$\mathcal{H}_\infty$ Optimal Estimation for Linear Coupled PDE Systems

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***Abstract*— In this work, we present a Linear Matrix Inequality (LMI) based method to synthesize an optimal $\mathcal{H}_\infty$ estimator for a large class of linear coupled partial differential equations (PDEs) utilizing only finite dimensional measurements. Our approach extends the newly developed framework for representing and analyzing distributed parameter systems using operators on the space of square integrable functions that are equipped with multipliers and kernels of semi-separable class. We show that by redefining the state, the PDEs can be represented using operators that embed the boundary conditions and input-output relations explicitly. The optimal estimator synthesis problem is formulated as a convex optimization subject to LMIs that require no approximation or discretization. A scalable algorithm is presented to synthesize the estimator. The algorithm is illustrated by suitable examples.**

I. INTRODUCTION

Partial differential equations (PDEs) are essential to model dynamic processes where physical quantities not only evolve over time but also over a spatial domain. Due to the infinite dimensional nature of the such models, the problem of synthesizing observers is difficult to solve and implement. In practice, such processes are usually equipped with a finite number of sensors. Therefore, it is a key problem to estimate non-observed or non-measured variables of these systems on the basis of (noise-corrupted) measurements inferred from the sensors. This paper addresses the problem of synthesizing an estimator for linear systems described by coupled PDEs. The estimator causally maps sensor information to estimates of the non-observed output (in many applications the non-observed outputs coincide with the state of the system) and achieves $\mathcal{H}_\infty$ optimal performance such that the effect of disturbances to the estimation error is bounded in $\mathcal{H}_\infty$ sense.

Regarding synthesizing such an observer for a system described by a set of coupled PDEs, the available methods can be broadly classified into two approaches: a) early-lumping and b) late-lumping. For the method of ealry-lumping, the PDEs are approximated by a finite dimensional model governed by a set of coupled ordinary differential equations (ODEs). For more details see [1] and the references therein. The synthesis of an $\mathcal{H}_\infty$ optimal observer is then carried out on the basis of the finite dimensional model. Specifically, if the finite dimensional model is represented as

$$\dot{x} = Ax + Bw, \quad y = Cx + Dw, \quad z = Ex,$$

then the estimator is given by

$$\dot{\hat{x}} = A\hat{x} + L(\hat{y} - y), \quad \hat{y} = C\hat{x}, \quad \hat{z} = E\hat{x}.$$

Here, matrices $L = P^{-1}Z$ with $P \succ 0$ and Z satisfy the following Linear Matrix Inequalities (LMIs) for a suitable and minimal performance level γ

$$\begin{bmatrix} PA + ZC + (PA + ZC)^\top & -PB - ZD & E^\top \\ -(PB + ZD)^\top & -\gamma^2 I & 0 \\ E & 0 & I \end{bmatrix} \prec 0.$$

In that case, we achieve $||\hat{z} - z|| \leq \gamma ||w||$. Key disadvantage of early-lumping is discretization techniques for approximating an infinite-dimensional system with a finite-dimensional model are prone to numerical instabilities. Moreover, increased demands on accuracy require the dimension of the system to be large which make them computationally demanding. Furthermore, these finite-dimensional approximations of PDEs do not necessarily represent the behavior of the original system to a quantified level of accuracy [2]. As a consequence, the performance level (e.g. in $\mathcal{H}_\infty$ sense) does not represent the achieved performance of the estimator when interconnected with the system of PDEs, in fact, the achieved performance is unknown a priori (c.f. [3], [4]).

On the other hand, the late-lumping approach directly utilizes the original PDEs to synthesize estimator without any discretization. Using spectral analysis optimal observers for specific PDEs (e.g. wave equation, coupled heat-wave equations, etc.) have been proposed in [5], [6]. Design of interval observers without finite element approximations have been proposed in [7] for non-homogeneous heat equation. Backstepping offers a systematic approach for observer synthesis of PDEs without any approximation. Here, integral transformations are typically used to convert the PDE model to the target PDEs with desired stability properties. Using backstepping, observers for parabolic PDEs in more than one dimensions have been presented in [8]. For semilinear PDEs, Luenberger type observers have been presented in [9]. Using the extension of Lyapunov theory for infinite dimensional space [10], Sum-of-Squares (SOS) optimization can be used for synthesizing optimal observer and controller that matches a specific performance criteria and does not depend on any approximation. For time-delay systems an $\mathcal{H}_\infty$ observer has been designed in [11]. Similarly, in [12] an SOS optimization based observer has been designed for parabolic PDEs in one spatial dimension. It is important to note that most of the late-lumping approaches are application-specific and not scalable. A generic framework for the synthesis of $\mathcal{H}_\infty$ optimal estimators is, therefore, largely missing and a challenging problem.

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Manuscript 1206 submitted to 2019 IEEE Conference on Decision and Control (CDC).
Received March 17, 2019.

In this paper, we establish a generic and scalable algorithm for the synthesis of an $\mathcal{H}_\infty$ optimal estimator that uses finite measurements and does not require any discretization. The estimator design involves solving a suitable set of LMIs that translates to a convex optimization problem which can be solved in a computationally effective manner. We propose to utilize the infinite dimensional model of the PDEs and develop a synthesis procedure for estimators that yield explicit guarantees on $\mathcal{H}_\infty$ performance of the interconnected PDE-estimator system.

To this end, we consider the following class of linear PDEs in one spatial dimension:

$$\begin{aligned}\begin{bmatrix}\dot{x}_1(s,t)\\ \dot{x}_2(s,t)\\ \dot{x}_3(s,t)\end{bmatrix} =&A_0(s)\begin{bmatrix}x_1(s,t)\\ x_2(s,t)\\ x_3(s,t)\end{bmatrix}+A_1(s)\frac{\partial}{\partial s}\begin{bmatrix}x_2(s,t)\\ x_3(s,t)\end{bmatrix}\\ &+A_2(s)\frac{\partial^2}{\partial s^2}\left[x_3(s,t)\right]+B_1(s)w(t). \qquad (1)\end{aligned}$$

Here, the state variables are partitioned so that we can consider both parabolic and hyperbolic types of PDEs with $x_i(s,t) : [a,b]\times\mathbb{R}^+ \rightarrow \mathbb{R}^{n_i}$. Uncertainty due to unmodeled dynamics or noise is incorporated by the signal $w(t)\in\mathbb{R}^m$. Here, $A_0(s), A_1(s), A_2(s), B_1(s)$ are matrix valued polynomials of appropriate dimensions. The PDEs satisfy the following boundary condition:

$$B_c x_b(t) = 0. \qquad (2)$$

Here, $x_b(t)$ is defined by

$$\mathrm{col}\Big(x_2(a,t), x_2(b,t), x_3(a,t), x_3(b,t), \frac{\partial x_3(a,t)}{\partial s}, \frac{\partial x_3(b,t)}{\partial s}\Big).$$

The PDE model is well-posed for a suitable matrix $B_c\in\mathbb{R}^{(n_2+2n_3)\times 2(n_2+2n_3)}$ that has a full row-rank.

The measured output is finite dimensional and corrupted by noise taking the following form:

$$\begin{aligned}y(t) =&C_1x_b(t)+\int_a^b C_a(s)\begin{bmatrix}x_1(s,t)\\ x_2(s,t)\\ x_3(s,t)\end{bmatrix}\mathrm{d}s\\ &+\int_a^b C_b(s)\frac{\partial}{\partial s}\begin{bmatrix}x_2(s,t)\\ x_3(s,t)\end{bmatrix}\mathrm{d}s+D_1w(t). \qquad (3)\end{aligned}$$

The regulated (to-be-estimated) output is also finite dimensional and of the following form:

$$\begin{aligned}z(t) =&E_1x_b(t)+\int_a^b E_a(s)\begin{bmatrix}x_1(s,t)\\ x_2(s,t)\\ x_3(s,t)\end{bmatrix}\mathrm{d}s\\ &+\int_a^b E_b(s)\frac{\partial}{\partial s}\begin{bmatrix}x_2(s,t)\\ x_3(s,t)\end{bmatrix}\mathrm{d}s. \qquad (4)\end{aligned}$$

Here, $y(t)\in\mathbb{R}^p$ and $z(t)\in\mathbb{R}^q$, C_1, D_1, E_1 are constant matrices of appropriate dimensions. $C_a(s), C_b(s), E_a(s)$ and $E_b(s)$ are matrix valued polynomials of appropriate dimensions.

Illustrative Example: Consider a one dimensional string of length L attached at one end and controlled via damping at the other end. Based on [10], [13], the following hyperbolic PDE model is given in terms of the wave displacement $u(s,t)$ over the domain $[0,L]$

$$\frac{\partial^2 u(s,t)}{\partial t^2}=\frac{\partial^2 u(s,t)}{\partial s^2}+B_1(s)w(t).$$

The boundary conditions are

$$u(0,t)=0, \qquad \frac{\partial u(s,t)}{\partial s}\mid_{s=L}=-\frac{\partial u(s,t)}{\partial t}\mid_{s=L}.$$

The noise corrupted measured output is $y(t)=\int_a^b \frac{\partial u(s,t)}{\partial t}\mathrm{d}s+D_1w(t)$. The regulated output is $z(t)=\frac{\partial u(s,t)}{\partial t}\mid_{s=L}$.

From the boundary conditions, we choose the states to be $x_2(s,t) := \mathrm{col}\Big(\frac{\partial u(s,t)}{\partial s}, \frac{\partial u(s,t)}{\partial t}\Big)$. Based on the above framework, we obtain

$$\dot{x}_2(s,t)=\underbrace{\begin{bmatrix}0&1\\1&0\end{bmatrix}}_{:=A_1}\frac{\partial}{\partial s}[x_2(s,t)]+B_1(s)w(t).$$

Here, $A_0=0, A_2$ is void. The boundary conditions are

$$\underbrace{\begin{bmatrix}0&1&0&0\\0&0&1&1\end{bmatrix}}_{:=B_c}\underbrace{\begin{bmatrix}x_2(0,t)\\x_2(L,t)\end{bmatrix}}_{x_b(t)}=0.$$

The measure output and the regulated output are

$$\begin{aligned}y(t)&=\int_a^b\underbrace{\begin{bmatrix}0&1\end{bmatrix}}_{:=C_a}x_2(s,t)\mathrm{d}s+D_1w(t),\\ z(t)&=\underbrace{\begin{bmatrix}0&0&0&1\end{bmatrix}}_{:=E_1}x_b(t).\end{aligned}$$

Moreover, C_1, C_b, E_a, E_b are zeros of appropriate dimensions. □

Using the PDE model and a finite dimensional measured output $y(t)$, we aim to synthesize an observation operator $\mathcal{L}$ that corrects the deviation of the estimated output from the measured output. The observer is of the following form:

$$\begin{aligned}\begin{bmatrix}\dot{\hat{x}}_1(s,t)\\ \dot{\hat{x}}_2(s,t)\\ \dot{\hat{x}}_3(s,t)\end{bmatrix} =&A_0(s)\begin{bmatrix}\hat{x}_1(s,t)\\ \hat{x}_2(s,t)\\ \hat{x}_3(s,t)\end{bmatrix}+A_1(s)\frac{\partial}{\partial s}\begin{bmatrix}\hat{x}_2(s,t)\\ \hat{x}_3(s,t)\end{bmatrix}\\ &+A_2(s)\frac{\partial^2}{\partial s^2}\left[\hat{x}_3(s,t)\right]+\underbrace{\mathcal{L}(\hat{y}(t)-y(t))},\\ B_c\hat{x}_b(t) =&0. \qquad (5)\end{aligned}$$

The measured and the regulated estimates are

$$\begin{aligned}\hat{y}(t) =&C_1\hat{x}_b(t)+\int_a^b C_a(s)\begin{bmatrix}\hat{x}_1(s,t)\\ \hat{x}_2(s,t)\\ \hat{x}_3(s,t)\end{bmatrix}\mathrm{d}s\\ &+\int_a^b C_b(s)\frac{\partial}{\partial s}\begin{bmatrix}\hat{x}_2(s,t)\\ \hat{x}_3(s,t)\end{bmatrix}\mathrm{d}s, \qquad (6)\\ \hat{z}(t) =&E_1\hat{x}_b(t)+\int_a^b E_a(s)\begin{bmatrix}\hat{x}_1(s,t)\\ \hat{x}_2(s,t)\\ \hat{x}_3(s,t)\end{bmatrix}\mathrm{d}s\\ &+\int_a^b E_b(s)\frac{\partial}{\partial s}\begin{bmatrix}\hat{x}_2(s,t)\\ \hat{x}_3(s,t)\end{bmatrix}\mathrm{d}s. \qquad (7)\end{aligned}$$

The aim will be to synthesize the operator $\mathcal{L}$ such that, for estimation error $z_e = \hat{z} - z$, the performance satisfies

$$\sup_{w \in L_2} \frac{||z_e||_{L_2}}{||w||_{L_2}} < \gamma$$

with $\gamma > 0$ being sufficiently small.

In this paper, we show that instead of considering $\mathrm{col}(x_1, x_2, x_3)$ as the state variables, the states $\mathrm{col}(x_1, \frac{\partial x_2}{\partial s}, \frac{\partial^2 x_3}{\partial s^2})$ can uniquely represent the PDE model in (1)-(3) while no additional constraint in terms of the boundary conditions is necessary. The $\mathcal{H}_\infty$ optimal estimation problem can then be formulated in terms of these *fundamental states* $\mathrm{col}(x_1, \frac{\partial x_2}{\partial s}, \frac{\partial^2 x_3}{\partial s^2})$. Owing to this reformulation of the estimation problem, features of the synthesis procedure that we present are the following:

- The feasibility conditions are expressed as LMIs;
- The framework is generic to handle both parabolic and hyperbolic linear PDEs with Dirichtlet, Neumann and mixed boundary conditions;
- The conditions are prima facie provable and they are certified using Lyapunov functionals that are parameterized by bounded operators with multipliers and integral kernels of semi-separable class;
- The design procedure includes an efficient numerical scheme for implementation.

Remainder of this paper is organized as follows. Section II provides a preliminary discussion on the notations, the class of operators $\mathcal{P}\{^{P,\, Q_1,\, Q_2}_{S,\, R_1,\, R_2}\}$ and the PDE model. In Section III, the concept of *fundamental states* is introduced and the PDE model is reformulated with the help of the *fundamental states*. Using the *fundamental states* based representation, the $\mathcal{H}_\infty$ estimation problem is formulated in Section IV and using the LMI formulation on the positivity of the class of operators $\mathcal{P}\{^{P,\, Q_1,\, Q_2}_{S,\, R_1,\, R_2}\}$ in Section V, LMI conditions for synthesizing such estimators are derived Section VI. In Section VII, a numerical method for real-time implementation of the designed estimator is explained. In Section VIII, we numerically illustrate the methodology for example PDE systems in MATLAB. At last, Section IX provides conclusions and directions for future research.

II. PRELIMINARIES

A. Notation

For convenience, we denote $x_s := \frac{\partial x}{\partial s}$ and $x_{ss} := \frac{\partial^2 x}{\partial s^2}$. $[]$ denotes empty matrices. Let $\mathbb{N}$ denote the field of integers. We use $\mathbb{S}^m \subset \mathbb{R}^{m\times m}$ to denote the set of symmetric matrices. We define the space of square integrable $\mathbb{R}^m$-valued functions on X as $L_2^m(X)$. $L_2^m(X)$ is equipped with the inner product $\langle x, y\rangle_{L_2} = \int_a^b x^\top(s) y(s) \mathrm{d}s$ and the norm $||x||_{L_2} = \sqrt{\int_a^b x^\top(s)x(s)\mathrm{d}s}$. For denoting the Euclidean inner product in $\mathbb{R}^n$, we use $\langle x, y\rangle_{\mathbb{R}^n} = x^\top y$. Sobolov spaces are denoted by $W_n^{q,p}(X) := \{x \in L_p^n(X) \mid \frac{\partial^k x}{\partial s^k} \in L_p^n(X) \text{ for all } k \le q\}$. For an inner product space X, an operator $\mathcal{P} : X \to X$ is called positive, if for all $x \in X$, we have $\langle x, \mathcal{P}x\rangle_X \ge 0$. We use $\mathcal{P} \succcurlyeq 0$ to indicate that $\mathcal{P}$ is a positive operator. We say that $\mathcal{P} : X \to X$ is coercive if there exists some $\epsilon > 0$ such that $\langle x, \mathcal{P}x\rangle_X \ge \epsilon \|x\|_X^2$ for all $x \in X$.

B. A Class of Linear Operators

Let $[a, b]$ denote the spatial manifold with $a < b$. A class of linear operators $\mathcal{P}\{^{P,\, Q_1,\, Q_2}_{S,\, R_1,\, R_2}\} : \mathbb{R}^m \times L_2^n[a, b] \to \mathbb{R}^m \times L_2^n[a, b]$ is parametrized by multiplier and kernels of semi-separable class and takes the following form:

$$\left(\mathcal{P}\{^{P,\, Q_1,\, Q_2}_{S,\, R_1,\, R_2}\} \begin{bmatrix} x \\ z \end{bmatrix}\right)(s) := \tag{8}$$
$$\begin{bmatrix} Px + \int_a^b Q_1(s)z(s)ds \\ Q_2(s)x + S(s)z(s) + \int_a^s R_1(s,\eta)z(\eta)\mathrm{d}\eta + \int_s^b R_2(s,\eta)z(\eta)\mathrm{d}\eta \end{bmatrix}.$$

Here, $P \in \mathbb{R}^{m\times m}$ is a matrix and $Q_1 : [a, b] \to \mathbb{R}^{m\times n}$, $Q_2 : [a, b] \to \mathbb{R}^{n\times m}$, $S : [a, b] \to \mathbb{R}^{n\times n}$, and $R_1, R_2 : [a, b] \times [a, b] \to \mathbb{R}^{n\times n}$ are matrix-valued polynomials with $x \in \mathbb{R}^m$ and $z \in L_2^n[a, b]$. The formulae for composition and adjoint of such operators can be found in [14] (pp. 4).

Note: We take the convention that P, Q_1 and Q_2 can be void in the definition (8). In that case, we define following subclass of operators that we denote as $\mathcal{P}_{\{S,R_1,R_2\}} : L_2^n[a, b] \to L_2^n[a, b]$.

$$[\mathcal{P}_{\{S,R_1,R_2\}}z](s) := S(s)z(s) + \int_a^s R_1(s,\eta)z(\eta)\mathrm{d}\eta + \int_s^b R_2(s,\eta)z(\eta)\mathrm{d}\eta. \tag{9}$$

For convenience, we denote the composition of two operators $\mathcal{P}_{\{R_0,R_1,R_2\}} = \mathcal{P}_{\{B_0,B_1,B_2\}}\mathcal{P}_{\{N_0,N_1,N_2\}}$ by $(R_0, R_1, R_2) = (B_0, B_1, B_2) \times (N_0, N_1, N_2)$. The adjoint $\mathcal{P}^*_{\{M_1,N_1,N_2\}}$ is again of the form $\mathcal{P}_{\{\hat{M}_1,\hat{N}_1,\hat{N}_2\}}$ and denoted as $(\hat{M}, \hat{N}_1, \hat{N}_2) = (M, N_1, N_2)^*$.

C. Linear Coupled PDE Systems

The class of systems described by coupled PDEs in (1) – (4) can also be represented by the following equations

$$\begin{aligned} \dot{x}_{\mathrm{p}}(t) &= \mathcal{A}x_{\mathrm{p}}(t) + \mathcal{B}w(t), \\ y(t) &= \mathcal{C}x_{\mathrm{p}}(t) + \mathcal{D}w(t), \\ z(t) &= \mathcal{E}x_{\mathrm{p}}(t), \\ x_{\mathrm{p}}(0) &= 0. \end{aligned} \tag{10}$$

Here, $\mathbf{x}_{\mathrm{p}} := \mathrm{col}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ with $\mathbf{x}_i : [a, b]\times\mathbb{R}^+ \to \mathbb{R}^{n_i}$ $(i = 1, 2, 3)$, $w : \mathbb{R}^+ \to \mathbb{R}^m$, $y : \mathbb{R}^+ \to \mathbb{R}^p$ and $z : \mathbb{R}^+ \to \mathbb{R}^q$. For convenience, let $n_x := n_1 + n_2 + n_3$. The system operators $\mathcal{A} : L_2^{n_x}[a, b] \supset D_{\mathcal{A}} \to L_2^{n_x}[a, b]$, $\mathcal{B} : \mathbb{R}^m \to L_2^{n_x}[a, b]$, $\mathcal{C} : L_2^{n_x}[a, b] \to \mathbb{R}^p$, $\mathcal{D} : \mathbb{R}^m \to \mathbb{R}^p$ and $\mathcal{E} : L_2^{n_x}[a, b] \to \mathbb{R}^q$ are specified by the following definitions:

$$(\mathcal{A}\mathbf{x}_{\mathrm{p}})(s) := A_0(s)\begin{bmatrix}\mathbf{x}_1(s)\\ \mathbf{x}_2(s)\\ \mathbf{x}_3(s)\end{bmatrix} + A_1(s)\begin{bmatrix}\mathbf{x}_2(s)\\ \mathbf{x}_3(s)\end{bmatrix}_s + A_2(s)\begin{bmatrix}\mathbf{x}_3(s)\end{bmatrix}_{ss}$$
$$(\mathcal{B}w)(s) := B_1(s)w, \qquad \mathcal{D}w := D_1 w, \tag{11}$$
$$\mathcal{C}\mathbf{x}_{\mathrm{p}} := C_1\mathbf{x}_b + \int_a^b C_a(s)\begin{bmatrix}\mathbf{x}_1(s)\\ \mathbf{x}_2(s)\\ \mathbf{x}_3(s)\end{bmatrix}\mathrm{d}s + \int_a^b C_b(s)\begin{bmatrix}\mathbf{x}_2(s)\\ \mathbf{x}_3(s)\end{bmatrix}_s \mathrm{d}s,$$

Manuscript 1206 submitted to 2019 IEEE Conference on Decision and Control (CDC).
Received March 17, 2019.

$$\mathcal{E}\mathbf{x}_{\rm p} := E_1\mathbf{x}_b + \int_a^b E_a(s)\begin{bmatrix}\mathbf{x}_1(s)\\ \mathbf{x}_2(s)\\ \mathbf{x}_3(s)\end{bmatrix}\mathrm{d}s + \int_a^b E_b(s)\begin{bmatrix}\mathbf{x}_2(s)\\ \mathbf{x}_3(s)\end{bmatrix}_s \mathrm{d}s.$$

In (11), $A_0(s), A_1(s), A_2(s), B_1(s), C_a(s), C_b(s)$ are matrix valued functions of appropriate dimensions. D_1, C_1, E_1 are constant real-valued matrices of appropriate dimensions.

The domain of $\mathbf{x}_{\rm p}$ is

$$\begin{aligned} D_{\mathcal{A}} := \{\mathbf{x}_{\rm p} \in L_2^{n_1}[a,b] \times W_{n_2}^{1,2}[a,b] \times W_{n_3}^{2,2}[a,b] :\\ B_c\mathbf{x}_b = 0\}, \end{aligned} \tag{12}$$
$$\mathbf{x}_b := \mathrm{col}\big(\mathbf{x}_2(a), \mathbf{x}_2(b), \mathbf{x}_3(a), \mathbf{x}_3(b), \mathbf{x}_{3s}(a), \mathbf{x}_{3s}(b)\big).$$

Evidently, the solution to (10) with the given definitions (11) and its domain (12) satisfy the PDEs in (1)-(4) and vice-versa.

III. FUNDAMENTAL STATES AND BOUNDARY-FREE REPRESENTATION OF PDE SYSTEMS

The model in (10)-(12) is defined by the states $\mathbf{x}_{\rm p} := \mathrm{col}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ which we refer to as the *primal states*. Using the class of operators $\mathcal{P}_{\{S,R_1,R_2\}}$, the primal states can be expressed in terms of the *fundamental states* $\mathbf{x}_{\rm f} := \mathrm{col}(\mathbf{x}_1, \mathbf{x}_{2s}, \mathbf{x}_{3ss})$ that belong to $L_2^{n_x}[a,b]$ and do not require the constraint (12).

Lemma 0.1: Suppose $\mathbf{x}_{\rm p} \in L_2^{n_1}[a,b] \times W_{n_2}^{1,2}[a,b] \times W_{n_3}^{2,2}[a,b]$ and $B_c\mathbf{x}_b = 0$ where $\mathbf{x}_b$ is defined in (12) with B_c of full row-rank. Then for $\mathbf{x}_{\rm f} := \mathrm{col}(\mathbf{x}_1, \mathbf{x}_{2s}, \mathbf{x}_{3ss}) \in L_2^{n_x}[a,b]$, the solution to the following equations:

$$\begin{aligned} \mathbf{x}_{\rm p}(t) &= \mathcal{F}\mathbf{x}_{\rm f}(t),\\ \dot{\mathbf{x}}_{\rm p}(t) &= \mathcal{A}_{\rm f}\mathbf{x}_{\rm f}(t) + \mathcal{B}w(t),\\ y(t) &= \mathcal{C}_{\rm f}\mathbf{x}_{\rm f}(t) + \mathcal{D}w(t),\\ z(t) &= \mathcal{E}_{\rm f}\mathbf{x}_{\rm f}(t),\\ \mathbf{x}_{\rm p}(0) &= 0 \end{aligned} \tag{13}$$

is also the solution to (10)-(12) as well as (1)-(4). Here,

$$\begin{aligned} \mathcal{F} &= \mathcal{P}_{\{G_0,G_1,G_2\}},\\ \mathcal{A}_{\rm f} &= \mathcal{P}_{\{H_0,H_1,H_2\}},\\ \mathcal{C}_{\rm f} &= \mathcal{P}_{\{0,C_2,C_2\}},\\ \mathcal{E}_{\rm f} &= \mathcal{P}_{\{0,E_2,E_2\}},\\ (\mathcal{B}w)(s) &= B_1(s)w, \quad \mathcal{D}w = D_1 w, \end{aligned} \tag{14}$$

with

$$\begin{aligned} (H_0,H_1,H_2) =& (A_0,0,0)\times(G_0,G_1,G_2)\\ &+ (A_1,0,0)\times(G_3,G_4,G_5)\\ &+ (\begin{bmatrix}0 & 0 & A_2(s)\end{bmatrix},0,0),\\ (0,C_2,C_2) =& (0,C_a^\top,C_a^\top)^*\times(G_0,G_1,G_2)\\ &+ (0,C_b^\top,C_b^\top)^*\times(G_3,G_4,G_5) + (C_1,0,0)\times\\ &(-T,0,0)\times(0,(B_cT)^{-1}B_cQ,(B_cT)^{-1}B_cQ)\\ &+ (C_1,0,0)\times(0,Q,Q),\\ (0,E_2,E_2) =& (0,E_a^\top,E_a^\top)^*\times(G_0,G_1,G_2)\\ &+ (0,E_b^\top,E_b^\top)^*\times(G_3,G_4,G_5) + (E_1,0,0)\times\\ &(-T,0,0)\times(0,(B_cT)^{-1}B_cQ,(B_cT)^{-1}B_cQ)\\ &+ (E_1,0,0)\times(0,Q,Q). \end{aligned} \tag{15}$$

Moreover,

$$\begin{aligned} &G_0(s) = \begin{bmatrix} I & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \quad G_3(s) = \begin{bmatrix} 0 & I & 0\\ 0 & 0 & 0\end{bmatrix},\\ &G_2(s,\eta) = -K(s)(BT)^{-1}BQ(s,\eta),\\ &G_5(s,\eta) = -V(BT)^{-1}BQ(s,\eta),\\ &G_1(s,\eta) = \begin{bmatrix} 0 & 0 & 0\\ 0 & I & 0\\ 0 & 0 & (s-\eta)I\end{bmatrix} + G_2(s,\eta),\\ &G_4(s,\eta) = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & I\end{bmatrix} + G_5(s,\eta),\\ &K(s) = \begin{bmatrix} 0 & 0 & 0\\ I & 0 & 0\\ 0 & I & (s-a)I\end{bmatrix}, \quad V = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & I\end{bmatrix}\\ &T = \begin{bmatrix} I & 0 & 0\\ I & 0 & 0\\ 0 & I & 0\\ 0 & I & (b-a)I\\ 0 & 0 & I\\ 0 & 0 & I\end{bmatrix}, \quad Q(s,\eta) = \begin{bmatrix} 0 & 0 & 0\\ 0 & I & 0\\ 0 & 0 & 0\\ 0 & 0 & (b-\eta)I\\ 0 & 0 & 0\\ 0 & 0 & I\end{bmatrix}. \end{aligned} \tag{16}$$

□

Proof: Using fundamental theorem of calculus ([15], pp. 3-4), the following identities can be proven:

$$\mathbf{x}_{\rm p} = \mathcal{P}_{\{G_0,G_1,G_2\}}\mathbf{x}_{\rm f}, \tag{17}$$
$$\mathbf{x}_{\rm h} = \mathcal{P}_{\{G_3,G_4,G_5\}}\mathbf{x}_{\rm f}. \tag{18}$$

Here, $\mathbf{x}_{\rm p} := \mathrm{col}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, $\mathbf{x}_{\rm h} := \mathrm{col}(\mathbf{x}_{2s}, \mathbf{x}_{3s})$ and $\mathbf{x}_{\rm f} := \mathrm{col}(\mathbf{x}_1, \mathbf{x}_{2s}, \mathbf{x}_{3ss})$. The rest of the proof can be derived by substituting (17)-(18) in the definitions specified in (11). ■

IV. $\mathcal{H}_\infty$ ESTIMATOR SYNTHESIS ON FUNDAMENTAL STATES

For any observation operator $\mathcal{L} : \mathbb{R}^p \to L_2^{n_x}[a,b]$ in (5), the observer dynamics can also be represented in terms of its fundamental state as

$$\begin{aligned} \hat{\mathbf{x}}_{\rm p}(t) &= \mathcal{F}\hat{\mathbf{x}}_{\rm f}(t)\\ \dot{\hat{\mathbf{x}}}_{\rm p}(t) &= \mathcal{A}_{\rm f}\hat{\mathbf{x}}_{\rm f}(t) + \mathcal{L}(\mathcal{C}_{\rm f}\hat{\mathbf{x}}_{\rm f}(t) - y(t)),\\ \hat{z}(t) &= \mathcal{E}_{\rm f}\hat{\mathbf{x}}_{\rm f}(t)\\ \hat{\mathbf{x}}_{\rm p}(0) &= 0. \end{aligned} \tag{19}$$

Defining the estimation error in primal state $\mathbf{e}_{\rm p}(t) = \hat{\mathbf{x}}_{\rm p}(t) - \mathbf{x}_{\rm p}(t)$, the estimation error in the fundamental state $\mathbf{e}_{\rm f}(t) = \hat{\mathbf{x}}_{\rm f}(t) - \mathbf{x}_{\rm f}(t)$ yields the following representation of the error dynamics in terms of the fundamental states:

$$\begin{aligned} \mathbf{e}_{\rm p}(t) &= \mathcal{F}\mathbf{e}_{\rm f}(t),\\ \dot{\mathbf{e}}_{\rm p}(t) &= (\mathcal{A}_{\rm f} + \mathcal{L}\mathcal{C}_{\rm f})\mathbf{e}_{\rm f}(t) - (\mathcal{B} + \mathcal{L}\mathcal{D})w(t),\\ z_e(t) &= \mathcal{E}_{\rm f}\mathbf{e}_{\rm f}(t),\\ \mathbf{e}_{\rm p}(0) &= 0. \end{aligned} \tag{20}$$

The $\mathcal{H}_\infty$ estimation problem is then equivalently expressed as the problem to find $\mathcal{L}$ in (20) such that

$$\|z_e\|_{L_2} \le \gamma\|w\|_{L_2}$$

Manuscript 1206 submitted to 2019 IEEE Conference on Decision and Control (CDC).
Received March 17, 2019.

for all square integrable function w and for $\gamma > 0$ being sufficiently small. Now, we have the following result:

Theorem 1: Suppose there exists a coercive linear operator $\mathcal{T} : L_2^{n_x}[a,b] \to L_2^{n_x}[a,b]$ and a linear operator $\mathcal{Z} : \mathbb{R}^p \to L_2^{n_x}[a,b]$ such that

$$\begin{aligned}&\left\langle (\mathcal{T}\mathcal{A}_\mathrm{f} + \mathcal{Z}\mathcal{C}_\mathrm{f})\mathbf{e}_\mathrm{f}, \mathcal{F}\mathbf{e}_\mathrm{f} \right\rangle_{L_2} + \left\langle \mathcal{F}\mathbf{e}_\mathrm{f}, (\mathcal{T}\mathcal{A}_\mathrm{f} + \mathcal{Z}\mathcal{C}_\mathrm{f})\mathbf{e}_\mathrm{f} \right\rangle_{L_2} \\ &- \left\langle (\mathcal{T}\mathcal{B} + \mathcal{Z}\mathcal{D})w, \mathcal{F}\mathbf{e}_\mathrm{f} \right\rangle_{L_2} - \left\langle \mathcal{F}\mathbf{e}_\mathrm{f}, (\mathcal{T}\mathcal{B} + \mathcal{Z}\mathcal{D})w \right\rangle_{L_2} \\ &- \gamma^2 \langle w, w\rangle_{\mathbb{R}^m} + \langle z_e, z_e\rangle_{\mathbb{R}^q} \\ &\le -\epsilon \langle \mathbf{e}_\mathrm{f}, \mathbf{e}_\mathrm{f}\rangle_{L_2} \end{aligned} \tag{21}$$

for some $\epsilon > 0$, for all $\mathbf{e}_\mathrm{f} \in L_2^{n_x}[a,b]$. Then $\mathcal{T}^{-1}$ exists and is a bounded linear operator. Moreover, for $\mathcal{L} = \mathcal{T}^{-1}\mathcal{Z}$ and any square integrable function $w \in \mathbb{R}^m$, the solution to (20) satisfies

$$\| z_e \|_{L_2} \le \gamma \| w \|_{L_2} . \tag{22}$$

□

Proof: Since $\mathcal{T}$ is coercive and $\mathcal{T} : L_2^{n_x}[a,b] \to L_2^{n_x}[a,b]$, $\mathcal{T}^{-1}$ exists and it is a bounded linear operator with $\mathcal{T}^{-1} : L_2^{n_x}[a,b] \to L_2^{n_x}[a,b]$.

Let $V(\mathbf{e}_\mathrm{p}) := \langle \mathbf{e}_\mathrm{p}, \mathcal{T}\mathbf{e}_\mathrm{p}\rangle_{L_2}$ be the storage function. Since, $\mathcal{T}$ is coercive, $V(\mathbf{e}_\mathrm{p}) \ge \delta\langle \mathbf{e}_\mathrm{p}, \mathbf{e}_\mathrm{p}\rangle_{L_2}$ for some $\delta > 0$. Differentiating the storage function with respect to time t and using (20) we obtain:

$$\begin{aligned}&\dot{V}(\mathbf{e}_\mathrm{p}(t)) = \\ &\left\langle (\mathcal{T}\mathcal{A}_\mathrm{f} + \mathcal{Z}\mathcal{C}_\mathrm{f})\mathbf{e}_\mathrm{f}(t), \mathcal{F}\mathbf{e}_\mathrm{f}(t) \right\rangle_{L_2} + \left\langle \mathcal{F}\mathbf{e}_\mathrm{f}(t), (\mathcal{T}\mathcal{A}_\mathrm{f} + \mathcal{Z}\mathcal{C}_\mathrm{f})\mathbf{e}_\mathrm{f}(t) \right\rangle_{L_2} \\ &- \left\langle (\mathcal{T}\mathcal{B} + \mathcal{Z}\mathcal{D})w(t), \mathcal{F}\mathbf{e}_\mathrm{f}(t) \right\rangle_{L_2} - \left\langle \mathcal{F}\mathbf{e}_\mathrm{f}(t), (\mathcal{T}\mathcal{B} + \mathcal{Z}\mathcal{D})w(t) \right\rangle_{L_2}\end{aligned}$$

The inequality (21) implies

$$\begin{aligned}&\dot{V}(\mathbf{e}_\mathrm{p}(t)) = \\ &\left\langle (\mathcal{T}\mathcal{A}_\mathrm{f} + \mathcal{Z}\mathcal{C}_\mathrm{f})\mathbf{e}_\mathrm{f}(t), \mathcal{F}\mathbf{e}_\mathrm{f}(t) \right\rangle_{L_2} + \left\langle \mathcal{F}\mathbf{e}_\mathrm{f}(t), (\mathcal{T}\mathcal{A}_\mathrm{f} + \mathcal{Z}\mathcal{C}_\mathrm{f})\mathbf{e}_\mathrm{f}(t) \right\rangle_{L_2} \\ &- \left\langle (\mathcal{T}\mathcal{B} + \mathcal{Z}\mathcal{D})w(t), \mathcal{F}\mathbf{e}_\mathrm{f}(t) \right\rangle_{L_2} - \left\langle \mathcal{F}\mathbf{e}_\mathrm{f}(t), (\mathcal{T}\mathcal{B} + \mathcal{Z}\mathcal{D})w(t) \right\rangle_{L_2} \\ &< \gamma^2 \langle w(t), w(t)\rangle_{\mathbb{R}^m} - \langle z_e(t), z_e(t)\rangle_{\mathbb{R}^q}\end{aligned}$$

Integrating both sides of the above inequality with respect to time t from 0 to ∞, we obtain

$$\begin{aligned}&\int_0^\infty \left(\dot{V}(\mathbf{e}_\mathrm{p}(t)) + \langle z_e(t), z_e(t)\rangle_{\mathbb{R}^q} - \gamma^2 \langle w(t), w(t)\rangle_{\mathbb{R}^m} \right) \mathrm{d}t < 0, \\ &\implies V(\mathbf{e}_\mathrm{p}(\infty)) - V(\mathbf{e}_\mathrm{p}(0)) + \|z_e\|_{L_2}^2 - \gamma^2 \|w\|_{L_2}^2 < 0.\end{aligned}$$

Since $V(\mathbf{e}_\mathrm{p}(t)) \ge \delta\langle \mathbf{e}_\mathrm{p}(t), \mathbf{e}_\mathrm{p}(t)\rangle_{L_2}$ for some $\delta > 0$, $\lim_{t\to\infty} V(\mathbf{e}_\mathrm{p}(t)) > 0$. Also recall, $\mathbf{e}_\mathrm{p}(0) = 0$, hence, $V(\mathbf{e}_\mathrm{p}(0)) = 0$. As a result, for $\gamma > 0$, we obtain (22). ■

Remark 1.1: Based on Lemma 0.1, the system definitions in (13) and (11)-(12) as well as (1)-(4) are equivalent. Therefore, any $\mathcal{L} : \mathbb{R}^p \to L_2^{n_x}[a,b]$ that satisfies Theorem 1 and achieves minimum γ-value for observer (19) also achieves the minimum γ-value for the observer (5)-(7).

Lemma 1.1: Suppose there exists a coercive operator $\mathcal{P}_{\{M,N_1,N_2\}} : L_2^{n_x}[a,b] \to: L_2^{n_x}[a,b]$ and a linear operator $\mathcal{P}_{\{\Phi,0,0\}} : \mathbb{R}^p \to L_2^{n_x}[a,b]$ for matrix valued polynomial functions $M : [a,b] \to \mathbb{R}^{n_x \times n_x}$, $N_1, N_2 : [a,b]\times[a,b] \to \mathbb{R}^{n_x\times n_x}$ and $\Phi : [a,b] \to \mathbb{R}^{p\times n_x}$ such that

$$\left\langle \begin{bmatrix} w \\ \mathbf{e}_\mathrm{f}\end{bmatrix}, \mathcal{P}_{\left\{\substack{P,\, Q,\, Q^\top \\ S,\, R_1,\, R_2}\right\}} \begin{bmatrix} w \\ \mathbf{e}_\mathrm{f}\end{bmatrix} \right\rangle_{\mathbb{R}^p \times L_2} \le 0, \tag{23}$$

with

$$\begin{aligned} P =& -\gamma^2 I, \\ (S, R_1, R_2) =& (H_0, H_1, H_2)^* \times (M, N_1, N_2) \times (G_0, G_1, G_2) \\ &+ (G_0, G_1, G_2)^* \times (M, N_1, N_2) \times (H_0, H_1, H_2) \\ &+ (0, C_2, C_2)^* \times (\Phi, 0, 0)^* \times (G_0, G_1, G_2) \\ &+ (G_0, G_1, G_2)^* \times (\Phi, 0, 0) \times (0, C_2, C_2) \\ &+ (0, E_2, E_2)^* \times (0, E_2, E_2) + \epsilon(I, 0, 0), \\ Q^\top(s) =& -\int_a^s W_1(s,\eta)\mathrm{d}\eta - \int_s^b W_2(s,\eta)\mathrm{d}\eta, \\ (0, W_1, W_2) =& (G_0, G_1, G_2)^* \times (M, N_1, N_2) \times (B_1, 0, 0) \\ &+ (G_0, G_1, G_2)^* \times (\Phi, 0, 0) \times (D_1, 0, 0). \end{aligned}$$

Then for some $\epsilon > 0$ and all $\mathbf{e}_\mathrm{f} \in L_2^{n_x}[a,b]$, $\mathcal{P}^{-1}_{\{M,N_1,N_2\}}$ exists and is a bounded linear operator. Moreover, for $\mathcal{L} := \mathcal{P}^{-1}_{\{M,N_1,N_2\}}\mathcal{P}_{\{\Phi,0,0\}}$ and any square integrable function $w \in \mathbb{R}^m$, the solution to (20) satisfies

$$\| z_e \|_{L_2} \le \gamma \| w \|_{L_2} .$$

□

Proof: Notice, $\mathcal{T} := \mathcal{P}_{\{M,N_1,N_2\}}$ and $\mathcal{Z} := \mathcal{P}_{\{\Phi,0,0\}}$. As a result, $\mathcal{P}^{-1}_{\{M,N_1,N_2\}} : L_2^{n_x}[a,b] \to L_2^{n_x}[a,b]$ exists and it is a bounded linear operator.

Now, applying the composition formulae of the operators, the inequality (21) in Theorem 1 can be rewritten as the operator inequality (23). ■

As an outcome of the operator inequality (23) in Lemma 1.1, the feasibility of solving the $\mathcal{H}_\infty$ optimal observer design amounts to verifying the Linear Operator Inequalities (LOIs) $\mathcal{P}_{\{M,N_1,N_2\}} \succ 0$ and $\mathcal{P}_{\left\{\substack{P,\, Q,\, Q^\top \\ S,\, R_1,\, R_2}\right\}} \preccurlyeq 0$.

V. POSITIVITY OF OPERATORS

In this section, we formulate LMI conditions for verifying positivity of the class of operators $\mathcal{P}_{\left\{\substack{P,\, Q,\, Q^\top \\ S,\, R_1,\, R_2}\right\}}$. We can then reformulate the operator inequality (23) in terms of LMIs to synthesize $\mathcal{H}_\infty$ optimal observers for coupled PDEs.

Theorem 2: For any square integrable functions $Z_1 : [a,b] \to \mathbb{R}^{d_1\times n}$, $Z_2 : [a,b]\times[a,b] \to \mathbb{R}^{d_2\times n}$ and $g : [a,b] \to \mathbb{R}^+$ define

$$\begin{aligned} P :=& T_{11}, \\ Q(s) :=& g(s)T_{12}Z_1(s) + \int_s^b g(\eta)T_{13}Z_2(\eta,s)\mathrm{d}\eta \\ &+ \int_a^s g(\eta)T_{14}Z_2(\eta,s)\mathrm{d}\eta, \\ S(s) :=& g(s)Z_1^\top(s)T_{22}Z_1(s), \\ R_1(s,\eta) :=& g(s)Z_1^\top(s)T_{23}Z_2(s,\eta) + g(\eta)Z_2^\top(\eta,s)T_{42}Z_1^\top(\eta) \\ &+ \int_s^b g(\theta)Z_2^\top(\theta,s)T_{33}Z_2(\theta,\eta)\mathrm{d}\theta \\ &+ \int_\eta^s g(\theta)Z_2^\top(\theta,s)T_{43}Z_2(\theta,\eta)\mathrm{d}\theta \\ &+ \int_a^\eta g(\theta)Z_2^\top(\theta,s)T_{44}Z_2(\theta,\eta)\mathrm{d}\theta, \end{aligned}$$

$$
\begin{aligned}
R_2(s,\eta) :=& g(s)Z_1^\top(s)T_{32}Z_2(s,\eta) + g(\eta)Z_2^\top(\eta,s)T_{24}Z_1^\top(\eta)\\
&+\int_\eta^b g(\theta)Z_2^\top(\theta,s)T_{33}Z_2(\theta,\eta)\mathrm{d}\theta\\
&+\int_s^\eta g(\theta)Z_2^\top(\theta,s)T_{34}Z_2(\theta,\eta)\mathrm{d}\theta\\
&+\int_a^s g(\theta)Z_2^\top(\theta,s)T_{44}Z_2(\theta,\eta)\mathrm{d}\theta,
\end{aligned}
$$

with

$$
T := \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14}\\ T_{21} & T_{22} & T_{23} & T_{24}\\ T_{31} & T_{32} & T_{33} & T_{34}\\ T_{41} & T_{42} & T_{43} & T_{44}\end{bmatrix} \succcurlyeq 0.
$$

Then, for all $\mathbf{z}\in\mathbb{R}^m\times L_2^n[a,b]$ with appropriate dimension of T, $\langle \mathbf{z}, \mathcal{P}\{{}^{P,\,Q,\,Q^\top}_{S,\,R_1,\,R_2}\}\mathbf{z}\rangle \geq 0$. □

Proof: It can be easily shown that $\mathcal{P}\{{}^{P,\,Q,\,Q^\top}_{S,\,R_1,\,R_2}\}$ is self-adjoint. Now, for any $z\in\mathbb{R}^m$ and $\mathbf{x}\in L_2^n[a,b]$, we choose:

$$
\left(\mathcal{Z}\begin{bmatrix} z\\ \mathbf{x}\end{bmatrix}\right)(s) := \begin{bmatrix} z\\ \sqrt{g(s)}Z_1(s)\mathbf{x}(s)\\ \int_a^s \sqrt{g(s)}Z_2(s,\eta)\mathbf{x}(\eta)\mathrm{d}\eta\\ \int_s^b \sqrt{g(s)}Z_2(s,\eta)\mathbf{x}(\eta)\mathrm{d}\eta\end{bmatrix}.
$$

Then

$$
\begin{aligned}
\left\langle \begin{bmatrix} z\\ \mathbf{x}\end{bmatrix}, \mathcal{P}\{{}^{P,\,Q,\,Q^\top}_{S,\,R_1,\,R_2}\}\begin{bmatrix} z\\ \mathbf{x}\end{bmatrix}\right\rangle &= \left\langle \mathcal{Z}\begin{bmatrix} z\\ \mathbf{x}\end{bmatrix}, T\mathcal{Z}\begin{bmatrix} z\\ \mathbf{x}\end{bmatrix}\right\rangle\\
&= \left\langle T^{\frac{1}{2}}\mathcal{Z}\begin{bmatrix} z\\ \mathbf{x}\end{bmatrix}, T^{\frac{1}{2}}\mathcal{Z}\begin{bmatrix} z\\ \mathbf{x}\end{bmatrix}\right\rangle \geq 0.
\end{aligned}
$$

■

Note: We take the convention that P, Q_1 and Q_2 can be void in the definition (8). In that case, the positivity for the subclass of operators $\mathcal{P}_{\{S,R_1,R_2\}}$ can still be proven using Theorem 2 while considering the first row and column of blocks in T empty.

Remark 2.1: The function Z_1, Z_2 will be referred to as parametrized basis functions. For any choice of Z_1, Z_2, Theorem 2 relates the positivity of $\mathcal{P}\{{}^{P,\,Q,\,Q^\top}_{S,\,R_1,\,R_2}\}$ to the positivity of the matrix T. Hence, the verification of the conditions of Theorem 2 can be translated into a feasibility test involving LMIs. Here, a typical choice for Z_1, Z_2 is a vector of monomials of degree d_1, d_2 respectively. For $g(s)$, motivated by Positivstellensatz-type results, we combine the choices of $g(s)=1$ and $g(s)=(s-a)(b-s)$.

VI. LMI CONDITIONS FOR OPTIMAL $\mathcal{H}_\infty$ ESTIMATOR SYNTHESIS

Definition 2.1: Let $P\in\mathbb{R}^{m\times m}$ be a constant real-valued matrix and $Q:[a,b]\to\mathbb{R}^{m\times n}$, $S:[a,b]\to\mathbb{R}^{n\times n}$, and $R_1, R_2:[a,b]\times[a,b]\to\mathbb{R}^{n\times n}$ be matrix-valued polynomials. Then, the cone of the positive operators $\mathcal{P}\{{}^{P,\,Q,\,Q^\top}_{S,\,R_1,\,R_2}\}$ is parametrized by

$$
\left.\begin{aligned}
\Xi_{d_1,d_2} := \Big\{&(P,Q,Q^\top,S,R_1,R_2) : \text{the positivity condition}\\
&\text{in Theorem 2 is satisfied with}\\
&Z_1:[a,b]\to\mathbb{R}^{d_1\times n}, Z_2:[a,b]\times[a,b]\to\mathbb{R}^{d_2\times n}
\end{aligned}\right\} \quad (24)
$$

Theorem 3: Suppose there exist $\delta>0$, $d_1,d_2\in\mathbb{N}$, $\hat{M}:[a,b]\to\mathbb{R}^{n_x\times n_x}$, $\hat{N}_1,\hat{N}_2:[a,b]\times[a,b]\to\mathbb{R}^{n_x\times n_x}$, $\hat{\Phi}:[a,b]\to\mathbb{R}^{p\times n_x}$ and $\hat{\rho}>0$ that satisfies

$$
\hat{\rho} = \arg\min \rho, \tag{25}
$$

such that

$$
\begin{aligned}
([],[],[],\hat{M}-\delta I,\hat{N}_1,\hat{N}_2) &\in \Xi_{d_1,d_2}, &(26)\\
-(\hat{P},\hat{Q},\hat{Q}^\top,\hat{S},\hat{R}_1,\hat{R}_2) &\in \Xi_{d_1,d_2}, &(27)
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{P} =& -\hat{\rho}I,\\
(\hat{S},\hat{R}_1,\hat{R}_2) =& (H_0,H_1,H_2)^*\times(\hat{M},\hat{N}_1,\hat{N}_2)\times(G_0,G_1,G_2)\\
&+(G_0,G_1,G_2)^*\times(\hat{M},\hat{N}_1,\hat{N}_2)\times(H_0,H_1,H_2)\\
&+(0,C_2,C_2)^*\times(\hat{\Phi},0,0)^*\times(G_0,G_1,G_2)\\
&+(G_0,G_1,G_2)^*\times(\hat{\Phi},0,0)\times(0,C_2,C_2)\\
&+(0,E_2,E_2)^*\times(0,E_2,E_2)+\epsilon(I,0,0),\\
\hat{Q}^\top(s) =& -\int_a^s \hat{W}_1(s,\eta)\mathrm{d}\eta - \int_s^b \hat{W}_2(s,\eta)\mathrm{d}\eta,\\
(0,\hat{W}_1,\hat{W}_2) =& (G_0,G_1,G_2)^*\times(\hat{M},\hat{N}_1,\hat{N}_2)\times(B_1,0,0)\\
&+(G_0,G_1,G_2)^*\times(\hat{\Phi},0,0)\times(D_1,0,0).
\end{aligned}
$$

Then $\hat{\rho}$ is the minimum value for which the observer gain $\mathcal{L} := \mathcal{P}^{-1}_{\{\hat{M},\hat{N}_1,\hat{N}_2\}}\mathcal{P}_{\{\hat{\Phi},0,0\}}$ and any square integrable function $w\in\mathbb{R}^m$ define the observer (19) as well as (5)-(6) while satisfying

$$
\|\, z_e \,\|_{L_2} \leq \sqrt{\hat{\rho}}\,\|\, w \,\|_{L_2}\ .
$$

□

VII. IMPLEMENTATION

The optimization algorithm has been implemented in MATLAB® using an adapted version of the freely available package SOSTOOLS [16]. Given a user-defined PDE model as specified in $(1)-(4)$, the implemented algorithm tests the feasibility and computes the optimal γ-value (i.e. $\hat{\rho}$) for the $\mathcal{H}_\infty$ estimation problem without any discretization or approximation of the PDEs. This yields a verifiable test for the existence of an estimator that achieves $\|z_e\|_{L_2}\leq\gamma\|w\|_{L_2}$ for all $w\in L_2$. However, the implementation of the resulting observer (5)-(6) in digital hardware involves evaluation of continuous functions and requires suitable discretization schemes.

A. Numerical Discretization of the Estimator

Recall that, with $\mathcal{L}=\mathcal{T}^{-1}\mathcal{Z}$, the estimator is

$$
\begin{aligned}
\dot{\hat{\mathbf{x}}}_{\mathrm{p}}(t) &= \mathcal{A}_{\mathrm{f}}\hat{\mathbf{x}}_{\mathrm{f}}(t) + \mathcal{T}^{-1}\mathcal{Z}(\mathcal{C}_{\mathrm{f}}\hat{\mathbf{x}}_{\mathrm{f}}(t)-y(t)),\\
\hat{z}(t) &= \mathcal{E}_{\mathrm{f}}\hat{\mathbf{x}}_{\mathrm{f}}(t).
\end{aligned} \tag{28}
$$

A closed form expression for the inverse of $\mathcal{T} := \mathcal{P}_{\{M,N_1,N_2\}}$ is not yet possible. Instead, using the invertibility of $\mathcal{T}$, we may reformulate the estimator by pre-multiplying the both sides of (28) with $\mathcal{T}$. This yields the equivalent estimator

$$\mathcal{T}\dot{\hat{\mathbf{x}}}_{\mathrm{p}}(t) = \mathcal{T}\mathcal{A}_{\mathrm{f}}\hat{\mathbf{x}}_{\mathrm{f}}(t) + \mathcal{Z}(\mathcal{C}_{\mathrm{f}}\hat{\mathbf{x}}_{\mathrm{f}}(t) - y(t)),\\ \hat{z}(t) = \mathcal{E}_{\mathrm{f}}\hat{\mathbf{x}}_{\mathrm{f}}(t). \tag{29}$$

To actually process the measurements $y(t)$ to the outputs $\hat{z}(t)$, we apply a numerical discretization scheme for (29). To do so, we consider that the spatial domain is discretized in N equidistant intervals with grid sample $\Delta_s > 0$. In (29), we apply central difference at the sequence of points $s_i \in S_d \subset [a,b]$, $i \in \{1,\cdots,N+1\}$ to approximate $\hat{\mathbf{x}}_{\mathrm{f}}$ using $\hat{\mathbf{x}}_{2s}(s_i) \approx \frac{\hat{\mathbf{x}}_2(s_{i+1})-\hat{\mathbf{x}}_2(s_{i-1})}{2\Delta_s}$ and $\hat{\mathbf{x}}_{3ss}(s_i) \approx \frac{\hat{\mathbf{x}}_3(s_{i+1})-2\hat{\mathbf{x}}_3(s_i)+\hat{\mathbf{x}}_3(s_{i-1})}{(\Delta_s)^2}$. Based on the number of grid points, the discretization yields the following finite dimensional state space model in terms of $\hat{\mathbf{x}}_{\mathrm{d}} := \mathrm{col}(\hat{\mathbf{x}}(s_2),\cdots,\hat{\mathbf{x}}(s_i),\cdots,\hat{\mathbf{x}}(s_N))$:

$$\bar{T}\dot{\hat{\mathbf{x}}}_{\mathrm{d}} = \left(\bar{T}\bar{A}_{\mathrm{f}} + \bar{\Phi}\bar{C}_{\mathrm{f}}\right)\bar{L}\hat{\mathbf{x}}_{\mathrm{d}} - \bar{Z}_1\ y_{\mathrm{d}},\\ \hat{z} = \bar{E}\hat{\mathbf{x}}_{\mathrm{d}}. \tag{30}$$

Here, $\bar{T},\bar{A}_{\mathrm{f}} \in \mathbb{R}^{n(N-1)\times n(N-1)}$ are the discretized versions of $\mathcal{T}$ and $\mathcal{A}_{\mathrm{f}}$ respectively. $\bar{\Phi} \in \mathbb{R}^{n(N-1)\times p}$ and $\bar{C}_{\mathrm{f}} \in \mathbb{R}^{p\times n(N-1)}$ are discretized versions of $\mathcal{Z}$ and $\mathcal{C}_{\mathrm{f}}$ respectively. $\bar{L} \in \mathbb{R}^{n(N-1)\times n(N-1)}$ contains the coefficients due to the central difference of $\mathbf{x}_{2s}$ and $\mathbf{x}_{3ss}$. The input to the discretized estimator is $y_{\mathrm{d}} := \mathrm{col}(y_m,\hat{\mathbf{x}}_1(a),\hat{\mathbf{x}}_2(a),\hat{\mathbf{x}}_3(a),\hat{\mathbf{x}}_1(b),\hat{\mathbf{x}}_2(b),\hat{\mathbf{x}}_3(b))$ with $\bar{Z}_1$ being a matrix of appropriate dimension containing the discretized coefficients for the output. Here, y_m is the measured output and $\mathrm{col}(\hat{\mathbf{x}}_1(a),\hat{\mathbf{x}}_2(a),\hat{\mathbf{x}}_3(a),\hat{\mathbf{x}}_1(b),\hat{\mathbf{x}}_2(b),\hat{\mathbf{x}}_3(b))$ are evaluated from the boundary conditions. $\bar{E} \in \mathbb{R}^{q\times n(N-1)}$ is the discretized $\mathcal{E}_{\mathrm{f}}$. As $\mathcal{T}$ is a positive operator, $\bar{T}$ should be invertible and (30) amounts to solving a set of linear finite dimensional differential equations that can be efficiently solved using any stable time-marching method.

B. Numerical Discretization of the Operators $\mathcal{P}_{\{S,R_1,R_2\}}$

The class of operators $\mathcal{P}_{\{S,R_1,R_2\}}$ accepts the inner product $\langle \mathbf{x}, \mathcal{P}_{\{S,R_1,R_2\}}\mathbf{x}\rangle$ for $\mathbf{x} \in L_2^n[a,b]$. We evaluate the inner product also at the sequence of points $(s_i,\eta_j) \in S_d \times S_d \subset [a,b]\times[a,b]$, $i \in \{1,\cdots,N+1\}$, $j \in \{1,\cdots,N+1\}$. To approximate the integration, we apply the following formula for trapezoidal Reimann sum:

$$\int_a^b f(s)\mathrm{d}s \approx \frac{\Delta_s}{2}\sum_{i=1}^{N}(f(s_i)+f(s_{i+1})). \tag{31}$$

1) Approximating the Multiplier: The approximation of $\langle \mathbf{x}, \mathcal{P}_{\{S,[],[]\}}\mathbf{x}\rangle_{L_2}$ at the sequence of points $s_i \in S_d \subset [a,b]$, $i \in \{1,\cdots,N+1\}$ yields:

$$\int_a^b \mathbf{x}^\top(s)M(s)\mathbf{x}(s)\mathrm{d}s \\ \approx \frac{\Delta_s}{2}\sum_{i=1}^{N}\mathbf{x}^\top(s_i)M(s_i)\mathbf{x}(s_i) \\ + \frac{\Delta_s}{2}\sum_{i=1}^{N}\mathbf{x}^\top(s_{i+1})M(s_{i+1})\mathbf{x}(s_{i+1}).$$

2) Approximating the Kernels of Semi-separable Class: The approximation of $\langle \mathbf{x}, \mathcal{P}_{\{[],R_1,[]\}}\mathbf{x}\rangle_{L_2}$ at the sequence of points $(s_i,\eta_j) \in S_d\times S_d \subset [a,b]\times[a,b]$, $i \in \{1,\cdots,N+1\}$, $j \in \{1,\cdots,N+1\}$ yields:

$$\int_a^b \mathbf{x}^\top(s)\Big[\int_a^s R_1(s,\eta)\mathbf{x}(\eta)\mathrm{d}\eta\Big]\mathrm{d}s \\ \approx \frac{\Delta_s^2}{4}\sum_{i=1}^{N}\mathbf{x}^\top(s_i)\sum_{j=1}^{i-1}R_1(s_i,\eta_j)\mathbf{x}(\eta_j) \\ + \frac{\Delta_s^2}{4}\sum_{i=1}^{N}\mathbf{x}^\top(s_i)\sum_{j=1}^{i-1}R_1(s_i,\eta_{j+1})\mathbf{x}(\eta_{j+1}) \\ + \frac{\Delta_s^2}{4}\sum_{i=1}^{N}\mathbf{x}^\top(s_{i+1})\sum_{j=1}^{i}R_1(s_{i+1},\eta_j)\mathbf{x}(\eta_j) \\ + \frac{\Delta_s^2}{4}\sum_{i=1}^{N}\mathbf{x}^\top(s_{i+1})\sum_{j=1}^{i}R_1(s_{i+1},\eta_{j+1})\mathbf{x}(\eta_{j+1}).$$

Similarly, we can approximate $\langle \mathbf{x}, \mathcal{P}_{\{[],[],R_2\}}\mathbf{x}\rangle_{L_2}$ at the sequence of points $(s_i,\eta_j) \in S_d \times S_d \subset [a,b]\times[a,b]$, $i \in \{1,\cdots,N+1\}$, $j \in \{1,\cdots,N+1\}$.

Combining them, we obtain an finite dimensional approximation of $\langle \mathbf{x}, \mathcal{P}_{\{S,R_1,R_2\}}\mathbf{x}\rangle_{L_2}$ as $\bar{\mathbf{x}}^\top\bar{P}\bar{\mathbf{x}}$. Here, $\bar{\mathbf{x}} := \mathrm{col}(\mathbf{x}(a),\cdots,\mathbf{x}(s_i),\cdots,\mathbf{x}(b))$ and $\bar{P} \in \mathbb{R}^{n(N+1)\times n(N+1)}$. Note that, if the original $\mathcal{P}_{\{S,R_1,R_2\}}$ is self-adjoint, its discretization retains its symmetry.

Discussion: Due to the lack of analytic expression for the inversion of $\mathcal{T}$, approximating the inversion by discretization is essential. Such approximation causes additional uncertainty on the accuracy of implementing the estimator. However, such problem is solely an implementation related aspect and by no means, offers any conservatism on either synthesizing the operators $\mathcal{T}$, $\mathcal{Z}$ or guaranteeing optimal performance bound in terms of $\hat{\rho}$-value in the original setting.

VIII. NUMERICAL ILLUSTRATIONS

In this section, we illustrate the $\mathcal{H}_\infty$ estimator with a hyperbolic and a parabolic PDE on the domain $[0,1]$.

A. Hyperbolic PDE: Wave Equation with Boundary Control

Here, we consider the one dimensional wave equation with boundary feedback which has been discussed in Section I as an illustrative example. Using the developed algorithm, we obtain the minimum γ-value, $\hat{\rho} = 0.164$. After implementing the discretized estimator on 50 grid points, Figure 1 shows the disturbance suppression on the evolution of the discretized $z_e(t) := \hat{z}(t) - z(t)$ over time.

B. Parabolic PDE: Coupled Diffusion-Reaction Equation

Here, we consider the following coupled PDEs

$$\begin{bmatrix}\dot{x}_1(s,t)\\ \dot{x}_2(s,t)\end{bmatrix} = \lambda\begin{bmatrix}1 & 0.3\\ 0.1 & 1\end{bmatrix}\begin{bmatrix}x_1(s,t)\\ x_2(s,t)\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1(s,t)\\ x_2(s,t)\end{bmatrix}_{ss} \\ + \begin{bmatrix}s-s^2\\ 0\end{bmatrix}w(t),$$

$$y(t) = \int_a^b \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1(s,t)\\ x_2(s,t)\end{bmatrix}\mathrm{d}s,\ z(t) = \int_a^b \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1(s,t)\\ x_2(s,t)\end{bmatrix}\mathrm{d}s. \tag{32}$$

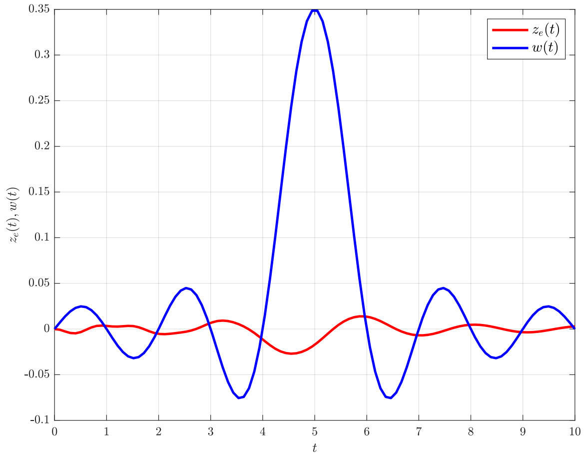

Fig. 1: For Example 1: Time evolution of $z_e(t)$ and $w(t)$ where $w(t)$ is generated by using `sinc` function.

The boundary conditions are of Dirichlet type, i.e.

$$\begin{bmatrix} x_1(a,t) \\ x_2(a,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1(b,t) \\ x_2(b,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{33}$$

Here, we synthesize the estimator for two cases with $\lambda = 5$ and $\lambda = 10$. For $\lambda = 10$, the original model is unstable. The algorithm yields estimators with minimum γ-value (i.e. $\hat{\rho}$) as well as the operators $\mathcal{T}$ and $\mathcal{Z}$. The obtained $\hat{\rho}$ values are a) $\underline{\hat{\rho} = 0.2886}$, for $\lambda = 5$, and b) $\underline{\hat{\rho} = 0.4745}$ for $\lambda = 10$.

After implementing the discretized estimator on 100 grid points, Figure 2 shows the disturbance suppression on the evolution of the discretized $z_e(t) := \hat{z}(t) - z(t)$ over time for $\lambda = 5, 10$.

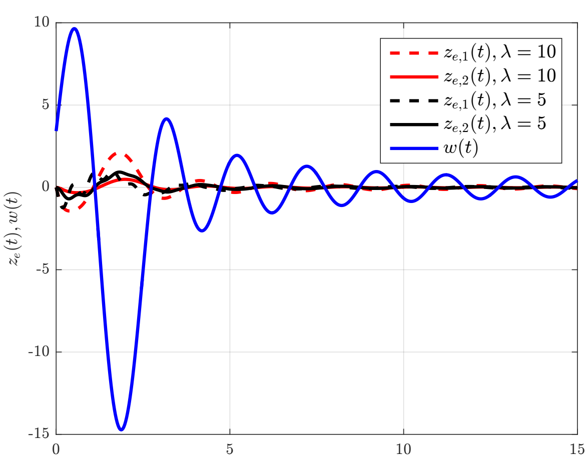

Fig. 2: For Example 2: Time evolution of $z_e(t)$ and $w(t)$ for $\lambda = 5, 10$ where $w(t)$ is generated by damped sinusoidal functions.

IX. CONCLUSIONS

In this paper, we have presented a LMI based framework to design an $\mathcal{H}_\infty$ optimal estimator for linear coupled PDE systems. Instead of using the conventional state definition for PDEs, a new set of fundamental states has been defined that offers a generic framework to describe linear PDEs of both parabolic and hyperbolic type without any explicit dependency on boundary conditions. Using a class of positive operators $\mathcal{P}\left\{\begin{smallmatrix} P, & Q, & Q^\top \\ S, & R_1, & R_2 \end{smallmatrix}\right\}$ that are equipped with multipliers and kernels of semi-separable class, scalable LMI conditions have been derived that determine the optimal $\mathcal{H}_\infty$ observer for coupled PDE systems with no approximation or discretization. A scalable algorithm has been implemented to synthesize the observer. By illustration, we have shown that developed $\mathcal{H}_\infty$ optimal estimator provides the desired performance for both parabolic and hyperbolic type of PDEs.

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Manuscript 1206 submitted to 2019 IEEE Conference on Decision and Control (CDC).
Received March 17, 2019.