that* (4.2) *holds. Then there exist $\epsilon > 0$ sufficiently small and $\sigma_i \succ 1$ such that $\underline{V}_i(s)$ satisfies*

$$c\underline{V}_i' \leq d_i\underline{V}_i'' + g_i(\underline{U}, \underline{V}), \quad \underline{V} = (\underline{V}_1, \dots, \underline{V}_n), \tag{4.4}$$

for $s \neq \underline{s}_i := -\frac{1}{\epsilon}\ln\sigma_i$, $i \in [n]$.

Proof. Recall that (i) $\kappa = (\kappa_j)$ is the unit positive eigenvector of $A_{\lambda_1,c}$, so that $A_{\lambda_1,c}\kappa = 0$, and (ii) α, σ_0 are specified in (4.2), so that $0 < \alpha < \lambda_1$. Now choose ϵ such that

$$0 < \epsilon < \min\{\alpha, \lambda_1, \overline{\lambda} - \underline{\lambda}\}, \tag{4.5}$$

where $\underline{\lambda}$ $(= \lambda_1)$ and $\overline{\lambda}$ are determined by Lemma 2.3(ii) with $P = G^0$. By Lemma 2.3(iii), $\Lambda_1(A_{\lambda_1+\epsilon,c}) < 0$ and we denote the corresponding unit positive eigenvector to be $\eta = (\eta_j)$, so that

$$(A_{\lambda_1+\epsilon,c}\eta)_j = \Lambda_1(A_{\lambda_1+\epsilon,c})\eta_j < 0. \tag{4.6}$$

Set $l_j := -(A_{\lambda_1+\epsilon,c}\eta)_j$, $\sigma_j = \eta_0\eta_j/\kappa_j$, $j \in [n]$, such that $l_j > 0$ by (4.6), where $\eta_0 > 0$ will be determined later. We can assume that $\underline{s}_i < \underline{s}_0 < 0$ by setting $\eta_0 \succ 1$. Here $\underline{s}_i \in \mathbb{R}$ is the nonsmooth point of $\underline{V}_i(s)$.

Having defined σ_j and thus $\underline{V}_j(s)$ according to (4.1), we proceed to show the differential inequality (4.4). First, we note that (4.4) is satisfied trivially whenever $\underline{V}_j(s) = 0$, i.e., $s > \underline{s}_i$. Denote $\underline{V}_j^*(s) = \kappa_j e^{\lambda_1 s}(1 - \sigma_j e^{\epsilon s})$, $j \in [n]$, yielding that $\underline{V}_j(s) = \underline{V}_j^*(s) > 0$ for $s < \underline{s}_j$ and that $\underline{V}_j(s) = 0 > \underline{V}_j^*(s)$ for $s > \underline{s}_j$. Observe that for each fixed $i \in [n]$ and $s < \underline{s}_i$, we have

$$\underline{V}_i(s) = \underline{V}_i^*(s) > 0 \quad \text{and} \quad \underline{V}_j(s) \geq \underline{V}_j^*(s) \quad \forall j \in [n].$$

In view of Taylor's theorem, we compute

$$
\begin{aligned}
& g_i(\underline{U},\underline{V}) \\
=\ & \sum_{j=1}^n g_{i,j}(\underline{U},0_n)\underline{V}_j + \frac{1}{2}\sum_{j,k=1}^n g_{i,jk}(P_0)\underline{V}_j\underline{V}_k \\
=\ & \sum_{j=1}^n \big[g_{i,j}(E_0) + g_{i,0j}(P_j)(\underline{U}-K)\big]\underline{V}_j + \frac{1}{2}\sum_{j,k=1}^n g_{i,jk}(P_0)\underline{V}_j\underline{V}_k \\
\geq\ & \sum_{j=1}^n g_{i,j}(E_0)\underline{V}_j^* + \sum_{j=1}^n g_{i,0j}(P_j)(\underline{U}-K)\underline{V}_j + \frac{1}{2}\sum_{j,k=1}^n g_{i,jk}(P_0)\underline{V}_j\underline{V}_k,
\end{aligned}
$$

for which

$$E_0 = (K, 0_n), \quad P_0 = (\underline{U}, \xi_0\underline{V}), \quad P_j = (\xi_j\underline{U}, 0_n), \quad \xi_0, \xi_j \in [0,1], \ \ j \in [n].$$

Then we have

$$
\begin{aligned}
& e^{-\lambda_1 s}\big[d_i\underline{V}_i'' - c\underline{V}_i' + g_i(\underline{U},\underline{V})\big] \\
& \geq \Big[(d_i\lambda_1^2 - c\lambda_1)\kappa_i + \sum_{j=1}^n g_{i,j}(E_0)\kappa_j\Big] \\
& \quad - e^{\epsilon s}\left[\Big(d_i(\lambda_1+\epsilon)^2 - c(\lambda_1+\epsilon)\Big)\kappa_i\sigma_i + \sum_{j=1}^n g_{i,j}(E_0)\kappa_j\sigma_j\right] \\
& \quad - \sigma_0 R_1(s)e^{\alpha s} + R_2(s)e^{\lambda_1 s} \\
& = (A_{\lambda_1,c}\kappa)_i - e^{\epsilon s}\eta_0(A_{\lambda_1+\epsilon,c}\eta)_i - \sigma_0 R_1(s)e^{\alpha s} + R_2(s)e^{\lambda_1 s} \\
& = -e^{\epsilon s}\eta_0(A_{\lambda_1+\epsilon,c}\eta)_i - \sigma_0 R_1(s)e^{\alpha s} + R_2(s)e^{\lambda_1 s} \\
& = e^{\epsilon s}\eta_0 l_i - \sigma_0 R_1(s)e^{\alpha s} + R_2(s)e^{\lambda_1 s},
\end{aligned}
$$

where

$$
\begin{aligned}
R_1(s) &= \sum_{j=1}^n g_{i,0j}(P_j)\kappa_j(1-\sigma_j e^{\epsilon s})_+, \\
R_2(s) &= \frac{1}{2}\sum_{j,k=1}^n g_{i,jk}(P_0)\kappa_j\kappa_k(1-\sigma_j e^{\epsilon s})_+(1-\sigma_k e^{\epsilon s})_+,
\end{aligned}
$$

and $\varphi(s)_+ := \max\{\varphi(s), 0\}$. Since $0 \leq (1-\sigma_j e^{\epsilon s})_+ \leq 1$, $j \in [n]$, $g_i(\cdot) \in C^2(\mathbb{R}_+^{n+1})$, there exists $M = M(\epsilon) > 0$ such that $|R_j(s)| < M(\epsilon)$, $j = 1, 2$. Then we have

$$
\begin{aligned}
& e^{-\lambda_1 s}\big[d_i\underline{V}_i'' - c\underline{V}_i' + g_i(\underline{U},\underline{V})\big] \\
& \geq \big[\eta_0 l_i - \sigma_0 R_1(s)e^{(\alpha-\epsilon)s} + R_2(s)e^{(\lambda_1-\epsilon)s}\big]e^{\epsilon s} \\
& > \big(\eta_0 l_i - \sigma_0 M - M\big)e^{\epsilon s} > 0,
\end{aligned}
$$

provided we choose $\eta_0 > 0$ such that $\eta_0 > \frac{(\sigma_0+1)M}{\min\{l_j : j \in [n]\}}$ and use (4.5) and $s < 0$. ◻

4.2. Existence of traveling semifronts. Note that $c > c_0^*$ in this section. For $a > 0$, we define $I_a = (-a, a)$, $\bar{I}_a = [-a, a]$, and

$$
\begin{aligned}
\Gamma_a = \{(U,V)(\cdot) \in C(\bar{I}_a, \mathbb{R}^{n+1}) :\ & \underline{U}(s) \leq U(s) \leq \overline{U}(s), \\
& \underline{V}_i(s) \leq V_i(s) \leq \overline{V}_i(s),\ i \in [n], s \in \bar{I}_a\}.
\end{aligned}
$$

Consider the following boundary-value problem:

$$\begin{cases} d_0U'' - cU' + f(U) - g_0(U,V) = 0, \quad s \in I_a, \\ d_jV_j'' - cV_j' + g_j(U,V) = 0, \quad j \in [n], \quad s \in I_a, \\ (U,V)(-a) = (\underline{U},\underline{V})(-a), \quad (U,V)(a) = 0. \end{cases} \tag{4.7}$$

LEMMA 4.4. *Boundary-value problem* (4.7) *has a solution*

$$(U,V)(\cdot) \in C^2(\bar{I}_a, \mathbb{R}^{n+1}) \cap \Gamma_a$$

for any large $a > 0$.

Proof. Set

$$\gamma = \max_{(u,v)\in\Gamma_*} \left(|\bar{g}_{0,0}(u,v)| + \sum_{i\in[n]} |g_{i,i}(u,v)| \right),$$

where

$$\bar{g}_0(u,v) = f(u) - g_0(u,v), \quad \Gamma_* = \left\{(u,v) \in \mathbb{R}^{n+1} : 0 \le u \le K,\, 0 \le v \le \overline{V}(a)\right\}.$$

Define the operator $\mathcal{T} : \Gamma_a \to C(\bar{I}_a, \mathbb{R}^{n+1})$ by $\mathcal{T}(U^0, V^0) = (U, V)$, where $(U,V)(s)$ is the unique solution to

$$\begin{cases} - d_0U'' + cU' + \gamma U = \gamma U^0 + \bar{g}_0(U^0,V^0) =: F_0(U^0,V^0), \quad s \in (-a,a), \\ - d_iV_i'' + cV_i' + \gamma V_i = \gamma V_i^0 + g_j(U^0,V^0) =: F_i(U^0,V^0), \quad i \in [n], \quad s \in (-a,a), \\ (U,V)(-a) = (\underline{U},\underline{V})(-a), \quad (U,V)(a) = 0. \end{cases} \tag{4.8}$$

A regularity estimate for elliptic equations shows that $(U,V) \in C^2(\bar{I}_a, \mathbb{R}^{n+1})$. From the choice of γ, we have, for all $(u,v) \in \Gamma_*$, that $F_0(u,v)$ is increasing in u and decreasing in v_j and that for each $i \in [n]$, $F_i(u,v)$ is increasing in both u and v_j.

CLAIM 2. *For each* $a \succ 1$, $\mathcal{T}(\Gamma_a) \subset \Gamma_a$.

Let $(U,V) = \mathcal{T}(U^0,V^0)$ for some $(U^0,V^0) \in \Gamma_a$, and let $a > 0$ be large enough such that $-a < \underline{s}_0 < a$, where $\underline{s}_0$ is defined in Lemma 4.2. Define $\phi(s) = U(s) - \underline{U}(s)$. We claim that $\bar{\phi}(s)$ satisfies in the weak sense

$$\begin{cases} -d_0\phi'' + c\phi' + \gamma\phi \ge 0 \quad \text{for } s \in I_a, \\ \phi(\pm a) \ge 0. \end{cases} \tag{4.9}$$

It is obvious that $\phi(\pm a) = 0$. Next, Lemma 4.2 and the first equality of (4.7) show that

$$-d_0\phi'' + c\phi' + \gamma\phi \ge F_0(U^0,V^0) - F_0(\underline{U},\overline{V}) \ge 0, \quad s \in (-a, \underline{s}_0) \cup (\underline{s}_0, a).$$

This and the fact that $\phi'(\underline{s}_0-) \ge \phi'(\bar{s}_0+)$ show that the differential inequality in (4.9) holds in the weak sense. i.e., $\phi(s)$ is a weak supersolution (see, e.g., [7, section 4.2] for the definition of weak super- and subsolutions). Since the coefficient of the zeroth-order term, γ, in (4.9) is nonnegative, we conclude that $\phi(s) \ge 0$ for $s \in \bar{I}_a$. By arguing similarly, one may show that $U(s) \le \overline{U}(s), \underline{V}(s) \le V(s) \le \overline{V}(s)$ for $s \in [-a,a]$ and thus that $(U,V) \in \Gamma_a$. The proof of this claim is completed.

Elliptic estimates imply that $\mathcal{T} : \Gamma_a \to \Gamma_a$ is continuous and compact. Obviously, Γ_a is closed and convex. Then Schauder's fixed-point theorem shows that $\mathcal{T}$ has a fixed point in Γ_a, which is a nonnegative solution of (4.7). ∎

LEMMA 4.5. *System* (3.1) *has a positive solution* $(U,V)(s)$, $s \in \mathbb{R}$, *satisfying boundary condition* (1.5).

Proof. Lemma 4.4 shows that (3.1) has a solution $(U^k,V^k)(\cdot) \in C^2([-k,k],\mathbb{R}^{n+1}) \cap \Gamma_k$ for any positive integer k. Elliptic estimates show, by passing to (diagonal) subsequence, that $(U^k(s),V^k(s)) \to (U_*^\infty(s),V_*^\infty(s))$ in $C^2_{loc}(\mathbb{R},\mathbb{R}^{n+1})$, where $(U_*^\infty,V_*^\infty)(\cdot) \in \Gamma_\infty$ is a nonnegative solution of (3.1). Since

$$\underline{U} \le U_*^\infty \le \overline{U} \quad \text{and} \quad \underline{V} \le V_*^\infty \le \overline{V} \quad \text{in } \mathbb{R},$$

it follows from $(\underline{U},\underline{V})(-\infty) = E_0 = (\overline{U},\overline{V})(-\infty)$ that $(U_*^\infty,V_*^\infty)(s)$ satisfies (1.5). Since $(U_*^\infty,V_*^\infty)(s) \gg 0$ for $s \prec -1$, Lemma 2.1(ii) shows that $(U_*^\infty,V_*^\infty)(s)$ is positive on $\mathbb{R}$. ◻

4.3. Boundedness of traveling semifronts. The boundedness plays an important role in studying TWSs. In this subsection, we will show the boundedness of traveling semifronts obtained in Lemma 4.5. For this purpose, let G_0 and $\mathbb{M}$ be given by (1.2) (note that $\Lambda_1(G_0) > 0 > \Lambda_1(\mathbb{M})$, i.e., (A1)), so that $c_0^* > 0$ is given by (3.1). Lemma 4.5 says that, for each $c > c_0^*$, system (3.1) has a positive solution (U_c,V_c) on $\mathbb{R}$.

LEMMA 4.6. *For each open bounded interval* $I \subset (c_0^*,+\infty)$, *there exists* $C > 0$ *such that*

$$\sup_{c\in I} \|(U_c,V_c)\|_{C(\mathbb{R})} \le C,$$

where (U_c,V_c) *is a positive solution of* (3.1) *with wave speed* c *obtained in Lemma* 4.5.

Proof. By the construction of the super- and subsolutions in (4.1), $0 \le U_c(s) \le K$ for all $c > c_0^*$ and $s \in \mathbb{R}$. So if we suppose to the contrary that this lemma is false, then there exist a sequence of wave speeds $c_k \in I$ and corresponding solution (U_{c_k},V_{c_k}) of system (3.1) such that (for the notation $\|\cdot\|$ see section 1.1)

$$c_k \to c_\infty \in [c_0^*,+\infty) \quad \text{and} \quad \mathcal{M}_k := \sup_{s\in\mathbb{R}} \|V_{c_k}(s)\| \to +\infty.$$

Again by the construction in (4.1),

$$\|V_{c_k}(s)\| \le \sum_{j=1}^n \kappa_{k,j} \exp(\underline{\lambda}_k s) = \exp(\underline{\lambda}_k s) \quad \text{for } s \in \mathbb{R}, \tag{4.10}$$

where $\underline{\lambda}_k > 0$ is the $\underline{\lambda}$ in (the last case of) Lemma 2.3(ii) with $c = c_k$ and $P = G_0$ $(\kappa_{k,1},\ldots,\kappa_{k,n})$ is the unit positive eigenvector of $H_{\underline{\lambda}_k,c_k} + G^0$ associated with $\underline{\lambda}_k$. Hence

$$\underline{\lambda}_k \to \underline{\lambda}_\infty, \tag{4.11}$$

where $\underline{\lambda}_\infty > 0$ is the $\underline{\lambda}$ in (the last or second to last case of) Lemma 2.3(ii) with $c = c_\infty$ and $P = G_0$.

Step 1. If $\|V_{c_k}(s_k)\| \to +\infty$ for some sequence s_k, then

$$\frac{g_j^0(U_{c_k},V_{c_k})(\cdot + s_k)}{\|V_{c_k}(s_k)\|} \to 0 \quad \text{in } C_{loc}(\mathbb{R}). \tag{4.12}$$

We discuss two cases separately: (i) $\frac{g_0(U_{c_k},V_{c_k})}{U_{c_k}}\big|_{s=s_k} \to +\infty$ or (ii) $\frac{g_0(U_{c_k},V_{c_k})}{U_{c_k}}\big|_{s=s_k}$ remains bounded in k.

For case (i), one may infer, by Harnack's inequality (Lemma 2.1(ii)) applied to equation of V_{c_k}, that if $(V_{c_k})_{j'}(s_k) \to +\infty$ for some j', then $\inf_{[s_k-L,s_k+L]}(V_{c_k})_{j'} \to +\infty$ for each $L > 0$. Hence we deduce that, for each $L > 0$, $\inf_{[s_k-L,s_k+L]} \frac{g_0(U_{c_k},V_{c_k})}{U_{c_k}} \to +\infty$. Then, for any $\epsilon > 0$, take a test function $\overline{u}(s) \in C^2([-L, L])$ satisfying

$$\overline{u}(\pm L) = K, \quad \overline{u}(s) > 0 \text{ in } [-L, L], \quad \text{and} \quad \overline{u}(s) = \epsilon \text{ in } [-L/2, L/2].$$

It follows from the fact that $\inf_{[s_k-L,s_k+L]} \frac{g_0(\bar{u},V_{c_k})}{\bar{u}} \to +\infty$ that

$$d_0\bar{u}'' - c\bar{u}' + \delta(K - \bar{u}) - \bar{u}\frac{g_0(\bar{u}, V_{c_k}(s + s_k))}{\bar{u}} \le 0 \quad \text{for } s \in (-L, L),$$

provided k is large enough. Denote $w := \bar{u}(s) - U_{c_k}(s+s_k)$, so that $w(\pm L) \ge 0$. Then by the first equality of (3.1) we get for all $s \in (-L, L)$ and large k that

$$\begin{aligned} &d_0\bar{w}'' - c\bar{w}' - \delta w - g_0(\bar{u}, V_{c_k}(s + s_k)) + g_0(U_{c_k}(s + s_k), V_{c_k}(s + s_k)) \\ &= d_0\bar{w}'' - c\bar{w}' - [\delta + D_u g_0(\hat{u}, V_{c_k}(s + s_k))]w \le 0, \end{aligned}$$

where $\hat{u}$ is between $\bar{u}$ and $U_{c_k}(s+s_k)$. Since $D_u g_0(\hat{u}, V_{c_k}(s+s_k)) \ge 0$, the comparison principle shows that, for sufficiently large k, $U_{c_k}(s + s_k) \le \overline{u}(s)$ in $s \in [-L, L]$ and thus $U_{c_k}(s + s_k) \le \epsilon$ in $s \in [-L/2, L/2]$. Since ϵ and L are arbitrary, we show that $U_{c_k}(s + s_k) \to 0$ (as $k \to \infty$) in $C_{loc}(\mathbb{R})$. By the definition of g_j^0, this implies (4.12).

For case (ii), $\frac{g_0(U_{c_k},V_{c_k})}{U_{c_k}}\big|_{s=s_k}$ remains bounded even if $\|V_{c_k}(s_k)\| \to +\infty$. It follows from the definition of g_0 (see system (1.1)) and Harnack's inequality (Lemma 2.1(ii)) that the family $\{g_0(U_{c_k}, V_{c_k})(\cdot + s_k)\}$ remains bounded in any compact subinterval of $\mathbb{R}$. By the definitions of g_0 and g_j^0, we have $0 \le g_j^0(u, v) \le g_0(u, v)$, so the same holds for $\{g_j^0(U_{c_k}, V_{c_k})(\cdot + s_k)\}$. Combining with the fact that $\|V_{c_k}(s_k)\| \to +\infty$, we obtain (4.12).

Step 2. Let $c = c_\infty$ and $P = \mathbb{M}$, let $\overline{\lambda}_* > 0$ be the corresponding $\overline{\lambda}$ in Lemma 2.3(i), and let $\Gamma(c)$ and $\Lambda(c)$ be given in (2.4) and (2.3), respectively. Then

$$\Gamma(c) \cap [0, \infty) = \Gamma(c) \cap [\sup \Lambda(c), \infty), \quad \Gamma(c) \cap (-\infty, 0] = \Gamma(c) \cap (-\infty, \inf \Lambda(c)], \tag{4.13}$$

where $\inf \Lambda(c) < 0 < \sup \Lambda(c)$, and $\sup \Lambda(c) = \overline{\lambda}_*$. Furthermore, $\overline{\lambda}_* > \underline{\lambda}_\infty$, in particular, $\overline{\lambda}_* > \frac{1}{2}(\overline{\lambda}_* + \underline{\lambda}_\infty) > \underline{\lambda}_k$ for all k sufficiently large, where $\underline{\lambda}_k$ and $\underline{\lambda}_\infty$ are defined after (4.10) and (4.11).

By hypothesis, $\Lambda_1(\mathbb{M}) < 0$, so that by Lemma 2.3(i), $\inf \Lambda(c) < 0 < \sup \Lambda(c)$. Hence Lemma 2.9 says that $\Gamma(c) \cap [0, \infty) = \Gamma(c) \cap [\sup \Lambda(c), \infty)$. The second part of (4.13) is similar. Also, it follows by the definition (when $c = c_\infty$ and $P = \mathbb{M}$) that $\sup \Lambda(c) = \overline{\lambda}_*$.

The strict inequality $\overline{\lambda}_* > \underline{\lambda}_\infty$ follows from the fact that $G_0 > \mathbb{M}$ and Lemma 2.3(iv) (see Figure 2). The inequality $\overline{\lambda}_* > \frac{1}{2}(\overline{\lambda}_* + \underline{\lambda}_\infty) > \underline{\lambda}_k$ follows from (4.11). This proves Step 2.

Step 3. It is impossible that there exists a sequence $s_k \in \mathbb{R}$ such that

$$\|V_{c_k}(s_k)\| \to +\infty, \quad \|V''_{c_k}(s_k)\| \le 0. \tag{4.14}$$

We suppose to the contrary that (4.14) holds. But if we define

$$\tilde{V}_k(s) := \frac{V_{c_k}(s + s_k)}{\|V_{c_k}(s_k)\|},$$

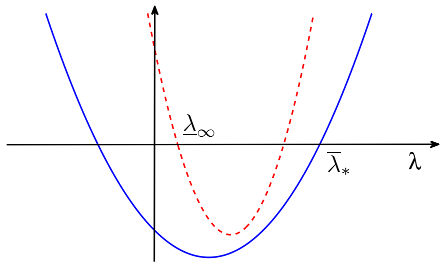

FIG. 2. *Diagram illustrating the definition of $\underline{\lambda}_\infty < \overline{\lambda}_*$. The solid curve is $\lambda \mapsto \Lambda_1(H_{\lambda,c_\infty} + \mathbb{M})$; the dashed curve is $\lambda \mapsto \Lambda_1(H_{\lambda,c_\infty} + G^0)$.*

then, passing to a subsequence, we deduce that $\tilde{V}_k \to \tilde{V}_\infty$ in $C^2_{loc}(\mathbb{R})$. Moreover, by the arguments of Step 1, $\tilde{V}_\infty$ is a nonnegative solution of

$$c_\infty \tilde{V}'_\infty = \mathrm{diag}(d_j)\tilde{V}''_\infty + \mathbb{M}\tilde{V}_\infty \quad \text{in } \mathbb{R} \quad \text{and} \quad \|\tilde{V}_\infty(0)\| = 1,$$

where the coefficient matrix $\mathbb{M}$ is defined in (1.2). However, by Proposition 2.4(i), we deduce that $\tilde{V}_\infty(s) = \sum_{\lambda\in\Gamma(c)\cap\mathbb{R}} c_\lambda e^{\lambda s}\zeta_\lambda$, where $c_\lambda\zeta_\lambda \geq 0$. By Step 2, $0 \notin \Gamma(c)\cap\mathbb{R}$. Therefore, $\|\tilde{V}''_\infty(0)\| > 0$, which contradicts (4.14). This completes Step 3.

Step 4. There exists $s'_k \in \mathbb{R}$ such that for k sufficiently large, $\|V_{c_k}(s)\|$ is strictly increasing in $[s'_k, +\infty)$, and $\|V_{c_k}(s'_k)\| \to +\infty$.

By Step 3 and the fact that $\mathcal{M}_k \to +\infty$, $\mathcal{M}_k \neq \max_{s\in\mathbb{R}} \|V_{c_k}(s)\|$ for all large k. Then we deduce that $\mathcal{M}_k = \limsup_{s\to+\infty} \|V_{c_k}(s)\|$ since (U_{c_k}, V_{c_k}) satisfies (1.5). If for k sufficiently large, $\|V_{c_k}(s)\|$ is not strictly increasing for $s \succ 1$, then there exists a sequence $s_k \in \mathbb{R}$ such that (4.14) holds, contradicting Step 3. Step 4 is proved.

Step 5. Let $\overline{\lambda}_*$ be as in Step 2. Then

$$\liminf_{k\to\infty} \left[\inf_{[s'_k+k,\infty)} \frac{V'_{c_k}(s)}{V_{c_k}(s)} \right] \geq \overline{\lambda}_*. \tag{4.15}$$

Now, let s''_k be any sequence such that $s''_k \geq s'_k + k$ for all k. By Step 4, we have $\|V_{c_k}(s''_k)\| \to +\infty$. Hence we may define $\hat{V}_k(s) := V_{c_k}(s + s''_k)/\|V_{c_k}(s''_k)\|$ and pass to the limit $\hat{V}_k \to \hat{V}_\infty$ in $C^2_{loc}(\mathbb{R})$ as in Step 3, where $\hat{V}_\infty$ is a nonnegative solution to

$$c_\infty \hat{V}'_\infty = \mathrm{diag}(d_j)\hat{V}''_\infty + \mathbb{M}\hat{V}_\infty \quad \text{in } \mathbb{R} \quad \text{and} \quad \|\hat{V}_\infty(0)\| = 1.$$

By Proposition 2.4 and Step 2,

$$\hat{V}_\infty(s) = \sum_{\lambda\in\Gamma(c)\cap\mathbb{R}} c_\lambda e^{\lambda s}\zeta_\lambda, \tag{4.16}$$

where the constant coefficients satisfy $c_\lambda\zeta_\lambda \geq 0$ and $\Gamma(c)$ is given in (2.4) with $c = c_\infty$ and $P = \mathbb{M}$.

By Step 4, $\|V_{c_k}(s)\|$ is nondecreasing in $[s'_k, +\infty)$, so that $\|\hat{V}_k(s)\|$ (resp., $\|\hat{V}_\infty(s)\|$) is nondecreasing in $[-k, +\infty)$ (resp., $\mathbb{R}$). Hence the sum in (4.16) is taken over $\lambda \in \Gamma(c)\cap[0,\infty)$ only. Combining with $\Gamma(c)\cap[0,\infty) = \Gamma(c)\cap[\overline{\lambda}_*,\infty)$ (Step 2), we have

$$\hat{V}_\infty(s) = \sum_{\lambda\in\Gamma(c)\cap[\overline{\lambda}_*,\infty)} c_\lambda e^{\lambda s}\zeta_\lambda.$$

This implies that

$$\lim_{k\to\infty} \frac{\|V'_{c_k}(s''_k)\|}{\|V_{c_k}(s''_k)\|} = \frac{\|\hat{V}'_\infty(0)\|}{\|\hat{V}_\infty(0)\|} \geq \overline{\lambda}_*.$$

Since this is true for arbitrary sequence $s''_k \geq s'_k + k$, this proves Step 5.

Finally, since $0 < \frac{1}{2}(\overline{\lambda}_* + \underline{\lambda}_\infty) < \overline{\lambda}_*$, Step 5 shows that for all large k,

$$\|V_{c_k}(s)\| \geq \|V_{c_k}(s'_k + k)\| \exp\left(\frac{1}{2}(\overline{\lambda}_* + \underline{\lambda}_\infty)(s - s'_k - k)\right) \quad \text{for } s \geq s'_k + k. \tag{4.17}$$

Since (by Step 2) $\frac{1}{2}(\overline{\lambda}_* + \underline{\lambda}_\infty) > \underline{\lambda}_k$ for k large, (4.17) is in contradiction to (4.10). ◻

5. Existence of traveling semifront of (1.1) with $c = c_0^*$. Take a sequence $c_j \searrow c_0^*$, and let $\Phi_j(s) := (U^{(j)}, V^{(j)})(s)$ be a positive solution of (3.1) with wave speed c_j. Lemma 4.6 shows that $\Phi_j(s) := (U^{(j)}, V^{(j)})(s)$ is uniformly bounded with respect to j and $s \in \mathbb{R}$. In this section, we will pass to the limit $j \to \infty$ to obtain a traveling wave with the critical wave speed c_0^*.

Lemma 5.1.

$$\limsup_{j\to\infty} \left[\sup_{s\in\mathbb{R}} (K - U^{(j)}(s))\right] > 0$$

or

$$\limsup_{j\to\infty} \left[\sup_{s\in\mathbb{R}} \|V^{(j)}(s)\|\right] > 0.$$

Proof. Suppose for contradiction that, as $j \to \infty$,

$$\sup_{s\in\mathbb{R}} (K - U^{(j)}(s)) \to 0 \quad \text{and} \quad \sup_{s\in\mathbb{R}} \|V^{(j)}(s)\| \to 0. \tag{5.1}$$

Lemma 2.1(i) yields $V^{(j)}(s) \to 0$ in $C^2(\mathbb{R})$ as $j \to \infty$.

We claim that there exists j_0 such that for $j > j_0$ and $k \in [n]$, $V_k^{(j)}(s)$ are nondecreasing with respect to $s \in \mathbb{R}$. We suppose by passing to a subsequence (in j) to the contrary that there exist $k_1 \in [n]$ and $s_j \in \mathbb{R}$ such that

$$V_{k_1}^{(j)\prime}(s_j) = 0, \quad V_{k_1}^{(j)\prime\prime}(s_j) \leq 0, \quad j \geq 1.$$

It is evident that the second part of (3.1) can be rewritten as

$$c_j V^{(j)\prime} = \mathrm{diag}(d_l) V^{(j)\prime\prime} + G^0 V^{(j)} + o(\|V^{(j)}(s)\|). \tag{5.2}$$

Define

$$\tilde{V}^{(j)}(s) := \frac{(V_1^{(j)}(s_j + s), \ldots, V_n^{(j)}(s_j + s))^T}{\|V^{(j)}(s_j)\|}.$$

Then it is easy to see by passing to a subsequence that $\tilde{V}^{(j)}(\cdot) \to \tilde{V}^{(\infty)}(\cdot)$ in $C^2_{loc}(\mathbb{R})$, where $\tilde{V}^{(\infty)}(s)$ is a nonnegative solution of

$$c_0^* V' = \mathrm{diag}(d_l) V'' + G^0 V$$

with $\|\tilde{V}^{(\infty)}(0)\| = 1$, $\tilde{V}_{k_1}^{(\infty)\prime}(0) = 0$. However, it follows from Lemma 2.3(ii) and Proposition 2.4(ii) that $\tilde{V}^{(\infty)}(s) = e^{\lambda_1 s}\zeta_{\lambda_1}$, where $\zeta_{\lambda_1} \gg 0$ is a unit vector, contradicting $\tilde{V}_{k_1}^{(\infty)\prime\prime}(0) \leq 0$. In conclusion, there exists j_0 such that for all $j > j_0$ and $k \in [n]$, $V_k^{(j)}(s)$ is nondecreasing with respect to $s \in \mathbb{R}$.

From the above claim, we deduce that for $j > j_0$, the limit $V^{(j)}(+\infty)$ exists, and hence $U^{(j)}(+\infty)$ also exists. Obviously, $(U^{(j)}(+\infty), V^{(j)}(+\infty))$ is a sequence of positive equilibria of (5.2), that is,

$$G^0 V^{(j)}(+\infty) = o(\|V^{(j)}(+\infty)\|).$$

Furthermore, by (5.1) we have

$$(U^{(j)}, V^{(j)})(+\infty) \to E_0(K, 0_n).$$

Define, for each j, the constant vectors

$$\tilde{V}_\infty^{(j)} := \frac{(V_1^{(j)}(+\infty), \dots, V_n^{(j)}(+\infty))^T}{\|V^{(j)}(+\infty)\|},$$

so that $\tilde{V}_\infty^{(j)} \to \tilde{V}_\infty^{(\infty)}$ by passing to a subsequence, where

$$G^0 \tilde{V}_\infty^{(\infty)} = 0, \quad \|\tilde{V}_\infty^{(\infty)}\| = 1, \quad \tilde{V}_\infty^{(\infty)} > 0.$$

Since G^0 is irreducible, it follows from Theorem 1.2 that $\Lambda_1(G^0) = 0$, contradicting assumption (A1). ◻

Lemma 5.2. *System* (3.1) *with* $c = c_0^*$ *has a bounded positive solution* $(U, V)(s)$ *satisfying* (1.5).

Proof. Lemma 5.1 yields that

$$\lim_{j\to\infty} \sup_{s\in\mathbb{R}} (K - U^{(j)}(s)) > 0 \quad \text{or} \quad \lim_{j\to\infty} \sup_{s\in\mathbb{R}} \|V^{(j)}(s)\| > 0.$$

Set $\epsilon > 0$ small enough. Since $(U^{(j)}, V^{(j)})(s)$ satisfies (1.5), by possible translations we can suppose that

$$U^{(j)}(s) > K - \epsilon, \quad \|V^{(j)}(s)\| < \epsilon \quad \forall s < 0$$

and that

$$U^{(j)}(0) = K - \epsilon \quad \text{or} \quad \|V^{(j)}(0)\| = \epsilon$$

holds. It follows by elliptic estimate and by passing to a subsequence that

$$(U^{(j)}, V^{(j)})(\cdot) \to (U, V)(\cdot)$$

in $C^2_{loc}(\mathbb{R})$, where $(U, V)(\cdot)$ is a nonnegative solution of (3.1) with $c = c_0^*$ such that

$$U(s) \geq K - \epsilon, \quad \|V(s)\| \leq \epsilon \quad \forall s < 0 \tag{5.3}$$

and that

$$U(0) = K - \epsilon \quad \text{or} \quad \|V(0)\| = \epsilon \tag{5.4}$$

holds. Lemma 2.1(ii) yields $U(s) > 0$ for all $s \in \mathbb{R}$.

Claim 3. $V(s) \gg 0$ *for all* $s \in \mathbb{R}$.

If $\|V(0)\| = 0$, then $U(0) = K - \epsilon$ and Lemma 2.1(ii) shows that $V(s) \equiv 0$, $s \in \mathbb{R}$. Hence $U(s)$ satisfies

$$c_0^* U' = d_0 U'' + f(U). \tag{5.5}$$

We claim that $U'(s) \leq 0$, $s \in [0, +\infty)$. Assume to the contrary that this does not hold. Then there exists $s_0 \geq 0$ such that $U'(s_0) = 0, U''(s_0) \geq 0$, $U(s_0) < K$, implying, together with (5.5), that $0 = U''(s_0) + f(U(s_0)) > 0$, a contradiction. Consequently, $U'(s) \leq 0$, $s \in [0, +\infty)$. Then (5.5) gives that $U''(s) < 0$, $s \in (0, +\infty)$, yielding that $U(+\infty) = -\infty$, a contradiction to U being nonnegative. This shows that $\|V(0)\| > 0$, and Harnack's inequality [1, Theorem 2.2] gives $V(s) \gg 0$ for all $s \in \mathbb{R}$.

So far we have shown that $0 < U(s) < K$, $V(s) \gg 0$ for all $s \in \mathbb{R}$ and that (5.3) and (5.4) hold. Define $\epsilon_j := \epsilon/j$, $j \in [n]$. It follows from the arbitrariness of the above small ϵ that there exists a positive solution $(U_*^j, V_*^j)(\cdot)$ to (3.1) with $c = c_0^*$ such that

$$U_*^{(j)}(s) \geq K - \epsilon_j, \quad \|V_*^{(j)}(s)\| \leq \epsilon_j \quad \forall s < 0$$

and that one of the following holds:

$$U_*^{(j)}(0) = K - \epsilon_j, \quad \|V_*^{(j)}(0)\| = \epsilon_j.$$

Claim 4. *For large j, each entry of $V_*^{(j)}(s)$ is monotonic with respect to $s \prec -1$.*

Assume to the contrary that this claim does not hold. By passing to a subsequence, there exist $k_0 \in [n]$ and j_0 such that $V_{*k_0}^{(j)}(s)$ is not monotonic with respect to $s \prec -1$ for $j > j_0$. Then there exists $s_j \to -\infty$ such that

$$V_{*k_0}^{(j)\prime}(s_j) = 0, \quad V_{*k_0}^{(j)\prime\prime}(s_j) \leq 0, \quad j > j_0. \tag{5.6}$$

Since $\epsilon_j \to 0$, then by passing to a subsequence, we have $(U_*^{(j)}, V_*^{(j)})(s) \to E_0(K, 0_n)$ in $C_{loc}^2((-\infty, 0))$. Define

$$\tilde{V}_*^{(j)}(s) := \frac{V_*^{(j)T}(s_j + s)}{\|V_*^{(j)}(s_j)\|}.$$

Then, similar to the proof of Lemma 5.1, we have $\tilde{V}_*^{(j)}(s) \to e^{\lambda_1 s}\zeta_{\lambda_1}$ in $C_{loc}^2(\mathbb{R})$, where $\zeta_{\lambda_1} \gg 0$, contradicting (5.6).

It follows from the above claim that $(U_*^{(j)}, V_*^{(j)})(-\infty)$ exists for large j and is a sequence of equilibria of (3.1) such that $(U_*^{(j)}, V_*^{(j)})(-\infty) \to E_0(K, 0_n)$. Since $E_0(K, 0_n)$ is an isolated equilibrium, we deduce that for all j sufficiently large,

$$(U_*^{(j)}, V_*^{(j)})(-\infty) = E_0.$$

This completes the proof of this lemma. ◻

6. Persistence of traveling semifronts. Note that we assume that assumption (A1) holds in this section.

Theorem 6.1. *System* (1.1) *has a persistent traveling semifront* $\Phi(x + ct)$ *if* $c \geq c_0^*$.

Remark 6.2. It follows from Theorems 3.2 and 6.1 that c_0^* is the minimal wave speed of system (1.1) if $\Lambda_1(G^0) > 0$.

Proof. Let $(U(s), V(s))$ be the bounded traveling semifront in Lemma 4.5 or Lemma 5.2. We claim that $\liminf_{s\to+\infty} U(s) > 0$. Suppose to the contrary that

$$\liminf_{s\to+\infty} U(s) = 0.$$

Then there exists a sequence $s_i \to +\infty$ such that $U(s_i) \to 0$. Lemma 2.1 shows that $U(s+s_i) \to 0$ in $C^2_{loc}(\mathbb{R})$ by passing to a subsequence. It follows from the boundedness of $V(s)$ and the first equality of (3.1) that $\delta K = 0$, a contradiction. We thus have $\liminf_{s\to+\infty} U(s) > 0$.

Next, we show that $\liminf_{s\to+\infty} V(s) \gg 0$. Assume to the contrary that there exists $j_0 \in [n]$ such that $\liminf_{s\to+\infty} V_{j_0}(s) = 0$, implying there exists a sequence $s_i \to +\infty$ such that $V_{j_0}(s_i) \to 0, V'_{j_0}(s_i) \leq 0$. We can rewrite $g_i(U, V)$, $i \in [n]$, as

$$(g_1(U, V), \ldots, g_n(U, V))^T = G(s)V,$$

where $G(s)$ is an $n \times n$ matrix. It follows from $\liminf_{s\to+\infty} U(s) > 0$ that there exists an irreducible matrix G^* such that $G(s) \geq G^*$ for large s. Then Harnack's inequality [1, Theorem 2.2] yields that $V(s_i + \cdot) \to 0$ in $C_{loc}(\mathbb{R})$. Lemma 2.1(i) implies that $V(s_i + \cdot) \to 0$ in $C^2_{loc}(\mathbb{R})$. It follows from the first equality of (3.1) (possibly by passing to a subsequence) that $U(s_i+\cdot) \to U^*(\cdot)$ in $C^2_{loc}(\mathbb{R})$, where $U^*(s)$ is a solution to

$$cU^{*\prime} = d_0 U^{*\prime\prime} + \delta(K - U^*).$$

However, any solution of this equation can be expressed as $U^*(s) = c_1 e^{\lambda_1 s} + c_2 e^{\lambda_2 s} + K$, where $\lambda_1 < 0 < \lambda_2$ are the zeros to $d_0\lambda^2 - c\lambda - \delta = 0$. Since $U^*(s) \leq K$ is bounded in $\mathbb{R}$, it follows that $U^*(s) \equiv K$ and $U(s_i + s) \to K$ in $C^2_{loc}(\mathbb{R})$. Define

$$\tilde{V}^{(i)}(s) := \frac{V(s + s_i)}{|V(s_i)|}, \quad s \in \mathbb{R},$$

and thus $|\tilde{V}^{(i)}(0)| = 1, \tilde{V}^{(i)\prime}_{j_0}(0) \leq 0$. Then it follows by passing to a subsequence and from Lemma 2.1(i) that $\tilde{V}^{(i)}(\cdot)$ converges to some $\tilde{V}_*(\cdot)$ in $C^2_{loc}(\mathbb{R})$, where $\tilde{V}_*(\cdot)$ is a nonnegative solution of (2.5) with $P = G^0$ and satisfies $(\tilde{V}'_*)_{j_0}(0) \leq 0$. Proposition 2.4 and Lemma 2.3(ii) imply that $\tilde{V}_*(s) = c_{\underline{\lambda}} e^{\underline{\lambda} s}\zeta_{\underline{\lambda}} + c_{\overline{\lambda}} e^{\overline{\lambda} s}\zeta_{\overline{\lambda}}$, where

$$0 < \underline{\lambda} \leq \overline{\lambda}, \quad \zeta_{\underline{\lambda}} \gg 0, \quad \zeta_{\overline{\lambda}} \gg 0, \quad c_{\underline{\lambda}} \geq 0, \quad c_{\overline{\lambda}} \geq 0, \quad \text{and} \quad c_{\underline{\lambda}} + c_{\overline{\lambda}} > 0.$$

Then $\tilde{V}'_*(0) = c_{\underline{\lambda}}\underline{\lambda}\zeta_{\underline{\lambda}} + c_{\overline{\lambda}}\overline{\lambda}\zeta_{\overline{\lambda}} \gg 0$, contradicting $(\tilde{V}'_*)_{j_0}(0) \leq 0$. Hence we have proved $\liminf_{s\to+\infty} V(s) \gg 0$. ∎

7. Applications. The TWSs in Theorem 6.1 connect the disease-free equilibrium $E_0(K, 0_n)$ at $s = -\infty$ and are persistent at $s = +\infty$. In this section, we will apply Theorems 3.2 and 6.1 for system (1.1) to system (1.3) and show that the TWSs of system (1.3) connect the endemic equilibrium at $s = +\infty$ (this is a more detailed result than persistence property). In addition, it will be shown that this method for system (1.3) can be applied to a class of specific disease-transmission models.

For system (1.3), define

$$\mathcal{F} = \begin{bmatrix} \beta_1 K & \beta_2 K & \cdots & \beta_n K \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \mathcal{V} = \begin{bmatrix} \psi_1 & -\phi_{12} & \cdots & -\phi_{1n} \\ -\phi_{21} & \psi_2 & \cdots & -\phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\phi_{n1} & -\phi_{n2} & \cdots & \psi_n \end{bmatrix}.$$

Then $\mathcal{V}$ is a nonsingular M-matrix and its inverse $\mathcal{V}^{-1}$ is nonnegative (see [17, p. 267]). Define $\mathscr{R}_0 := \Lambda_1(\mathcal{F}\mathcal{V}^{-1})$, which is the basic reproduction number. It follows from the expression of $\mathcal{F}$ and the nonnegativity of $\mathcal{V}^{-1}$ that $\mathscr{R}_0 = (\mathcal{F}\mathcal{V}^{-1})_{11}$. Note that the linearization of the reaction terms in I_i, $i \in [n]$, of (1.3) at the equilibrium $P_0(K, 0_n)$ is $G^0 := \mathcal{F} - \mathcal{V}$. Indeed, it is possible that G^0 is irreducible and yet $-\mathcal{V}$ (which plays the role of $\mathbb{M}$ in the notations of earlier sections) is not irreducible.

THEOREM 7.1. *Assume G^0 is irreducible. System* (1.3) *has no bounded positive TWS connecting P_0 at $s = -\infty$ if $\mathscr{R}_0 < 1$. Now let $\mathscr{R}_0 > 1$. Then* (1.3) *has a unique endemic equilibrium $P^*(S^*, I^*)$. Furthermore, there exists a constant $c_1^* > 0$ such that* (1.3) *has a positive TWS $\Phi(x + ct)$ satisfying boundary conditions*

$$\Phi(-\infty) = P_0, \quad \Phi(+\infty) = P^* \tag{7.1}$$

if and only if $c \geq c_1^*$.

Proof. We first study the relation between $\Lambda_1(G^0)$ and $\mathscr{R}_0$.

CLAIM 5. $\Lambda_1(G^0) = 0$ *if and only if* $\mathscr{R}_0 = 1$.

We first consider the necessity and suppose $\Lambda_1(G^0) = 0$. Since G^0 is irreducible, Theorem 1.2 gives that there exists a positive eigenvector $\nu \gg 0$ such that $G^0\nu = \mathcal{F}\nu - \mathcal{V}\nu = 0$. Then the expression of $\mathcal{F}$ shows that $\mathcal{V}\nu = \mathcal{F}\nu = (\kappa^*, 0, \ldots, 0)^T$, where $\kappa^* > 0$. This yields that

$$0 = (G^0\nu)_1 = (G^0\mathcal{V}^{-1}\mathcal{V}\nu)_1 = (G^0\mathcal{V}^{-1})_{11}\kappa^*,$$

and thus $(G^0\mathcal{V}^{-1})_{11} = 0$. Since $\mathcal{V}^{-1}$ and, therefore, $\mathcal{F}\mathcal{V}^{-1}$ are nonnegative, we have

$$0 = (G^0\mathcal{V}^{-1})_{11} = (\mathcal{F}\mathcal{V}^{-1} - I_{n\times n})_{11} = (\mathcal{F}\mathcal{V}^{-1})_{11} - 1 = \mathscr{R}_0 - 1,$$

where $I_{n\times n}$ is the identity matrix.

Now suppose $\mathscr{R}_0 = 1$. Since $\mathscr{R}_0 = (\mathcal{F}\mathcal{V}^{-1})_{11}$, it is easy to show that

$$G^0\mathcal{V}^{-1}(1, 0, \ldots, 0)^T = (\mathcal{F}\mathcal{V}^{-1} - I_{n\times n})(1, 0, \ldots, 0)^T = 0.$$

Since $\mathcal{V}^{-1}(1, 0, \ldots, 0)^T$ is a nonnegative and nonzero vector, Theorem 1.2 implies that $\Lambda_1(G^0) = 0$. This claim is proved.

Next we show that $\mathscr{R}_0 > 1$ if and only if $\Lambda_1(G^0) > 0$. It follows from the expression of $\mathcal{F}$ and the nonnegativity of $\mathcal{V}^{-1}$ that

$$\mathscr{R}_0 = (\mathcal{F}\mathcal{V}^{-1})_{11} = K\sum_{i=1}^{n}\beta_i(\mathcal{V}^{-1})_{i1}.$$

Thus $\mathscr{R}_0$ is strictly increasing with respect to $K > 0$. Since G^0 is irreducible, the Perron–Frobenius theorem shows that $\Lambda_1(G^0)$ is also strictly increasing with respect to $K > 0$. Then the monotonicity of $\mathscr{R}_0$ and $\Lambda_1(G^0)$ with respect to $K > 0$ and Claim 5 show that $\mathscr{R}_0 > 1$ if and only if $\Lambda_1(G^0) > 0$.

Similarly to the proof of Theorem 3.2, we can show that system (1.3) has no bounded positive TWS connecting P_0 at $s = -\infty$ if $\mathscr{R}_0 < 1$. In the following, we assume $\mathscr{R}_0 > 1$, i.e., $\Lambda_1(G^0) > 0$. Let c_1^* be the c^* in Lemma 2.3(ii) with $P = G^0$. Since $\mathcal{V}$ is a nonsingular M-matrix, we have $\Lambda_1(-\mathcal{V}) < 0$ [3, p. 135]. This means that the corresponding assumption (A1) for (1.3) holds. Completely similar to the proofs of Theorems 3.2 and 6.1, it can be shown that (1.3) has a persistent positive TWS $(S(x + ct), I(x + ct))$ connecting P_0 if and only if $c \geq c_1^*$.

Next, suppose $c \geq c_1^*$. To complete the proof, we need to show $(S, I)(+\infty) = P^*$. Following the idea of [10, 8, 12, 39], a Lyapunov function, motivated by [17], will be constructed. It is obvious that $(S, I)(s)$, $s = x + ct$, satisfies

$$\begin{cases} S' = W_0, \\ d_0 W_0' = cW_0 - G_0(S, I), \\ I_i' = W_i, \\ d_i W_i' = cW_i - G_i(S, I), \quad i \in [n], \end{cases} \tag{7.2}$$

where

$$G_0(S, I) = \delta_0(K - S) - S \sum_{j=1}^{n} \beta_j I_j, \quad G_1(S, I) = S \sum_{j=1}^{n} \beta_j I_j + \sum_{j=1}^{n} \phi_{1j} I_j - \psi_1 I_1,$$

$$G_i(S, I) = \sum_{j=1}^{n} \phi_{ij} I_j - \psi_i I_i, \quad i = 2, \ldots, n.$$

Define

$$L(s) := \sum_{j=0}^{n} \sigma_j L_j(s),$$

where positive constants σ_j will be determined later and

$$L_0(s) = c \int_{S^*}^{S} 1 - \frac{S^*}{\xi} \mathrm{d}\xi - d_0 W_0 \left(1 - \frac{S^*}{S}\right),$$

$$L_i(s) = c \int_{I_i^*}^{I_i} 1 - \frac{I_i^*}{\xi} \mathrm{d}\xi - d_i W_i \left(1 - \frac{I_i^*}{I_i}\right), \quad i \in [n].$$

Trivial calculations give

$$\begin{aligned} \frac{dL_0(s)}{ds}\Big|_{(7.2)} &= [cW_0 - d_0 W_0'] \frac{S - S^*}{S} - \frac{d_0 W_0 S^* S'}{S^2} \\ &= G_0(S, I) \frac{S - S^*}{S} - \frac{d_0 S^* W_0^2}{S^2} \\ &=: \mathcal{J}_{01} - \mathcal{J}_{02}. \end{aligned}$$

It can be similarly shown that

$$\frac{dL_i(s)}{ds} = G_i(S, I) \frac{I_i - I_i^*}{I_i} - \frac{d_i I_i^* W_i^2}{I_i^2} =: \mathcal{J}_{i1} - \mathcal{J}_{i2}, \quad i \in [n].$$

It is evident that $\mathcal{J}_{k2} \geq 0$, $0 \leq k \leq n$, and that the conditions of this theorem imply those of Theorem 5.1 of [17]. Then, from the proof of Theorem 5.1 of [17], there exist positive constants σ_j, $j = 0, 1, \ldots, n$, such that $\sum_{j=0}^{n} \sigma_j \mathcal{J}_{j1} \leq 0$. This means that $L'(s) \leq 0$ and that the only invariant set in the set $\{L'(s) = 0\}$ is the singleton

$$(S, I)(s) \equiv (S^*, I^*), \quad W_j(s) \equiv 0, \quad 0 \leq j \leq n.$$

Since $(S, I)(s)$ is persistent and $W_j(s)$, $0 \leq j \leq n$, are bounded in $[0, +\infty)$, we have that $L(s)$ is bounded in $[0, +\infty)$. Then LaSalle's invariance principle gives $(S, I)(+\infty) = P^*$. ∎

Remark 7.2. Since the saturation condition ((A5)(II)) is required for the model in [36], the methods in this paper can be applied to the model in [36] under a weaker (unsaturated) condition.

By the proof of Theorem 7.1, we know that the Lyapunov function $L(s)$ for system (7.2) is constructed based on that in [17] for the ODE or nondiffusive model corresponding to diffusive model (1.3) (i.e., model (1.3) with $d_i = 0$ for $i = 0, 1, \ldots, n$). Generally speaking, a Lyapunov function for the traveling-wave system (3.1) can be constructed if the corresponding ODE or nondiffusive model has a Lyapunov function. We can thus obtain a theorem similar to Theorem 7.1 for the models or special cases in [21, 30, 29, 23, 13, 11, 40].

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