Chance Constraint based Design of Controllers for Linear Uncertain Systems

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***Abstract*— This paper considers the problem of state-to-state transition with state and control constraints, for a linear system with model parameter uncertainties. Polynomial chaos is used to transform the stochastic model to a deterministic surrogate model. This surrogate model is used to pose a chance constrained optimal control problem where the state constraints and the residual energy cost are represented in terms of the mean and variance of the stochastic states. The resulting convex optimization is illustrated on the problem of rest-to-rest maneuver of the benchmark floating oscillator.**

I. INTRODUCTION

Control of flexible structures under assumptions of uncertainties in the model has been a topic of research interest for a while and a number of researchers have made contributions to the field ([1], [2], [3]). One of the approaches of dealing with model parameter uncertainty has been to reduce the sensitivity of cost function in the proximity of the nominal model by forcing the local sensitivity to zero [1], [2]. The other popular approach has been to design controllers so as to take care of worst case scenarios [4]. But this method can often lead to very conservative results which may not be practical.

A way to mitigate the issues of both these methods is to use the information available in the probability distributions of the uncertain variables to address robustness. In doing so, it often leads to posing optimization problems (i.e. optimal control problems) with probabilistic or chance constraints ([5], [6]). These chance constraints can in fact be written as deterministic constraints to solve the optimization problem. For example, Calafiore and El Ghaoui in the article [7] investigate linear chance constraints which are robust to distributions of the random variables. The article allows one to write any linear chance constraint as a deterministic constraint based on the available random variable information.

Mesbah et al. in [8] presents a generic framework for implementing linear chance inequality constraints in Model Predictive Control (MPC) for non-linear systems with parametric uncertainties. Polynomial Chaos (PC) is used to determine the first 2 moments of the stochastic states which are then used to enforce these chance constraints. However, the non-linear inequality constraints remain deterministic and are limited to the nominal trajectories of the states. The cost in the MPC framework is also considered to be non-probabilistic.

In this paper, the focus is on linear systems with linear constraints and a residual energy cost which has a quadratic form, making the entire problem convex in a deterministic framework. In a probabilistic framework, most often, the worst-case state trajectories have significantly low associated probabilities of realization [9]. It is also common knowledge that one needs to trade-off performance for robustness or vice versa when dealing with uncertain systems. Consequently, considering a probabilistic cost function permits a simple approach for the trade-off analysis. The chance constraint corresponding to a linear constraint for an uncertain system results in a cone constraint, but does not permit posing a quadratic cost using the same framework. This prompts using a l_∞ or a l_1 norm approximation of the l_2 norm which permits using the chance constraint to pose a convex optimization problem for the design of controllers.

The structure of the paper is as follows: Section I introduces the problem being solved, provides background on the type of existing literature and motivates the need for chance constraints in dynamic systems. Section II and III provides a brief review of control framework used in the work and presents a simple example problem that runs throughout the document respectively. Then, Section IV gives a short overview of Polynomial Chaos and shows its application in estimating moments of stochastic states. Using results in this section, solution to the example problem for chance state constraints and chance energy constraints are presented in Sections V and VI. Finally, the paper is completed with concluding remarks in Section VII.

II. CONTROL OBJECTIVE

This section in the paper is used to present the control problem clearly and elaborate the framework on which the work has been developed.
The state space model for a lumped parameter spring-mass-dashpot system is:,

$$\underbrace{\begin{Bmatrix} \dot{\boldsymbol{z}} \\ \ddot{\boldsymbol{z}} \end{Bmatrix}}_{\dot{\boldsymbol{Z}}} = \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -M^{-1}K & -M^{-1}C \end{bmatrix}}_{A'} \underbrace{\begin{Bmatrix} \boldsymbol{z} \\ \dot{\boldsymbol{z}} \end{Bmatrix}}_{\boldsymbol{Z}} + \underbrace{\begin{bmatrix} \boldsymbol{0} \\ M^{-1}D \end{bmatrix}}_{B'} \boldsymbol{u}. \quad (1)$$

where M, C, K and D are mass, damping, stiffness and control influence matrices respectively.

Assuming $\boldsymbol{z} \in \mathbb{R}^n$ (i.e. $\boldsymbol{z} = [z_1, \ldots, z_n]^T$), $\boldsymbol{Z} \in \mathbb{R}^{2n}$. The discrete time representation of equation (1) is:

$$\boldsymbol{Z}(\tilde{k}+1) = A\boldsymbol{Z}(\tilde{k}) + B\boldsymbol{u}(\tilde{k}) \quad (2)$$

which will be used for controller design, where $\tilde{k}$ represents the $\tilde{k}^{th}$ time step. The control objective is to determine $\boldsymbol{u}(\tilde{k})$ which can be used to drive the system from an initial state ($\boldsymbol{Z}(0)$ at time $t = 0$) to a final desired state ($\boldsymbol{Z_d}(T_f)$ at time $t = T_f$) with constraints on the states and control input during the transition. However, since the system is considered to have parametric uncertainties, it is impossible to find an open loop control $\boldsymbol{u}(\tilde{k})$ which assures all realizations of the dynamic system reach $\boldsymbol{Z_d}(T_f)$. To make all the realizations reach as close to the desired value as possible, a quantity of measure is necessary which characterizes this closeness. A popular choice for this quantity in the literature has been the residual energy (and thus, the same is chosen for this work). The residual energy is defined as

$$E_r = \frac{1}{2}\dot{\boldsymbol{x}}^T M \dot{\boldsymbol{x}} + \frac{1}{2}\boldsymbol{x}^T K \boldsymbol{x} \tag{3}$$

where the residual states are defined by

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}(T_f) \\ \dot{\boldsymbol{x}}(T_f) \end{bmatrix} = \boldsymbol{Z}(T_f) - \boldsymbol{Z_d}(T_f). \tag{4}$$

$\boldsymbol{Z}(T_f)$ is the terminal value of the states of any realization of the model in equation (2).

Equation (2) can also be written as a linear function of only the control inputs and the initial conditions

$$\boldsymbol{Z}(\tilde{k}+1) = A^{\tilde{k}}\boldsymbol{Z}(0) + \sum_{i=0}^{\tilde{k}} A^{\tilde{k}-i} B \boldsymbol{u}(i). \tag{5}$$

This linear mapping can then be used to write state constraints at any instant in time (and if need be at all instants in time i.e. $\forall \tilde{k}$) with the help of an appropriate output matrix. Finally, the optimal control problem can be posed as:

$$\begin{array}{lll} \text{minimize}_{\boldsymbol{u},f} & f & \\ \text{subject to} & E_r^{(i)} \leqslant f & \\ & u_{lb} \leqslant u(\tilde{k}) \leqslant u_{ub} & \forall k \\ & \text{State Constraints}^{(i)} & \forall k \\ & for \quad i = 1, 2, \ldots, p. & \end{array}$$

where p represents the number of different realizations of the uncertain system. If the state constraints are linear, then the optimization can be shown to be convex. Barring the residual energy constraint, all other constraints as well as the cost function are linear. Moreover, the residual energy constraint is a quadratic one which can be written as a LMI. Therefore, the optimization problem is convex.

In fact, the same problem can be posed as a Linear Programming (LP) problem if the residual energy constraint is slightly modified and written as a linear function of the states. Two such formulations (the l_1 and the l_∞ formulation) have been shown in [10] and have been summarised here.

If a new set of states are defined by

$$\boldsymbol{Y} = \begin{bmatrix} \sqrt{K} & \boldsymbol{0} \\ \boldsymbol{0} & \sqrt{M} \end{bmatrix} \boldsymbol{X} \tag{6}$$

the l_1 formulation of the residual energy can be written as: $||\boldsymbol{Y}||_1 \leq f$, which is equivalent to 2^{2n} linear constraints

$$\pm y_1 \pm y_2 \ldots \pm y_{2n} \leq f \tag{7}$$

where $\boldsymbol{Y} = [y_1, \ldots, y_{2n}]^T$. Similarly the l_∞ formulation ($||\boldsymbol{Y}||_\infty \leq f$) can be written as $4n$ linear constraints

$$-f \leq y_i \leq f; \quad \forall i = 1 \ \ldots \ 2n. \tag{8}$$

Other linear approximations of the residual energy function can be found in [11]. Note that in case the stiffness matrix K is positive semi-definite, a pseudo energy function is added to the cost function to make it positive definite.

III. EXAMPLE PROBLEM

This section presents the 2 mass floating oscillator problem (Figure. 1) which runs throughout the document. All control problems have been solved on this example for illustration. The M, C, K and D matrices are given by

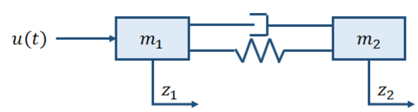


Fig. 1. 2 Mass Spring Damper System

$$M = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}; \ C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \ K = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \tag{9}$$

and $D = [1, 0]^T$.

It is assumed that k is uncertain. The control objective in this example is to find the control trajectory $u(\tilde{k})$ which can move the system from an initial rest state $\boldsymbol{Z}(0) = [0, 0, 0, 0]^T$ to a final desired rest state $\boldsymbol{Z_d}(T_f) = [1, 1, 0, 0]^T$ under control constraints: $|u(\tilde{k})| \leq 1, \quad (\forall \, \tilde{k})$ and state constraints

$$|z_1(\tilde{k}) - z_2(\tilde{k})| \leq 0.15, \quad \forall \, \tilde{k}. \tag{10}$$

All simulations in this work were done with a $T_f = 15$. The sampling time for discretization used was $T_s = 0.1$. Therefore the total number of steps were $N_t = T_f/T_s + 1 = 151$.

IV. POLYNOMIAL CHAOS

This section presents a tool which allows one to characterize the evolution of the uncertainty for stochastic systems by expressing the stochastic states as a polynomial function of the uncertain parameters of the model.

First investigated by Norbert Wiener in [12], his work approximated states of a Gaussian process with an infinite series expansion of orthogonal Hermite polynomials. Then, it was shown that this expansion in terms of the Hermite polynomials converged for any process characterized by a finite variance [13]. These results were used to solve stochastic differential equations by Ghanem and Spanos in their book [14]. The infinite series expansion was truncated to a finite number of terms following which a Galerkin projection was done to formulate a set of deterministic equations. The solution to these deterministic equations yielded the coefficients of the truncated series expansion. However, PC was generalised by Xiu et al. [15] where they showed that any stochastic process could be approximated by an infinite series expansion. The basis functions however, had to be selected to be appropriate orthogonal polynomials (given by the Wiener-Askey scheme) for exponential convergence. A formulation of this concept (generalised PC (gPC)) has been illustrated on the floating oscillator problem.

A. Methodology

Let a stochastic linear dynamical system be expressed in the form

$$\dot{\boldsymbol{x}}(t,\boldsymbol{\xi}) = A(\boldsymbol{\xi})\boldsymbol{x}(t) + B(\boldsymbol{\xi})\boldsymbol{u}(t) \text{ and } \boldsymbol{x}(t_0,\boldsymbol{\xi}) = \boldsymbol{x_0} \quad (11)$$

where, $\boldsymbol{x} \in \mathbb{R}^{\tilde{n}}$ is the state vector, $\boldsymbol{\xi} \in \mathbb{R}^m$, the vector of random variables, and $\boldsymbol{u}(t)$ the control input.
From gPC, the states can be expressed as

$$\boldsymbol{x}(t,\boldsymbol{\xi}) = \sum_{i=0}^{\infty} \boldsymbol{x_{\bar{i}}}(t)\Psi_i(\boldsymbol{\xi}) \quad (12)$$

where, $\Psi_i(\boldsymbol{\xi})$ is a complete set of multivariate orthogonal (w.r.t the pdf of $\boldsymbol{\xi}$) polynomials and $\boldsymbol{x_{\bar{i}}} \in \mathbb{R}^{\tilde{n}}$ is the time varying coefficient vector of $\Psi_i(\boldsymbol{\xi})$. The selection of the set of orthogonal polynomials for popular distributions is given by the Wiener-Askey scheme [15].
As an approximation, the expansion is usually truncated to a finite number of terms (depending on the desired accuracy) [15]. Hence, equation (12) is rewritten as

$$\boldsymbol{x}(t,\boldsymbol{\xi}) \approx \sum_{i=0}^{N} \boldsymbol{x_{\bar{i}}}(t)\Psi_i(\boldsymbol{\xi}) \quad (13)$$

The objective is to evaluate the unknown vectors $\boldsymbol{x_{\bar{i}}}(t)$ over time. Equation (13) is substituted in equation (11) to get

$$\sum_{i=0}^{N} \dot{\boldsymbol{x}}_{\bar{\boldsymbol{i}}}(t)\Psi_i(\boldsymbol{\xi}) = A(\boldsymbol{\xi})\left[\sum_{i=0}^{N} \boldsymbol{x_{\bar{i}}}(t)\Psi_i(\boldsymbol{\xi})\right] + B(\boldsymbol{\xi})\boldsymbol{u}(t) \quad (14)$$

Using the Galerkin projection approach after truncating the PC expansion to $N = 5$ and recognising that for the example $\tilde{n} = 2n = 4$, $(N+1)\tilde{n} = 24$ deterministic equations are formed which are used to evaluate the PC coefficients. Therefore, if the states are expanded as

$$\boldsymbol{Z} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{bmatrix} = \begin{bmatrix} Z_{10} \\ Z_{20} \\ Z_{30} \\ Z_{40} \end{bmatrix}\Psi_0 + \begin{bmatrix} Z_{11} \\ Z_{21} \\ Z_{31} \\ Z_{41} \end{bmatrix}\Psi_1 + \ldots + \begin{bmatrix} Z_{15} \\ Z_{25} \\ Z_{35} \\ Z_{45} \end{bmatrix}\Psi_5 \quad (15)$$

where $\boldsymbol{Z} = [z_1, z_2, \dot{z_1}, \dot{z_2}]^T$ then the deterministic equations formed are given by

$$\begin{bmatrix} \dot{Z}_{10}\langle\Psi_0,\Psi_0\rangle \\ \vdots \\ \dot{Z}_{45}\langle\Psi_5,\Psi_5\rangle \end{bmatrix} = G\begin{bmatrix} Z_{10} \\ \vdots \\ Z_{45} \end{bmatrix} + Hu \quad (16)$$

Since, the control problem has been posed in the discrete domain, equation (16) is discretized as

$$\boldsymbol{Z_C}(\tilde{k}+1) = \bar{A}\boldsymbol{Z_C}(\tilde{k}) + \bar{B}u(\tilde{k}) \quad (17)$$

where $\boldsymbol{Z_C} = [Z_{10}, \ldots, Z_{15}, Z_{20}, \ldots, Z_{25}, Z_{30}, \ldots, Z_{45}]^T$.

V. CONTROL PROBLEM WITH NON-PROBABILISTIC ENERGY BUT PROBABILISTIC STATE CONSTRAINTS

As mentioned previously, [7] provides an approach to rewrite linear probabilistic inequalities as non-probabilistic inequalties. In their work, they prove that if a and b are random variables with known means and variances, then the constraint:

$$\text{Prob}\{a^T x + b \leq 0\} \geq 1 - \epsilon \quad (18)$$

is equivalent to the constraint

$$\sqrt{\frac{1-\epsilon}{\epsilon}}\{var[a^T x + b]\}^{1/2} + E[a^T x + b] \leq 0 \quad (19)$$

where ϵ represents the risk level i.e. the probability with which the constraint is permitted to be violated. It should be noted that the constraint is conservative since it subsumes all distributions with the same mean and variance. This means that the equality sign of equation (18) is going to be active for only a particular distribution (which is unknown).
Therefore, this simple formulation now allows us to enforce linear probabilistic constraints as long as the mean and the variance of the random variables are known. To illustrate that different distributions yield different results, 3 separate distributions with same mean and variance for k in the example problem are considered.
The first distribution is a uniform one and is defined in terms of the r.v. $\xi_1 \in U[-1,1]$. Therefore, we have $k = 1 + 0.2324\xi_1$.
The second distribution is defined via a beta distributed r.v. $\xi_2 \in [-1,1]$ with parameters $a = 1$ and $b = 1$ making $k = 1 + 0.3\xi_2$.
The final distribution is chosen from the article [16]. The r.v. $\xi_3 \in [-1,1]$ and has a pdf given by

$$p(\xi_3) = 1 - W\sum_{i=0}^{1} A_i|\xi_3|^{2-i+1} \quad (20)$$

where $W = -(3)!$; $A_i = \frac{(-1)^{i\,1}C_i}{2-i+1}$; and ${}^1C_i = \frac{1!}{i!(1-i)!}$. k is written in terms of ξ_3 as $k = 1 + 0.3674\xi_3$.
The mean and the variance of all 3 different k descriptions are 1 and 0.018 respectively. A PC expansion for each of these 3 cases are done to determine the mean and the variance of the states at each time instant. Then these means and variances are used to implement a probabilistic state constraint where the probabilistic state constraint is written in the form of equation (19). In this section, the optimal control problem in section II is chosen to be solved with a l_∞ residual energy formulation for the example problem. A linear version of the residual energy need not be chosen though and any of the other formulations are acceptable. The control constraints are the same as section III. The relative displacement state constraint is however modified to the probabilistic constraints

$$\text{Prob}\{Z_1(\tilde{k}) - Z_2(\tilde{k}) - 0.15 \leq 0\} \geq 1 - \epsilon \quad (21)$$

$$\text{Prob}\{-Z_1(\tilde{k}) + Z_2(\tilde{k}) - 0.15 \leq 0\} \geq 1 - \epsilon \quad (22)$$

$\forall\, \tilde{k}$. Equation (21) is equivalent to

$$k_e\{var[Z_1 - Z_2 - 0.15]\}^{1/2} + E[Z_1 - Z_2 - 0.15] \leq 0. \quad (23)$$

where $k_e = \sqrt{\frac{1-\epsilon}{\epsilon}}$. If a vector is defined as $C_{con} = [1, -1, 0, 0]^T$, then the constraint can be simplified to

$$k_e\{var[C_{con}^T \boldsymbol{Z}(\tilde{k}) - 0.15]\}^{1/2} + E[C_{con}^T \boldsymbol{Z}(\tilde{k}) - 0.15] \leq 0. \quad (24)$$

$\boldsymbol{Z}(\tilde{k})$ can be represented as a linear function of the control input as

$$\boldsymbol{Z}(\tilde{k}) = \Psi A_{eq}(\tilde{k}) U \quad (25)$$

where

$$\underbrace{\Psi}_{(\tilde{n}\times\tilde{n}(N+1))} = \begin{bmatrix} \Psi_0 & \ldots & \Psi_N & & & \mathbf{0} \\ & & & \ddots & & \\ \mathbf{0} & & & \Psi_0 & \ldots & \Psi_N \end{bmatrix}; \quad (26)$$

$$\underbrace{A_{eq}(\tilde{k})}_{(\tilde{n}(N+1)\times N_t)} = \begin{bmatrix} \bar{A}^{\tilde{k}-1}\bar{B} & \bar{A}^{\tilde{k}-2}\bar{B} & \ldots & \bar{B} & \mathbf{0}^T \end{bmatrix} \quad (27)$$

and $U = [u(0), \ldots, u(\tilde{k}), \ldots, u(150)]^T$. One must be aware that $\boldsymbol{Z_C}$ in equation (17) is basically $\boldsymbol{Z_C}(\tilde{k}) = A_{eq}(\tilde{k})U$.
It should be noted that Ψ is different for the 3 distinct distributions. The basis functions (Ψ_0 through Ψ_N) for the uniform distribution are Legendre polynomials. The basis functions for the beta distribution are Jacobi polynomials and the basis functions for the final distribution are generated using the Gram – Schmidt orthogonalization. Therefore, different values of $\bar{A}$, $\bar{B}$, A_{eq} and Ψ are calculated for each distribution. However, the order of PC is chosen to be $N = 5$ to be the same for all the 3 cases.
Now, using equation (25) we get

$$E[C_{con}^T \boldsymbol{Z}(\tilde{k}) - 0.15] = C_{con}^T E[\Psi A_{eq}(\tilde{k}) U] - 0.15. \quad (28)$$

Since, the only random variable is Ψ, the equation reduces to: $E[C_{con}^T \boldsymbol{Z}(\tilde{k}) - 0.15] = C_{con}^T E[\Psi] A_{eq}(\tilde{k}) U - 0.15$.
Similarly, var$[C_{con}^T \boldsymbol{Z}(\tilde{k}) - 0.15]$ can be reduced to

$$U^T A_{eq}^T \underbrace{(E[\Psi^T C_{con} C_{con}^T \Psi] - E[\Psi^T] C_{con} C_{con}^T E[\Psi])}_{S} A_{eq} U \quad (29)$$

$\forall\, \tilde{k}$. Defining a new matrix S (S is real symmetric positive semidefinite) in equation (29), we can derive

$$\text{var}[C_{con}^T \boldsymbol{Z}(\tilde{k}) - 0.15]^{1/2} = ||\Lambda^{1/2} V^T A_{eq} U||_2 \quad (30)$$

where an SVD decomposition of S is done as: $S = V \Lambda V^T$. Once again, the matrices comprising expected values ($E[\Psi]$ and S) are distribution dependent. Finally, the constraints described in equation (21) and (22) can be summarised by the convex constraints

$$\begin{aligned} &k_e ||\Lambda^{1/2} V^T A_{eq} U||_2 + C_{con}^T E[\Psi] A_{eq}(\tilde{k}) U - 0.15 \leq 0 \\ &k_e ||\Lambda^{1/2} V^T A_{eq} U||_2 - C_{con}^T E[\Psi] A_{eq}(\tilde{k}) U - 0.15 \leq 0. \end{aligned} \quad (31)$$

Equation (31) is used to enforce the state constraints when solving for U, in the example problem.

TABLE I
COMPARISON ACROSS DISTRIBUTIONS FROM 100000 MC SIMULATIONS

	Uniform		Beta		Custom	
ϵ	f	Max. V	f	Max. V	f	Max. V
0.2	0.0147	3.46	0.0209	5.17	0.0276	5.66
0.5	0.0105	22.32	0.0161	20.37	0.0229	19.01
0.8	0.0090	51.39	0.0143	47.75	0.0210	45.12

Results

Figure 2(a) shows a plot with MC realizations of the relative displacement (for $\epsilon = 0.5$, i.e. for a 50% constraint violation allowance) for a beta distribution. It can be seen that the state constraints are in fact violated. Figure. 2(b) shows the percentage of times these violations take place from 100000 simulations.

Although a 50% violation was allowed, a maximum violation of only 20.37% is observed. This is not an anomaly since we must remember that the probabilistic constraint is a conservative one and that the violations are a function of the distribution. Table I presents different maximum percentage violations (Max. V) that were observed for the 3 distributions.

Another interesting observation about the results can be made from the optimal value of the cost (i.e. f). For an l_∞ E_r, in the deterministic case (where state constraints are non-probabilistic) the cost f is seen to be 0.0248; while the probabilistic case with a beta distribution is seen to have a cost of 0.0161 (Figure 2(c)).
A lower cost for the probabilistic state constraints is expected since a probabilistic constraint is not a hard one. Therefore, if certain violations are allowed, the final residual energy cost is bound to improve. This trend of decreasing optimal cost with increased probability constraint violations can be seen across all distributions (Table I).

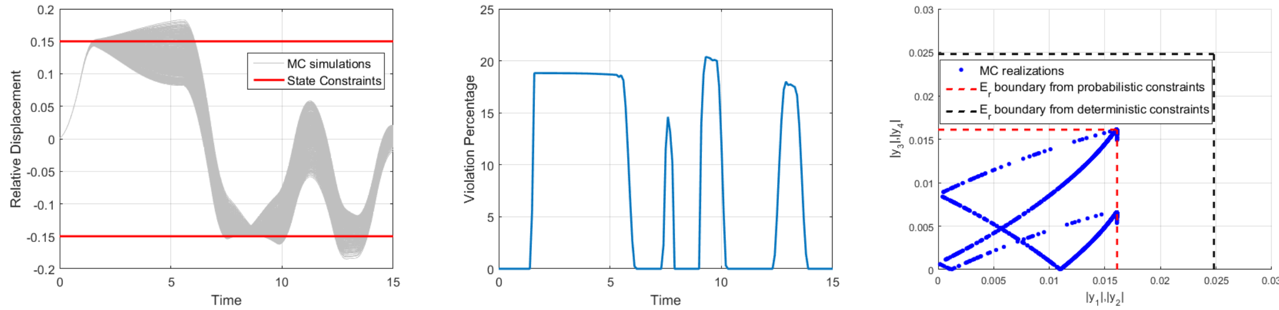
VI. CONTROL PROBLEM WITH PROBABILISTIC ENERGY AND PROBABILISTIC STATE CONSTRAINTS

An approach similar to the probabilistic state constraints is used to enforce the probabilistic terminal residual energy constraint. Since, the formulation only allows for linear constraints, only the linear versions of the residual energy (l_∞ and l_1 formulations) can be incorporated.
Therefore, in this section, the optimal control problem in section II is chosen to be solved with probabilistic state constraints (same as equations (21) and (22)) and the probabilistic l_∞ residual energy constraints

$$\text{Prob}\{\pm y_i - f \leq 0\} \geq 1 - \epsilon \text{ for } i = 1, \ldots, 4 \quad (32)$$

where y_i are defined through equation (6). It should be noted that equation (32) represents a total of 8 constraints for the fourth order system. The control constraints are chosen to be the same as before, given in section III.
y_i has been explicitly expanded for the example problem in the following equations

$$y_1 = K_1 C_k \boldsymbol{X}; \quad y_2 = K_2 C_k \boldsymbol{X}; \quad (33)$$



(a) (b) (c)

Fig. 2. (a) Relative displacement vs Time (b) Percentage Violation vs Time (c) Residual Energy Plot

$$y_3 = M_1 C_m \boldsymbol{X}; \quad y_4 = M_2 C_m \boldsymbol{X} \tag{34}$$

where $\boldsymbol{X} = \boldsymbol{Z}(T_f) - \boldsymbol{Z_d}(T_f)$;

$$C_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}; \; C_m = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \tag{35}$$

$$\sqrt{K} = [K_1^T, K_2^T]^T; \; \sqrt{M} = [M_1^T, M_2^T]^T. \tag{36}$$

One must remember that K_1 and K_2 are still random row vectors since K is a random matrix. The development of just one of the constraints is shown. The other constraints can be derived in an identical fashion. The constraint shown is

$$\text{Prob}\{y_1 - f \leq 0\} \geq 1 - \epsilon. \tag{37}$$

Similar to the previous section, equation (37) is equivalent to

$$\sqrt{\frac{1-\epsilon}{\epsilon}}\{var[y_1 - f]\}^{1/2} + E[y_1 - f] \leq 0. \tag{38}$$

$$\text{Now, } E[y_1 - f] = E[K_1 C_k \boldsymbol{Z}(T_f) - K_1 C_k \boldsymbol{Z_d}(T_f) - f]. \tag{39}$$

But on substituting $\boldsymbol{Z}(T_f) = \Psi A_{eq}(\tilde{k}_f)U$ (where $\tilde{k}_f$ is the final time iteration number), equation (39) can be written as

$$E[y_1 - f] = E[K_1 C_k \Psi] A_{eq}(\tilde{k}_f)U - E[K_1] C_k \boldsymbol{Z_d}(T_f) - f. \tag{40}$$

Moreover, $var[y_1 - f] = var[y_1]$ since f is not a random variable. After substituting y_1 and simplifying the equation we get

$$var[y_1] = var[K_1 C_k \boldsymbol{Z}(T_f) - K_1 C_k \boldsymbol{Z_d}(T_f)] \tag{41}$$

$$\begin{aligned} &= E[(K_1 C_k \boldsymbol{Z}(T_f) - K_1 C_k \boldsymbol{Z_d}(T_f))(K_1 C_k \boldsymbol{Z}(T_f) - \\ &K_1 C_k \boldsymbol{Z_d}(T_f))^T] - E[K_1 C_k \boldsymbol{Z}(T_f) - \\ &K_1 C_k \boldsymbol{Z_d}(T_f)] E[K_1 C_k \boldsymbol{Z}(T_f) - K_1 C_k \boldsymbol{Z_d}(T_f)]^T. \end{aligned} \tag{42}$$

Equation (42) can be simplified to $var[y_1] = U^T P U + 2QU + R$ where

$$\begin{aligned} P = A_{eq}(\tilde{k}_f)^T (E[\Psi^T C_k^T K_1^T K_1 C_k \Psi] - \\ E[\Psi^T C_k^T K_1^T] E[K_1 C_k \Psi]) A_{eq}(\tilde{k}_f); \end{aligned} \tag{43}$$

$$Q = \boldsymbol{Z_d}^T C_k^T (E[K_1^T] E[K_1 C_k \Psi] - E[K_1^T K_1 C_k \Psi]) A_{eq}(\tilde{k}_f) \tag{44}$$

and $R = \boldsymbol{Z_d}^T C_k (E[K_1^T K_1] - E[K_1^T] E[K_1]) C_k \boldsymbol{Z_d}$.
Since, $var[y_1 - f] \geq 0$, a factorization exists of the form

$$var[y_1 - f] = (MU + D)^T (MU + D), \tag{45}$$

in which case we get, $var[y_1 - f]^{1/2} = ||MU + D||_2$. Therefore, the probability constraint (equation (37)) finally becomes the convex constraint

$$\begin{aligned} &k_e ||MU + D||_2 + \\ &E[K_1 C_k \Psi] A_{eq}(\tilde{k}_f)U - E[K_1] Ck \boldsymbol{Z_d}(T_f) - f \leq 0. \end{aligned} \tag{46}$$

Similarly, the probability constraint

$$\text{Prob}\{-y_1 - f \leq 0\} \geq 1 - \epsilon \tag{47}$$

amounts to the convex constraint

$$\begin{aligned} &k_e ||MU + D||_2 - \\ &E[K_1 C_k \Psi] A_{eq}(\tilde{k}_f)U + E[K_1] Ck \boldsymbol{Z_d}(T_f) - f \leq 0 \end{aligned} \tag{48}$$

(from a development similar to equations (37) through (46)). The other probability constraints to enforce the residual energy can be done in the same way.

Results

After the formulation of all the constraints (both state and terminal residual energy), the example problem was solved. Once again, results from the Beta distribution is presented. In all subsequent notations, ϵ_1 has been used to indicate the risk levels for the state constraints and ϵ_2 has been used to indicate the risk levels for the residual energy constraints. Since, the constraints are all conservative, a maximum state constraint violation of only 5.06% is observed (although the allowance was 20%). Figure. 3 shows a plot of the l_∞ residual energy. The red dotted line indicates the cost determined for the case of probabilistic state and energy constraints. The black dotted line is the deterministic counterpart and is the same as Figure 2(c). It can be seen that the cost ($f = 0.0082$) has significantly reduced since the constraints are no longer hard (as limited violations are permitted) compared to the deterministic case ($f = 0.0248$). Furthermore, it is seen that the cost is less even compared to when only state constraints were probabilistic ($f = 0.0209$ for $\epsilon = 0.2$ in Table I) as expected. The blue dots represent the MC realizations when both constraints (state and energy)

TABLE II

COMPARISON ACROSS DISTRIBUTIONS FROM 100000 MC SIMULATIONS

		Uniform			Beta			Custom		
ϵ_1	ϵ_2	f	Max. V_s	Max. V_e	f	Max. V_s	Max. V_e	f	Max. V_s	Max. V_e
0.2	0.2	0.0154	3.68	6.16	0.0161	5.06	5.84	0.0167	5.27	5.53
0.2	0.5	0.0078	3.48	21.57	0.0082	5.06	19.24	0.0085	5.15	17.98
0.2	0.8	0.0039	3.39	44.62	0.0041	5.11	43.41	0.0043	5.12	42.44
0.5	0.2	0.0110	22.66	7.51	0.0118	20.08	6.45	0.0125	19.14	5.91
0.5	0.5	0.0055	22.69	21.49	0.0059	20.29	18.94	0.0063	19.16	17.65
0.5	0.8	0.0027	22.23	50.28	0.0030	20.27	49.22	0.0032	19.19	46.68
0.8	0.2	0.0091	51.77	7.93	0.0100	46.83	6.62	0.0107	45.09	5.95
0.8	0.5	0.0046	51.77	20.69	0.0050	47.34	18.13	0.0054	44.80	17.11
0.8	0.8	0.0023	51.88	53.17	0.0025	47.32	46.67	0.0027	45.22	44.77

are probabilistic. From 10000 MC simulations, the maximum percentage violation of an energy constraint was seen to be 23.2% (well within the allowed 50%). This means that the majority of the blue dots lie within red dotted lines.

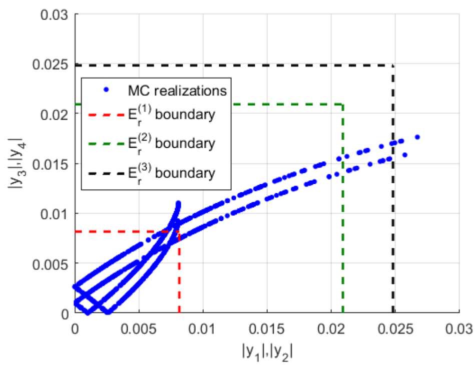


Fig. 3. Residual Energy Boundary: $E_r^{(1)}$: probabilistic state and energy constraints, $E_r^{(2)}$: probabilistic state constraints, $E_r^{(3)}$: deterministic state and energy constraints

The violations are all distribution dependent and Table II has been used to list the results for the 3 distributions for various combinations of ϵ_1 and ϵ_2. Max. V_s and Max. V_e represents the maximum violations observed for the state and the energy constraints respectively.

VII. CONCLUSIONS

In this paper, a design approach is presented which results in a convex optimization problem for the design of controllers for linear systems with model parameter uncertainties. The probabilistic representation of model parameter uncertainties is suited to the use of polynomial chaos to convert the stochastic model to a deterministic surrogate model, which permits evaluation of the mean and variance of the evolving states. The chance constraint are used to formulate a convex optimization problem as a function of an acceptable level of constraint violation. Since the problem formulation is agnostic to the distribution of the uncertainty if their mean and variance match, three distributions are selected to illustrate the relative performance for the benchmark floating oscillator problem. It is seen that the constraint violation are always significantly smaller than the bounds that are imposed in the optimal control problem.

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