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J. Differential Equations 267 (2019) 5084–5114

Journal of Differential Equations

www.elsevier.com/locate/jde

A spatial SEIRS reaction-diffusion model in heterogeneous environment ☆

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Received 12 December 2018; accepted 16 May 2019

Available online 23 May 2019

Abstract

We propose a susceptible-exposed-infected-recovered-susceptible (SEIRS) reaction-diffusion model, where the disease transmission and recovery rates can be spatially heterogeneous. The basic reproduction number (R_0) is connected with the principal eigenvalue of a linear cooperative elliptic system. Threshold-type results on the global dynamics in terms of R_0 are established. The monotonicity of R_0 with respect to the diffusion rates of the exposed and infected individuals, which does not hold in general, is established in several cases. Finally, the asymptotic profile of the endemic equilibrium is investigated when the diffusion rate of the susceptible individuals is small. Our results reveal the importance of the movement of the exposed and recovered individuals in disease dynamics, as opposed to most of previous works which solely focused on the movement of the susceptible and infected individuals.

MSC: 35J55; 35B32

Keywords: SEIRS epidemic model; Reaction-diffusion equation; Basic reproduction number; Persistence/extinction; Endemic equilibrium

☆ Research partially supported by the China Scholarship Council grant 201706280333 (PS), NSF grant DMS-1411476 (YL), NSFC grants 11631012 and 11571273 (YX), and National Mega-project of Science Research 2017ZX10201101-002-002 (YX).

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https://doi.org/10.1016/j.jde.2019.05.022

1. Introduction

In epidemiology, there is increasing evidence that environmental heterogeneity and individual motility have significant impact on the spread of infectious diseases ([4,32]). In recent years, a number of reaction-diffusion models have been proposed to investigate the roles of movement and environmental heterogeneity on the transmission of diseases across the habitat ([2,5,6,9, 12,17,22,26–29,33–36,42,43,45]). Among these works, Allen et al. [2] proposed the following susceptible-infected-susceptible (SIS) reaction-diffusion system:

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \Delta S - \frac{\beta(x)SI}{S+I} + \gamma(x)I, & x \in \Omega, t > 0, \\ \frac{\partial I}{\partial t} = d_I \Delta I + \frac{\beta(x)SI}{S+I} - \gamma(x)I, & x \in \Omega, t > 0, \\ \frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{1.1}$$

Here, Ω is a bounded domain in R^k with smooth boundary $\partial\Omega$, where n is the outward normal unit vector on $\partial\Omega$ and the homogeneous Neumann boundary conditions mean that no individuals cross the boundary. $S(x,t)$ and $I(x,t)$ denote the density of susceptible and infected populations at location x and time t, d_S and d_I represent the diffusion coefficients for susceptible and infected individuals, and $\beta(x)$, $\gamma(x)$ are transmission and recovery rates at x, respectively.

The main results of [2] concern the properties of the basic production number (R_0), threshold-type results on the global dynamics in terms of R_0, and particularly the existence, uniqueness and asymptotic behaviors of the endemic equilibrium as the diffusion rate of the susceptible individuals approaches zero; See also [33–35]. Peng and Zhao ([36]) considered the same SIS reaction-diffusion model, but the disease transmission and recovery rates are assumed to be spatially heterogeneous and temporally periodic. In [9,43], the authors investigated an SIS model with mass action infection. In [26], Li et al. provided qualitative analysis on an SIS reaction-diffusion system with a linear source term. Ge et al. introduced a free boundary model for characterizing the spreading front of the disease in [17]. The effects of diffusion and advection for SIS epidemic models in heterogeneous environment were studied in [5,6]. Dynamics and asymptotic profiles of endemic equilibrium for two frequency-dependent SIS epidemic models with cross-diffusion was considered in [27]. We also refer to [1,8,12,15,22] and the references therein for related works.

However, these models did not include the class of exposed individuals and ignored the movement of exposed (latently infected) individuals. For some epidemic diseases, infected individuals can experience incubation before showing symptoms, e.g. malaria, West Nile virus, HIV/AIDS. The travel of exposed individuals showing no symptoms can spread the disease geographically, which makes disease harder to control ([16]). Therefore, it seems imperative to include the exposed subclass and explore the influences of exposed individuals' movement on disease spread. Mathematically, this is related to the dependence of the basic reproduction number on the diffusion rates of exposed individuals. There were some previous results on this aspect in discrete-space multi-patch models ([16,44]). In this paper, we will extend continuous-space model (1.1) to include the exposed and recovery classes, and analyze the corresponding SEIRS reaction-diffusion model.

1.1. SEIRS reaction-diffusion model

To model the progress of infectious diseases in populations, we divide the individuals into four different compartments: susceptible (S), exposed (E), infectious (I), recovered (immune by vaccination, R). The susceptible individuals are infected by infectious individuals with a rate of β, and become exposed; exposed individuals become infectious with a rate σ; infected individuals are recovered with a rate γ; recovery individuals lose immunity and go back into the susceptible class with a rate of α. Thus the SEIRS (suspected-exposed-infected-recovered-suspected) epidemic reaction-diffusion model are given as follows:

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \Delta S - \frac{\beta(x)SI}{S+I+E+R} + \alpha R, & x \in \Omega, t > 0, \\ \frac{\partial E}{\partial t} = d_E \Delta E + \frac{\beta(x)SI}{S+I+E+R} - \sigma E, & x \in \Omega, t > 0, \\ \frac{\partial I}{\partial t} = d_I \Delta I + \sigma E - \gamma(x) I, & x \in \Omega, t > 0, \\ \frac{\partial R}{\partial t} = d_R \Delta R + \gamma(x) I - \alpha R, & x \in \Omega, t > 0, \\ \frac{\partial S}{\partial n} = \frac{\partial E}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial R}{\partial n} = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{1.2}$$

Here $S(x,t)$, $E(I,t)$, $I(x,t)$ and $R(x,t)$ denote the density of susceptible, exposed, infected and recovered individuals at location x and time t, and d_S, d_E, d_I, d_R represent the diffusion coefficients for susceptible, exposed, infected and recovered populations, respectively. Throughout this paper, we assume that the disease transmission rate $\beta(x)$ and recovery rate $\gamma(x)$ are environmentally dependent and could be spatially heterogeneous, and they are assumed to be positive, Hölder continuous functions on $\overline{\Omega}$. The latent period $1/\sigma$ and the rate of loss of immunity α are not associated with external environment but usually depend on an individual itself, and thus σ and α are always assumed to be constants in this paper.

Throughout the paper, we assume that the initial conditions satisfy

(A1) $S(x,0), E(x,0), I(x,0), R(x,0) \geq 0$ for $x \in \overline{\Omega}$ and $\int_\Omega I(x,0)dx > 0$.

It is easy to verify that $\frac{\beta(x)SI}{S+I+E+R}$ is a Lipschitz continuous function of S and I, therefore we define it to be zero whenever $S = 0$ or $I = 0$. By the regularity theory for parabolic equations ([20]) and assumption **(A1)**, system (1.2) admits a unique classical solution $S, E, I, R \in C^{2,1}(\overline{\Omega} \times (0,\infty))$. Moreover, it follows from the strong maximum principle for parabolic equations ([37]) that S, E, I and R are positive for $x \in \overline{\Omega}$ and $t > 0$. We define the total population size at time t as

$$N(t) = \int_\Omega (S(x,t) + E(x,t) + I(x,t) + R(x,t))dx,$$

and assume that the total population size at the initial time $t = 0$ is a fixed positive constant, denoted by N_0. By system (1.2), we have $N'(t) = 0$ for $t > 0$. Thus

$$N(t) = N_0 \text{ for any } t \geq 0. \tag{1.3}$$

This paper also concerns non-negative equilibrium solutions of (1.2) which satisfy

$$\begin{cases} d_S \Delta \widetilde{S} - \frac{\beta(x)\widetilde{S}\widetilde{I}}{\widetilde{S}+\widetilde{I}+\widetilde{E}+\widetilde{R}} + \alpha \widetilde{R} = 0, & x \in \Omega, \\ d_E \Delta \widetilde{E} + \frac{\beta(x)\widetilde{S}\widetilde{I}}{\widetilde{S}+\widetilde{I}+\widetilde{E}+\widetilde{R}} - \sigma \widetilde{E} = 0, & x \in \Omega, \\ d_I \Delta \widetilde{I} + \sigma \widetilde{E} - \gamma(x)\widetilde{I} = 0, & x \in \Omega, \\ d_R \Delta \widetilde{R} + \gamma(x)\widetilde{I} - \alpha \widetilde{R} = 0, & x \in \Omega, \\ \frac{\partial \widetilde{S}}{\partial n} = \frac{\partial \widetilde{E}}{\partial n} = \frac{\partial \widetilde{I}}{\partial n} = \frac{\partial \widetilde{R}}{\partial n} = 0, & x \in \partial\Omega, \\ \int_\Omega (\widetilde{S} + \widetilde{E} + \widetilde{I} + \widetilde{R}) dx = N_0, \end{cases} \tag{1.4}$$

where $\widetilde{S}$, $\widetilde{E}$, $\widetilde{I}$, $\widetilde{R}$ denote the density of susceptible, exposed, infected and recovered individuals at equilibrium, respectively. A *disease-free equilibrium* (DFE) is a solution of (1.4) satisfying $\widetilde{I}(x) = 0$ for every $x \in \Omega$; An *endemic equilibrium* (EE) is a solution of (1.4) for which $\widetilde{I}(x) > 0$ for some $x \in \Omega$. It is easy to verify that the disease free equilibrium is unique, given by $E_0 = (\frac{N_0}{|\Omega|}, 0, 0, 0)$, where $|\Omega|$ is the Lebesgue measure of Ω. By the strong maximum principle, for any endemic equilibrium, $\widetilde{S}(x)$, $\widetilde{E}(x)$, $\widetilde{I}(x)$, $\widetilde{R}(x)$ are positive for any $x \in \overline{\Omega}$.

1.2. *Statement of main results*

The goal of this paper is to investigate the impact of population movement and environmental heterogeneity on the persistence or extinction of infectious diseases. We will focus on the dynamics of model (1.2), the properties of the basic production number, and the asymptotic behaviors of the endemic equilibria as the diffusion rate of the susceptible individuals approaches zero.

For infectious disease models, the basic reproduction number, defined as the expected number of secondary cases produced in a completely susceptible population by an infective individual, is one of the most significant concepts in studying the transmission of infectious disease ([3,10]). More importantly, it often determines the threshold behavior for many epidemic models. It is often the case that a disease dies out if the basic reproduction number is less than unity and the disease is established in the population if it is greater than unity. We refer to [10,11,41] for the approach of next generation operators for the basic reproduction number and to [28,29,40,42,45] for related works.

By adopting the theory developed in [40,42], we characterize the basic reproduction number of system (1.2), denoted by R_0, via the relationship $R_0 = \frac{1}{\mu_0}$ (see Lemma 2.2), where μ_0 is the unique positive eigenvalue with a positive eigenfunction for the linear problem

$$\begin{cases} -d_E \Delta \varphi_E + \sigma \varphi_E = \mu \beta(x) \varphi_I, & x \in \Omega, \\ -d_I \Delta \varphi_I + \gamma(x)\varphi_I - \sigma \varphi_E = 0, & x \in \Omega, \\ \frac{\partial \varphi_E}{\partial n} = \frac{\partial \varphi_I}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{1.5}$$

Threshold-type dynamics for system (1.2) in terms of R_0 can be stated as follows:

Theorem 1.1. (i) *If* $R_0 \le 1$, *then* $E(x,t), I(x,t), R(x,t) \to 0$ *uniformly in* $\overline{\Omega}$ *as* $t \to \infty$, *and* $\int_\Omega S(x,t)dx \to N_0$ *as* $t \to \infty$;

(ii) *If $R_0 > 1$, there exists some constant $\epsilon_0 > 0$ such that any positive solution of* (1.2) *satisfies*

$$\liminf_{t\to\infty} \|(S(t,\cdot), E(t,\cdot), I(t,\cdot), R(t,\cdot)) - (\frac{N_0}{|\Omega|}, 0, 0, 0)\|_{L^\infty(\Omega)} > \epsilon_0. \tag{1.6}$$

Moreover, system (1.2) *admits at least one endemic equilibrium.*

To explore the influence of population movement on the persistence of infectious diseases, we proceed to investigate the dependence of R_0 on d_E, d_I. It should be noted that R_0 is independent of d_S and d_R. It is shown in [2] that R_0 for model (1.1) is decreasing in d_I. However, for $SEIRS$ system (1.2), it is not always the case, as the movement of exposed individuals makes the monotonicity of R_0 more subtle. An underlying reason is that for SIS model (1.1), R_0 is the principal eigenvalue of a self-adjoint elliptic operator and it has a variational characterization, which implies that R_0 is decreasing in d_I. However, for $SEIRS$ system (1.2), the lack of variational structure for the eigenvalue problem (1.5) makes the situation here more sophisticated and the analysis more challenging.

The asymptotic properties of R_0 when d_E, d_I tend to 0 or infinity are given in Theorem 3.1. The following result addresses the monotonicity of R_0 with respect to d_E, d_I:

Theorem 1.2. *If either β or γ is a constant function, then R_0 is a monotone decreasing function of d_E and d_I. Moreover, the strict monotonicity holds if and only if one of them is non-constant.*

Theorem 1.2 may fail to hold if both β and γ are non-constant; See Theorem 3.4 and discussion section for further details. We conjecture that, for any β and γ, there exists a constant $\bar{d}$ independent of d_E, d_I such that if $0 < d_E < \bar{d}$ or $0 < d_I < \bar{d}$, R_0 is monotone decreasing in d_E, d_I.

If the habitat is one dimensional, we have the following result:

Theorem 1.3. *Assume Ω is one dimensional, i.e., $k = 1$, and one of β and γ is monotone decreasing and the other is monotone increasing. Then R_0 is a monotone decreasing function of d_E, d_I, and the strict monotonicity holds if and only if β or γ is non-constant.*

Theorem 1.3 may even fail to hold if both $\beta(x)$ and $\gamma(x)$ are monotone increasing; see Theorem 3.5 and discussion section for further details.

Finally, to understand the effect of the suspected population movement on the spatial distribution of populations, we investigate the asymptotic profiles of the endemic equilibrium of system (1.2) when d_S tends to zero. We assume that $R_0 > 1$ so that system (1.2) admits at least one endemic equilibrium by Theorem 1.1. To this end, consider the linear eigenvalue problem

$$-d_R\Delta\phi + \alpha(1 - \tfrac{\gamma}{\beta})\phi = \lambda\phi \ \text{ in } \ \Omega, \quad \tfrac{\partial\phi}{\partial n}|_{\partial\Omega} = 0, \tag{1.7}$$

and denote the smallest eigenvalue of (1.7) by $\lambda_1(-d_R\Delta + \alpha(1 - \frac{\gamma}{\beta}))$.

Theorem 1.4. *Assume $R_0 > 1$ and $\lambda_1(-d_R\Delta + \alpha(1 - \frac{\gamma}{\beta})) < 0$. Then the following assertions hold:*

(i) *There exist positive constants* C_1, C_2, *independent of* d_S, *such that for sufficiently small* d_S,

$$C_1 \le \frac{\widetilde{E}}{d_S}, \frac{\widetilde{I}}{d_S}, \frac{\widetilde{R}}{d_S} \le C_2;$$

(ii) *As* $d_S \to 0$, *subject to a sequence,*

$$\widetilde{S} \to \widetilde{S}^* = \frac{N_0(1-M^*(x))}{\int_\Omega (1-M^*(x))dx} \quad in \ \ C^1(\overline{\Omega}),$$

for some $M^*(x)$ *satisfying* $0 \le M^* \le 1$ *in* Ω *and* $|\{x \in \Omega : M^*(x) = 1\}| \in (0, |\Omega|)$.

In particular, part (i) of Theorem 1.4 implies that $\widetilde{E}, \widetilde{I}, \widetilde{R} \to 0$ uniformly in Ω as $d_S \to 0$, and part (ii) of Theorem 1.4 implies that $\widetilde{S}^* \ge 0$, $\widetilde{S}^* \not\equiv 0$, and the set $\{x \in \Omega : \widetilde{S}^*(x) = 0\}$ is nonempty. Biologically, Theorem 1.4 implies that restricting the movement of susceptible population can effectively control the number of exposed and infected individuals, and contain susceptible individuals in some subregion within the habitat.

Theorem 1.4 may fail to hold if $\lambda_1(-d_R\Delta + \alpha(1-\frac{\gamma}{\beta})) > 0$; See discussion section for details.

This paper is organized as follows. In section 2, we establish the wellposedness of model (1.2), define R_0 and study the dynamics of system (1.2) in terms of R_0. In section 3, we investigate the asymptotic properties and monotonicity of R_0 with respect to diffusion coefficients d_E and d_I. The asymptotic profiles of the endemic equilibrium as d_S tends to zero is considered in section 4. Finally in section 5, we discuss our main results and present some numerical results.

2. Wellposedness, basic reproduction number and threshold dynamics of model (1.2)

To start with, the following uniform bound for the solution of system (1.2) is established.

Lemma 2.1. *There exist some positive constants* C_1, *independent of initial values, and* $T > 0$ *such that the solution* $(S, E, I, R) \in C^{2,1}(\overline{\Omega} \times (0, \infty))$ *of system* (1.2) *satisfies*

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|E(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} + \|R(\cdot,t)\|_{L^\infty(\Omega)} \le C_1, \quad \forall t > T. \tag{2.1}$$

The proof of Lemma 2.1 is standard. By (1.3), $\|S(\cdot,t)\|_{L^1(\Omega)}$, $\|E(\cdot,t)\|_{L^1(\Omega)}$, $\|I(\cdot,t)\|_{L^1(\Omega)}$ and $\|R(\cdot,t)\|_{L^1(\Omega)}$ are uniformly bounded. This and Lemma 2.1 in [13] (due to [25]) with $\sigma = p_0 = 1$, along with the positiveness of S, E, I and R, imply (2.1).

We now adopt the theory developed in [42] to derive the basic reproduction number of system (1.2). The linearization of system (1.2) at E_0 is given by

$$\begin{cases} \frac{\partial \bar{S}}{\partial t} = d_S\Delta\bar{S} - \beta(x)\bar{I} + \alpha\bar{R}, & x \in \Omega, t > 0, \\ \frac{\partial \bar{E}}{\partial t} = d_E\Delta\bar{E} + \beta(x)\bar{I} - \sigma\bar{E}, & x \in \Omega, t > 0, \\ \frac{\partial \bar{I}}{\partial t} = d_I\Delta\bar{I} + \sigma\bar{E} - \gamma(x)\bar{I}, & x \in \Omega, t > 0, \\ \frac{\partial \bar{R}}{\partial t} = d_R\Delta\bar{R} + \gamma(x)\bar{I} - \alpha\bar{R}, & x \in \Omega, t > 0, \\ \frac{\partial \bar{S}}{\partial n} = \frac{\partial \bar{E}}{\partial n} = \frac{\partial \bar{I}}{\partial n} = \frac{\partial \bar{R}}{\partial n} = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{2.2}$$

Note that the infected compartments are E and I in system (1.2). Besides, L, $F(x)$ and $V(x)$ in [42] can be defined as $L = diag(-d_E\Delta, -d_I\Delta)$,

$$F(x) = \begin{pmatrix} 0 & \beta(x) \\ 0 & 0 \end{pmatrix}, \quad V(x) = \begin{pmatrix} \sigma & 0 \\ -\sigma & \gamma(x) \end{pmatrix}. \tag{2.3}$$

Note that assumptions (**A1**)-(**A6**) in [42] hold for model (1.2). Here we point out that $F(x)$ is

$$\begin{pmatrix} 0 & \beta(x) \\ 0 & 0 \end{pmatrix} \text{ rather than } \begin{pmatrix} 0 & \beta(x) \\ \sigma & 0 \end{pmatrix},$$

as $F(x)$ represents the infection process. In our model, infection process occurs only in S to E and σ is associated with the transient process from E to I.

Lemma 2.2. *The eigenvalue problem* (1.5) *admits a unique positive eigenvalue, denoted by μ_0, with a positive eigenfunction. Moreover, the basic reproduction number of system* (1.2)*, denoted by R_0, satisfies*

$$R_0 = \frac{1}{\mu_0}. \tag{2.4}$$

Proof. By Theorem 5.1 in [31], there exists a positive eigenvalue of (1.5) with a positive eigenfunction. To prove the uniqueness we assume that there exist positive eigenvalue μ_1 with positive eigenfunction $\varphi_1 = (\varphi_{E,1}, \varphi_{I,1})^T$ and positive eigenvalue μ_2 with positive eigenfunction $\varphi_2^* = (\varphi_{E,2}^*, \varphi_{I,2}^*)^T$ such that

$$L\varphi_1 + V\varphi_1 = \mu_1 F\varphi_1 \text{ in } \Omega, \quad \tfrac{\partial\varphi_1}{\partial n}|_{\partial\Omega} = 0, \tag{2.5}$$

and

$$L\varphi_2^* + V^T\varphi_2^* = \mu_2 F^T\varphi_2^* \text{ in } \Omega, \quad \tfrac{\partial\varphi_2^*}{\partial n}|_{\partial\Omega} = 0. \tag{2.6}$$

We now multiply the equation in (2.5) by $(\varphi_2^*)^T$ and the equation in (2.6) by φ_1^T, subtract the two resulting equations, and integrate by parts over Ω to give

$$(\mu_1 - \mu_2)\int_\Omega \beta\varphi_{E,2}^*\varphi_{I,1}dx = 0.$$

Since $\beta, \varphi_{E,2}^*$ and $\varphi_{I,1}$ are positive, we obtain $\mu_1 = \mu_2$. This establishes the uniqueness. In view of Theorem 3.2 in [42] and the uniqueness of positive eigenvalue with a positive eigenfunction for (1.5), we obtain (2.4). □

Next we study the stability of E_0 in terms of R_0. We first consider the eigenvalue problem

$$\begin{cases} -d_E\Delta\phi_E - \beta(x)\phi_I + \sigma\phi_E = \lambda\phi_E, & x \in \Omega, \\ -d_I\Delta\phi_I + \gamma(x)\phi_I - \sigma\phi_E = \lambda\phi_I, & x \in \Omega, \\ \frac{\partial\phi_E}{\partial n} = \frac{\partial\phi_I}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.7}$$

By the Krein-Rutman theorem ([23]), the eigenvalue problem (2.7) has a unique principal eigenvalue λ_1, that is, a real and simple eigenvalue with positive eigenfunctions ϕ_E, ϕ_I, and it is strictly less than the real parts of all other eigenvalues.

Lemma 2.3. *The following relationship holds:*

$$sign(1 - R_0) = sign(\lambda_1). \tag{2.8}$$

Proof. Consider the principal eigenvalue corresponding to the adjoint problem of (2.7), i.e.

$$\begin{cases} -d_E \Delta\phi_E^* + \sigma\phi_E^* - \sigma\phi_I^* = \lambda_1\phi_E^*, & x \in \Omega, \\ -d_I \Delta\phi_I^* - \beta(x)\phi_E^* + \gamma(x)\phi_I^* = \lambda_1\phi_I^*, & x \in \Omega, \\ \frac{\partial\phi_E^*}{\partial n} = \frac{\partial\phi_I^*}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.9}$$

We multiply the first equation in (1.5) by ϕ_E^* and the first equation in (2.9) by φ_E, subtract the two resulting equations, and integrate by parts to give

$$\lambda_1 \int_\Omega \varphi_E\phi_E^* dx = \int_\Omega (\frac{1}{R_0}\beta(x)\varphi_I\phi_E^* - \sigma\phi_I^*\varphi_E)dx. \tag{2.10}$$

Moreover, it follows from multiplying the second equation in (1.5) by ϕ_I^* and multiplying the second equation in (2.9) by φ_I, subtracting the two resulting equations, and integrating by parts to find

$$\lambda_1 \int_\Omega \varphi_I\phi_I^* dx = -\int_\Omega (\beta(x)\varphi_I\phi_E^* - \sigma\phi_I^*\varphi_E)dx. \tag{2.11}$$

Adding two equations (2.10) and (2.11) yields

$$\lambda_1 \int_\Omega (\varphi_E\phi_E^* + \varphi_I\phi_I^*)dx = \frac{1 - R_0}{R_0} \int_\Omega \beta(x)\varphi_I\phi_E^* dx.$$

Since $\varphi_E, \phi_E^*, \varphi_I, \phi_I^*, \beta$ are positive, we have $sign(1 - R_0) = sign(\lambda_1)$. □

Lemma 2.4. *The disease-free equilibrium E_0 in system (1.2) is locally asymptotically stable if $R_0 < 1$, unstable if $R_0 > 1$.*

Proof. Let Λ be the spectrum of the following eigenvalue problem:

$$\begin{cases} d_S\Delta\phi_S - \beta(x)\phi_I + \alpha\phi_R + \lambda\phi_S = 0, & x \in \Omega, \\ d_E\Delta\phi_E + \beta(x)\phi_I - \sigma\phi_E + \lambda\phi_E = 0, & x \in \Omega, \\ d_I\Delta\phi_I + \sigma\phi_E - \gamma(x)\phi_I + \lambda\phi_I = 0, & x \in \Omega, \\ d_R\Delta\phi_R + \gamma(x)\phi_I - \alpha\phi_R + \lambda\phi_R = 0, & x \in \Omega, \\ \frac{\partial\phi_S}{\partial n} = \frac{\partial\phi_E}{\partial n} = \frac{\partial\phi_I}{\partial n} = \frac{\partial\phi_R}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.12}$$

We first show that if $R_0 < 1$, then $\inf\{Re\lambda,\ \lambda \in \Lambda\} > 0$. Note that $\Lambda = \Lambda_{\{(\phi_E,\phi_I)\neq 0\}} \cup \Lambda_{\{(\phi_E,\phi_I)=0\}}$. It is easy to verify $\inf\{Re\lambda, \lambda \in \Lambda_{\{(\phi_E,\phi_I)=0\}}\} > 0$. Moreover, we have $\Lambda_{\{(\phi_E,\phi_I)\neq 0\}} \subseteq \sigma(L+V-F)$, where $\sigma(L+V-F)$ denotes the spectrum of the operator $L+V-F$ and L, V, F are defined in (2.3). Note that $\inf\{Re\lambda, \lambda \in \sigma(L+V-F)\} = \lambda_1 > 0$, where λ_1 is the principal eigenvalue of eigenvalue problem (2.7). By Lemma 2.3, we obtain $\inf\{Re\lambda, \lambda \in \Lambda_{\{(\phi_E,\phi_I)\neq 0\}}\} > 0$. Therefore, by Theorem 5.1.1 in [20], E_0 is locally asymptotically stable if $R_0 < 1$.

To prove E_0 is linearly unstable when $R_0 > 1$, we show that there exists a non-trivial solution of (2.12) such that $Re\lambda < 0$. Let $\lambda = \lambda_1 < 0$, where λ_1 is the principal eigenvalue of (2.7), and choose (ϕ_E, ϕ_I) as the eigenfunction of (2.7) associated with λ_1. In view of [18], ϕ_S, ϕ_R in (2.12) are uniquely solvable. Therefore, Theorem 5.1.3 in [20] yields that E_0 is unstable if $R_0 > 1$. □

Now we proceed to the proof of Theorem 1.1.

2.1. *Proof of Theorem 1.1(i)*

We prove (i) by constructing a Lyapunov functional and applying LaSalle's invariance principle (Theorem 1 in [19]) for infinite dimensional dynamical systems. Let $X = C(\overline{\Omega}; R^4)$ with the supremum norm $\|\|_\infty$, then X is an ordered Banach space with the cone P consisting of all nonnegative functions in X, and X has nonempty interior, denoted by $int(P)$. Set

$$X_0 = \{u = (u_s, u_e, u_i, u_r) \in X | \int_\Omega (u_s + u_e + u_i + u_r)dx = N_0\}$$

and $U = P \cap X_0$. It is easy to verify that (1.2) coupled with (1.3) defines a dynamic system on U. Denote the unique solution of system (1.2) with initial value $(s_0, e_0, i_0, r_0) \in U$ by $\Phi_t(s_0, e_0, i_0, r_0) = (S(\cdot,t), E(\cdot,t), I(\cdot,t), R(\cdot,t))$ for any $t > 0$. It follows from parabolic L^p estimates and Sobolev inequalities that for each $\tau \in (0,1)$, there exists some positive constant C_2 such that

$$\|(S, E, I, R)\|_{C^{\tau,\frac{\tau}{2}}(\overline{\Omega}\times[t_0-\frac{1}{2},t_0+1))} \le C_2 \|(S, E, I, R)\|_{L^\infty(\overline{\Omega}\times[t_0-1,t_0+1])}$$

for each $t_0 \ge 1$. Since C_2 is independent of t_0, then we obtain by Lemma 2.1 that

$$\|(S(\cdot,t), E(\cdot,t), I(\cdot,t), R(\cdot,t))\|_{C^\tau(\bar{\Omega})} \le C_1 C_2, \quad t \ge 1, \tag{2.13}$$

where C_1 is defined in (2.1). Therefore Φ_t is compact, and for each $u_0 \in U$, the orbit of u_0 under the dynamical system generated by (1.2) has compact closure in U.

Define the functional

$$L(u) = \int_\Omega (u_e \phi_E^* + u_i \phi_I^*)dx$$

for $u \in U$, where (ϕ_E^*, ϕ_I^*) is the eigenfunction corresponding to the principal eigenvalue λ_1 associated with the eigenvalue problem (2.9). Now we prove $L(u)$ is a Lyapunov functional for system (1.2). For an arbitrary solution $u = (S, E, I, R)$ of system (1.2) coupled with (1.3), we have

$$\begin{aligned}\frac{d}{dt}L(u(\cdot,t)) &= \int_\Omega (E_t\phi_E^* + I_t\phi_I^*)dx \\ &= \int_\Omega ((d_E\Delta E + \tfrac{\beta SI}{S+I+E+R} - \sigma E)\phi_E^* + (d_I\Delta I + \sigma E - \gamma I)\phi_I^*)dx \\ &= -\int_\Omega \beta\phi_E^* I \tfrac{E+I+R}{S+I+E+R}dx - \lambda_1\int_\Omega (E\phi_E^* + I\phi_I^*)dx.\end{aligned} \tag{2.14}$$

By Lemma 2.3, $R_0 \le 1$ yields that $\lambda_1 \ge 0$. Besides, S, E, I, R are nonnegative, and $\beta, \phi_E^*, \phi_I^*$ are positive. Hence, $\frac{d}{dt}L(u(\cdot,t)) \le 0$, which implies $L(u)$ is a Lyapunov functional of system (1.2).

Next define

$$\dot{L}(u_0) := \frac{d}{dt}L(u(\cdot,t))|_{t=0} \text{ and } M = \{u_0 \in U | \dot{L}(u_0) = 0\},$$

where $u = (S, E, I, R)$ is the unique solution of (1.2) with initial condition $u_0 = (s_0, e_0, i_0, r_0) \in U$. By (2.14), we have $M = \{u_0 = (s_0, e_0, i_0, r_0) \in U | i_0 = 0\}$ if $\lambda_1 = 0$, and $M = \{u_0 = (s_0, e_0, i_0, r_0) \in U | e_0 = i_0 = 0\}$ if $\lambda_1 > 0$. It follows from (1.2) that for $\lambda_1 \ge 0$, the maximal invariant set in M is given by

$$\hat{M} := \{u_0 = (s_0, e_0, i_0, r_0) \in U | e_0 = i_0 = 0\}.$$

Therefore, by the LaSalle invariant principle (Theorem 1 in [19]), we obtain

$$(E(x,t), I(x,t)) \to (0,0) \text{ in } [L^\infty(\Omega)]^2, \text{ as } t \to \infty,$$

which together with (1.2) imply $R(x,t) \to 0$ uniformly in $\overline{\Omega}$ as $t \to \infty$. Therefore, thanks to (1.3), we obtain $\int_\Omega S(x,t)dx \to N_0$ as $t \to \infty$.

2.2. *Proof of Theorem 1.1 (ii)*

We appeal to the uniform persistence theory developed in [30,45]. Denote

$$\begin{aligned}U_0 &:= \{(s_0, e_0, i_0, r_0) \in U | e_0 \ne 0 \text{ and } i_0 \ne 0\}, \\ \partial U_0 &:= \{(s_0, e_0, i_0, r_0) \in U | e_0 = 0 \text{ or } i_0 = 0\}.\end{aligned}$$

Note that $U = U_0 \cup \partial U_0$. Moreover, U_0 and ∂U_0 are relatively open and closed subsets of U, respectively, and U_0 is convex. Denote the unique solution of (1.2) with initial value $(s_0, e_0, i_0, r_0) \in U$ by $\Phi_t(s_0, e_0, i_0, r_0) = (S(\cdot,t), E(\cdot,t), I(\cdot,t), R(\cdot,t))$ for $t > 0$. Φ_t is continuous and compact for $t > 0$. By Lemma 2.1, Φ_t is pointwisely dissipative. Therefore, Φ_t has a global attractor ([45]).

Step 1. We have $\Phi_t U_0 \subset U_0$ for all $t > 0$. This is a direct result of the strong maximum principle for parabolic equations.

Step 2. Let A_∂ be the maximal positively invariant set for $\Phi(t)$ in ∂U_0, i.e.

$$A_\partial := \{(s_0, e_0, i_0, r_0) \in U | \Phi(t)(s_0, e_0, i_0, r_0) \in \partial U_0, t \geq 0\}.$$

It is easy to verify that $A_\partial = \{u_0 = (s_0, e_0, i_0, r_0) \in U | e_0 = i_0 = 0\}$.

Denote $\omega((s_0, e_0, i_0, r_0))$ as the ω-limit set of (s_0, e_0, i_0, r_0) in U (see [45]) and

$$\hat{A}_\partial := \cup_{\{(s_0,e_0,i_0,r_0)\in A_\partial\}} \omega((s_0, e_0, i_0, r_0)).$$

We now prove $\hat{A}_\partial = \{E_0\}$. For any $(s_0, e_0, i_0, r_0) \in A_\partial$, i.e. $e_0 = i_0 = 0$, then $E(x,t) = I(x,t) = 0$ for all $x \in \overline{\Omega}, t \geq 0$, and system (1.2) becomes

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \Delta S + \alpha R, & x \in \Omega, t > 0, \\ \frac{\partial R}{\partial t} = d_R \Delta R - \alpha R, & x \in \Omega, t > 0, \\ \frac{\partial S}{\partial n} = \frac{\partial R}{\partial n} = 0, & x \in \partial\Omega, \end{cases}$$

which implies $R(\cdot,t) \to 0, S(\cdot,t) \to \frac{N_0}{|\Omega|}$ uniformly as $t \to \infty$. Hence, $\hat{A}_\partial = \{E_0\}$. Therefore, $\{E_0\}$ is a compact and isolated invariant set for Φ_t restricted in A_∂.

Step 3. We prove that there exists some constant $\epsilon_1 > 0$ independent of initial values such that

$$\limsup_{t\to\infty} \| \Phi_t(s_0, e_0, i_0, r_0) - (\frac{N_0}{|\Omega|}, 0, 0, 0) \| > \epsilon_1.$$

Assume, on the contrary, that for any $\epsilon_2 > 0$, there exists some initial value $(s_0^*, e_0^*, i_0^*, r_0^*)$ such that

$$\limsup_{t\to\infty} \| \Phi_t(s_0^*, e_0^*, i_0^*, r_0^*) - (\frac{N_0}{|\Omega|}, 0, 0, 0) \| \leq \frac{\epsilon_2}{2}. \tag{2.15}$$

Given any small $\epsilon_3 > 0$ and let $\lambda_1(\epsilon_3)$ be the unique principal eigenvalue of the following eigenvalue problem with a positive eigenfunction (ϕ_E, ϕ_I):

$$\begin{cases} -d_E \Delta\phi_E - \frac{\beta(x)(\frac{N_0}{|\Omega|}+\epsilon_3)}{\frac{N_0}{|\Omega|}+4\epsilon_3}\phi_I + \sigma\phi_E = \lambda\phi_E, & x \in \Omega, \\ -d_I \Delta\phi_I + \gamma(x)\phi_I - \sigma\phi_E = \lambda\phi_I, & x \in \Omega, \\ \frac{\partial\phi_E}{\partial n} = \frac{\partial\phi_I}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

Note that $\lim_{\epsilon_3\to 0} \lambda_1(\epsilon_3) = \lambda_1 < 0$, where λ_1 is the principal eigenvalue of eigenvalue problem (2.7). Therefore, we can choose ϵ_3 such that $\lambda_1(\epsilon_3) < 0$. Since ϵ_2 is arbitrary, choose $\epsilon_2 = \epsilon_3$. By (2.15), there exists $T > 0$ such that $S^* \leq \frac{N}{|\Omega|} + \epsilon_3$, $E^*, I^*, R^* \leq \epsilon_3$ for any $x \in \overline{\Omega}, t \geq T$. By the strong maximum principal of parabolic equations, $(S^*(\cdot,t), E^*(\cdot,t), I^*(\cdot,t), R^*(\cdot,t)) \in int(P)$ for all $t > 0$. Then we can find a small positive constant c_* such that $E^*(x,T) \geq c_*\phi_E$, $I^*(x,T) \geq c_*\phi_I$. It is easy to verify that $(E^*(x,t), I^*(x,t))$ is a supersolution of the problem

$$\begin{cases} \frac{\partial \hat{E}}{\partial t} = d_E \Delta \hat{E} + \frac{\beta(x)(\frac{N_0}{|\Omega|}+\epsilon_3)}{\frac{N_0}{|\Omega|}+4\epsilon_3} \hat{I} - \sigma \hat{E}, & x \in \Omega, t > T, \\ \frac{\partial \hat{I}}{\partial t} = d_I \Delta \hat{I} + \sigma \hat{E} - \gamma(x) \hat{I}, & x \in \Omega, t > T, \\ \frac{\partial \hat{E}}{\partial n} = \frac{\partial \hat{I}}{\partial n} = 0, & x \in \partial\Omega, t > T, \\ \hat{E}(x,T) = c_* \phi_E, \hat{I}(x,T) = c_* \phi_I, \end{cases} \tag{2.16}$$

where $(c_* e^{-\lambda_1(\epsilon_3)(t-T)} \phi_E, c_* e^{-\lambda_1(\epsilon_3)(t-T)} \phi_I)$ is the unique solution to system (2.16). Note that $\lambda_1(\epsilon_3) < 0$, therefore $E^*(x,t) \geq c_* e^{-\lambda_1(\epsilon_3)(t-T)} \phi_E, I^*(x,t) \geq c_* e^{-\lambda_1(\epsilon_3)(t-T)} \phi_I \to \infty$ uniformly in $\overline{\Omega}$ as $t \to \infty$. This contradiction finishes the proof of step 3.

The result of step 3 implies that $\{E_0\}$ is an isolated invariant set for Φ_t in U, and $W^S(\{E_0\}) \cap U_0$ is an empty set, where $W^S(\{E_0\})$ is the stable set of $\{E_0\}$ for Φ_t.

Finally, by steps 1-3 and Theorem 1.3.1 in [45], Φ_t is uniformly persistent with respect to $(U, \partial U_0)$. Moreover, by Theorem 1.3.7 in [45], (1.2) admits at least one endemic equilibrium. □

3. Properties of basic reproduction number R_0

We have in previous section established threshold dynamics of system (1.2) in terms of R_0. In this section, to explore the influence of population movement on the persistence of infectious diseases, we will investigate the asymptotic properties and monotonicity of R_0 with respect to d_E, d_I.

3.1. Asymptotic properties of R_0 with respect to d_E, d_I

By Lemma 2.2, $\frac{1}{R_0}$ is the unique principal eigenvalue of (1.5), thus we have

$$\begin{cases} -d_E \Delta \varphi_E + \sigma \varphi_E = \frac{1}{R_0} \beta(x) \varphi_I, & x \in \Omega, \\ -d_I \Delta \varphi_I + \gamma(x) \varphi_I - \sigma \varphi_E = 0, & x \in \Omega, \\ \frac{\partial \varphi_E}{\partial n} = \frac{\partial \varphi_I}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Moreover, $\frac{1}{R_0}$ is the unique principal eigenvalue of the adjoint problem of (1.5), i.e.

$$\begin{cases} -d_E \Delta \varphi_E^* + \sigma \varphi_E^* - \sigma \varphi_I^* = 0, & x \in \Omega, \\ -d_I \Delta \varphi_I^* + \gamma(x) \varphi_I^* = \frac{1}{R_0} \beta(x) \varphi_E^*, & x \in \Omega, \\ \frac{\partial \varphi_E^*}{\partial n} = \frac{\partial \varphi_I^*}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{3.2}$$

where $(\varphi_E^*, \varphi_I^*)$ is an eigenfunction corresponding to the unique principal eigenvalue of the adjoint problem of (1.5). Now we give an estimate of R_0.

Lemma 3.1. *For any $d_E > 0, d_I > 0$, the following inequalities hold:*

$$min\{\frac{\beta(x)}{\gamma(x)}, x \in \overline{\Omega}\} \leq R_0 \leq max\{\frac{\beta(x)}{\gamma(x)}, x \in \overline{\Omega}\}. \tag{3.3}$$

Proof. It follows from adding two equations of (3.1) that

$$-d_E\Delta\varphi_E - d_I\Delta\varphi_I + \gamma(x)\varphi_I = \tfrac{1}{R_0}\beta(x)\varphi_I \text{ in } \Omega, \quad \tfrac{\partial\varphi_E}{\partial n}|_{\partial\Omega} = \tfrac{\partial\varphi_I}{\partial n}|_{\partial\Omega} = 0. \tag{3.4}$$

Integrating (3.4) by parts over Ω yields

$$\int_\Omega \gamma(x)(R_0 - \frac{\beta(x)}{\gamma(x)})\varphi_I dx = 0.$$

Since $\gamma(x)$ and φ_I are positive, we obtain (3.3). □

Lemma 3.1 implies that if $\frac{\beta}{\gamma}$ is constant, then R_0 is independent of d_E, d_I.

Theorem 3.1. (i) *Fix $d_I > 0$. Then $R_0 \to \frac{1}{\mu_1}$ as $d_E \to 0$, and $R_0 \to \frac{1}{|\Omega|}\int_\Omega (-d_I\Delta + \gamma)^{-1}\beta\, dx$ as $d_E \to \infty$, where μ_1 is the smallest eigenvalue of the problem*

$$-d_I\Delta\bar{\varphi}_I + \gamma\bar{\varphi}_I = \mu\beta\bar{\varphi}_I \text{ in } \Omega, \quad \tfrac{\partial\bar{\varphi}_I}{\partial n}|_{\partial\Omega} = 0; \tag{3.5}$$

(ii) *Fix $d_E > 0$. Then $R_0 \to \frac{1}{\mu_2}$ as $d_I \to 0$ and $R_0 \to \frac{\int_\Omega \beta\, dx}{\int_\Omega \gamma\, dx}$ as $d_I \to \infty$, where μ_2 is the smallest eigenvalue of the problem*

$$-d_E\Delta\bar{\varphi}_E + \sigma\bar{\varphi}_E = \mu\tfrac{\sigma\beta}{\gamma}\bar{\varphi}_E, \text{ in } \Omega, \quad \tfrac{\partial\bar{\varphi}_E}{\partial n}|_{\partial\Omega} = 0; \tag{3.6}$$

(iii) *As $d_E, d_I \to 0$, then $R_0 \to max\{\frac{\beta(x)}{\gamma(x)}, x \in \overline{\Omega}\}$;*
(iv) *As $d_E \to \infty$ and $d_I \to 0$, then $R_0 \to \frac{1}{|\Omega|}\int_\Omega \frac{\beta}{\gamma}dx$.*

3.1.1. Proof of Theorem 3.1 (i,ii)

We only prove (i) here as (ii) can be established by similar arguments. We first consider the case $d_E \to 0$. Given $\epsilon \in (0,1)$, since $A = \{u \in C^2(\overline{\Omega}) | \frac{\partial u}{\partial n} = 0\}$ is dense in $C(\overline{\Omega})$, we can choose $\beta_1^*(x), \beta_2^*(x) \in A$ such that

$$\frac{\beta(x)}{1+\epsilon} < \beta_1^*(x) < \beta(x) < \beta_2^*(x) < \frac{\beta(x)}{1-\epsilon}.$$

Set

$$(\hat{\varphi}_E, \hat{\varphi}_I) = (\frac{\mu_1\beta_1^*\bar{\varphi}_I}{\sigma}, \bar{\varphi}_I), \quad (\check{\varphi}_E, \check{\varphi}_I) = (\frac{\mu_1\beta_2^*\bar{\varphi}_I}{\sigma}, \bar{\varphi}_I).$$

For any $\epsilon \in (0,1)$, there exists δ such that $0 < d_E < \delta$,

$$-d_E\Delta\hat{\varphi}_E + \sigma(1 - \frac{\beta}{\beta_1^*(1+\epsilon)})\hat{\varphi}_E \geq 0 \text{ for } x \in \Omega, \quad \frac{\partial\hat{\varphi}_E}{\partial n} = 0 \text{ for } x \in \partial\Omega, \tag{3.7}$$

and

$$-d_E\Delta\check{\varphi}_E+\sigma(1-\frac{\beta}{\beta_2^*(1-\epsilon)})\check{\varphi}_E\le 0 \text{ for } x\in\Omega, \quad \frac{\partial\check{\varphi}_E}{\partial n}=0 \text{ for } x\in\partial\Omega. \tag{3.8}$$

It follows from (3.5) and (3.7) that

$$\begin{cases} -d_E\Delta\hat{\varphi}_E+\sigma\hat{\varphi}_E\ge\frac{\mu_1}{1+\epsilon}\beta(x)\hat{\varphi}_I, & x\in\Omega,\\ -d_I\Delta\hat{\varphi}_I+\gamma(x)\hat{\varphi}_I-\sigma\hat{\varphi}_E\ge 0, & x\in\Omega,\\ \frac{\partial\hat{\varphi}_E}{\partial n}=\frac{\partial\hat{\varphi}_I}{\partial n}=0, & x\in\partial\Omega. \end{cases} \tag{3.9}$$

Multiplying the first equation in (3.9) by φ_E^* and the first equation in (3.2) by $\hat{\varphi}_E$, subtracting the resulting functions and integrating the results by parts over Ω yield

$$\int_\Omega(\frac{\mu_1}{1+\epsilon}\beta(x)\hat{\varphi}_I\varphi_E^*-\sigma\varphi_I^*\hat{\varphi}_E)dx\le 0. \tag{3.10}$$

Similarly, multiplying the second equation in (3.9) by φ_I^* and the second equation in (3.2) by $\hat{\varphi}_I$, subtracting the resulting functions and integrating by parts over Ω we have

$$\int_\Omega(\frac{1}{R_0}\beta(x)\hat{\varphi}_I\varphi_E^*-\sigma\varphi_I^*\hat{\varphi}_E)dx\ge 0. \tag{3.11}$$

Thus by (3.10) and (3.11) we get

$$(\frac{1}{R_0}-\frac{\mu_1}{1+\epsilon})\int_\Omega\beta(x)\hat{\varphi}_I\varphi_E^*dx\ge 0,$$

which implies that $R_0\le\frac{1+\epsilon}{\mu_1}$. Similar procedures yield

$$(\frac{1}{R_0}-\frac{\mu_1}{1-\epsilon})\int_\Omega\beta(x)\check{\varphi}_I\varphi_E^*dx\le 0,$$

from which it follows that $\frac{1-\epsilon}{\mu_1}\le R_0$. This proves that $R_0\to 1/\mu_1$ as $d_E\to 0$.

Next we consider the case $d_E\to\infty$. It follows from Lemma 3.1 that, passing to a sequence if necessary, $R_0\to\tilde{R}_0>0$ as $d_E\to\infty$. Without loss of generality, we may assume $\|\varphi_E\|_{L^\infty(\Omega)}+\|\varphi_I\|_{L^\infty(\Omega)}=1$. By L^p estimate, for any $p>1$, $\|\varphi_E\|_{W^{2,p}(\Omega)}$, $\|\varphi_I\|_{W^{2,p}(\Omega)}$ are uniformly bounded. Thus by Sobolev embedding theorem, $\|\varphi_E\|_{C^{1,\tau}(\Omega)}$, $\|\varphi_I\|_{C^{1,\tau}(\Omega)}$ are uniformly bounded. Passing to a sequence if necessary, $\varphi_E\to\tilde{\varphi}_E, \varphi_I\to\tilde{\varphi}_I$ in $C^1(\overline{\Omega})$ as $d_E\to\infty$. Therefore, $\tilde{\varphi}_I$ is a H^1 weak solution of

$$-d_I\Delta\tilde{\varphi}_I+\gamma(x)\tilde{\varphi}_I-\sigma\tilde{\varphi}_E=0 \text{ for } x\in\Omega \text{ and } \frac{\partial\tilde{\varphi}_I}{\partial n}=0 \text{ for } x\in\partial\Omega.$$

By first equation of (3.1) and elliptic regularity ([18]), $\tilde{\varphi}_E$ is constant satisfying $\tilde{\varphi}_E=\frac{\int_\Omega\beta(x)\tilde{\varphi}_I dx}{\sigma\tilde{R}_0|\Omega|}$. Thus the weak solution $\tilde{\varphi}_I$ is actually a classical solution, i.e. $\tilde{\varphi}_I\in C^2(\overline{\Omega})$ satisfies

$$-d_I\Delta\tilde{\varphi}_I + \gamma(x)\tilde{\varphi}_I - \frac{\int_\Omega \beta(x)\tilde{\varphi}_I dx}{\tilde{R}_0|\Omega|} = 0 \text{ for } x \in \Omega \text{ and } \frac{\partial\tilde{\varphi}_I}{\partial n} = 0 \text{ for } x \in \partial\Omega. \tag{3.12}$$

Thus, $\tilde{R}_0 = \frac{1}{|\Omega|}\int_\Omega(-d_I\Delta + \gamma(x))^{-1}\beta(x)dx$. This completes the proof of (i).

3.1.2. Proof of Theorem 3.1(iii)

Denote $\tau_0 = max\{\frac{\beta(x)}{\gamma(x)}, x \in \bar{\Omega}\}$. It follows from Lemma 3.1 that, passing to a sequence if necessary, $R_0 \to \tilde{R}_0 > 0$ as $d_E, d_I \to 0$. Then, for any small positive ϵ, there exists a positive constant δ, such that if $0 < d_E, d_I < \delta$, then $\tilde{R}_0 - \epsilon < R_0 < \tilde{R}_0 + \epsilon$.

Consider the eigenvalue problem

$$Lu - \frac{Fu}{\tau} + Vu = \lambda_1(\tau)u \text{ in } \Omega, \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \tag{3.13}$$

where τ is a positive parameter and $\lambda_1(\tau)$ is the principal eigenvalue. Let $\varphi = (\varphi_E, \varphi_I)^T$ in (3.1), then it follows from (3.1) that

$$L\varphi - \frac{F\varphi}{\tilde{R}_0 - \epsilon} + V\varphi \le 0 \le L\varphi - \frac{F\varphi}{\tilde{R}_0 + \epsilon} + V\varphi, \quad x \in \Omega.$$

In view of the comparison principle corresponding to the principal eigenvalue for irreducible cooperative elliptic systems (Proposition 3.4 in [24]), we have

$$\lambda_1(\tilde{R}_0 - \epsilon) \le \lambda_1(R_0) = 0 \le \lambda_1(\tilde{R}_0 + \epsilon). \tag{3.14}$$

Moreover, in view of [24], $\lambda_1(\tau)$ satisfies

$$\lambda_1(\tau) \to \lambda_1^*(\tau) = -\max_{x\in\bar{\Omega}} \Lambda_1(\frac{F(x)}{\tau} - V(x)), \tag{3.15}$$

as $d_E, d_I \to 0$, where $\Lambda_1(\frac{F(x)}{\tau} - V(x))$ is the principal eigenvalue of the cooperative matrix $\frac{F(x)}{\tau} - V(x)$ ([14]) at position x. It is easy to verify that

$$\Lambda_1(\frac{F(x)}{\tau} - V(x)) = \frac{-\sigma - \gamma(x) + \sqrt{(\sigma + \gamma(x))^2 + 4\frac{\sigma\beta(x)}{\tau} - 4\sigma\gamma(x)}}{2}. \tag{3.16}$$

If $\tau > (=, <)\tau_0$, it follows from (3.16) that $\max_{x\in\bar{\Omega}} \Lambda_1(\frac{F(x)}{\tau} - V(x)) < (=, >)0$, and thus $sign(\tau - \tau_0) = sign(\lambda_1^*(\tau))$. Note that (3.14) implies $\lambda_1^*(\tilde{R}_0 - \epsilon) \le 0 \le \lambda_1^*(\tilde{R}_0 + \epsilon)$. Hence, $\tau_0 - \epsilon \le \tilde{R}_0 \le \epsilon + \tau_0$. This establishes (iii). □

3.1.3. Proof of Theorem 3.1 (iv)

Without loss of generality, we may assume $\|\varphi_E\|_{L^\infty(\Omega)} + \|\varphi_I\|_{L^\infty(\Omega)} = 1$. By L^p estimate, for any $p > 1$, $\|\varphi_E\|_{W^{2,p}(\Omega)}$ is uniformly bounded. Thus by Sobolev embedding theorem, $\|\varphi_E\|_{C^{1,\tau}(\Omega)}$ is uniformly bounded. Passing to a sequence if necessary, $\varphi_E \to \tilde{\varphi}_E$ in $C^1(\overline{\Omega})$ as $d_E \to \infty, d_I \to 0$, where $\tilde{\varphi}_E$ is a non-negative constant. Therefore, for any small ϵ, there exists δ_ϵ such that for any $0 < d_I, \frac{1}{d_E} < \delta_\epsilon$, we have

$$\sigma(\tilde{\varphi}_E - \epsilon) < -d_I \Delta\varphi_I + \gamma(x)\varphi_I < \sigma(\tilde{\varphi}_E + \epsilon),$$

which yields that $\varphi_I \to \frac{\sigma\tilde{\varphi}_E}{\gamma}$ in $L^\infty(\Omega)$ as $d_E \to \infty, d_I \to 0$. By $\|\varphi_E\|_{L^\infty(\Omega)} + \|\varphi_I\|_{L^\infty(\Omega)} = 1$, we obtain $\tilde{\varphi}_E > 0$. It follows from integrating the first equation of (3.1) and passing to limits that

$$\tilde{\varphi}_E|\Omega| = \lim_{\substack{d_E\to\infty\\ d_I\to 0}} \frac{1}{R_0}\int_\Omega \frac{\beta\tilde{\varphi}_E}{\gamma}dx.$$

Since $\tilde{\varphi}_E$ is a positive constant, we have $R_0 \to \frac{1}{|\Omega|}\int_\Omega \frac{\beta}{\gamma}dx$ as $d_E \to \infty, d_I \to 0$. □

3.2. Monotonicity of R_0 with respect to d_E, d_I

In what follows, we explore some cases that R_0 has monotonicity with respect to d_E, d_I. By the same arguments as [4] and Lemma 15.1 in [21], we can show that the basic reproduction number R_0 and the corresponding eigenfunctions (φ_E, φ_I), $(\varphi_E^*, \varphi_I^*)$ are differentiable functions of d_E, d_I. For further purposes, we differentiate both sides of the equations in (3.1) by d_E, d_I, we obtain

$$\begin{cases} -d_E\Delta\varphi_E' - \Delta\varphi_E + \sigma\varphi_E' = \frac{1}{R_0}\beta(x)\varphi_I' - \frac{R_0'}{R_0^2}\beta(x)\varphi_I, & x\in\Omega,\\ -d_I\Delta\varphi_I' + \gamma(x)\varphi_I' - \sigma\varphi_E' = 0, & x\in\Omega,\\ \frac{\partial\varphi_E'}{\partial n} = \frac{\partial\varphi_I'}{\partial n} = 0, & x\in\partial\Omega, \end{cases} \tag{3.17}$$

and

$$\begin{cases} -d_E\Delta\varphi_E' + \sigma\varphi_E' = \frac{1}{R_0}\beta(x)\varphi_I' - \frac{R_0'}{R_0^2}\beta(x)\varphi_I, & x\in\Omega,\\ -d_I\Delta\varphi_I' - \Delta\varphi_I + \gamma(x)\varphi_I' - \sigma\varphi_E' = 0, & x\in\Omega,\\ \frac{\partial\varphi_E'}{\partial n} = \frac{\partial\varphi_I'}{\partial n} = 0, & x\in\partial\Omega, \end{cases} \tag{3.18}$$

respectively. Here, for convenience, the prime notation denotes differentiation by d_E or d_I since no confusion will happen in further proofs.

Theorem 3.2. *If $\beta(x)$ is constant on Ω, then R_0 is monotone decreasing function of d_E, d_I. Moreover, the strict monotonicity holds if $\gamma(x)$ is nonconstant on Ω.*

Proof of Theorem 3.2. To begin with, we show that R_0 is monotone decreasing with respect to d_E. We now multiply the first equation in (3.17) by φ_E and the first equation in (3.1) by φ_E', subtract the two resulting equations, and then integrate by parts over Ω to give

$$\frac{R_0'}{R_0^2}\int_\Omega \beta(x)\varphi_I\varphi_E dx = -\int_\Omega |\nabla\varphi_E|^2 dx + \frac{1}{R_0}\int_\Omega \beta(x)(\varphi_I'\varphi_E - \varphi_I\varphi_E')dx.$$

Similarly, we multiply the second equation in (3.17) by φ_I and multiply the second equation in (3.1) by φ_I', subtract the two resulting equations, and then integrate by parts over Ω to give (as σ is constant)

$$\int_\Omega (\varphi_I' \varphi_E - \varphi_I \varphi_E')dx = 0.$$

If $\beta(x)$ is constant on Ω, we can obtain that

$$\frac{\beta R_0'}{R_0^2} \int_\Omega \varphi_I \varphi_E dx = -\int_\Omega |\nabla \varphi_E|^2 dx.$$

Since φ_I, φ_E are positive functions, we obtain $R_0' \le 0$. Furthermore, the equality is possible only if φ_E is constant on Ω. This fact together with the first equation of (3.1) yield that φ_I must be constant, which along with the second equation of (3.1) imply $\gamma(x)$ must be constant. Therefore, R_0 is monotone decreasing with respect to d_E and the strict monotonicity holds if and only if $\gamma(x)$ is nonconstant on Ω.

We next show that R_0 is monotone decreasing with respect to d_I. We now multiply the first equation in (3.18) by φ_E and the first equation in (3.1) by φ_E', subtract the two resulting equations, and then integrate by parts over Ω to give

$$\frac{R_0'}{R_0} \int_\Omega \beta(x)\varphi_I \varphi_E dx = \int_\Omega \beta(x)(\varphi_I' \varphi_E - \varphi_I \varphi_E')dx.$$

Similarly, we multiply the second equation in (3.18) by φ_I and multiply the second equation in (3.1) by φ_I', subtract the two resulting equations, and then integrate by parts over Ω to give

$$\int_\Omega |\nabla \varphi_I|^2 dx + \sigma \int_\Omega (\varphi_I' \varphi_E - \varphi_I \varphi_E')dx = 0.$$

If $\beta(x)$ is constant on Ω. We obtain

$$\frac{\sigma R_0'}{R_0} \int_\Omega \varphi_I \varphi_E dx = -\int_\Omega |\nabla \varphi_I|^2 dx.$$

By the same arguments as before, R_0 is strictly monotone decreasing with respect to d_I if and only if $\gamma(x)$ is nonconstant. □

Theorem 3.3. *If $\gamma(x)$ is constant on Ω, then R_0 is monotone decreasing function of d_E, d_I and the strict monotonicity holds if and only if $\beta(x)$ is nonconstant on Ω.*

Proof of Theorem 3.3. To start with, we show that R_0 is monotone decreasing with respect to d_E. Multiplying the first equation in (3.17) by φ_I and integrating by parts over Ω yield

$$\begin{aligned}\frac{R_0'}{R_0^2}\int_\Omega \beta(x)\varphi_I^2 dx &= \int_\Omega \varphi_I \Delta\varphi_E dx + \frac{1}{R_0}\int_\Omega \beta(x)\varphi_I'\varphi_I dx - \int_\Omega(-d_E\Delta\varphi_E' + \sigma\varphi_E')\varphi_I dx \\ &= \int_\Omega \varphi_E \Delta\varphi_I dx + \frac{1}{R_0}\int_\Omega \beta(x)\varphi_I'\varphi_I dx - \int_\Omega(-d_E\Delta\varphi_I + \sigma\varphi_I)\varphi_E' dx.\end{aligned} \tag{3.19}$$

It follows (3.17) and (3.1) that

$$\varphi_E' = \frac{-d_I\Delta\varphi_I' + \gamma\varphi_I'}{\sigma}, \quad \varphi_E = \frac{-d_I\Delta\varphi_I + \gamma\varphi_I}{\sigma}, \tag{3.20}$$

respectively. Thus, by substituting (3.20) into the last equality of (3.19) and integrating by parts over Ω, we obtain

$$\begin{aligned} \tfrac{R_0'}{R_0^2}\int_\Omega \beta(x)\varphi_I^2 dx &= \tfrac{1}{\sigma}\int_\Omega(-d_I|\Delta\varphi_I|^2 - \gamma|\nabla\varphi_I|^2)dx + \tfrac{1}{R_0}\int_\Omega \beta(x)\varphi_I'\varphi_I dx \\ &\quad - \tfrac{1}{\sigma}\int_\Omega(-d_E\Delta\varphi_I + \sigma\varphi_I)(-d_I\Delta\varphi_I' + \gamma\varphi_I')dx. \end{aligned} \tag{3.21}$$

Multiplying the first equation in (3.1) by φ_I' and integrating by parts over Ω claim

$$\frac{1}{R_0}\int_\Omega \beta(x)\varphi_I'\varphi_I dx - \int_\Omega(-d_E\Delta\varphi_I' + \sigma\varphi_I')\varphi_E dx = 0. \tag{3.22}$$

Moreover, $\varphi_E = (-d_I\Delta\varphi_I + \gamma\varphi_I)/\sigma$ gives

$$\begin{aligned} \int_\Omega(-d_E\Delta\varphi_I' + \sigma\varphi_I')\varphi_E dx &= \tfrac{1}{\sigma}\int_\Omega(-d_E\Delta\varphi_I' + \sigma\varphi_I')(-d_I\Delta\varphi_I + \gamma\varphi_I)dx \\ &= \tfrac{1}{\sigma}\int_\Omega(-d_E\Delta\varphi_I + \sigma\varphi_I)(-d_I\Delta\varphi_I' + \gamma\varphi_I')dx, \end{aligned}$$

where the second equality holds because γ and σ are constants. This together with (3.22) yield

$$\frac{1}{R_0}\int_\Omega \beta(x)\varphi_I'\varphi_I dx - \frac{1}{\sigma}\int_\Omega(-d_E\Delta\varphi_I + \sigma\varphi_I)(-d_I\Delta\varphi_I' + \gamma\varphi_I')dx = 0. \tag{3.23}$$

It follows from (3.21) and (3.23) that

$$\frac{\sigma R_0'}{R_0^2}\int_\Omega \beta(x)\varphi_I^2 dx = \int_\Omega(-d_I|\Delta\varphi_I|^2 - \gamma|\nabla\varphi_I|^2)dx.$$

Therefore, $R_0' \le 0$. The same argument as Theorem 3.2 shows the strict monotonicity holds if $\beta(x)$ is nonconstant on Ω.

Next, we show that R_0 is monotone decreasing with respect to d_I. It follows from multiplying the first equation in (3.18) by φ_I and integrating by parts over Ω that

$$\begin{aligned} \tfrac{R_0'}{R_0^2}\int_\Omega \beta(x)\varphi_I^2 dx &= \tfrac{1}{R_0}\int_\Omega \beta(x)\varphi_I'\varphi_I dx - \int_\Omega(-d_E\Delta\varphi_E' + \sigma\varphi_E')\varphi_I dx \\ &= \tfrac{1}{R_0}\int_\Omega \beta(x)\varphi_I'\varphi_I dx - \int_\Omega(-d_E\Delta\varphi_I + \sigma\varphi_I)\varphi_E' dx. \end{aligned} \tag{3.24}$$

In view of the second equation of (3.18), we have

$$\varphi_E' = \frac{-d_I\Delta\varphi_I' + \gamma\varphi_I' - \Delta\varphi_I}{\sigma}. \tag{3.25}$$

By (3.24), (3.25) and integrating by parts over Ω, we have

$$\begin{aligned}\frac{R_0'}{R_0^2}\int_\Omega \beta(x)\varphi_I^2 dx &= \frac{1}{\sigma}\int_\Omega(-d_E|\Delta\varphi_I|^2-\sigma|\nabla\varphi_I|^2)dx+\frac{1}{R_0}\int_\Omega\beta(x)\varphi_I'\varphi_I dx\\ &\quad -\frac{1}{\sigma}\int_\Omega(-d_E\Delta\varphi_I+\sigma\varphi_I)(-d_I\Delta\varphi_I'+\gamma\varphi_I')dx.\end{aligned} \tag{3.26}$$

By the same arguments as before, we obtain

$$\frac{1}{R_0}\int_\Omega \beta(x)\varphi_I'\varphi_I dx-\frac{1}{\sigma}\int_\Omega(-d_E\Delta\varphi_I+\sigma\varphi_I)(-d_I\Delta\varphi_I'+\gamma\varphi_I')dx=0. \tag{3.27}$$

By equations (3.26) and (3.27), we obtain

$$\frac{\sigma R_0'}{R_0^2}\int_\Omega \beta(x)\varphi_I^2 dx=\int_\Omega(-d_E|\Delta\varphi_I|^2-\sigma|\nabla\varphi_I|^2)dx.$$

Hence, $R_0'\le 0$ and strict monotonicity holds if $\beta(x)$ is nonconstant on Ω. □

3.3. One dimensional habitat

In this part, we assume the habitat is a bounded open interval and prove Theorem 1.3.

Proof of Theorem 1.3. We first consider the case $\beta(x)$ is monotone decreasing in x and $\gamma(x)$ is monotone increasing, and show that R_0 is monotone decreasing with respect to d_E. Without loss of generality, let $\Omega=(0,1)$. We now multiply the first equation in (3.17) by φ_E^* and the first equation in (3.2) by φ_E', subtract the two resulting equations, and then integrate by parts over Ω to give

$$\frac{R_0'}{R_0^2}\int_0^1 \beta(x)\varphi_I\varphi_E^* dx=\int_0^1\varphi_E^*\frac{d^2\varphi_E}{dx^2}dx+\int_0^1(\frac{1}{R_0}\beta(x)\varphi_I'\varphi_E^*-\sigma\varphi_I^*\varphi_E')dx. \tag{3.28}$$

By multiplying the second equation in (3.17) by φ_I^* and multiplying the second equation in (3.2) by φ_I', we obtain

$$\int_0^1(\frac{1}{R_0}\beta(x)\varphi_I'\varphi_E^*-\sigma\varphi_I^*\varphi_E')dx=0. \tag{3.29}$$

It follows from (3.28) and (3.29) that

$$\frac{R_0'}{R_0^2}\int_0^1 \beta(x)\varphi_I\varphi_E^* dx=\int_0^1\varphi_E^*\frac{d^2\varphi_E}{dx^2}dx=-\int_0^1\frac{d\varphi_E^*}{dx}\frac{d\varphi_E}{dx}dx. \tag{3.30}$$

Next, we show that $\frac{d\varphi_E^*}{dx}\frac{d\varphi_E}{dx} \geq 0$ on Ω. We differentiate (3.1) with respect to x to obtain

$$\begin{cases} -d_E \frac{d^3\varphi_E}{dx^3} + \sigma \frac{d\varphi_E}{dx} - \frac{1}{R_0}\beta(x)\frac{d\varphi_I}{dx} = \frac{1}{R_0}\varphi_I \frac{d\beta(x)}{dx}, & x \in \Omega, \\ -d_I \frac{d^3\varphi_I}{dx^3} + \gamma(x)\frac{d\varphi_I}{dx} - \sigma \frac{d\varphi_E}{dx} = -\varphi_I \frac{d\gamma(x)}{dx}, & x \in \Omega, \\ \frac{d\varphi_E}{dx}(0) = \frac{d\varphi_E}{dx}(1) = \frac{d\varphi_I}{dx}(0) = \frac{d\varphi_I}{dx}(1) = 0, \end{cases} \tag{3.31}$$

and (3.2) with respect to x to give

$$\begin{cases} -d_E \frac{d^3\varphi_E^*}{dx^3} + \sigma \frac{d\varphi_E^*}{dx} - \sigma \frac{d\varphi_I^*}{dx} = 0, & x \in \Omega, \\ -d_I \frac{d^3\varphi_I^*}{dx^3} + \gamma(x)\frac{d\varphi_I^*}{dx} - \frac{1}{R_0}\beta(x)\frac{d\varphi_E^*}{dx} = -\frac{d\gamma(x)}{dx}\varphi_I^* + \frac{1}{R_0}\frac{d\beta(x)}{dx}\varphi_E^*, & x \in \Omega, \\ \frac{d\varphi_E^*}{dx}(0) = \frac{d\varphi_E^*}{dx}(1) = \frac{d\varphi_I^*}{dx}(0) = \frac{d\varphi_I^*}{dx}(1) = 0. \end{cases} \tag{3.32}$$

Denote $L = diag(-d_E \frac{d^2}{dx^2}, -d_I \frac{d^2}{dx^2})$ and

$$M(x) = \begin{pmatrix} -\sigma & \frac{\beta(x)}{R_0} \\ \sigma & -\gamma(x) \end{pmatrix}.$$

Let $\lambda_1(L_N - M), \lambda_1(L_D - M)$ be the unique principal eigenvalue of $L - M$ under the Neumann and Dirichlet conditions, respectively. Besides, $\lambda_1(L_N - M^T), \lambda_1(L_D - M^T)$ are defined as the unique principal eigenvalue corresponding to the adjoint operator of $L - M$ under the Neumann and Dirichlet conditions, respectively. It can be seen from (3.1) and (3.2) that $\lambda_1(L_N - M) = 0$, $\lambda_1(L_N - M^T) = 0$. In view of Proposition 3.4 in [24], we obtain

$$\lambda_1(L_D - M) > 0 \;\text{ and }\; \lambda_1(L_D - M^T) > 0. \tag{3.33}$$

Since $\beta(x)$ is monotone decreasing in x and $\gamma(x)$ is monotone increasing, we have $\frac{d\beta(x)}{dx} \leq 0$, $\frac{d\gamma(x)}{dx} \geq 0$. Then, it follows from (3.31), (3.32) (3.33) and the maximum principle for cooperative elliptic systems (Theorem 1.1 in [39]) that $\frac{d\varphi_E}{dx}, \frac{d\varphi_I}{dx}, \frac{d\varphi_E^*}{dx}, \frac{d\varphi_I^*}{dx} \leq 0$ for $x \in \Omega$. This together with (3.30) implies $R_0' \leq 0$. By the same arguments as the proof of Theorem 3.2, the equality holds if and only if both $\beta(x)$ and $\gamma(x)$ are constants.

If $\beta(x)$ is monotone increasing in x and $\gamma(x)$ is monotone decreasing, by similar arguments we obtain $\frac{d\varphi_E}{dx}, \frac{d\varphi_I}{dx}, \frac{d\varphi_E^*}{dx}, \frac{d\varphi_I^*}{dx} \geq 0$ for $x \in \Omega$. The rest arguments are similar.

Next we show that R_0 is monotone decreasing with respect to d_I. By the same arguments as before, we show that

$$\frac{R_0'}{R_0^2}\int_0^1 \beta(x)\varphi_I\varphi_E^* dx = -\int_0^1 \frac{d\varphi_I^*}{dx}\frac{d\varphi_I}{dx}dx,$$

and the rest arguments are similar.

3.4. Non-monotonicity of R_0 in d_E, d_I

In previous subsections, we have proved in some cases, R_0 is monotone decreasing in d_E, d_I. In this subsection, we will show R_0 is not always monotone decreasing associated with d_E, d_I.

Theorem 3.4. *There exist d_E^0 and $d_I^1 < d_I^2$ such that $R_0(d_E^0, d_I^1) < R_0(d_E^0, d_I^2)$, if*

$$\frac{\int_\Omega \beta(x)dx}{\int_\Omega \gamma(x)dx} > \frac{1}{|\Omega|}\int_\Omega \frac{\beta(x)}{\gamma(x)}dx.$$

Proof of Theorem 3.4. In view of Theorem 3.1, we know for any fixed $d_E > 0$, $R_0 \to \frac{\int_\Omega \beta(x)dx}{\int_\Omega \gamma(x)dx}$ as $d_I \to \infty$, and $R_0 \to \frac{1}{|\Omega|}\int_\Omega \frac{\beta(x)}{\gamma(x)}dx$ as $d_I \to 0, d_E \to \infty$. Then, for any positive small ϵ, there exists $C_1(\epsilon)$ large enough, such that for any $\frac{1}{d_I^1}, d_E \ge C_1(\epsilon)$, we have

$$R_0(d_E, d_I^1) \le \frac{1+\epsilon}{|\Omega|}\int_\Omega \frac{\beta(x)}{\gamma(x)}dx.$$

Moreover, there exists $C_2(\epsilon, d_E)$ such that for any $d_I^2 \ge C_2(\epsilon, d_E)$,

$$R_0(d_E, d_I^2) \ge (1-\epsilon)\frac{\int_\Omega \beta(x)dx}{\int_\Omega \gamma(x)dx}.$$

Since

$$\frac{\int_\Omega \beta(x)dx}{\int_\Omega \gamma(x)dx} > \frac{1}{|\Omega|}\int_\Omega \frac{\beta(x)}{\gamma(x)}dx,$$

we can choose ϵ_0 small enough such that

$$(1-\epsilon_0)\frac{\int_\Omega \beta(x)dx}{\int_\Omega \gamma(x)dx} > \frac{1+\epsilon_0}{|\Omega|}\int_\Omega \frac{\beta(x)}{\gamma(x)}dx,$$

and let $d_E^0 = C_1(\epsilon_0)$, $d_I^1 = \frac{1}{C_1(\epsilon_0)}$, $d_I^2 = C_2(\epsilon_0, d_E^0)$, we have $R_0(d_E^0, d_I^1) < R_0(d_E^0, d_I^2)$. □

Theorem 3.5. *Let $\nu_0 = \frac{\int_\Omega \gamma(x)dx}{\int_\Omega \beta(x)dx}$ and φ_1, ϕ_1 be the unique solutions of*

$$-\Delta\varphi_1 = \nu_0\beta - \gamma, \quad x \in \Omega \quad \textit{with} \quad \frac{\partial\varphi_1}{\partial n} = 0, \quad x \in \partial\Omega$$

and

$$-\Delta\phi_1 = \frac{\nu_0}{|\Omega|}\int_\Omega \beta dx - \gamma, \quad x \in \Omega \quad \textit{with} \quad \frac{\partial\phi_1}{\partial n} = 0, \quad x \in \partial\Omega,$$

respectively. If

$$\int_\Omega (\gamma - \nu_0\beta)(\varphi_1 - \phi_1)dx > 0, \tag{3.34}$$

there exist d_I^0 and $d_E^1 < d_E^2$ such that $R_0(d_E^1, d_I^0) < R_0(d_E^2, d_I^0)$.

Proof of Theorem 3.5. Consider the principal eigenvalues, denoted by μ and ν respectively, of the following two eigenvalue problems:

$$-d_I\Delta\varphi + \gamma(x)\varphi = \mu\beta(x)\varphi \ \text{ in } \ \Omega, \quad \tfrac{\partial\varphi}{\partial n}|_{\partial\Omega} = 0, \tag{3.35}$$

and

$$-d_I\Delta\phi + \gamma(x)\phi = \tfrac{\nu}{|\Omega|}\int_\Omega \beta(x)\phi dx \ \text{ in } \ \Omega, \quad \tfrac{\partial\phi}{\partial n}|_{\partial\Omega} = 0 \tag{3.36}$$

with $\int_\Omega \varphi^2 dx = \int_\Omega \phi^2 dx = |\Omega|$. Now we take $\epsilon = \frac{1}{d_I}$ and the expansions on (φ, μ) and (ϕ, ν) to give

$$\begin{aligned} \varphi(x) &= \varphi_0(x) + \epsilon\varphi_1(x) + \epsilon^2\varphi_2(x, \epsilon), \\ \phi(x) &= \phi_0(x) + \epsilon\phi_1(x) + \epsilon^2\phi_2(x, \epsilon), \\ \mu &= \mu_0 + \epsilon\mu_1 + \epsilon^2\mu_2(\epsilon), \\ \nu &= \nu_0 + \epsilon\nu_1 + \epsilon^2\nu_2(\epsilon). \end{aligned} \tag{3.37}$$

Our goal is to prove $\mu > \nu$ when ϵ is small under the condition (3.34). By direct calculation, we obtain $\varphi_0 = \phi_0 = 1$, $\mu_0 = \nu_0 = \frac{\int_\Omega \gamma(x)dx}{\int_\Omega \beta(x)dx}$ and φ_1, ϕ_1 satisfy

$$-\Delta\varphi_1 = \mu_0\beta - \gamma \ \text{ in } \ \Omega, \quad \tfrac{\partial\varphi_1}{\partial n}|_{\partial\Omega} = 0$$

and

$$-\Delta\phi_1 = \tfrac{\nu_0}{|\Omega|}\int_\Omega \beta dx - \gamma \ \text{ in } \ \Omega, \quad \tfrac{\partial\phi_1}{\partial n}|_{\partial\Omega} = 0,$$

respectively. Furthermore, we have

$$\int_\Omega \gamma\varphi_1 dx = \mu_0 \int_\Omega \beta\varphi_1 dx + \mu_1 \int_\Omega \beta dx \tag{3.38}$$

and

$$\int_\Omega \gamma\phi_1 dx = \nu_0 \int_\Omega \beta\phi_1 dx + \nu_1 \int_\Omega \beta dx. \tag{3.39}$$

Therefore, by condition (3.34), (3.38), (3.39) and $\mu_0 = \nu_0$, we obtain

$$(\mu_1 - \nu_1)\int_\Omega \beta dx = \int_\Omega (\gamma - \nu_0\beta)(\varphi_1 - \phi_1)dx > 0.$$

Thus $\mu > \nu$ for large d_I. It follows from Theorem 3.1 that for any fixed $d_I > 0$, $R_0 \to \frac{1}{\mu}$ as $d_E \to 0$ and $R_0 \to \frac{1}{\nu}$ as $d_E \to \infty$. Therefore, we can find d_I^0 large, and d_E^2 large, d_E^1 small such that $R_0(d_E^1, d_I^0) < R_0(d_E^2, d_I^0)$. □

Remark: We give a case such that condition (3.34) holds. Let $\Omega = (0, 1)$, $\beta = \sqrt{x} + 0.001$ and $\gamma = x + 0.001$, then direct calculation yields

$$\int_\Omega (\gamma - \nu_0\beta)(\varphi_1 - \phi_1)dx = 0.0021 > 0.$$

4. Asymptotic properties of endemic equilibrium

Throughout this section, we assume that **(A1)** holds, N_0 is fixed and $R_0 > 1$ so that system (1.2) admits at least one endemic equilibrium by Theorem 1.1. To further understand the effect of the suspected population movement on the spatial distribution of the individuals of system (1.2) in heterogeneous environment, we will investigate the asymptotic profiles of the endemic equilibria when d_S approaches zero.

For later purposes, we start by rewriting the endemic equilibria problem (1.4). Denote $\xi = d_S\widetilde{S} + d_E\widetilde{E} + d_I\widetilde{I} + d_R\widetilde{R}$, and set $S = \frac{\widetilde{S}}{\xi}$, $E = \frac{\widetilde{E}}{\xi}$, $I = \frac{\widetilde{I}}{\xi}$, $R = \frac{\widetilde{R}}{\xi}$. It follows from (1.4) that

$$\begin{cases} d_E\Delta E + \frac{\beta(x)SI}{S+I+E+R} - \sigma E = 0, & x \in \Omega, \\ d_I\Delta I + \sigma E - \gamma(x)I = 0, & x \in \Omega, \\ d_R\Delta R + \gamma(x)I - \alpha R = 0, & x \in \Omega, \\ d_S S + d_E E + d_I I + d_R R = 1, & x \in \Omega, \\ \frac{\partial E}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial R}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{4.1}$$

Therefore, the following results hold:

Lemma 4.1. *$(\widetilde{S}, \widetilde{E}, \widetilde{I}, \widetilde{R})$ is a solution of (1.4) if and only if (S, E, I, R) is a solution of (4.1). Moreover, $\widetilde{S} = \xi S$, $\widetilde{E} = \xi E$, $\widetilde{I} = \xi I$, $\widetilde{R} = \xi R$ and*

$$\xi = \frac{N_0}{\int_\Omega (S + E + I + R)dx}.$$

We now investigate the asymptotic profiles of the endemic equilibria when d_S tends to zero. Recall that $\lambda_1(-d_R\Delta + \alpha(1 - \frac{\gamma}{\beta}))$ is the smallest eigenvalue of the eigenvalue problem (1.7).

Theorem 4.1. *Assume that $R_0 > 1$. Then the following assertions hold:*

(i) *As $d_S \to 0$, subject to a sequence, E, I, R converge to E^*, I^*, R^* in $C^1(\overline{\Omega})$, respectively, for some $E^* \geq 0$, $I^* > 0$, $R^* > 0$;*

(ii) *The set* $J^+ := \{x | M^*(x) = 1, x \in \overline{\Omega}\}$ *has positive Lebesgue measure, where* $M^*(x) := d_E E^* + d_I I^* + d_R R^*$*;*
(iii) *If further assume* $\lambda_1(-d_R\Delta + \alpha(1 - \frac{\gamma}{\beta})) < 0$*, then the set* $J^- := \{x | M^*(x) < 1, x \in \overline{\Omega}\}$ *has positive Lebesgue measure.*

Theorem 4.2. *Assume* $R_0 > 1$ *and* $\lambda_1(-d_R\Delta + \alpha(1 - \frac{\gamma}{\beta})) < 0$*. Then the following assertions hold:*

(i) *As* $d_S \to 0$*, subject to a sequence,*

$$\frac{\xi}{d_S} \to \frac{N_0}{\int_\Omega (1 - M^*(x))dx} \ \ \textit{and} \ \ \widetilde{S} \to \widetilde{S}^* = \frac{N_0(1 - M^*(x))}{\int_\Omega (1 - M^*(x))dx} \ \ \textit{in} \ \ C^1(\overline{\Omega});$$

(ii) *There exist positive constants* C_1, C_2*, independent of* d_S *such that for sufficiently small* d_S*,*

$$C_1 \le \frac{\widetilde{E}}{d_S}, \frac{\widetilde{I}}{d_S}, \frac{\widetilde{R}}{d_S} \le C_2.$$

4.1. Proof of Theorem 4.1

We first prove part (i). Note that $E(x), I(x), R(x) > 0$ for any $x \in \Omega, d_S > 0$. In view of $d_S S + d_E E + d_I I + d_R R = 1$, $\frac{\beta(x)SI}{S+E+I+R}$ is uniformly bounded for any $d_S > 0$. It follows from L^p estimate ([18]) that $\|E\|_{W^{2,p}}$ is bounded for any $p > 1$. Thus, $\|E\|_{C^{1,\tau}}$ is bounded for any $\tau \in (0, 1)$ by Sobolev embedding theorem. Passing to a subsequence if necessary, $E \to E^*$ in $C^1(\overline{\Omega})$ as $d_S \to 0$ where $E^*(x) \ge 0$ for $x \in \Omega$ and $\frac{\partial E^*}{\partial n} = 0$ for $x \in \partial\Omega$. By similar arguments, $I \to I^*$, $R \to R^*$ in $C^1(\overline{\Omega})$ as $d_S \to 0$ where $I^*(x), R^*(x) \ge 0$ for $x \in \Omega$, which satisfy

$$\begin{cases} d_I \Delta I^* + \sigma E^* - \gamma(x) I^* = 0, & x \in \Omega, \\ d_R \Delta R^* + \gamma(x) I^* - \alpha R^* = 0, & x \in \Omega, \\ \frac{\partial I^*}{\partial n} = \frac{\partial R^*}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{4.2}$$

Now we show that $E^*(x) \not\equiv 0$ on Ω by contradiction argument. If $E^* = 0$, then we obtain by (4.2) that $I^* = R^* = 0$, which implies that $S \to \infty$ $a.e$ as $d_S \to 0$. Thus $\frac{\beta(x)S}{S+E+I+R} \to \beta(x)$ $a.e$ as $d_S \to 0$. Define

$$K = \|E\|_{L^\infty(\Omega)} + \|I\|_{L^\infty(\Omega)} + \|R\|_{L^\infty(\Omega)}, \quad \hat{E} = \frac{E}{K}, \quad \hat{I} = \frac{I}{K}, \quad \hat{R} = \frac{R}{K}.$$

Note that $\hat{E}, \hat{I}, \hat{R} > 0$ and $\|\hat{E}\|_{L^\infty(\Omega)} + \|\hat{I}\|_{L^\infty(\Omega)} + \|\hat{R}\|_{L^\infty(\Omega)} = 1$. Then as before, by a standard compactness argument for elliptic equations, after passing to a further subsequence if necessary, we have $\hat{E} \to \hat{E}^*, \hat{I} \to \hat{I}^*, \hat{R} \to \hat{R}^*$ in $C^1(\overline{\Omega})$ as $d_S \to 0$, where $\hat{E}^*(x), \hat{I}^*(x), \hat{R}^*(x) \ge 0$ for $x \in \Omega$ and

$$\|\hat{E}^*\|_{L^\infty(\Omega)} + \|\hat{I}^*\|_{L^\infty(\Omega)} + \|\hat{R}^*\|_{L^\infty(\Omega)} = 1 \tag{4.3}$$

with $\frac{\partial \hat{E}^*}{\partial n} = \frac{\partial \hat{I}^*}{\partial n} = \frac{\partial \hat{R}^*}{\partial n} = 0$ for $x \in \partial\Omega$. It follows from $\frac{\beta(x)S}{S+E+I+R} \to \beta(x)$ $a.e$ as $d_S \to 0$ that $\hat{E}^*$ is a weak solution of

$$d_E\Delta\hat{E}^* - \sigma\hat{E}^* + \beta(x)\hat{I}^* = 0 \quad \text{in } \Omega, \quad \frac{\partial\hat{E}^*}{\partial n}|_{\partial\Omega} = 0.$$

By elliptic regularity, we have $\hat{E}^* \in C^2(\overline{\Omega})$, which gives

$$\begin{cases} d_E\Delta\hat{E}^* - \sigma\hat{E}^* + \beta(x)\hat{I}^* = 0, & x \in \Omega, \\ d_I\Delta\hat{I}^* + \sigma\hat{E}^* - \gamma(x)\hat{I}^* = 0, & x \in \Omega, \\ d_R\Delta\hat{R}^* + \gamma(x)\hat{I}^* - \alpha\hat{R}^* = 0, & x \in \Omega, \\ \frac{\partial\hat{E}^*}{\partial n} = \frac{\partial\hat{I}^*}{\partial n} = \frac{\partial\hat{R}^*}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{4.4}$$

It follows from maximum principle together with (4.3) that $\hat{E}^*(x), \hat{I}^*(x), \hat{R}^*(x) > 0$, which implies that $R_0 = 1$. This contradiction yields $E^*(x) \not\equiv 0$. Therefore, again by maximum principle together with (4.2), we obtain $I^*, R^* > 0$.

Next we prove $|J^+| > 0$ by contradiction. If $|J^+| = 0$, then $S \to \infty$ $a.e$ as $d_S \to 0$ and thus $\frac{\beta(x)SI}{S+E+I+R} \to \beta(x)I^*$ $a.e$ as $d_S \to 0$. Therefore, E^* is a H^1 weak solution of

$$d_E\Delta E^* - \sigma E^* + \beta(x)I^* = 0 \quad \text{in } \Omega, \quad \frac{\partial E^*}{\partial n}|_{\partial\Omega} = 0.$$

By elliptic regularity, we have $E^* \in C^2(\overline{\Omega})$, which yields

$$\begin{cases} d_E\Delta E^* - \sigma E^* + \beta(x)I^* = 0, & x \in \Omega, \\ d_I\Delta I^* + \sigma E^* - \gamma(x)I^* = 0, & x \in \Omega, \\ d_R\Delta R^* + \gamma(x)I^* - \alpha R^* = 0, & x \in \Omega, \\ \frac{\partial E^*}{\partial n} = \frac{\partial I^*}{\partial n} = \frac{\partial R^*}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{4.5}$$

It follows from (4.5) together with $I^* > 0$, $R^* > 0$ that $E^* > 0$. Thus, $R_0 = 1$. This contradiction implies $|J^+| > 0$.

We prove part (iii) by contradiction. Now assume that $|J^-| = 0$. Denote $h(x) := \frac{\beta(x)SI}{S+E+I+R} - \alpha R$ and choose $\varphi \in C^1(\overline{\Omega})$ such that $\varphi \geq 0$ on Ω. Multiplying the first three equations in (4.1) by φ, adding them together and integrating on Ω, we have

$$-\int_\Omega \nabla\varphi \cdot \nabla(d_E E + d_I I + d_R R)dx + \int_\Omega \varphi h(x)dx = 0. \tag{4.6}$$

As $|J^-| = 0$, $M^*(x) = 0$ $a.e.$ in Ω. Thus, we obtain

$$\int_\Omega h(x)\varphi dx \to 0 \text{ as } d_S \to 0 \tag{4.7}$$

for any $\varphi \in C^1(\overline{\Omega})$, $\varphi \geq 0$ on Ω. Hence, (4.7) holds for any $\varphi \in C(\overline{\Omega})$ such that $\varphi \geq 0$ on Ω.

Let ϕ_0 be a positive eigenfunction of $\lambda_1(-d_R\Delta + \alpha(1 - \frac{\gamma}{\beta}))$, i.e.

$$-d_R\Delta\phi_0 + \alpha(1 - \frac{\gamma}{\beta})\phi_0 = \lambda_1\phi_0 \quad \text{in } \Omega, \quad \frac{\partial\phi_0}{\partial n}|_{\partial\Omega} = 0. \tag{4.8}$$

Since $d_R\Delta R+\gamma(x)I-\alpha R=0$ and $S, E, I, R>0$ on Ω, we have

$$-d_R\Delta R+\alpha(1-\frac{\gamma(x)}{\beta(x)})R>\frac{\gamma(x)h(x)}{\beta(x)}, \quad x\in\Omega. \tag{4.9}$$

Multiplying (4.9) by ϕ_0, integrating by parts over Ω and applying (4.8), we obtain

$$\lambda_1\int_\Omega \phi_0 R dx>\int_\Omega \frac{\gamma(x)h(x)}{\beta(x)}\phi_0 dx.$$

Let $d_S\to 0$, it follows from (4.7) that $\lambda_1\int_\Omega \phi_0 R^* dx\geq 0$. Since $\phi_0, R^*>0$ on Ω, we see that $\lambda_1\geq 0$. This contradiction yields (iii). □

4.2. *Proof of Theorem 4.2*

We first prove part (i). Denote for further purposes $M(x):=d_E E+d_I I+d_R R$. By (4.1), we have

$$\begin{aligned} N_0 &= \textstyle\int_\Omega(\widetilde{S}+\widetilde{E}+\widetilde{I}+\widetilde{R})dx \\ &= \textstyle\frac{\xi}{d_S}(\int_\Omega d_S(E+I+R)dx+\int_\Omega(1-M(x))dx). \end{aligned}$$

It follows from $S, E, I, R>0$ and $d_S S+d_E E+d_I I+d_R R=1$ that E, I, R are uniformly bounded with respect to d_S. Thus,

$$\int_\Omega d_S(E+I+R)dx\to 0 \ \text{ as } \ d_S\to 0.$$

In view of Theorem 4.1 (i) (ii),

$$\int_\Omega (1-M(x))dx\to\int_\Omega(1-M^*(x))dx>0 \ \text{ as } \ d_S\to 0.$$

Therefore,

$$\frac{\xi}{d_S}\to\frac{N_0}{\int_\Omega(1-M^*(x))dx} \ \text{ as } \ d_S\to 0. \tag{4.10}$$

Moreover, (4.1) yields $\widetilde{S}=\frac{\xi}{d_S}(1-M(x))$. By (4.10) together with Theorem 4.1 (i),

$$\widetilde{S}\to\widetilde{S}^*=\frac{N_0(1-M^*(x))}{\int_\Omega(1-M^*(x))dx}$$

in $C^1(\overline{\Omega})$ as $d_S\to 0$.

Next we prove part (ii). It follows from $d_S S+d_E E+d_I I+d_R R=1$, and $\widetilde{E}=\frac{\xi}{d_S}d_S E$, $\widetilde{I}=\frac{\xi}{d_S}d_S I$, $\widetilde{R}=\frac{\xi}{d_S}d_S R$ that

$$0 < \frac{\widetilde{E}}{d_S}, \frac{\widetilde{I}}{d_S}, \frac{\widetilde{R}}{d_S} < \frac{\xi}{d_S} \max\{\frac{1}{d_E}, \frac{1}{d_I}, \frac{1}{d_R}\}.$$

Hence, (i) implies

$$\limsup_{d_S\to 0} \sup_\Omega \frac{\widetilde{E}}{d_S}, \limsup_{d_S\to 0} \sup_\Omega \frac{\widetilde{I}}{d_S}, \limsup_{d_S\to 0} \sup_\Omega \frac{\widetilde{R}}{d_S} \le \frac{N_0}{\int_\Omega (1-M^*(x))dx} \max\{\frac{1}{d_E}, \frac{1}{d_I}, \frac{1}{d_R}\}. \quad (4.11)$$

Now we prove

$$\min\{\inf_\Omega \widetilde{E}, \inf_\Omega \widetilde{I}, \inf_\Omega \widetilde{R}\}/d_S \nrightarrow 0, \quad \text{as } d_S \to 0 \quad (4.12)$$

by contradiction. Assume that $\min\{\inf_\Omega \widetilde{E}, \inf_\Omega \widetilde{I}, \inf_\Omega \widetilde{R}\} = o(d_S)$. By Lemma 2.3 in [7] and (1.4), there exists a positive constant δ such that

$$\begin{aligned}
&\inf_\Omega \widetilde{E} \ge \delta \int_\Omega \frac{\beta(x)\widetilde{S}\widetilde{I}}{\widetilde{S}+\widetilde{I}+\widetilde{E}+\widetilde{R}} dx = \delta\sigma \int_\Omega \widetilde{E}dx,\\
&\inf_\Omega \widetilde{I} \ge \delta\sigma \int_\Omega \widetilde{E}dx,\\
&\inf_\Omega \widetilde{R} \ge \delta\alpha \int_\Omega \widetilde{R}dx = \delta\sigma \int_\Omega \widetilde{E}dx.
\end{aligned}$$

Hence $\int_\Omega \widetilde{E}dx = o(d_S)$. In view of

$$\alpha \int_\Omega \widetilde{R}dx = \int_\Omega \gamma(x)\widetilde{I}dx = \sigma \int_\Omega \widetilde{E}dx,$$

we obtain $\int_\Omega \widetilde{I}dx, \int_\Omega \widetilde{R}dx = o(d_S)$, which implies

$$\int_\Omega \frac{d_E\widetilde{E} + d_I\widetilde{I} + d_R\widetilde{R}}{d_S} dx \to 0 \text{ as } d_S \to 0. \quad (4.13)$$

Note that

$$N_0 = \int_\Omega \frac{\xi}{d_S} dx - \int_\Omega \frac{d_E\widetilde{E} + d_I\widetilde{I} + d_R\widetilde{R}}{d_S} dx + \int_\Omega (\widetilde{E} + \widetilde{I} + \widetilde{R}) dx.$$

Let $d_S \to 0$, it follows from (i), (4.11) and (4.13) that

$$N_0 = \frac{N_0|\Omega|}{\int_\Omega (1-M^*(x))dx},$$

which yields that $|J^-| = 0$. This contradiction implies (ii). □

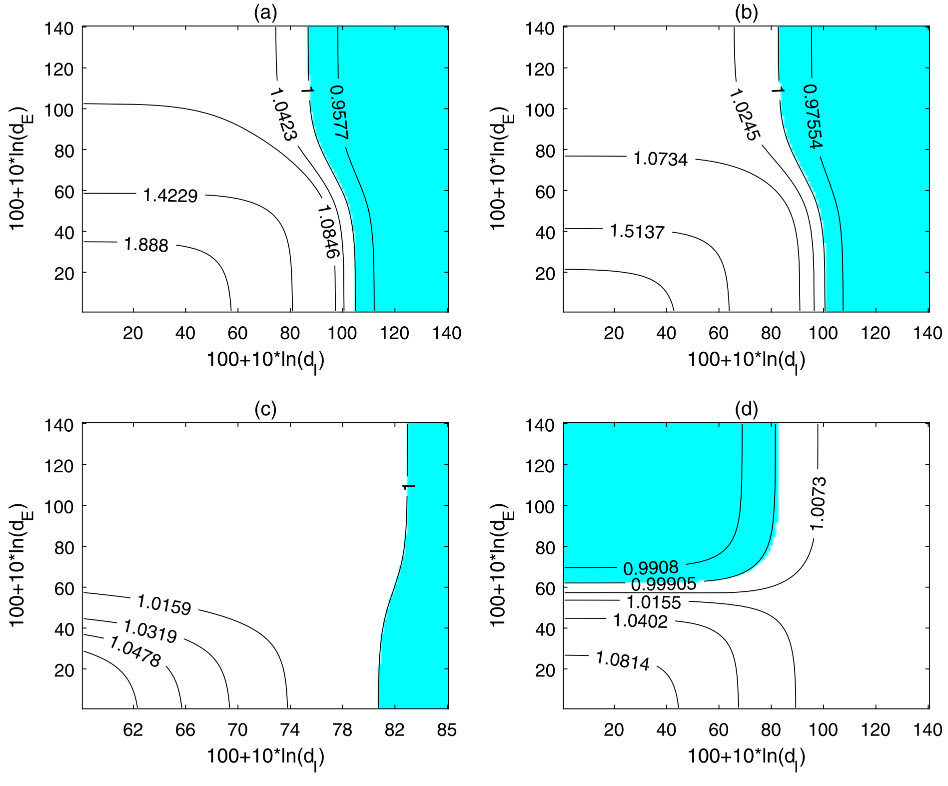

Fig. 1. $\Omega = (0, 1.4)$, $\sigma = 0.1$. The dyed part means $R_0 < 1$. Contour of $R_0(d_E, d_I)$ in phase (d_E, d_I) under: (a) $\beta(x) = 2.44\cos(\pi x/2.8)$, $\gamma(x) = x + 1$; (b) $\beta(x) = 0.916(x + 1)$, $\gamma(x) = 1 + \cos(\pi x/2.8)$; (c) $\beta(x) = x + 1$, $\gamma(x) = 0.74(x + 1 + \sin(\pi x/2.8))$; (d) $\beta(x) = 1/(x + 1) + \cos(\pi x/2.8)$, $\gamma = 2/(x + 1)$.

5. Numerical simulation and discussion

In this section, we use numerical results to demonstrate our theoretical findings and explore the effect of exposed and recovered individuals' movement on disease persistence. We refer to [38] for more extensive numerical results.

We first illustrate by numerical examples that the movement of exposed individuals makes the monotonicity of the basic reproduction number R_0 more complex. Fig. 1(a, b) shows that if one of $\beta(x)$ and $\gamma(x)$ is increasing and the other is decreasing, then R_0 is monotone decreasing in d_E, d_I, which is in agreement with the results in Theorem 1.3. However, if both $\beta(x)$ and $\gamma(x)$ are increasing or decreasing, we can see from the curve $R_0 = 1$ in Fig. 1(c) that R_0 is no longer decreasing in d_E which agrees with the result in Theorem 3.5. Interestingly, Fig. 1(d) shows when $d_E > e^{-2}$, R_0 is an increasing function of d_I. Fig. 1(d) is probably more close to the real epidemic situation and potentially explains the relationship between the basic reproduction number and the movement of infected individuals, as the movement of exposed individuals may not be restricted during the epidemic. The faster the infected individuals move, the more infections happen.

Fig. 2 represents the contour of $R_0(d_E, d_I)$ in phase (d_E, d_I) for non-monotone β, γ, where the dyed part means $R_0 < 1$. It can be observed from Fig. 2 that when d_E is small, R_0 is a

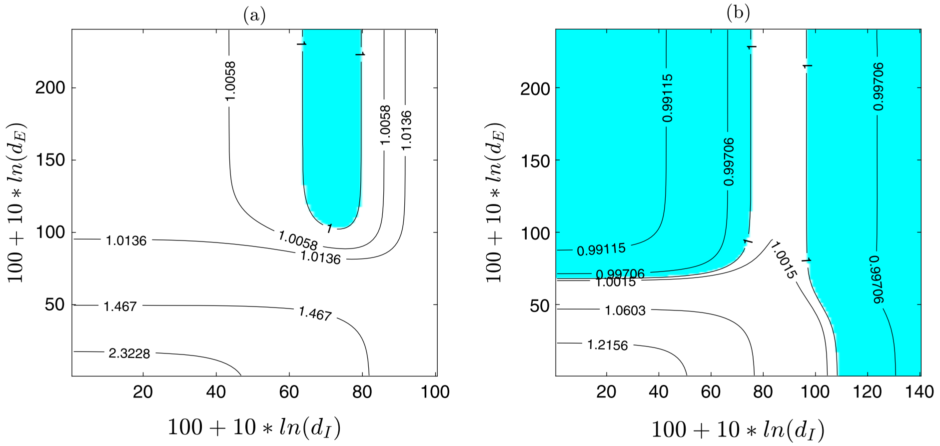

Fig. 2. $\Omega = (0, 1.4)$, $\sigma = 0.1$. The dyed part means $R_0 < 1$. Contour of $R_0(d_E, d_I)$ in phase (d_E, d_I) under: (a) $\beta(x) = 2.58(1 - \sin(1.05\pi x/1.4))(x+1)$, $\gamma(x) = x + 1$; (b) $\beta(x) = 3.74x(1.6 - x)$, $\gamma(x) = x + 1 + 0.1\sin(\pi x/1.4)$.

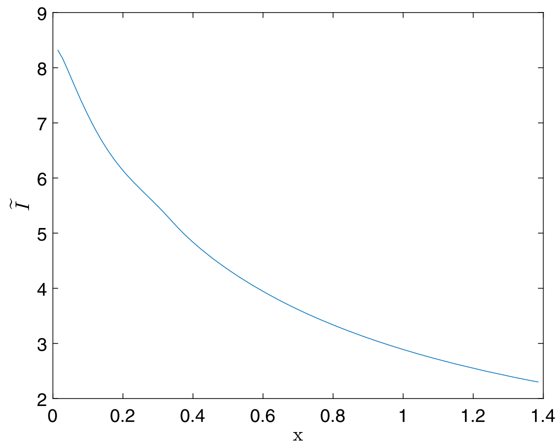

Fig. 3. An endemic equilibrium $\widetilde{I}(x)$ of system (1.2). $\Omega = (0, 1.4)$, $d_S \approx 0, d_E = d_I = e^{-9}, d_R = 1, \alpha = \sigma = 0.1$, $\beta(x) = (1 + (x - 0.1)(x - 0.2))\gamma(x)$, $\gamma(x) = 2x + 1$, $S(x, 0) = 10000$, $E(x, 0) = I(x, 0) = 1$, $R(x, 0) = 0$.

monotone decreasing function of d_I. However, as d_E increases, R_0 loses the monotonicity with respect to d_I and reveals complicated dependences upon d_E and d_I.

Next we give a numerical example to show that the movement of recovered individuals may increase the number of infected individuals and enhance the endemic. For the parameters given in Fig. 3, we can calculate $R_0 \approx 2.2846 > 1$, $\lambda_1(-d_R\Delta + \alpha(1 - \frac{\gamma}{\beta})) \approx 0.0251 > 0$. Fig. 3 suggests that $\widetilde{I}$ does not converge to zero as $d_S \to 0$, which is in contrast with conclusions of Theorem 1.4.

Our theoretical and numerical results suggest that the travel of exposed individuals could have an important impact on the persistence of disease and the movement of recovered individuals may enhance the endemic. Accordingly, a good understanding of the behaviors of the exposed and recovered individuals could also be important in designing effective disease control measures.

Acknowledgment

This research was done during the visit of PS to the Department of Mathematics at Ohio State University. He wishes to thank the department for the hospitality.

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