

GAUSSIAN COOLING AND $O^*(n^3)$ ALGORITHMS FOR VOLUME
AND GAUSSIAN VOLUME*BEN COUSINS[†] AND SANTOSH VEMPALA[†]

Abstract. We present an $O^*(n^3)$ randomized algorithm for estimating the volume of a well-rounded convex body (e.g., $K \subseteq \mathbb{R}^n$ if $B_n \subseteq K$ and $\mathbb{E}_{X \sim K}(\|X\|^2) = O^*(n)$) given by a membership oracle, improving on the previous best complexity of $O^*(n^4)$. The new algorithmic ingredient is an accelerated cooling schedule where the rate of cooling increases with the temperature. Previously, the known approach for potentially achieving this asymptotic complexity relied on a positive resolution of the Kannan–Lovász–Simonovits (KLS) hyperplane conjecture, a central open problem in convex geometry. We also obtain an $O^*(n^3)$ randomized algorithm for integrating a standard Gaussian distribution over an arbitrary convex set containing the unit ball. Both the volume and the Gaussian volume algorithms use an improved algorithm for sampling a Gaussian distribution restricted to a convex body. In this latter setting, as we show, the KLS conjecture holds and for a spherical Gaussian distribution with variance σ^2 , the sampling complexity is $O^*(\max\{n^3, \sigma^2 n^2\})$ for the first sample and $O^*(\max\{n^2, \sigma^2 n^2\})$ for every subsequent sample.

Key words. volume computation, integration, random walks, high-dimensional algorithms, ball walk, Gaussian isoperimetry

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1. Introduction. Computing the volume of a convex body is an ancient and fundamental problem; it is also a *difficult* problem, as evidenced by both the $\#P$ -hardness of computing the volume of an explicit polytope [11] and exponential lower bounds for deterministic algorithms in the general oracle model, even to approximate the volume to within an exponential factor in the dimension [3, 4]. Against this backdrop, the breakthrough result of Dyer, Frieze, and Kannan [13, 14] established a randomized polynomial-time algorithm for estimating the volume to within any desired accuracy. In the quarter-century since then, the quest for faster volume algorithms has revealed an array of powerful and elegant techniques for the design and analysis of algorithms and has influenced the development of asymptotic convex geometry [1, 20, 12, 21, 18, 19, 7, 24, 22, 16].

The Dyer–Frieze–Kannan (DFK) algorithm for computing the volume of a convex body K in \mathbb{R}^n given by a membership oracle uses a sequence of convex bodies $K_0, K_1, \dots, K_m = K$, starting with the unit ball fully contained in K and ending with K . Each successive body $K_i = 2^{i/n} B_n \cap K$ is a slightly larger ball intersected with K . Using random sampling, the algorithm estimates the ratios of volumes of consecutive bodies. The product of these ratios times the volume of the unit ball was the estimate of the volume of K . Sampling is achieved by a random walk in the convex body. There were many technical issues to be addressed, but the central challenge was to show a random walk that “mixed” rapidly, i.e., converged to its stationary distribution in a polynomial number of steps. The overall complexity of the algorithm was $O^*(n^{23})$.

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[†]College of Computing, Georgia Tech, Atlanta, GA 30308 (bcousins3@gatech.edu, vempala@gatech.edu).

oracle calls.¹

Since then, researchers have improved the complexity of volume computation and sampling for convex bodies considerably, to $O^*(n^4)$ for volume estimation and for obtaining the first random sample [24, 22] and to $O^*(n^3)$ per sample for subsequent samples [22, 23]. These improvements rely on continuous random walks, the use of affine transformations, improved isoperimetric inequalities, and several other developments. However, throughout the course of these developments, the outer DFK algorithm using a chain of bodies remained unchanged until the most recent improvement in 2003 [24]. The Lovász–Vempala (LV) algorithm [24] relies on sampling a sequence of logconcave distributions, akin to simulated annealing, starting with one that is highly concentrated around a point deep inside the convex body and ending with the uniform distribution (we will discuss these ideas in more detail presently). The total number of random points needed is only $O^*(n)$, down from $\Omega(n^2)$ needed by all previous algorithms. Combining this with the $O^*(n^3)$ complexity for each sample yielded the overall $O^*(n^4)$ complexity for volume computation. Before running this algorithm, there is a preprocessing step where the convex body is placed in nearly isotropic position, ensuring in particular that most of the body is contained in a ball of radius $O(\sqrt{n})$. Crucially, this well-roundedness property is maintained during the course of the algorithm.

Is there a faster algorithm? In 1995, Kannan, Lovász, and Simonovits, while analyzing the convergence of the ball walk for sampling, proposed a beautiful geometric conjecture now known as the Kannan–Lovász–Simonovits (KLS) hyperplane conjecture [18]. Roughly speaking, it says that the worst-case isoperimetric ratio for a subset of a convex body is achieved by a hyperplane to within a constant factor. They were able to show that hyperplanes are within $O(\sqrt{n})$ of the minimum. The convergence of the ball walk depends on the square of the reciprocal of the isoperimetric ratio; thus the KLS conjecture had the potential to improve the sampling time by a factor of n to $O^*(n^2)$ per sample and thereby indicated the possibility of an $O^*(n^3)$ volume algorithm (such an algorithm would have to surmount other substantial hurdles).

The KLS hyperplane conjecture remains unresolved, in spite of intensive efforts and partial progress towards its resolution [2, 17, 16]. Indeed, it captures two well-known and much older conjectures from convex geometry, the slicing (or hyperplane) conjecture and the thin-shell conjecture (these were all shown to be equivalent in a certain sense recently [15, 16]) and thus has effectively evaded resolution for nearly a half-century.

Our first result is an $O^*(n^2)$ algorithm for sampling from the standard Gaussian distribution in \mathbb{R}^n restricted to a convex body containing the unit ball. To achieve this complexity, we prove the KLS conjecture for such distributions. We then show that the *Gaussian volume* or Gaussian measure, i.e., the integral of a standard Gaussian over a convex body, can be computed in $O^*(n^3)$ queries, provided the convex body contains the unit ball.

Our main finding is an $O^*(n^3)$ algorithm for computing the volume of any convex body containing a unit ball and mostly contained in a ball of radius $O^*(\sqrt{n})$. Equivalently, it suffices to have $\mathbb{E}(\|X\|^2) = O^*(n)$ for a uniform random point X from the body. Assuming the body is well-rounded (or sandwiched) in this sense, no further affine transformation is used, and there is no need to assume or maintain near-isotropy during the course of the volume algorithm.

To describe the main ideas behind the improvement, we recall the LV algorithm

¹The O^* notation suppresses error terms and logarithmic factors.

in more detail. It uses a sequence of $O^*(\sqrt{n})$ exponential distributions, starting with a distribution that is concentrated inside the unit ball contained in K and then “flattening” this distribution to the uniform by adjusting a multiplicative factor in the exponent.² In each phase, samples from the previous distribution are used to estimate the ratio of the integrals of two consecutive exponential functions (by simply averaging the ratio of the function values at the sample points). It is crucial to keep the variance of this ratio estimator bounded, and to do this, the distributions could be cooled by a factor of $1 + \frac{1}{\sqrt{n}}$ in each phase. This leads to $O^*(\sqrt{n})$ phases in total and to $O^*(\sqrt{n})$ samples per phase. Along with the sample complexity of $O^*(n^3)$ per sample, this gives the bound of $O^*(n^4)$.

The improved complexity for Gaussian volume estimation is achieved by using a sequence of Gaussians (rather than exponentials as in LV), starting with a highly concentrated Gaussian centered inside K and ending with the standard Gaussian. The cooling schedule is the same as in the LV algorithm, but each sample takes only $O^*(n^2)$ time. For a Gaussian with covariance $\sigma^2 I$, the mixing time is $O^*(\max\{\sigma^2, 1\}n^2)$ (see Theorem 1.5 below). Since the starting σ is small and the last σ is 1, this bound is $O^*(n^2)$ throughout the algorithm. (We encounter additional technical issues such as maintaining a warm start for the random walks.)

Returning to the usual Lebesgue volume, how could we possibly improve the LV algorithm without relying on the KLS conjecture? We will also use Gaussian cooling, starting with a highly concentrated Gaussian and flattening it (i.e., increasing σ) until we reach the uniform distribution. In the beginning, this is similar to the algorithm of [9]. But after σ becomes higher than 1 (or some constant), we no longer have quadratic sampling time, as the mixing time of the ball work grows as $\max\{\sigma^2, 1\}n^2$. Moreover, we need to go until $\sigma^2 = \Omega(n)$, so cooling at the fixed rate of $1 + 1/n$ would be too slow. The main new idea is that for $\sigma > 1$, the cooling rate can be made higher, in fact about $1 + \sigma^2/n$ instead of only $1 + 1/n$. This means that the number of phases to double σ^2 is only n/σ^2 . It can be shown that the number of samples per “doubling” phase is only $O^*(1)$, giving n/σ^2 samples in total. Multiplying by the sampling time, we have $\frac{n}{\sigma^2} \cdot \sigma^2 n^2 = n^3$, a cubic algorithm! The key technical component of the analysis is to show that the variance of the ratio estimator remains bounded even at this higher cooling rate of $1 + \sigma^2/n$.

We now formally state the problems.

PROBLEM 1.1 (volume). *Given a membership oracle for a convex set K in \mathbb{R}^n containing the unit ball B_n , and error parameter $\varepsilon > 0$, give an algorithm that computes a number V such that with probability at least 3/4,*

$$(1 - \varepsilon)\text{vol}(K) \leq V \leq (1 + \varepsilon)\text{vol}(K).$$

We denote the Gaussian density function as $\gamma(x) = (2\pi)^{-n/2} \cdot \exp(-\|x\|^2/2)$.

PROBLEM 1.2 (Gaussian volume). *Given a membership oracle for a convex set K in \mathbb{R}^n containing the unit ball B_n , and error parameter $\varepsilon > 0$, give an algorithm that computes a number V such that with probability at least 3/4,*

$$(1 - \varepsilon) \int_K \gamma(x) dx \leq V \leq (1 + \varepsilon) \int_K \gamma(x) dx.$$

²In the original description, the algorithm first created a “pencil” using an extra dimension, but this can be avoided [22].

1.1. Main results. Our main result can be stated more precisely as follows, which solves Problem 1.1 in $O^*(n^3)$ assuming the input body K is well-rounded. We note that the roundness condition can be achieved for any convex body by a preprocessing step consisting of an affine transformation. It is a significantly weaker condition than isotropic position.

THEOREM 1.1. *There is an algorithm that, for any $\varepsilon > 0$, $p > 0$ and any convex body K in \mathbb{R}^n that contains the unit ball and has $\mathbb{E}_K(\|X\|^2) = O(n)$, with probability $1 - p$, approximates the volume of K within relative error ε and has complexity*

$$O\left(\frac{n^3}{\varepsilon^2} \cdot \log^2 n \log^2 \frac{1}{\varepsilon} \log^2 \frac{n}{\varepsilon} \log \frac{1}{p}\right) = O^*(n^3)$$

in the membership oracle model.

More generally, if $\mathbb{E}_K(\|X\|^2) = R^2$, then the algorithm has complexity

$$O\left(\frac{\max\{R^2 n^2, n^3\}}{\varepsilon^2} \cdot \log^2 n \log^2 \frac{1}{\varepsilon} \log^2 \frac{n}{\varepsilon} \log \frac{1}{p}\right) = O^*(\max\{R^2 n^2, n^3\}).$$

The current best complexity for achieving well-roundedness, i.e., $R^2 = O^*(n)$, for a convex body is $O^*(n^4)$ [24]. In previous work, the complexity of generating the first nearly uniform random point was always significantly higher than for later points. Here, using a faster cooling schedule, we can generate the first random point in $O^*(n^3)$ steps under the same assumption that K is well-rounded. Any subsequent uniform random points also require $O^*(n^3)$ steps.

THEOREM 1.2. *There is an algorithm that, for any $\varepsilon > 0$, $p > 0$ and any convex body K in \mathbb{R}^n that contains the unit ball and has $\mathbb{E}_K(\|X\|^2) = R^2$, with probability $1 - p$, generates random points from a density ν that is within total variation distance ε from the uniform distribution on K . In the membership oracle model, the complexity of each random point, including the first, is*

$$O\left(\max\{R^2 n^2, n^3\} \log n \log^2 \frac{n}{\varepsilon} \log \frac{1}{p}\right) = O^*(\max\{R^2 n^2, n^3\}).$$

In addition to volume and uniform sampling, we also have an $O^*(n^3)$ algorithm for computing the Gaussian volume. This algorithm does not require a rounding preprocessing step and gives an $O^*(n^3)$ algorithm for any convex set $K \subseteq \mathbb{R}^n$ containing the unit ball.

THEOREM 1.3. *For any $\varepsilon > 0$, $p > 0$ and any convex set K in \mathbb{R}^n containing the unit ball, there is an algorithm that, with probability $1 - p$, approximates the Gaussian volume of K within relative error ε and has complexity*

$$O\left(\frac{n^3}{\varepsilon^2} \cdot \log^2(n) \log^2\left(\frac{n}{\varepsilon}\right) \log\left(\frac{1}{p}\right)\right) = O^*(n^3)$$

in the membership oracle model.

Both the uniform volume and the Gaussian volume algorithms utilize an improved sampling algorithm for Gaussian distributions restricted by convex sets.

THEOREM 1.4. *For any $\varepsilon > 0$, $p > 0$ and any convex set K in \mathbb{R}^n containing the unit ball, there is an algorithm that, with probability $1 - p$, can generate a random*

point within total variation distance ε of the Gaussian density $\mathcal{N}(0, \sigma^2 I)$ restricted to K . In the membership oracle model, the complexity of the first random point is

$$O\left(\max\{\sigma^2, 1\}n^3 \log(n) \log^2\left(\frac{n}{\varepsilon}\right) \log\left(\frac{1}{p}\right)\right) = O^*(\max\{\sigma^2, 1\}n^3).$$

For subsequent random points, the complexity is

$$O\left(\max\{\sigma^2, 1\}n^2 \log\left(\frac{n}{\varepsilon}\right) \log\left(\frac{1}{\varepsilon}\right)\right) = O^*(\max\{\sigma^2, 1\}n^2).$$

The set of random points will be ε -independent.

For two random variables X and Y , we say X and Y are ε -independent if

$$\sup_{A, B} |\Pr(X \in A, Y \in B) - \Pr(X \in A) \Pr(Y \in B)| \leq \varepsilon,$$

where A, B range over measurable subsets of the ranges of X, Y . The following theorem guarantees we can efficiently obtain Gaussian samples from a warm start.

THEOREM 1.5. *Let K be a convex set containing the unit ball, Q_0 be a starting distribution, and Q be the target Gaussian density $\mathcal{N}(0, \sigma^2 I)$ restricted to $K \cap 4\sigma\sqrt{n}B_n$. For any $\nu > 0, p > 0$, the lazy Metropolis ball walk with δ -steps for $\delta = \min\{\sigma, 1\}/(4096\sqrt{n \log n/\nu})$, starting from Q_0 , satisfies $d_{tv}(Q_t, Q) \leq \nu$ after*

$$t \geq C \cdot M(Q_0, Q) \cdot \max\{\sigma^2, 1\} \cdot n^2 \log\left(\frac{n}{\nu}\right) \log\left(\frac{M(Q_0, Q)}{\nu}\right)$$

expected steps for an absolute constant C .

Here $M(Q_0, Q)$ is a measure of how close Q_0 is to Q (also called the *warm start* parameter) and is defined as $M(Q_0, Q) = \sup_{S \subseteq K} \frac{Q_0(S)}{Q(S)}$. In other words, the theorem says that the ball walk mixes in $O^*(\max\{\sigma^2, 1\}n^2)$ steps from a warm start.

Ball Walk(δ, f)

At point x :

1. Pick a random point y from $x + \delta B_n$.
2. Go to y with probability $\min\{1, f(y)/f(x)\}$; otherwise, stay at x .

FIG. 1. The ball walk with a Metropolis filter.

2. Algorithm. At a high level, the algorithm relies on sampling random points from a sequence of distributions using the ball walk with a Metropolis filter. For a target density proportional to the function f , the ball walk with δ -steps is defined in Figure 1.

After a suitable number of steps, the point x obtained will be from a distribution close to the one whose density is proportional to f . However, this process is slightly complicated by the fact that we only know that the point is mixed once a certain number of *proper* steps have been taken, i.e., steps where $y \in K$ or alternatively where $f(y) \neq 0$.

The algorithm in Figure 2 starts with a Gaussian of variance $1/(4n)$, with the mean at the center of the unit ball inside K . This variance is increased over a sequence

Volume(K, ε) We assume $B_n \subseteq K \subseteq C\sqrt{n}B_n$.

1. Initialize: $\nu = \left(\frac{\varepsilon}{n}\right)^{16}$, $\sigma_0^2 = \frac{1}{4n}$, $k = \frac{512 \log C^2 n}{\varepsilon^2}$, $i = 0$; x_0 is a random point from $N(0, \sigma_0^2 I) \cap K$.
Define
$$\beta(\sigma) = \begin{cases} 1 + \frac{1}{n} & \text{if } \sigma^2 \leq 1, \\ 1 + \frac{\sigma^2}{2C^2 n} & \text{otherwise.} \end{cases}$$
2. While $\sigma_i^2 \leq C^2 n$:
 - (a) Get k points $\{X_1, \dots, X_k\}$ using the **Ball Walk** with
$$\begin{cases} \delta = \min\{\sigma_i, 1\}/(4096\sqrt{n \log n/\varepsilon}) & \text{ball radius,} \\ f_i = f(\sigma_i^2, K \cap 4\sigma_i\sqrt{n}B_n) & \text{target density,} \\ 10^{16} \max\{\sigma^2, 1\} n^2 \cdot \log(1/\nu) & \text{proper steps.} \end{cases}$$
 - (b) Set $\sigma_{i+1}^2 = \sigma_i^2 \cdot \beta(\sigma_i)$; if $\sigma_{i+1}^2 > C^2 n$, set $\sigma_{i+1}^2 = \infty$.
 - (c) Compute the ratio estimate
$$W_{i+1} = \frac{1}{k} \cdot \sum_{j=1}^k \frac{f_{i+1}(X_j)}{f_i(X_j)}.$$
 - (d) Increment i .
3. Return $(2\pi\sigma_0^2)^{n/2} W_1 \dots W_i$ as the volume estimate for K .

FIG. 2. *The volume algorithm.*

of phases until the distribution becomes uniform over K . Until the variance σ^2 reaches 1, it is increased by a fixed factor of $1 + 1/n$ in each phase. After the variance reaches 1, the variance accelerates, increasing by a factor of $1 + \sigma^2/(2C^2 n)$ where σ^2 is the current variance. This process is continued until the variance becomes linear in $C^2 n$, at which point one final phase can be used to jump to the uniform distribution. In each phase, we pick a sample of random points from the current distribution and compute the average of the ratio of the current density to the next density for each point. The product of these ratios times a fixed term to account for the integral of the initial function is the estimate output by the algorithm.

Let $f(\sigma^2, K)$ be the function that assigns value $\exp(-\|x\|^2/(2\sigma^2))$ to points in a convex set K and zero to points outside. The algorithm below uses a series of such functions.

3. Outline of analysis.

3.1. Outline of sampling analysis. To show that the random walk quickly reaches its stationary distribution, we will use the standard method of bounding the conductance. For the ball walk, this runs into a hurdle, namely that the local conductance of points near sharp corners of the body can be arbitrarily small, so the walk can get stuck and waste a large number of steps. To avoid this, we could start the walk from a random point chosen from a distribution sufficiently close to the target distribution. But how do we generate random points from such a starting distribution? We do this by considering a sequence of distributions, each providing a warm start for

the next. The very first distribution is chosen to be a highly concentrated Gaussian so that it lies almost entirely inside the unit ball (inside K). Thus sampling from the initial distribution is easy by standard rejection sampling. Each successive Gaussian is “flatter,” with the final one being the target distribution, e.g., standard Gaussian or uniform distribution.

The next challenge is to show that, from a warm start, the expected number of steps to converge to the stationary distribution is only $O^*(n^2)$. This is usually done by bounding the conductance of the Markov chain. The conductance, ϕ , of a Markov chain with state space K and next-step distribution P_x is defined as

$$\phi = \min_{S \subset K} \frac{\int_S P_x(K \setminus S) dQ(x)}{\min Q(S), Q(K \setminus S)}.$$

Unfortunately, for the ball walk, this can be arbitrarily small, e.g., for points near corners (but also for points in the interior). To utilize the warm start, we use an idea from [19], namely the *speedy walk*. We emphasize that the speedy walk cannot be implemented efficiently and is only a tool for analysis. It is defined as follows.

At current point x , do the following:

1. Pick random point y from $K \cap x + \delta B_n$.
2. Go to y with probability $\min\{1, f(y)/f(x)\}$; otherwise, stay at x .

To capture the stationary distribution of the speedy walk with a Metropolis filter, we need another parameter. The *local conductance* at x for the speedy walk, without a filter, is defined as follows:

$$\ell(x) = \frac{\text{vol}(K \cap x + \delta B_n)}{\text{vol}(\delta B_n)}.$$

The following lemma can be proved using detailed balance.

LEMMA 3.1. *The stationary distribution of the speedy walk with a Metropolis filter applied with a function f has density proportional to $\ell(x)f(x)$.*

For the speedy walk with $\delta = O(1/\sqrt{n})$, we can show that the conductance is $\Omega(1/(\sigma n))$, and so the total number of steps needed is only $O^*(\sigma^2 n^2)$. This is a factor n faster than previous best bounds. We do this by establishing a stronger (and nearly optimal) isoperimetric inequality.

As noted, the speedy walk cannot actually be implemented efficiently. To bound the Metropolis ball walk, we can view it as an interleaving of a speedy walk with wasted steps. Let the Markov chain for the original walk be $w_0, w_1, \dots, w_i, \dots$. The subsequence w_{i_1}, w_{i_2}, \dots , where we record x if the point y chosen by the Metropolis ball walk is in K , corresponds to the speedy walk. We then need to estimate the number of wasted steps from a warm start. We will show that this is at most a constant factor higher than the number of proper steps. The key ingredient of this analysis is the (known) fact that, for a body containing the unit ball, the average local conductance is high for ball radius $\delta = O(1/\sqrt{n})$. Even within the speedy walk, there are “null” steps due to the Metropolis filter. However, by restricting the walk to a large ball, we ensure that the probability of rejection by the filter is bounded by a constant, and therefore the number of wasted steps within the speedy walk is at most a constant fraction of all steps. Also, the speedy walk converges to a distribution proportional to $\ell(x)f(x)$, but we can map this to a random sample from f with rejection sampling routine (see section 6.4).

To sample efficiently, we need a warm start for each phase. For two probability

distributions P and Q with state space K , the M -warmness of P and Q is defined as

$$(3.1) \quad M(P, Q) = \sup_{S \subseteq K} \frac{P(S)}{Q(S)}.$$

To keep this parameter bounded by a constant, we use a finer-grained cooling schedule so that a random point from one phase is a warm start for the next phase. This cooling schedule is also different in the two parts. In the first part of the algorithm, we can cool at the rate of $1 + 1/n$ and use $O^*(n^2)$ steps to sample. In the second part, we cool at the rate of $1 + \sigma^2/(2C^2n)$, and this is fast enough to compensate for the higher sample complexity of $O^*(\sigma^2 n^2)$. Thus the overall time to obtain a warm start for every phase of the algorithm is also $O^*(n^3)$. We analyze this in full detail in section 6, including the proof that this cooling rate maintains a warm start from one phase to the next.

We can obtain uniform random samples from K given samples from a Gaussian with variance $\sigma^2 = C^2n$ via a simple rejection sampling routine. Since $K \subseteq C\sqrt{n}B_n$, the two distributions will be within a constant factor of each other, and therefore we can use $O(1)$ expected samples from the Gaussian distribution to obtain a uniform random point.

3.2. Outline of volume analysis. The sampling time when the variance is σ^2 is $\max\{1, \sigma^2\}n^2$. If we cooled at a rate of $1 + 1/n$ throughout the algorithm, we would get an $O^*(n^4)$ algorithm since the last doubling phase, i.e., the set of phases until σ^2 doubles, takes $\Omega(n)$ samples, each mixing for $\Omega(n^3)$ steps. The main insight that speeds up our algorithm is the cooling rate of $1 + \sigma^2/(2C^2n)$ once $\sigma^2 > 1$. Cooling at a faster rate once $\sigma^2 > 1$ will allow us to compute volume in time $O^*(n^3)$ by having fewer phases when the mixing time of the ball walk increases.

The volume algorithm proceeds as a series of phases, where each phase seeks to estimate a ratio of Gaussian integrals over the convex body K . More precisely, let

$$f(\sigma^2, x) = \begin{cases} \exp(-\|x\|^2/(2\sigma^2)) & \text{if } x \in K, \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$F(\sigma^2) = \int_{\mathbb{R}^n} f(\sigma^2, x) dx.$$

Define μ_i as the probability distribution proportional to $f(\sigma_i^2, x)$; that is, μ_i is a symmetric Gaussian distribution with variance σ_i^2 restricted to K . Let X be a random sample point from μ_i , and let $Y = f(\sigma_{i+1}^2, X)/f(\sigma_i^2, X)$. We see that the expectation of Y is the ratio of $F(\sigma_{i+1}^2)/F(\sigma_i^2)$:

$$\begin{aligned} \mathbb{E}(Y) &= \int_K \exp\left(\frac{\|x\|^2}{2\sigma_i^2} - \frac{\|x\|^2}{2\sigma_{i+1}^2}\right) d\mu_i(x) \\ &= \int_K \exp\left(\frac{\|x\|^2}{2\sigma_i^2} - \frac{\|x\|^2}{2\sigma_{i+1}^2}\right) \cdot \frac{\exp(-\|x\|^2/(2\sigma_i^2))}{F(\sigma_i^2)} dx \\ &= \frac{1}{F(\sigma_i^2)} \cdot \int_K \exp\left(-\frac{\|x\|^2}{2\sigma_{i+1}^2}\right) dx = \frac{F(\sigma_{i+1}^2)}{F(\sigma_i^2)}. \end{aligned}$$

Our goal is to estimate $\mathbb{E}(Y)$ within some target relative error. The algorithm estimates the quantity $\mathbb{E}(Y)$ by taking random sample points X_1, \dots, X_k and computing

the empirical estimate for $\mathbb{E}(Y)$ from the corresponding Y_1, \dots, Y_k :

$$W = \frac{1}{k} \sum_{j=1}^k Y_j = \frac{1}{k} \sum_{j=1}^k \frac{f_{i+1}(X_j)}{f_i(X_j)}.$$

The variance of Y divided by its expectation squared will give a bound on how many independent samples X_i are needed to estimate $\mathbb{E}(Y)$ within the target accuracy. Thus we seek to bound $\mathbb{E}(Y^2)/\mathbb{E}(Y)^2$. We have that

$$\mathbb{E}(Y^2) = \frac{\int_K \exp\left(\frac{\|x\|^2}{2\sigma_i^2} - \frac{\|x\|^2}{\sigma_{i+1}^2}\right) dx}{\int_K \exp\left(-\frac{\|x\|^2}{2\sigma_i^2}\right) dx} = \frac{F\left(\frac{\sigma_{i+1}^2 \sigma_i^2}{2\sigma_i^2 - \sigma_{i+1}^2}\right)}{F(\sigma_i^2)}$$

and

$$\frac{\mathbb{E}(Y^2)}{\mathbb{E}(Y)^2} = \frac{F(\sigma_i^2)F\left(\frac{\sigma_{i+1}^2 \sigma_i^2}{2\sigma_i^2 - \sigma_{i+1}^2}\right)}{F(\sigma_{i+1}^2)^2}.$$

If we let $\sigma^2 = \sigma_{i+1}^2$ and $\sigma_i^2 = \sigma^2/(1+\alpha)$, then we can further simplify to

$$\frac{\mathbb{E}(Y^2)}{\mathbb{E}(Y)^2} = \frac{F\left(\frac{\sigma^2}{1+\alpha}\right)F\left(\frac{\sigma^2}{1-\alpha}\right)}{F(\sigma^2)^2}.$$

The algorithm has two parts, and the cooling rate α_i is different for them. In the first part, starting with a Gaussian of variance $\sigma^2 = 1/(4n)$, which has almost all its measure inside the ball contained in K , we increase σ^2 by a fixed factor of $1 + 1/n$ in each phase until the variance σ^2 reaches 1. When cooling at the fixed rate of $1 + 1/n$, Lemma 7.12 implies that the variance of the ratio estimator is small enough that only $O^*(1)$ samples suffice. For each σ , we sample random points from the corresponding distribution and estimate the ratio of the densities for the current phase and the next phase by averaging over samples. The total complexity for the first part is thus

$$O^*(n) \text{ phases} \times O^*(1) \text{ samples per phase} \times O^*(n^2) \text{ steps per sample} = O^*(n^3).$$

In the second part, we increase the variance until it reaches C^2n , after which one final phase suffices to compare with the target uniform distribution. However, we cannot afford to cool at the same rate of $1 + 1/n$ because the time per sample goes to $O^*(\sigma^2 n^2)$ for $\sigma > 1$. By the end of this part, we would be using $O^*(n^3)$ per sample, and the overall complexity would be $O^*(n^4)$. Instead we observe that we can cool at a faster rate of $1 + \sigma^2/(2C^2n)$ and still maintain that the variance of the ratio estimator is a constant. The following bound on the variance, proved in section 7.1, allows us to cool at a faster rate as σ increases and overcome the increased sampling cost of $O^*(\sigma^2 n^2)$.

LEMMA 3.2. *Let $K \subseteq C\sqrt{n}B_n$ and $\alpha \leq 1/2$. Then*

$$\frac{F\left(\frac{\sigma^2}{1+\alpha}\right)F\left(\frac{\sigma^2}{1-\alpha}\right)}{F(\sigma^2)} \leq \exp\left(2 \cdot \frac{C^2 \alpha^2 n}{\sigma^2}\right).$$

Note that the above right-hand side is $\leq 1 + \sigma^2/(Cn)$ if we select $\alpha = \sigma^2/(2C^2n)$. With this rate, the number of phases needed to double the variance is only $O(C^2 n / \sigma^2)$,

and the number of samples per phase will be $O^*(1)$. Together, they compensate for the higher complexity of obtaining each sample. The complexity of the second part of the algorithm is thus

$$\begin{aligned} O^*\left(\frac{C^2 n}{\sigma^2}\right) \text{ phases} \times O^*(1) \text{ samples per phase} \times O^*(\sigma^2 n^2) \text{ steps per sample} \\ = O^*(C^2 n^3). \end{aligned}$$

In section 7.1, we prove that cooling at this accelerated rate still keeps the variance of the ratio estimator appropriately bounded.

We note that with respect to estimating the volume using Lemma 3.2 there is a range of cooling rates for the second part of the algorithm that we could select to obtain an $O^*(C^2 n^3)$ algorithm. If we instead use the cooling rate $1 + \sigma/(C\sqrt{n})$, then the complexity of the algorithm is

$$\begin{aligned} O^*\left(\frac{C\sqrt{n}}{\sigma}\right) \text{ phases} \times O^*\left(\frac{C\sqrt{n}}{\sigma}\right) \text{ samples per phase} \times O^*(\sigma^2 n^2) \text{ steps per sample} \\ = O^*(C^2 n^3). \end{aligned}$$

We could select any α in the range $\sigma^2/(C^2 n) \leq \alpha \leq \sigma/(C\sqrt{n})$ and cool at the rate $1 + \alpha$, as there is a proportional tradeoff between the number of phases and samples per phase for this range of α . We select the cooling rate of $\alpha = \sigma^2/(2C^2 n)$ for simplicity of the algorithm since this cooling rate also maintains a warm start for the ball walk sampler, as shown in Lemma 6.8.

4. Preliminaries. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is *logconcave* if it has convex support and the logarithm of f , wherever f is nonzero, is concave. Equivalently, f is logconcave if, for any $x, y \in \mathbb{R}^n$ and any $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}.$$

Let $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be the density of the standard Gaussian $\mathcal{N}(0, I)$.

For two probability distributions P and Q with state space K , we will use $M(P, Q)$ to denote the M -warmness between P and Q as defined in (3.1) and $d_{tv}(P, Q)$ to denote the total variation distance between P and Q :

$$d_{tv}(P, Q) = \sup_{S \subseteq K} |P(S) - Q(S)|.$$

For a nonnegative function $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$, we define the f -distance between two points $u, v \in \mathbb{R}^n$ as

$$d_f(u, v) = \frac{|f(u) - f(v)|}{\max\{f(u), f(v)\}}.$$

5. Isoperimetry. The following theorem is due to Brascamp and Lieb.

THEOREM 5.1 (Theorem 5.1 in [5]). *Let $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be the standard Gaussian density in \mathbb{R}^n . Let $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be any logconcave function. Define the density function h over \mathbb{R}^n as follows:*

$$h(x) = \frac{f(x)\gamma(x)}{\int_{\mathbb{R}^n} f(y)\gamma(y) dy}.$$

Fix a unit vector $v \in \mathbb{R}^n$, and let $\mu = \mathbf{E}_h(x)$. Then, for any $\alpha \geq 1$,

$$\mathbf{E}_h(|v^T(x - \mu)|^\alpha) \leq \mathbf{E}_\gamma(|x_1|^\alpha).$$

We can use Theorem 5.1 to say that h inherits the following well-known Gaussian concentration inequality (see, e.g., [6]):

$$(5.1) \quad \Pr_{X \sim \mathcal{N}(0, I)} (\|X\|^2 \geq n + ct\sqrt{n}) \leq \exp(-t^2)$$

for an absolute constant c .

COROLLARY 5.2. *For h as defined in Theorem 5.1 and any $t \geq 1$,*

$$\Pr_h (\|x - \mu\|^2 \geq n + ct\sqrt{n}) \leq e^{-t^2}$$

for an absolute constant c .

The next lemma about one-dimensional isoperimetry is from [18].

LEMMA 5.3 (see [18]). *For any one-dimensional isotropic logconcave function f and any partition S_1, S_2, S_3 of the real line,*

$$\pi_f(S_3) \geq \ln(2) d(S_1, S_2) \pi_f(S_1) \pi_f(S_2).$$

THEOREM 5.4. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be any logconcave function and $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be the standard Gaussian density. Let π denote the probability distribution proportional to $h(x) = f(x)\gamma(x)$. Let S_1, S_2, S_3 partition \mathbb{R}^n such that, for any $u \in S_1, v \in S_2$, either $\|u - v\| \geq d/\ln(2)$ or $d_h(u, v) \geq 4d\sqrt{n}$. Then*

$$\pi(S_3) \geq \frac{d}{\sigma} \pi(S_1) \pi(S_2).$$

Proof. We prove the theorem for the case $\sigma = 1$ and then note that by applying the scaling $x = y/\sigma$ we get the general case.

Our main tool, as in previous work, is the localization lemma of Lovász and Simonovits [21]. Suppose the conclusion is false. Define $h(x) = f(x)\gamma(x)$. Then there exists a partition S_1, S_2, S_3 for which, for some positive real number A ,

$$\begin{aligned} \int_{S_1} h(x) dx &= A \int_{\mathbb{R}^n} h(x) dx, \\ \int_{S_3} h(x) dx &< dA \int_{S_2} h(x) dx. \end{aligned}$$

By the localization lemma, there must be a “needle” given by $a, b \in \mathbb{R}^n$ and a non-negative linear function $l : [0, 1] \rightarrow \mathbb{R}_+$ for which

$$\begin{aligned} &\int_{(1-t)a+tb \in S_1 \cap [0,1]} h((1-t)a+tb)l(t)^{n-1} dt \\ &= A \int_{(1-t)a+tb \in [0,1]} h((1-t)a+tb)l(t)^{n-1} dt \end{aligned}$$

and

$$\begin{aligned} &\int_{(1-t)a+tb \in S_3 \cap [0,1]} h((1-t)a+tb)l(t)^{n-1} dt \\ &< dA \int_{(1-t)a+tb \in S_2 \cap [0,1]} h((1-t)a+tb)l(t)^{n-1} dt. \end{aligned}$$

We can assume that $Z_i = \{t : (1-t)a + tb \in S_i\}$ are intervals that partition $[a, b]$; otherwise, we can apply a standard combinatorial argument as in the proof of Theorem 5.2 in [18] to reduce to the interval case. Thus, to reach a contradiction, it suffices to prove that for a one-dimensional logconcave function $h(t) = f((1-t)a + tb)\gamma((1-t)a + tb)$ with support $[a, b] \subset \mathbb{R}$ and $a \leq u \leq v \leq b$ the following statements hold:

$$(5.2) \quad \int_a^b h(t)l(t)^{n-1} dt \int_u^v h(t)l(t)^{n-1} dt \geq \frac{d_h(u, v)}{4\sqrt{n}} \int_a^u h(t)l(t)^{n-1} dt \int_v^b h(t)l(t)^{n-1} dt,$$

$$(5.3) \quad \int_a^b h(t)l(t)^{n-1} dt \int_u^v h(t)l(t)^{n-1} dt \geq \ln(2)\|u - v\| \int_a^u h(t)l(t)^{n-1} dt \int_v^b h(t)l(t)^{n-1} dt.$$

The first inequality (5.2) follows directly from Lemma 3.8 in [19] by taking $F = h$ and $g = \ell$. To see the second inequality (5.3), we first note that by applying Theorem 5.1, with $\alpha = 2$, we have that the variance of the distribution proportional to $h(t)l(t)^{n-1}$ is at most 1. This is because $h(t)l(t)^{n-1} = (f((1-t)a + tb)l(t)^{n-1})\gamma((1-t)a + tb)$ and the $f((1-t)a + tb)l(t)^{n-1}$ is itself a logconcave function. Now we note that by scaling down to increase the variance to exactly 1 the isoperimetric inequality still holds. Hence, the second inequality is implied by Lemma 5.3. \square

6. Sampling. The analysis of the sampling algorithm is divided into several parts: bounding the conductance of the speedy walk, bounding the warmth of the distribution from one phase to the next, the mixing time of the Metropolis ball walk from a warm start, and finally the complexity of sampling.

6.1. Conductance. For this section, let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ denote the Gaussian density function

$$f(x) = \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right).$$

For $x \in K$ and $S \subseteq K$, let $P_x(S)$ denote the probability that one step of the speedy walk from x is in S :

$$P_x(S) = \frac{\int_{S \cap (x + \delta B_n)} \min\left\{1, \frac{f(y)}{f(x)}\right\} dy}{\text{vol}(K \cap (x + \delta B_n))} = \frac{\int_{S \cap (x + \delta B_n)} \min\left\{1, \frac{f(y)}{f(x)}\right\} dy}{\ell(x)\text{vol}(\delta B_n)}.$$

We bound the rate of convergence of the random walk through a lower bound on the conductance. The conductance ϕ of a Markov chain with state space K and next-step distribution P_x is defined as

$$\phi = \min_{S \subseteq K} \frac{\int_S P_x(K \setminus S) dQ(x)}{\min\{Q(S), Q(K \setminus S)\}}.$$

We will make use of the following theorem of Lovász and Simonovits [21] to bound the total variation distance between the current distribution and the target distribution.

THEOREM 6.1 (see [21]). *Let Q_t be the distribution after t steps of a lazy Markov chain and Q be its stationary distribution. Suppose that Q_0 is M -warm with respect to Q . Then*

$$d_{tv}(Q_t, Q) \leq \sqrt{M} \left(1 - \frac{\phi^2}{2}\right)^t.$$

LEMMA 6.2. *For the speedy walk applied to a convex body $K \subseteq 4\sigma\sqrt{n}B_n$ with $\delta \leq \sigma/(8\sqrt{n})$, the acceptance probability of the Metropolis filter is at least $\frac{1}{e}$.*

Proof. Assume that $f(x) \leq f(u)$ (otherwise, the Metropolis filter always accepts). Then the acceptance probability is

$$\begin{aligned} \frac{f(x)}{f(u)} &= \exp\left(-\frac{\|x\|^2 - \|u\|^2}{2\sigma^2}\right) \\ &\geq \exp\left(-\frac{(\|u\| + \delta)^2 - \|u\|^2}{2\sigma^2}\right) \\ &= \exp\left(-\frac{2\delta\|u\| + \delta^2}{2\sigma^2}\right) \\ &\geq \exp\left(-\frac{\sigma^2 + \sigma^2/(64n)}{2\sigma^2}\right) \\ &\geq \frac{1}{e}. \end{aligned}$$

□

LEMMA 6.3. *Let $K \subseteq 4\sigma\sqrt{n}B_n$ be a convex body, and let $u, v \in K$ such that $\|u - v\| \leq \delta/\sqrt{n}$. If $\delta \leq \sigma/(8\sqrt{n})$, $n \geq 2$, and*

$$\frac{|\ell(u)f(u) - \ell(v)f(v)|}{\max\{\ell(u)f(u), \ell(v)f(v)\}} < \frac{1}{4},$$

then

$$\frac{|\ell(u) - \ell(v)|}{\max\{\ell(u), \ell(v)\}} < \frac{1}{2}.$$

Proof. Assume without loss of generality that $\ell(u)f(u) \geq \ell(v)f(v)$. It follows that

$$\frac{3f(u)\ell(u)}{4} < f(v)\ell(v).$$

Since $\|u - v\| \leq \delta/\sqrt{n}$, we have that $2/3 \leq f(v)/f(u) \leq 3/2$ for $n \geq 2$ (by an argument identical to that in Lemma 6.2). Thus

$$\ell(u) < \frac{4f(v)\ell(v)}{3f(u)} \leq 2\ell(v).$$

By assumption, we have that $f(u)\ell(u) > f(v)\ell(v)$. Thus we also have that

$$\ell(v) < 2\ell(u).$$

The lemma then follows. □

The following lemma bounds the overlap for a step of the speedy walk with respect to the speedy walk. We then show that the Gaussian weighting only hurts by a constant factor. For technical convenience, we define a slightly different one-step distribution that ignores the Gaussian filter:

$$P_x^{\text{unif}}(S) = \frac{\text{vol}(S \cap (x + \delta B_n))}{\ell(x)}$$

for $x \in K, S \subseteq K$.

LEMMA 6.4. *Let K be a convex set with $S \subseteq K$. Let $\bar{S} = K \setminus S$. Suppose that $d_\ell(u, v) < 1/2$ and $\|u - v\| \leq \delta/\sqrt{n}$. Then, for $u \in S$ and $v \in \bar{S}$,*

$$P_u^{\text{unif}}(\bar{S}) + P_v^{\text{unif}}(S) > \frac{1}{2(e+1)}.$$

Proof. Let $B_u = u + \delta B_n$, and let $C = B_u \cap B_v$. By Lemma 3.5 from [18], we know that

$$(6.1) \quad \text{vol}(K \cap C) \geq \frac{\text{vol}(\delta B_n)}{e+1} \min\{\ell(u), \ell(v)\}.$$

We have that

$$P_u^{\text{unif}}(\bar{S}) = \frac{\text{vol}(\bar{S} \cap B_u)}{\ell(u)\text{vol}(\delta B_n)} \geq \frac{\text{vol}(\bar{S} \cap C)}{\ell(u)\text{vol}(\delta B_n)},$$

and similarly for $P_v^{\text{unif}}(S)$. Assume that $\ell(u) \geq \ell(v)$, which implies that $\ell(u) \leq 2\ell(v)$. Therefore,

$$\begin{aligned} P_u^{\text{unif}}(\bar{S}) + P_v^{\text{unif}}(S) &\geq \frac{\text{vol}(\bar{S} \cap C)}{\ell(u)\text{vol}(\delta B_n)} + \frac{\text{vol}(S \cap C)}{\ell(v)\text{vol}(\delta B_n)} \\ &\geq \frac{\text{vol}(\bar{S} \cap C)}{2\ell(v)\text{vol}(\delta B_n)} + \frac{\text{vol}(S \cap C)}{\ell(v)\text{vol}(\delta B_n)} \\ &\geq \frac{\text{vol}(K \cap C)}{2\ell(v)\text{vol}(\delta B_n)} \\ &\geq \frac{1}{2(e+1)}. \end{aligned} \quad \square$$

It then follows that the Gaussian filter decreases the overlap by at most a constant factor.

COROLLARY 6.5. *Let S, \bar{S} be a partition of a convex body $K \subseteq 4\sigma\sqrt{n}B_n$ and $u \in S, v \in \bar{S}$ be such that $\|u - v\| < \delta/\sqrt{n}$ and $d_h(u, v) < 1/4$, where $h(x) = f(x)\ell(x)$. Then*

$$P_u(\bar{S}) + P_v(S) > \frac{1}{2e(e+1)} > \frac{1}{25}.$$

Proof. By Lemma 6.3, we know that $d_\ell(u, v) < 1/2$. We then apply Lemma 6.4, while noting that the Gaussian weighting affects the one-step distributions by at most a $1/e$ factor since that is a lower bound on the acceptance probability of the Metropolis filter (Lemma 6.2). \square

We can now prove the desired lower bound on the conductance of the speedy walk with respect to a Gaussian weighting over a convex set.

THEOREM 6.6. *Let K be a convex body such that $B_n \subseteq K \subseteq 4\sigma\sqrt{n}B_n$. The conductance of the speedy walk applied to K with Gaussian density $\mathcal{N}(0, \sigma^2 I)$ and $\delta \leq \sigma/8\sqrt{n}$ steps is $\Omega(\frac{\delta}{\sigma\sqrt{n}})$.*

Proof. Let $h(x) = \ell(x)f(x)$, and let π be the probability distribution proportional to h . Let $S \subset K$ be an arbitrary measurable set of K , and let $\bar{S} = K \setminus S$. Assume that $\pi(S) \leq 1/2$. Consider the following partition of K :

$$\begin{aligned} S_1 &= \left\{ x \in S : P_x(\bar{S}) < \frac{1}{25} \right\}, \\ S_2 &= \left\{ x \in \bar{S} : P_x(S) < \frac{1}{25} \right\}, \\ S_3 &= K \setminus S_1 \setminus S_2. \end{aligned}$$

By Corollary 6.5, we have that, for any $u \in S_1, v \in S_2$, either $\|u - v\| \geq \delta/\sqrt{n}$ or $d_h(u, v) \geq 1/4$.

We may assume that $\pi(S_1) \geq \pi(S)/2$ and $\pi(S_2) \geq \pi(\bar{S})/2$. If not, we can bound the conductance of S as follows (similarly for \bar{S}):

$$\begin{aligned} \phi(S) &= \frac{\int_S P_x(\bar{S})h(x) dx}{\pi(S)} \\ &= \frac{1}{2} \frac{\int_S P_x(\bar{S})h(x) dx + \int_{\bar{S}} P_x(S)h(x) dx}{\pi(S)} \\ &\geq \frac{1}{2} \frac{\int_{S_3} \frac{h(x)}{20} dx}{\pi(S)} \\ &= \frac{1}{50} \frac{\pi(S_3)}{\pi(S)} \\ &\geq \frac{1}{100}. \end{aligned}$$

Now we can apply Theorem 5.4 with

$$d = \min \left\{ \frac{\delta \ln 2}{\sqrt{n}}, \frac{1}{16\sqrt{n}} \right\}$$

to the partition S_1, S_2, S_3 to get

$$\pi(S_3) \geq \frac{d}{\sigma} \pi(S_1) \pi(S_2).$$

Using the above, we get that

$$\begin{aligned}\phi(S) &\geq \frac{1}{50} \frac{\pi(S_3)}{\pi(S_1)} \\ &\geq \frac{d}{50\sigma} \frac{\pi(S_1)\pi(S_2)}{\pi(S_1)} \\ &\geq \frac{d}{200\sigma} \\ &\geq \frac{\delta}{300\sigma\sqrt{n}},\end{aligned}$$

which proves the theorem. \square

6.2. Getting a warm start. The following two lemmas guarantee that the ball walk in the algorithm will always have a *warm start*, i.e., the M -warmness (3.1) is bounded by a constant. The first lemma bounds the warmness under the fixed cooling rate of $1 + 1/n$.

LEMMA 6.7. *Let $K \subseteq \mathbb{R}^n$, $\sigma_{i+1}^2 = \sigma_i^2(1 + 1/n)$, and $f_i(x) = \exp(-\|x\|^2/(2\sigma_i^2))$. Denote Q_i as the associated probability distribution of f_i over K . Then we can bound the warmness between successive*

$$M(Q_i, Q_{i+1}) \leq \sqrt{e}.$$

The following lemma bounds the warmness when the cooling schedule begins to accelerate under the roundness condition.

LEMMA 6.8. *Let $K \subseteq C\sqrt{n} \cdot B_n$, let $\sigma_{i+1}^2 = \sigma_i^2(1 + \sigma_i^2/(C^2n))$, and let $f_i(x) = \exp(-\|x\|^2/(2\sigma_i^2))$. Denote Q_i as the associated probability distribution of f_i over K . Then we can bound the warmness between successive phases as*

$$M(Q_i, Q_{i+1}) \leq \sqrt{e}.$$

Proof of Lemma 6.7. Let

$$A = \frac{\int_K f_{i+1}(x) dx}{\int_K f_i(x) dx}.$$

Then

$$\begin{aligned}M(Q_i, Q_{i+1}) &= \sup_{S \subseteq K} \frac{Q_i(S)}{Q_{i+1}(S)} \\ &\leq \sup_{x \in K} \frac{Q_i(x)}{Q_{i+1}(x)} \\ &= \sup_{x \in K} A \frac{f_i(x)}{f_{i+1}(x)} \\ &= A \cdot \sup_{x \in K} \exp\left(-\frac{\|x\|^2(1 + 1/n)}{2n}\right) \\ &= A,\end{aligned}$$

where the last line follows from the fact that $0 \in K$.

We will now bound A . First, we extend A to be over all \mathbb{R}^n instead of K and then argue that it can only decrease when restricted to K .

$$\begin{aligned} \frac{\int_{\mathbb{R}^n} \exp\left(-\frac{\|x\|^2}{2\sigma_{i+1}^2}\right) dx}{\int_{\mathbb{R}^n} \exp\left(-\frac{\|x\|^2}{2\sigma_i^2}\right) dx} &= \frac{\sigma_{i+1}^n}{\sigma_i^n} \cdot \frac{\frac{1}{\sigma_{i+1}^n} \int_{\mathbb{R}^n} \exp\left(-\frac{\|x\|^2}{2\sigma_{i+1}^2}\right) dx}{\frac{1}{\sigma_i^n} \int_{\mathbb{R}^n} \exp\left(-\frac{\|x\|^2}{2\sigma_i^2}\right) dx} \\ &= \frac{\sigma_{i+1}^n}{\sigma_i^n} \\ &= \left(1 + \frac{1}{n}\right)^{n/2} \\ &\leq \sqrt{e}. \end{aligned}$$

Let $\mu_K(r)$ be the proportion of the sphere of radius r centered at 0 that is contained in K . Since K is a convex body that contains 0, then $r_1 > r_2 \Rightarrow \mu_K(r_1) \leq \mu_K(r_2)$. Then

$$A = \frac{\int_0^\infty r^{n-1} \exp\left(-\frac{r^2}{2\sigma_{i+1}^2}\right) \mu_K(r) dr}{\int_0^\infty r^{n-1} \exp\left(-\frac{r^2}{2\sigma_i^2}\right) \mu_K(r) dr}.$$

Note that

$$\frac{r^{n-1} \exp\left(-\frac{r^2}{2\sigma_{i+1}^2}\right)}{r^{n-1} \exp\left(-\frac{r^2}{2\sigma_i^2}\right)}$$

is a monotonically increasing function in r . Since K is a convex body containing 0, we can partition K into infinitesimally small cones centered at 0. Consider an arbitrary cone C . $\mu_C(r)$ is 1 for $r \in [0, r']$ and then 0 for $r \in (r', \infty)$ since K is convex. Since the rest of the integrand is monotonically increasing, the integral over the cone only gets larger by extending $\mu_C(r)$ to be 1 for $r \in [0, \infty)$. Therefore,

$$\frac{\int_0^\infty r^{n-1} \exp\left(-\frac{r^2}{2\sigma_{i+1}^2}\right) \mu_C(r) dr}{\int_0^\infty r^{n-1} \exp\left(-\frac{r^2}{2\sigma_i^2}\right) \mu_C(r) dr} \leq \frac{\int_0^\infty r^{n-1} \exp\left(-\frac{r^2}{2\sigma_{i+1}^2}\right) dr}{\int_0^\infty r^{n-1} \exp\left(-\frac{r^2}{2\sigma_i^2}\right) dr} \leq \sqrt{e}.$$

Since C was an arbitrary cone from a partition of A , we have that $A \leq \sqrt{e}$. \square

Proof of Lemma 6.8. Note that

$$\begin{aligned} f_{i+1}(x) &= \exp\left(-\frac{\|x\|^2}{2\sigma_{i+1}^2}\right) \\ &= \exp\left(-\frac{\|x\|^2}{2\sigma_i^2(1 + \sigma_i^2/(C^2n))}\right) \\ &= \exp\left(-\frac{\|x\|^2}{2\sigma_i^2} \cdot \left(1 - \frac{\sigma_i^2/(C^2n)}{1 + \sigma_i^2/(C^2n)}\right)\right) \\ &= f_i(x) \cdot \exp\left(\frac{\|x\|^2}{2C^2n} \cdot \frac{1}{1 + \sigma_i^2/(C^2n)}\right) \\ &\leq f_i(x) \cdot \exp\left(\frac{\|x\|^2}{2C^2n}\right). \end{aligned}$$

We have that

$$\begin{aligned}
M(Q_i, Q_{i+1}) &= \sup_{S \subseteq K} \frac{Q_i(S)}{Q_{i+1}(S)} \\
&\leq \frac{\int_K f_{i+1}(x) dx}{\int_K f_i(x) dx} \cdot \sup_{x \in K} \frac{f_i(x)}{f_{i+1}(x)} \\
&= \frac{\int_K f_{i+1}(x) dx}{\int_K f_i(x) dx} \cdot \sup_{x \in K} \left(\exp \left(-\frac{\|x\|^2}{2C^2n} \cdot \frac{1}{1 + \sigma_i^2/(C^2n)} \right) \right) \\
&= \frac{\int_K f_{i+1}(x) dx}{\int_K f_i(x) dx} \\
&\leq \frac{\int_K f_i(x) \exp \left(\frac{\|x\|^2}{2C^2n} \right) dx}{\int_K f_i(x) dx} \\
&\leq \sup_{x \in K} \left(\exp \left(\frac{\|x\|^2}{2C^2n} \right) \right) \\
&\leq \sqrt{e}
\end{aligned}$$

since $\|x\| \leq C\sqrt{n}$. □

6.3. Bounding wasted steps. The speedy walk is defined as the *proper* steps of the ball walk, where the point the ball walk attempts to visit is contained in K . For convenience, we restate the definition of the speedy walk from earlier (Figure 3).

Speedy Walk(δ, f)

At current point $x \in K$, do the following:

1. Pick random point y from $K \cap (x + \delta B_n)$.
2. Go to y with probability $\min\{1, f(y)/f(x)\}$; otherwise, stay at x .

FIG. 3. *The speedy walk with a Metropolis filter.*

To prove convergence of the ball walk with a Metropolis filter, we prove convergence of the speedy walk and then bound the number of “wasted” steps. Note that the speedy walk cannot be implemented as described in Figure 3 but is an analysis tool to prove the mixing time of the ball walk.

Next, we bound the average number of wasted steps of the ball walk, i.e., when the ball walk tries to visit a point not in K . The average local conductance of the ball walk is defined as

$$\lambda(f) = \frac{\int_K \ell(x) f(x) dx}{\int_K f(x) dx}.$$

We say that a density function $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is *a-rounded* if any level set L

contains a ball of radius $a \cdot \mu_f(L)$. We now show that the average local conductance is large, i.e., at least a constant.

LEMMA 6.9. *For any a -rounded logconcave density function f in \mathbb{R}^n where $K = \text{supp}(f)$,*

$$\lambda(f) \geq 1 - 32 \frac{\delta^{1/2} n^{1/4}}{a^{1/2}}.$$

Proof. Define \hat{f} as the following smoothed version of f . Let D be a convex subset of δB_n such that $16 \cdot \text{vol}(D) = \text{vol}(\delta B_n)$.

$$\hat{f}(x) = \min_D \frac{\int_{y \in x+D} f(y) dy}{\text{vol}(D)}.$$

Now Lemma 6.3 from [25] shows that

$$\int_K \hat{f}(x) dx \geq 1 - 32 \frac{\delta^{1/2} n^{1/4}}{a^{1/2}}.$$

To complete the proof, we observe that, for any point x ,

$$\ell(x)f(x) \geq \hat{f}(x).$$

To see this, note that

$$\begin{aligned} \ell(x)f(x) &= f(x) \frac{\int_{K \cap (x+\delta B_n)} 1 dy}{\text{vol}(\delta B_n)} \\ &\geq \frac{\int_{x+\delta B_n} \min\{f(x), f(y)\} dy}{\int_{x+\delta B_n} 1 dy} \\ &\geq \frac{\int_{y \in x+\delta B_n: f(y) \leq f(x)} f(y) dy}{\int_{y \in x+\delta B_n: f(y) \leq f(x)} 1 dy} \\ &\geq \hat{f}(x). \end{aligned} \quad \square$$

LEMMA 6.10. *The Gaussian $\mathcal{N}(0, \sigma^2 I)$ restricted to K containing a unit ball centered at 0 is $\min\{\sigma, 1\}$ -rounded.*

Proof. The level sets of the distribution are balls restricted to K . For the distribution to be $\min\{\sigma, 1\}$ -rounded, we need that a level set of measure k contains a ball of radius $k \cdot \min\{\sigma, 1\}$. Consider the following function of t , which is an upper bound on the measure of the ball of radius $t \leq \min\{\sigma, 1\}$ since the unit ball is contained in K :

$$g(t) = \frac{\int_0^t x^{n-1} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx}{\int_0^{\min\{\sigma, 1\}} x^{n-1} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx}.$$

Consider the second derivative of g :

$$g''(t) = \left((n-1) - \frac{t^2}{\sigma^2} \right) \cdot \frac{t^{n-2} \exp\left(-\frac{t^2}{2\sigma^2}\right)}{\int_0^{\min\{\sigma, 1\}} x^{n-1} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx}.$$

For $g''(t)$ to be nonnegative, we need $\sigma^2(n-1) - t^2 \geq 0$, which it is for $n \geq 2, t \in [0, \min\{\sigma, 1\}]$. Since $g(0) = 0$, $g(\min\{\sigma, 1\}) = 1$, and the second derivative is nonnegative, we then have that $g(t \min\{\sigma, 1\}) \leq t$ for $t \in [0, 1]$, which proves the lemma. \square

We now show that for an appropriate selection of ball radius the ball walk has large average local conductance.

LEMMA 6.11. *If $\delta \leq \min\{\sigma, 1\}/(4096\sqrt{n})$, then the average local conductance, $\lambda(f)$, for the density function f proportional to the Gaussian $\mathcal{N}(0, \sigma^2 I_n)$ restricted to K containing the unit ball is at least $1/2$.*

Proof. Using Lemmas 6.9 and 6.10, we have that

$$\lambda(f) \geq 1 - 32 \frac{\min\{\sigma^{1/2}, 1\} n^{1/4}}{64n^{1/4} \min\{\sigma^{1/2}, 1\}} = \frac{1}{2}. \quad \square$$

The following lemma is shown in [9].

LEMMA 6.12. *If the average local conductance is at least λ , $M(Q_0, Q) \leq M$, and the speedy walk takes t steps, then the expected number of steps of the corresponding ball walk is at most Mt/λ .*

Proof. Since $M(Q_0, Q) \leq M$, we have that, for all $S \subseteq K$,

$$Q_0(S) \leq MQ(S),$$

and by induction on i we get that

$$Q_i(S) = \int_K P_x(S) dQ_{i-1}(x) \leq M \int_K P_x(S) dQ(x) = MQ(S).$$

For any point x , the expected number of steps until a proper step is made is $1/\ell(x)$. So, given a point from Q_i , the expected number of steps to obtain a point from Q_{i+1} is

$$\int_K \frac{1}{\ell(x)} dQ_i(x) \leq M \int_K \frac{1}{\ell(x)} dQ(x) = M \int_K \frac{1}{\lambda} d\hat{Q}(x) = \frac{M}{\lambda},$$

where \hat{Q} is the corresponding distribution for the ball walk with a Metropolis filter (i.e., with stationary distribution proportional to $f(x)$). If the speedy walk took t steps, then by linearity of expectation the expected number of steps for the ball walk is at most Mt/λ . \square

6.4. Mapping speedy distribution to target distribution. At the point when the speedy walk has converged, we obtain a point approximately from the speedy walk distribution $\ell(x)f(x)$. We will use a rejection routine to map a random point from this distribution to the target distribution $f(x)$ while incurring a small amount of additional sampling error. We adapt the proof of Theorem 4.16 of [19] to the Gaussian setting.

LEMMA 6.13. *Assume that $\|P - \hat{Q}\|_{tv} \leq \varepsilon$, $B_n \subseteq K$, $\varepsilon \leq 1/10$, and*

$$\delta \leq \frac{\min\{\sigma, 1\}}{8\sqrt{n \log(n/\varepsilon)}}.$$

There is an algorithm that will use a constant number of random samples from P , in expectation, to obtain a distribution R satisfying $\|R - Q\|_{tv} \leq 10\varepsilon$.

Proof. The rejection routine is as follows: let $c = 1 - 1/(2n)$. For a point u from distribution $\ell(x)f(x)$, let $v = (1/c)u$. Accept v with probability $f(v)/f(u)$. Repeat until we accept a v .

The correctness of the above routine follows from the following two facts: (i) with constant probability, the rejection sampling will succeed, and (ii) removing a thin shell around the boundary makes $\ell(x)$ look close to uniform on average.

Recall that \hat{Q} is the speedy walk distribution and Q is the ball walk distribution. Consider a level set $\mu_L = \{x : f(x) \geq L\}$. By logconcavity of f , μ_L is convex.

Recall that the level sets μ_L are balls intersected with K since f is a spherical Gaussian distribution. From [19], if μ_L contains the unit ball, then

$$\int_{\mu_L \cap cK} (1 - \ell(x)) dx \leq \varepsilon \text{vol}(\mu_L \cap cK).$$

If μ_L does not contain the unit ball, a standard calculation (using that $B_n \subseteq K$) shows that the local conductance is at least $1 - \varepsilon$ for every point, and thus

$$\int_{\mu_L \cap cK} (1 - \ell(x)) dx \leq \varepsilon \text{vol}(\mu_L \cap cK).$$

Using the above, we see that

$$\begin{aligned} \int_{cK} (1 - \ell(x)) f(x) dx &= \int_0^\infty \int_{\mu_L} (1 - \ell(x)) \{x \in \mathbb{1}_{cK}\} dx dL \\ &\leq \int_0^\infty \varepsilon \text{vol}(\mu_L \cap cK) dL \\ &= \varepsilon \int_{cK} f(x) dx. \end{aligned}$$

Then

$$\begin{aligned} (6.2) \quad \hat{Q}(cK) &= \frac{\int_{cK} \ell(x) f(x) dx}{\int_K \ell(x) f(x) dx} \\ &= \frac{\int_{cK} f(x) dx - \int_{cK} (1 - \ell(x)) f(x) dx}{\int_K \ell(x) f(x) dx} \\ &\geq \frac{(1 - \varepsilon) \int_{cK} f(x) dx}{\int_K \ell(x) f(x) dx} \\ &= \frac{(1 - \varepsilon) c^n \int_K f\left(\frac{x}{c}\right) dx}{\int_K \ell(x) f(x) dx} \\ &\geq \frac{(1 - \varepsilon) \int_K f(x) dx}{2 \int_K \ell(x) f(x) dx} \\ (6.3) \quad &\geq \frac{9}{20}. \end{aligned}$$

Similarly,

$$(6.4) \quad \hat{Q}(cS) \leq \frac{\int_{cS} f(x) dx}{\int_K \ell(x) f(x) dx}.$$

Let P' be the distribution of the first sample from P which satisfies $(1/c)x \in K$. Define

$$z(x) = \begin{cases} f(cx) & \text{if } x \in K, \\ 0 & \text{otherwise,} \end{cases}$$

and let Z be the probability distribution corresponding to z . Then

$$\begin{aligned} P'(S) - Z(S) &= \frac{P(cS)}{P(cK)} - \frac{Q(cS)}{Q(cK)} \\ &\leq \frac{\hat{Q}(cS) + \varepsilon}{\hat{Q}(cK) - \varepsilon} - \frac{Q(cS)}{Q(cK)} \\ &\leq \frac{\int_{cS} f(x) dx + \varepsilon \int_K \ell(x) f(x) dx}{(1 - \varepsilon) \int_{cK} f(x) dx - \varepsilon \int_K \ell(x) f(x) dx} - \frac{\int_{cS} f(x) dx}{\int_{cK} f(x) dx} \quad \text{by (6.2), (6.4)} \\ &\leq \frac{1 + 3\varepsilon}{1 - 3\varepsilon} - 1 \quad \text{by (6.3)} \\ &\leq 10\varepsilon. \end{aligned}$$

Then accept a point x with probability $f(x)/z(x)$, which is at least a constant since $\|x\| \leq 4\sigma\sqrt{n}$. The overall expected number of rejection steps is a constant since $\hat{Q}(cK) \geq 9/20$. \square

6.5. Proof of sampling theorems. We can now prove Theorems 1.4 and 1.5 for sampling a Gaussian distribution restricted to a convex body.

Proof of Theorem 1.5. By Theorems 6.6 and 6.1, we have that selecting $\delta = \min\{\sigma, 1\}/(4096\sqrt{n})$ implies that the speedy walk starting from a distribution that is M -warm will be within total variation distance ν of the target distribution in $O(\max\{\sigma^2, 1\}n^2 \log(n/\nu) \log(M/\nu))$ steps.

By Lemma 6.12, the ball walk will, in expectation, take at most $2M$ times as many steps since the average local conductance λ is at least $1/2$. Therefore, the total number of expected ball walk steps is $O(M \max\{\sigma^2, 1\}n^2 \log(n/\nu) \log(M/\nu))$. We then repeat this walk $O(1)$ times until we obtain a point from the proper target distribution using Lemma 6.13. \square

Proof of Theorem 1.4. Note that here we are analyzing the sampling phases of Figure 2 and only the phases when $\sigma^2 \leq 1$. For these phases, we restrict K to a ball radius $4\sigma\sqrt{n}$, which only ignores an exponentially small fraction of the Gaussian measure by Corollary 5.2 (note that since K is convex and contains the origin, the mean of the Gaussian restricted to K is at most $2\sigma\sqrt{n}$).

By Theorem 1.5, the ball walk will take $O(M \max\{\sigma^2, 1\}n^2 \log(n/\nu) \log(M/\nu))$ steps in expectation. By Lemma 6.7, each phase will always provide a warm start to the next, i.e., $M = O(1)$. By assigning a sampling error $\nu = (\varepsilon/n)^{16}$ to each phase,

we ensure that the overall sampling failure is at most ε by a straightforward union bound. Therefore, each sampling phase takes

$$O\left(n^2 \log^2\left(\frac{n}{\varepsilon}\right)\right)$$

expected steps of the ball walk. Adding up across phases introduces an additional $n \log n$ factor since we increase σ^2 by the rate of $1 + 1/n$ between phases.

If we want to instead run for a fixed number of steps, we can keep a global counter of the ball walk steps. Say the expected number of ball walk steps is T . If at any point the number of ball walk steps goes above $2T$, we abandon this run of the algorithm. The probability of a single run failing is at most $1/2$ by Markov's inequality. If we want an overall failure probability of at most p , then we can run $\log(1/p)$ iterations of the algorithm and, with probability $1 - p$, at least one of them will succeed. \square

Proof of Theorem 1.2. The proof of Theorem 1.2, which extends Gaussian sampling to uniform sampling, follows along the same lines as Theorem 1.4. When $\sigma^2 \leq 1$, the total expected ball walk steps is

$$O\left(n^3 \log(n) \log^2\left(\frac{n}{\varepsilon}\right)\right).$$

When $\sigma^2 > 1$, we additionally use Lemma 6.8, which implies that we can accelerate our cooling rate and still maintain a warm start. This accelerated rate allows us to overcome the increased mixing time of $O^*(\max\{\sigma^2, 1\}n^2)$ once $\sigma^2 \geq 1$. Now consider a “chunk” of phases as a set of phases until σ^2 doubles. There will be $O(C^2 n / \sigma^2)$ phases in a chunk, where each chunk has expected mixing time $O(\sigma^2 n^2 \log(n/\varepsilon))$. Since there are $O(\log n)$ chunks (provided $C = \text{poly}(n)$), the total number of expected ball walk steps when $\sigma^2 > 1$ is

$$O\left(C^2 n^2 \log(n) \log\left(\frac{n}{\varepsilon}\right)\right).$$

Note that this will yield a random sample with respect to a Gaussian with $\sigma^2 = C^2 n$ restricted to K . We can map this point to a uniform random point using simple rejection sampling, which will succeed with probability at least $1/e$ since $K \subseteq C\sqrt{n}$. If it fails, we can restart the algorithm. As with Theorem 1.4, we can repeat $\log(1/p)$ times to transform the expected ball walk steps into a fixed number of steps with success probability $1 - p$. \square

7. Analysis of the volume algorithm.

7.1. Accelerated cooling schedule. The goal of this section is to prove Lemma 3.2, which gives a bound on the variance of the random variable we use to estimate the ratio of Gaussian integrals in the volume algorithm in Figure 2. Here we will actually prove the inequality to be true for all logconcave functions but only apply it to an indicator function of a convex body. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a logconcave distribution such that $E_{X \sim f}(\|X\|^2) = R^2$.

Define

$$g(x, \sigma^2) = f(x) \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right),$$

and also define

$$G(\sigma^2) = \int_{\mathbb{R}^n} g(x, \sigma^2) dx.$$

Define μ_i as the probability distribution proportional to $g(x, \sigma_i^2)$. Let X be a random sample from μ_i , and let $Y = g(X, \sigma_{i+1}^2)/g(X, \sigma_i^2)$. From a standard calculation, we have that

$$\mathbb{E}(Y) = \frac{G(\sigma_{i+1}^2)}{G(\sigma_i^2)}.$$

The second moment of Y is given by

$$\begin{aligned} \mathbb{E}(Y^2) &= \int_{\mathbb{R}^n} \left(\frac{g(x, \sigma_{i+1}^2)}{g(x, \sigma_i^2)} \right)^2 d\mu_i(x) \\ &= \int_{\mathbb{R}^n} \left(\frac{g(x, \sigma_{i+1}^2)}{g(x, \sigma_i^2)} \right)^2 \cdot \frac{g(x, \sigma_i^2)}{G(\sigma_i^2)} dx \\ &= \frac{1}{G(\sigma_i^2)} \int_{\mathbb{R}^n} \frac{g(x, \sigma_{i+1}^2)^2}{g(x, \sigma_i^2)} dx \\ &= \frac{1}{G(\sigma_i^2)} \int_{\mathbb{R}^n} g\left(x, \frac{\sigma_{i+1}^2 \sigma_i^2}{2\sigma_i^2 - \sigma_{i+1}^2}\right) dx \\ &= \frac{G(\frac{\sigma_{i+1}^2 \sigma_i^2}{2\sigma_i^2 - \sigma_{i+1}^2})}{G(\sigma_i^2)}. \end{aligned}$$

To bound the number of samples X needed to estimate Y within a target relative error, we will bound $\mathbb{E}(Y^2)/\mathbb{E}(Y)^2$, which is given by

$$\frac{\mathbb{E}(Y^2)}{\mathbb{E}(Y)^2} = \frac{G(\frac{\sigma_{i+1}^2 \sigma_i^2}{2\sigma_i^2 - \sigma_{i+1}^2})G(\sigma_i^2)}{G(\sigma_{i+1}^2)^2}.$$

Then, letting $\sigma^2 = \sigma_{i+1}^2$ and $\sigma_i^2 = \sigma^2/(1+\alpha)$, we can further simplify to

$$\frac{\mathbb{E}(Y^2)}{\mathbb{E}(Y)^2} = \frac{G\left(\frac{\sigma^2}{1+\alpha}\right)G\left(\frac{\sigma^2}{1-\alpha}\right)}{G(\sigma^2)^2}.$$

The above n -dimensional inequality is difficult to analyze directly. We will reduce it to a simpler one-dimensional inequality via localization. Define an exponential needle $E = (a, b, \gamma)$ as a segment $[a, b] \subseteq \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ corresponding to the weight function $e^{\gamma t}$ applied to the segment $[a, b]$. The integral of an n -dimensional function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ over this one-dimensional needle is

$$\int_E h = \int_0^{|b-a|} h(a + tu) e^{\gamma t} dt, \quad \text{where } u = \frac{b-a}{|b-a|}.$$

We use the following theorem from [18].

THEOREM 7.1 (Theorem 2.7 in [18]). *Let f_1, f_2, f_3, f_4 be four nonnegative continuous functions defined on \mathbb{R}^n , and let $\alpha, \beta > 0$. Then the following are equivalent:*

1. *For every logconcave function F defined on \mathbb{R}^n with compact support,*

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} F(t) f_1(t) dt \right)^\alpha \left(\int_{\mathbb{R}^n} F(t) f_2(t) dt \right)^\beta \\ &\leq \left(\int_{\mathbb{R}^n} F(t) f_3(t) dt \right)^\alpha \left(\int_{\mathbb{R}^n} F(t) f_4(t) dt \right)^\beta. \end{aligned}$$

2. For every exponential needle E ,

$$\left(\int_E f_1\right)^\alpha \left(\int_E f_2\right)^\beta \leq \left(\int_E f_3\right)^\alpha \left(\int_E f_4\right)^\beta.$$

A crucial aspect of our proof is that we can restrict the support of our target logconcave function f , which then allows us to consider a restricted family of needles. Recall that we assumed $E_{X \sim f}(\|X\|^2) = R^2$. Set $R_1 = 2R \cdot \log(1/\varepsilon)$. By the following lemma from [25], if we restrict the support of f to be $R_1 \cdot B_n$, we only lose an $\varepsilon/2$ fraction of the mass.

LEMMA 7.2 (Lemma 5.17 in [25]). *Let $X \in \mathbb{R}^n$ be a random point from a log-concave distribution with $E(X^2) = R^2$. Then, for any $t > 1$, $\Pr(\|X\| > tR) < \exp(-R + 1)$.*

We can now reduce the desired inequality to a simpler form of exponential needles, which are restricted to lie in the interval $[-R_1, R_1]$.

LEMMA 7.3. *If, for all intervals $[\ell, u] \subseteq [-R_1, R_1]$ and $\gamma > 0$,*

$$\frac{\int_\ell^u \exp(\gamma t) \exp\left(-\frac{t^2(1+\alpha)}{2\sigma^2}\right) dt \cdot \int_\ell^u \exp(\gamma t) \exp\left(-\frac{t^2(1-\alpha)}{2\sigma^2}\right) dt}{\left(\int_\ell^u \exp(\gamma t) \exp\left(-\frac{t^2}{2\sigma^2}\right) dt\right)^2} \leq c,$$

then, for all logconcave functions f defined on \mathbb{R}^n whose support is a compact subset of $R_1 \cdot B_n$,

$$\frac{G(\frac{\sigma^2}{1+\alpha})G(\frac{\sigma^2}{1-\alpha})}{G(\sigma^2)^2} \leq c.$$

Proof. Define

$$\begin{cases} f_1(x) = \exp\left(-\frac{\|x\|^2(1+\alpha)}{2\sigma^2}\right) & \text{if } x \in R_1 B_n, \\ f_2(x) = \exp\left(-\frac{\|x\|^2(1-\alpha)}{2\sigma^2}\right) & \text{if } x \in R_1 B_n, \\ f_3(x) = f_4(x) = \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right) & \text{if } x \in R_1 B_n, \end{cases}$$

and define $f_i(x) = 0$ for all $x \notin R_1 B_n$, $i = 1, 2, 3, 4$. Applying Theorem 7.1, by setting $F(x) = f(x)$ and $\alpha = \beta = 1$ we have that

$$\frac{G(\frac{\sigma^2}{1+\alpha})G(\frac{\sigma^2}{1-\alpha})}{G(\sigma^2)^2} \leq c$$

if, for all exponential needles $E \subseteq \mathbb{R}^n$,

$$(7.1) \quad \frac{\int_E f_1(x) dx \int_E f_2(x) dx}{\left(\int_E f_3(x) dx\right)^2} \leq c.$$

To prove the lemma, we will show that we can reduce the inequality in (7.1) for an arbitrary exponential needle $E \subseteq \mathbb{R}^n$ to the simpler form. E is defined by an interval \mathcal{I} in \mathbb{R}^n and an arbitrary exponential function $\exp(\gamma t)$ on \mathcal{I} . Note that we may set

$\mathcal{I} = \mathcal{I} \cap R_1 B_n$ without altering inequality (7.1). Define z as the closest distance from the origin to the extension of the \mathcal{I} in both directions. Parameterize the interval \mathcal{I} in terms of t , where $t = 0$ gives the closest point along the extension of \mathcal{I} to the origin (note that $t = 0$ does not necessarily have to be on \mathcal{I}). Also define the minimum and maximum values of t on \mathcal{I} as ℓ and u , respectively, where $[\ell, u] \subseteq [-R_1, R_1]$ since $\mathcal{I} \subseteq R_1 B_n$. We then have that

$$\begin{aligned} \int_E g(\sigma^2, x) dx &= \int_\ell^u \exp(\gamma t) \exp\left(-\frac{t^2 + z^2}{2\sigma^2}\right) dt \\ &= \exp\left(-\frac{z^2}{2\sigma^2}\right) \cdot \int_\ell^u \exp(\gamma t) \exp\left(-\frac{t^2}{2\sigma^2}\right) dt. \end{aligned}$$

Note that in the integral ratio the terms with z cancel out since

$$\exp\left(-\frac{(1+\alpha)z^2}{2\sigma^2} - \frac{(1-\alpha)z^2}{2\sigma^2} + \frac{2z^2}{2\sigma^2}\right) = 1,$$

which then proves the lemma. \square

Before bounding the simpler form of the variance inequality obtained by Lemma 7.3, we first prove the following two helper lemmas.

LEMMA 7.4. *Let X be a random variable with $\mathbb{E}(X^4) < \infty$, and let $a \leq X \leq b$. Then*

$$\mathbb{E}(X^4) - \mathbb{E}(X^2)^2 \leq 4 \max\{a^2, b^2\} \text{Var}(X).$$

Proof. Let Y be an independent random variable drawn from the same distribution as X . Then

$$\begin{aligned} 2\text{Var}(X^2) &= \text{Var}(X^2) + \text{Var}(Y^2) \\ &= \mathbb{E}(X^4) - \mathbb{E}(X^2)^2 + \mathbb{E}(Y^4) - \mathbb{E}(Y^2)^2 \\ &= \mathbb{E}(X^4) - 2\mathbb{E}(X^2)\mathbb{E}(Y^2) + \mathbb{E}(Y^4) \\ &= \mathbb{E}((X^2 - Y^2)^2) \\ &= \mathbb{E}((X + Y)^2(X - Y)^2) \\ &\leq 4 \max\{a^2, b^2\} \mathbb{E}((X - Y)^2) \\ &= 4 \max\{a^2, b^2\} \mathbb{E}(X^2 - 2XY + Y^2) \\ &= 8 \max\{a^2, b^2\} \text{Var}(X). \end{aligned}$$

\square

LEMMA 7.5. *Let $[\ell, u] \subseteq [-R_1, R_1]$ and*

$$v(x) = \frac{\int_\ell^u t^2 \exp(\gamma t) \exp\left(-\frac{t^2 x}{2\sigma^2}\right) dt}{\int_\ell^u \exp(\gamma t) \exp\left(-\frac{t^2 x}{2\sigma^2}\right) dt}.$$

Then $v'(x) \geq -2R_1^2/x$.

Proof. For convenience, define

$$s(x, t) = \exp(\gamma t) \exp\left(-\frac{t^2 x}{2\sigma^2}\right).$$

We have that

$$v'(x) = \left(\frac{1}{2\sigma^2} \right) \cdot \frac{\left(\int_{\ell}^u t^2 s(x, t) dt \right)^2 - \int_{\ell}^u s(x, t) dt \int_{\ell}^u t^4 s(x, t) dt}{\left(\int_{\ell}^u s(x, t) dt \right)^2}.$$

Observe that the above quantity is the difference of moments of a truncated Gaussian distribution. We then have that

$$\begin{aligned} v'(x) &= \left(\frac{1}{2\sigma^2} \right) \cdot (\mathbb{E}(X^2)^2 - \mathbb{E}(X^4)), \quad \text{where } X \sim \mathcal{N} \left(\frac{\gamma\sigma^2}{x}, \frac{\sigma^2}{x} \right) \mid \ell \leq X \leq u \\ &\geq -\frac{2R_1^2}{\sigma^2} \cdot \text{Var}(X) \quad \text{by Lemma 7.4} \\ &\geq -\frac{2R_1^2}{\sigma^2} \cdot \frac{\sigma^2}{x} \quad \text{by Theorem 5.1} \\ &= -\frac{2R_1^2}{x}, \end{aligned}$$

where in the second-to-last step we used the fact that by truncating a Gaussian its variance can only go down. To see this, let $Z \sim \mathcal{N}(0, 1)$ and $Z_h \sim Z \cap [a, b]$ for any $[a, b] \subseteq \mathbb{R}$. Then Theorem 5.1 says that $\text{Var}(Z_h) \leq \text{Var}(Z)$ by letting $\alpha = 2$; the case for general Gaussians is then obtained by rescaling and shifting. \square

The following lemma now proves the variance bound.

LEMMA 7.6. *Let $[\ell, u] \subseteq [-R_1, R_1]$ and $\alpha \leq 1/2$. Then*

$$\frac{\int_{\ell}^u \exp(\gamma t) \exp\left(-\frac{t^2(1+\alpha)}{2\sigma^2}\right) dt \cdot \int_{\ell}^u \exp(\gamma t) \exp\left(-\frac{t^2(1-\alpha)}{2\sigma^2}\right) dt}{\left(\int_{\ell}^u \exp(\gamma t) \exp\left(-\frac{t^2}{2\sigma^2}\right) dt\right)^2} \leq \exp\left(2 \cdot \frac{R_1^2 \alpha^2}{\sigma^2}\right).$$

Proof. Again for convenience, define

$$s(x, t) = \exp(\gamma t) \exp\left(-\frac{t^2 x}{2\sigma^2}\right).$$

Define

$$h(\alpha) := \frac{\int_{\ell}^u s(1+\alpha, t) dt \cdot \int_{\ell}^u s(1-\alpha, t) dt}{\left(\int_{\ell}^u s(1, t) dt\right)^2}.$$

Note that the lemma is equivalent to bounding $h(\alpha)$. We first prove the following claim, from which the lemma will easily follow.

CLAIM 7.7. *For $\alpha \leq 1/2$,*

$$h'(\alpha) \leq \frac{4 \cdot \alpha R^2 \cdot h(\alpha)}{\sigma^2}.$$

Proof. First, observe that

$$\begin{aligned}\frac{\partial}{\partial \alpha} \left(s(1+\alpha, t) \right) &= \frac{\partial}{\partial \alpha} \left(\int_{\ell}^u \exp(\gamma t) \exp \left(-\frac{t^2(1+\alpha)}{2\sigma^2} \right) dt \right) \\ &= \frac{-1}{2\sigma^2} \left(\int_{\ell}^u t^2 \exp(\gamma t) \exp \left(-\frac{t^2(1+\alpha)}{2\sigma^2} \right) dt \right),\end{aligned}$$

and similarly

$$\frac{\partial}{\partial \alpha} \left(s(1-\alpha, t) \right) = \frac{1}{2\sigma^2} \left(\int_{\ell}^u t^2 \exp(\gamma t) \exp \left(-\frac{t^2(1-\alpha)}{2\sigma^2} \right) dt \right).$$

Then taking the derivative of $h(\alpha)$ with respect to α gives

$$\begin{aligned}\frac{\partial}{\partial \alpha} (h(\alpha)) &= \frac{\partial}{\partial \alpha} \left(\frac{\int_{\ell}^u s(1+\alpha, t) dt \cdot \int_{\ell}^u s(1-\alpha, t) dt}{\left(\int_{\ell}^u s(1, t) dt \right)^2} \right) \\ &= \frac{1}{2\sigma^2} \cdot \frac{\int_{\ell}^u s(1+\alpha, t) dt \cdot \int_{\ell}^u t^2 s(1-\alpha, t) dt - \int_{\ell}^u s(1-\alpha, t) dt \cdot \int_{\ell}^u t^2 s(1+\alpha, t) dt}{\left(\int_{\ell}^u s(1, t) dt \right)^2}.\end{aligned}$$

We now have that $h'(\alpha)/h(\alpha)$ is equal to

$$\frac{1}{2\sigma^2} \cdot \left(\frac{\int_{\ell}^u t^2 \exp(\gamma t) \exp \left(-\frac{t^2(1-\alpha)}{2\sigma^2} \right) dt}{\int_{\ell}^u \exp(\gamma t) \exp \left(-\frac{t^2(1-\alpha)}{2\sigma^2} \right) dt} - \frac{\int_{\ell}^u t^2 \exp(\gamma t) \exp \left(-\frac{t^2(1+\alpha)}{2\sigma^2} \right) dt}{\int_{\ell}^u \exp(\gamma t) \exp \left(-\frac{t^2(1+\alpha)}{2\sigma^2} \right) dt} \right).$$

Let

$$v(x) = \frac{\int_{\ell}^u t^2 \exp(\gamma t) \exp \left(-\frac{t^2 n x}{2\sigma^2} \right) dt}{\int_{\ell}^u \exp(\gamma t) \exp \left(-\frac{t^2 n x}{2\sigma^2} \right) dt}.$$

We then have that

$$\begin{aligned}\frac{h'(\alpha)}{h(\alpha)} &= -\frac{1}{2\sigma^2} \left(\int_{1-\alpha}^{1+\alpha} v'(x) dx \right) \\ &\leq \frac{1}{2\sigma^2} \left(\int_{1-\alpha}^{1+\alpha} \frac{2R_1^2}{x} dx \right) && \text{by Lemma 7.5} \\ &= \frac{R_1^2}{\sigma^2} \ln \left(\frac{1+\alpha}{1-\alpha} \right) \\ &\leq \frac{2R_1^2 \alpha}{\sigma^2 (1-\alpha)} && \text{using } \ln(1+x) \leq x \\ &\leq \frac{4R_1^2 \alpha}{\sigma^2}.\end{aligned}$$

□

By Claim 7.7, we then have a bound on $h(\alpha)$ as follows:

$$\begin{aligned}\ln h(\alpha) &= \ln h(0) + \int_0^\alpha \frac{d}{dx} (\ln h(x)) \, dx \\ &= \ln(1) + \int_0^\alpha \frac{h'(x)}{h(x)} \, dx \\ &\leq \int_0^\alpha \frac{4R_1^2 x}{\sigma^2} \, dx \\ &\leq \frac{2R_1^2 \alpha^2}{\sigma^2},\end{aligned}$$

and thus

$$h(\alpha) \leq \exp\left(\frac{2R_1^2 \alpha^2}{\sigma^2}\right). \quad \square$$

LEMMA 7.8. Suppose a logconcave function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ has support contained in $R_1 \cdot B_n$. Let $g(x, \sigma^2) = f(x) \exp(-\|x\|^2/(2\sigma^2))$. Let X be drawn from a distribution proportional to $g(x, \sigma^2)$, and let $Y = g(X, \sigma^2(1+\alpha))/g(X, \sigma^2)$. Then, for $\alpha \leq 1/2$,

$$\frac{\mathbb{E}(Y^2)}{\mathbb{E}(Y)^2} \leq \exp\left(\frac{2R_1^2 \alpha^2}{\sigma^2}\right).$$

Proof. It follows immediately from Lemmas 7.3 and 7.6. \square

The bound in Lemma 3.2 then follows by applying Lemma 7.8 with the indicator function of a convex body.

7.2. Proof of the main theorem. In this section, we prove Theorem 1.1 by analyzing the runtime of the algorithm in Figure 2 and also showing that the volume estimate it computes is accurate.

The following two lemmas say that the beginning and ending σ^2 for the algorithm are sufficient.

LEMMA 7.9. If $\sigma^2 \leq 1/(n + \sqrt{8n \ln(1/\varepsilon)})$ and $B_n \subseteq K$, then

$$\int_K \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right) \, dx \geq (1 - \varepsilon) \int_{\mathbb{R}^n} \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right) \, dx.$$

Proof. We will use the following concentration bound on a spherical Gaussian in \mathbb{R}^n with mean μ and variance σ^2 , which is valid for $t > 1$ (see, e.g., [6]):

$$\Pr(\|X - \mu\|^2 - \sigma^2 n > t\sigma^2 \sqrt{n}) \leq e^{-t^2/8}.$$

Selecting $\mu = 0$, $t = \sqrt{8 \ln(1/\varepsilon)}$, and $\sigma^2 = 1/(n + t\sqrt{n})$ gives

$$\Pr(\|X\|^2 > 1) \leq \varepsilon,$$

and therefore all but an ε -fraction of the Gaussian is contained inside B_n (and therefore K). \square

LEMMA 7.10. Let $K \subseteq C\sqrt{n}B_n$, $f_i(x) = \exp(-\frac{\|x\|^2}{2\sigma_i^2})$, $\sigma_i^2 \geq C^2 n$, and $\sigma_{i+1}^2 = \infty$. Then, for X drawn from a distribution proportional to $f_i \cap K$ and $Y = f_{i+1}(X)/f_i(X)$,

$$\frac{\mathbb{E}(Y^2)}{\mathbb{E}(Y)^2} \leq e^2.$$

Proof. Observe that $f_i(X) \geq 1/e$ since $\|X\|^2 \leq C^2 n$, and thus $Y \leq e$. Also note that $Y \geq 1$. Therefore, $\mathbb{E}(Y^2)/\mathbb{E}(Y)^2 \leq e^2$. \square

We now bound the variance when σ^2 is small. First, we will need the following lemma that is proved in [24].

LEMMA 7.11 (Lemma 3.2 in [24]). *Let $K \subseteq \mathbb{R}^n$ be a convex body and $f : K \rightarrow \mathbb{R}$ be a logconcave function. For any $a > 0$, define*

$$Z(a) = \int_K f(ax) dx.$$

Then $a^n Z(a)$ is a logconcave function of a .

LEMMA 7.12. *Assume $n \geq 3$. Let X be a random point in K with density proportional to $f_i(x) = \exp(-\frac{\|x\|^2}{2\sigma_i^2})$, $\sigma_{i+1}^2 = \sigma_i^2(1 + \alpha)$, and $Y = f_{i+1}(X)/f_i(X)$. If $\alpha = 1/n$, then*

$$\frac{\mathbb{E}(Y^2)}{\mathbb{E}(Y)^2} < 1 + \frac{2}{n}.$$

Proof. For convenience, let $\sigma = \sigma_{i+1}$. Then

$$\frac{\mathbb{E}(Y^2)}{\mathbb{E}(Y)^2} = \frac{\int_K \exp\left(-\frac{\|x\|^2(1-\alpha)}{2\sigma^2}\right) dx \int_K \exp\left(-\frac{\|x\|^2(1+\alpha)}{2\sigma^2}\right) dx}{\left(\int_K \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right) dx\right)^2}.$$

By Lemma 7.11, the function $z(a) = a^{n+1} \int_K \exp(-a\|x\|^2/2) dx$ is logconcave, and thus

$$z\left(\frac{1-\alpha}{\sigma^2}\right) z\left(\frac{1+\alpha}{\sigma^2}\right) \leq z\left(\frac{1}{\sigma^2}\right)^2.$$

Therefore,

$$\begin{aligned} & \int_K \exp\left(-\frac{\|x\|^2(1-\alpha)}{2\sigma^2}\right) dx \int_K \exp\left(-\frac{\|x\|^2(1+\alpha)}{2\sigma^2}\right) dx \\ & \leq \left(\frac{1}{1-\alpha^2}\right)^{n+1} \left(\int_K \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right) dx\right)^2. \end{aligned}$$

Setting $\alpha = 1/n$, we have that

$$\frac{\mathbb{E}(Y^2)}{\mathbb{E}(Y)^2} \leq \left(\frac{1}{1-1/n^2}\right)^{n+1} = \left(1 + \frac{1}{n^2-1}\right)^{n+1} \leq \exp\left(\frac{1}{n-1}\right) \leq 1 + \frac{2}{n}. \quad \square$$

We now show that the volume estimate computed in Algorithm 2 is accurate. Define R_i as the i th integral ratio, i.e.,

$$R_i := \frac{F(\sigma_{i+1}^2)}{F(\sigma_i^2)} = \frac{\int_K \exp(-\|x\|^2/(2\sigma_{i+1}^2)) dx}{\int_K \exp(-\|x\|^2/(2\sigma_i^2)) dx},$$

and let W_i denote the estimate of the algorithm for R_i .

For two random variables X, Y , we will measure their independence by the following:

$$\mu(X, Y) = \sup_{A, B} |P(X \in A, Y \in B) - P(X \in A)P(Y \in B)|,$$

where A, B range over measurable subsets of the ranges of X, Y .

We will give an argument similar to that in [24] and use the following lemmas that were proved there.

LEMMA 7.13 (Lemma 3.5 in [24]). *If f and g are two measurable functions, then*

$$\mu(f(X), g(Y)) \leq \mu(X, Y).$$

LEMMA 7.14 (Lemma 3.6 in [24]). *Let X, Y be random variables such that $0 \leq X \leq a$ and $0 \leq Y \leq b$. Then*

$$|\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)| \leq ab\mu(X, Y).$$

LEMMA 7.15 (Lemma 3.9 in [24]). *Let $X \geq 0$ be a random variable, let $a > 0$, and let $X' = \min(X, a)$. Then*

$$\mathbb{E}(X') \geq \mathbb{E}(X) - \frac{\mathbb{E}(X^2)}{4a}.$$

LEMMA 7.16. *With probability at least 4/5,*

$$(1 - \varepsilon)R_1 \dots R_m \leq W_1 \dots W_m \leq (1 + \varepsilon)R_1 \dots R_m.$$

Proof. Let $(X_0^i, X_1^i, X_2^i, \dots, X_k^i)$ be the sequence of sample points for the i th volume phase. The distribution of each X_i is approximately the correct distribution but slightly off based on the error parameter ν in each phase that bounds the total variation distance. We will define new random variables \bar{X}_j^i that have the correct distribution for each phase.

Note that X_j^0 would be sampled from the exact distribution and then rejected if outside of K . Therefore, $\Pr(X_j^0 = \bar{X}_j^0) = 1$. Suppose that the total number of sample points throughout the algorithm is t . Using induction and the definition of total variation distance, we see that

$$(7.2) \quad \Pr(X_i^j = \bar{X}_i^j \forall i, j) \geq 1 - t\nu.$$

Let

$$Y_j^i = \frac{\exp\left(-\frac{\|X_j^i\|^2}{2\sigma_{i+1}^2}\right)}{\exp\left(-\frac{\|X_j^i\|^2}{2\sigma_i^2}\right)} \quad \text{and} \quad \bar{W}_i = \frac{1}{k} \sum_{j=1}^k Y_j^i.$$

Note that, for a fixed i , all of the Y_j^i have the same expectation since they are from the exact distribution, and it is equal to $\mathbb{E}(\bar{W}_i)$. Suppose that we have $\mathbb{E}((Y_j^i)^2) \leq c_i \mathbb{E}(Y_j^i)^2$. Then

$$(7.3) \quad \begin{aligned} \mathbb{E}(\bar{W}_i^2) &= \frac{1}{k^2} \left(\sum_{j=1}^k \mathbb{E}((Y_j^i)^2) + k(k-1)R_i^2 \right) \\ &\leq \left(1 + \frac{c_i - 1}{k}\right) \cdot \mathbb{E}(\bar{W}_i)^2. \end{aligned}$$

When $\sigma^2 \leq 1$ and we use a fixed cooling rate, then $c_i \leq 1 + 2/n$ by Lemma 7.12, and (7.3) gives that

$$(7.4) \quad \mathbb{E}(\bar{W}_i^2) \leq \left(1 + \frac{2}{kn}\right) \mathbb{E}(\bar{W}_i)^2.$$

The following claim bounds the variance of our ratio estimator under a faster cooling rate, which we use to bound the variance when $1 \leq \sigma^2 \leq C^2n$. It follows from Lemma 3.2.

CLAIM 7.17. *Suppose that $K \subseteq C\sqrt{n}B_n$ and let $\alpha = \sigma^2/(2C^2n)$. Then*

$$\frac{\mathbb{E}((Y^i)^2)}{\mathbb{E}(Y^i)^2} < 1 + \frac{\sigma^2}{C^2n}.$$

Suppose that we had independence between samples, and consider bounding the cumulative error for all phases of the algorithm. When $\sigma^2 \leq 1$, we can bound the number of phases for the first part as $m_1 \leq 2n \log 4n$. When $\sigma^2 > 1$, we will analyze the phases in chunks, where a chunk is the set of phases until σ^2 doubles. Note that the number of phases in a chunk starting with variance σ^2 is at most $2C^2n/\sigma^2$. Also there are at most $\log(C^2n)$ chunks. Observe that for a single chunk with starting variance σ^2 , where i, j are the starting and ending phases of the chunk, we have

$$(7.5) \quad \frac{\mathbb{E}(\overline{W}_i^2 \dots \overline{W}_j^2)}{R_i^2 \dots R_j^2} \leq \left(1 + \frac{2\sigma^2}{kC^2n}\right)^{2C^2n/\sigma^2} \leq \left(1 + \frac{5}{k}\right).$$

Then there is one final phase when we switch to the uniform distribution, which has variance at most $1 + e^2$ by Lemma 7.9. Note that this, together with (7.4), (7.5), gives the following bound for all $1 \leq i \leq m$:

$$(7.6) \quad \mathbb{E}(\overline{W}_i^2) \leq \left(1 + \frac{e^2}{k}\right) \mathbb{E}(\overline{W}_i)^2.$$

Let m denote the total number of phases. Note that, from above, we have that $m \leq 4C^2n \log(C^2n)$. Consider bounding the product of the $\mathbb{E}(\overline{W}_i^2)$ terms:

$$(7.7) \quad \begin{aligned} \frac{\mathbb{E}(\overline{W}_1^2 \dots \overline{W}_m^2)}{R_1^2 \dots R_m^2} &\leq \left(1 + \frac{2}{kn}\right)^{m_1} \left(1 + \frac{5}{k}\right)^{\log C^2n} \left(1 + \frac{e^2}{k}\right) \\ &\leq \exp\left(\frac{2m_1}{kn} + \frac{5 \log(C^2n)}{k} + \frac{e^2}{k}\right) \\ &\leq \exp\left(\frac{\varepsilon^2}{50}\right). \end{aligned}$$

If we had independence between samples, then we could use Lemma 7.12 and Claim 7.17 with Chebyshev's inequality to bound the probability of failure:

$$\begin{aligned} \Pr\left(\frac{|\overline{W}_1 \dots \overline{W}_m - R_1 \dots R_m|}{R_1 \dots R_m} \geq \frac{\varepsilon}{2}\right) &\leq \frac{4\text{Var}(\overline{W}_1 \dots \overline{W}_m)}{\varepsilon^2 R_1^2 \dots R_m^2} \\ &= \frac{4}{\varepsilon^2} \left(\frac{\mathbb{E}(\overline{W}_1^2 \dots \overline{W}_m^2)}{R_1^2 \dots R_m^2} - 1 \right) \\ &\leq \frac{4}{\varepsilon^2} \left(\exp\left(\frac{\varepsilon^2}{50}\right) - 1 \right) \quad \text{by (7.7)} \\ &\leq \frac{4}{\varepsilon^2} \left(\left(1 + \frac{\varepsilon^2}{40}\right) - 1 \right) \\ &= \frac{1}{10}. \end{aligned}$$

However, subsequent samples are dependent, and we must carefully bound the dependence. The analysis is somewhat involved, but will follow essentially the same template as in [24, 9], which utilizes the following lemma to bound dependence between subsequent samples, where ν is the target total variation distance for each sample point. For convenience, denote the entire sequence of t samples points used in the algorithm as $(Z_0, Z_1, \dots, Z_{t-1})$.

LEMMA 7.18 (Lemma 4.3 in [24]).

- (a) For $0 \leq i < t$, the random variables Z_i and Z_{i+1} are ν -independent, and the random variables \bar{Z}_i and \bar{Z}_{i+1} are (3ν) -independent.
- (b) For $0 \leq i < t$, the random variables (Z_0, \dots, Z_i) and Z_{i+1} are (3ν) -independent.
- (c) For $0 \leq i < m$, the random variables $\bar{W}_1 \dots \bar{W}_i$ and \bar{W}_{i+1} are $(3km\nu)$ -independent.

The variables \bar{W}_i are not bounded, but we will introduce a new set of random variables based on \bar{W}_i that are bounded so we can later apply Lemma 7.14. Let

$$\alpha = \frac{\varepsilon^{1/2}}{8(m\mu)^{1/4}},$$

where

$$\mu = 3km\nu = \Theta\left(\frac{\varepsilon^{14} \log^2 C^2 n}{n^{15}}\right).$$

Note that α is much larger than one. Define

$$(7.8) \quad V_i = \min\{\bar{W}_i, \alpha \mathbb{E}(\bar{W}_i)\}.$$

It is clear that $\mathbb{E}(V_i) \leq \mathbb{E}(\bar{W}_i)$, and we also have

$$\begin{aligned} \mathbb{E}(V_i) &\geq \mathbb{E}(\bar{W}_i) - \frac{\mathbb{E}(\bar{W}_i^2)}{4\alpha \mathbb{E}(\bar{W}_i)} && \text{by Lemma 7.15} \\ &\geq \left(1 - \frac{1}{4\alpha} \left(1 + \frac{e^2}{k}\right)\right) \mathbb{E}(\bar{W}_i) && \text{by (7.6)} \\ (7.9) \quad &\geq \left(1 - \frac{1}{2\alpha}\right) \mathbb{E}(\bar{W}_i). \end{aligned}$$

Let $U_0 = 1$, and define recursively

$$(7.10) \quad U_i = \min\{U_{i-1} V_i, \alpha \mathbb{E}(V_1) \dots \mathbb{E}(V_i)\}.$$

We will now show that

$$(7.11) \quad \left(1 - \frac{i-1}{\alpha}\right) \mathbb{E}(V_1) \dots \mathbb{E}(V_i) \leq \mathbb{E}(U_i) \leq (1 + 2\mu\alpha^2 i) \mathbb{E}(V_1) \dots \mathbb{E}(V_i).$$

We first show the upper bound in (7.11). To see the independence of U_{i-1} and V_i , observe that $U_{i-1} = f(\bar{W}_1, \dots, \bar{W}_{i-1})$ for some function f . Note that $\alpha \mathbb{E}(V_1) \dots \mathbb{E}(V_{i-1})$ is a fixed constant. Similarly, $V_i = g(\bar{W}_i)$ for some function g . Thus, by Lemmas 7.13 and 7.18, the random variables U_{i-1} and V_i are μ -independent.

Then

$$\begin{aligned} | \mathbb{E}(U_{i-1} V_i) - \mathbb{E}(U_{i-1}) \mathbb{E}(V_i) | &\leq \mu \alpha \mathbb{E}(V_1) \dots \mathbb{E}(V_{i-1}) \alpha \mathbb{E}(\bar{W}_i) && \text{by Lemma 7.14} \\ (7.12) \quad &\leq 2\mu\alpha^2 \mathbb{E}(V_1) \dots \mathbb{E}(V_i) && \text{by (7.9) since } \alpha \geq 1. \end{aligned}$$

We can now get the upper bound on $\mathbb{E}(U_i)$ by induction:

$$\begin{aligned}
 \mathbb{E}(U_i) &\leq \mathbb{E}(U_{i-1}V_i) && \text{from (7.10)} \\
 &\leq \mathbb{E}(U_{i-1})\mathbb{E}(V_i) + 2\mu\alpha^2\mathbb{E}(V_1)\dots\mathbb{E}(V_i) && \text{by (7.12)} \\
 (7.13) \quad &\leq (1 + 2\mu\alpha^2i)\mathbb{E}(V_1)\dots\mathbb{E}(V_i) && \text{using an inductive hypothesis.}
 \end{aligned}$$

To obtain a lower bound in (7.11), we first upper bound the second moment by a similar sequence of steps. Observe that $\mathbb{E}(U_{i-1}^2)$ and $\mathbb{E}(V_i^2)$ are μ -independent. Therefore,

$$\begin{aligned}
 |\mathbb{E}(U_{i-1}^2V_i^2) - \mathbb{E}(U_{i-1}^2)\mathbb{E}(V_i^2)| &\leq \mu\alpha^2\mathbb{E}(V_1)^2\dots\mathbb{E}(V_{i-1})^2\alpha^2\mathbb{E}(\bar{W}_i)^2 \\
 &\leq 2\mu\alpha^2\mathbb{E}(V_1^2)\dots\mathbb{E}(V_i^2).
 \end{aligned}$$

And using induction we see that

$$(7.14) \quad \mathbb{E}(U_i^2) \leq \mathbb{E}(U_{i-1}^2V_i^2)$$

$$(7.15) \quad \leq (1 + 2\mu\alpha^4i)\mathbb{E}(V_1^2)\dots\mathbb{E}(V_i^2).$$

Now for the lower bound in (7.11) we have that

$$\begin{aligned}
 \mathbb{E}(U_i) &\geq \mathbb{E}(U_{i-1}V_i) - \frac{\mathbb{E}(U_{i-1}^2V_i^2)}{4\alpha\mathbb{E}(V_1)\dots\mathbb{E}(V_i)} && \text{by Lemma 7.15} \\
 (7.16) \quad &\geq \mathbb{E}(U_{i-1}V_i) - (1 + 2\mu\alpha^4i) \frac{\mathbb{E}(V_1^2)\dots\mathbb{E}(V_i^2)}{4\alpha\mathbb{E}(V_1)\dots\mathbb{E}(V_i)} && \text{by (7.15).}
 \end{aligned}$$

For $\alpha \geq 3k$, we have that

$$\begin{aligned}
 \mathbb{E}(V_1^2)\dots\mathbb{E}(V_i^2) &\leq \mathbb{E}(\bar{W}_1^2)\dots\mathbb{E}(\bar{W}_i^2) && \text{by (7.8)} \\
 &\leq \exp\left(\frac{\varepsilon^2}{50}\right) \mathbb{E}(\bar{W}_1^2)\dots\mathbb{E}(\bar{W}_i)^2 && \text{by (7.7)} \\
 &\leq \exp\left(\frac{\varepsilon^2}{50}\right) \frac{1}{(1 - 1/(2\alpha))^{2i}} \mathbb{E}(V_1)^2\dots\mathbb{E}(V_i)^2 && \text{by (7.9)} \\
 &\leq \exp\left(\frac{\varepsilon^2}{50} + \frac{8C^2n \log C^2n}{2\alpha}\right) \mathbb{E}(V_1)^2\dots\mathbb{E}(V_i)^2 \quad (i \leq 4C^2n \log C^2n) \\
 (7.17) \quad &\leq \exp\left(\frac{\varepsilon^2}{40}\right) \mathbb{E}(V_1)^2\dots\mathbb{E}(V_i)^2.
 \end{aligned}$$

Combining (7.16) and (7.17),

$$\begin{aligned}
 \mathbb{E}(U_i) &\geq \mathbb{E}(U_{i-1}V_i) - \frac{1}{4\alpha} (1 + 2\mu\alpha^4i) \exp\left(\frac{\varepsilon^2}{40}\right) \mathbb{E}(V_1)\dots\mathbb{E}(V_i) \\
 &\geq \mathbb{E}(U_{i-1}V_i) - \frac{1 + 2\mu\alpha^4i}{2\alpha} \mathbb{E}(V_1)\dots\mathbb{E}(V_i) \\
 &\geq \mathbb{E}(U_{i-1})\mathbb{E}(V_i) - \frac{1 + 3\mu\alpha^4i}{2\alpha} \mathbb{E}(V_1)\dots\mathbb{E}(V_i) && \text{by (7.12)} \\
 &\geq \mathbb{E}(U_{i-1})\mathbb{E}(V_i) - \frac{1}{\alpha} \mathbb{E}(V_1)\dots\mathbb{E}(V_i) && \text{since } 3\mu\alpha^4i < 1.
 \end{aligned}$$

Then, by induction on i ,

$$(7.18) \quad \mathbb{E}(U_i) \geq \mathbb{E}(V_1) \dots \mathbb{E}(V_i) - \frac{i-1}{\alpha} \mathbb{E}(V_1) \dots \mathbb{E}(V_i).$$

Putting (7.13) and (7.18) together, we now have a proof of (7.11). Thus

$$\begin{aligned} \mathbb{E}(U_m) &\leq (1 + 2\mu\alpha^2 m) \mathbb{E}(V_1) \dots \mathbb{E}(V_m) && \text{by (7.11)} \\ &\leq \left(1 + \frac{\varepsilon}{4}\right) \mathbb{E}(V_1) \dots \mathbb{E}(V_m) && \text{by the definition of } \alpha \\ &\leq \left(1 + \frac{\varepsilon}{4}\right) \mathbb{E}(\bar{W}_1) \dots \mathbb{E}(\bar{W}_m). \end{aligned}$$

We also have that

$$\begin{aligned} \mathbb{E}(U_m) &\geq \left(1 - \frac{m-1}{\alpha}\right) \mathbb{E}(V_1) \dots \mathbb{E}(V_m) && \text{by (7.11)} \\ &\geq \left(1 - \frac{m-1}{\alpha}\right) \left(1 - \frac{1}{2\alpha}\right)^m \mathbb{E}(\bar{W}_1) \dots \mathbb{E}(\bar{W}_m) && \text{by (7.9)} \\ &\geq \left(1 - \frac{\varepsilon}{4}\right) \mathbb{E}(\bar{W}_1) \dots \mathbb{E}(\bar{W}_m) && \text{since } \alpha \geq 4m/\varepsilon. \end{aligned}$$

We also have an upper bound on the second moment of U_m :

$$\begin{aligned} \mathbb{E}(U_m^2) &\leq (1 + 2\mu\alpha^4 m) \mathbb{E}(V_1^2) \dots \mathbb{E}(V_m^2) && \text{by (7.15)} \\ &\leq (1 + 2\mu\alpha^4 m) \exp\left(\frac{\varepsilon^2}{40}\right) \mathbb{E}(V_1)^2 \dots \mathbb{E}(V_m)^2 && \text{by (7.17)} \\ &\leq (1 + 2\mu\alpha^4 m) \exp\left(\frac{\varepsilon^2}{40}\right) \frac{1}{(1 - (m-1)/\alpha)^2} \mathbb{E}(U_m)^2 && \text{by (7.11)} \\ (7.19) \quad &\leq \left(1 + \frac{\varepsilon^2}{64}\right) \mathbb{E}(U_m)^2. \end{aligned}$$

Using Chebyshev's inequality, we see that

$$\begin{aligned} \Pr\left(|U_m - \mathbb{E}(U_m)| \leq \frac{\varepsilon}{2} \mathbb{E}(\bar{W}_1) \dots \mathbb{E}(\bar{W}_m)\right) &\geq 1 - \frac{4\text{Var}(U_m)}{\varepsilon^2 \mathbb{E}(\bar{W}_1)^2 \dots \mathbb{E}(\bar{W}_m)^2} \\ &\geq 1 - \frac{4\mathbb{E}(U_m)^2}{64\mathbb{E}(\bar{W}_1)^2 \dots \mathbb{E}(\bar{W}_m)^2} && \text{by (7.19)} \\ &\geq 0.9 && \text{by (7.11).} \end{aligned}$$

Then, applying Markov's inequality,

$$\begin{aligned} \Pr(U_{i+1} \neq U_i V_{i+1}) &= \Pr(U_i V_{i+1} > \alpha \mathbb{E}(V_1) \dots \mathbb{E}(V_{i+1})) \\ &\leq \frac{2}{\alpha} && \text{by (7.13).} \end{aligned}$$

Similarly,

$$\begin{aligned} \Pr(V_i \neq \bar{W}_i) &= \Pr(\bar{W}_i > \alpha \mathbb{E}(\bar{W}_i)) \\ &\leq \frac{1}{\alpha}. \end{aligned}$$

Applying a union bound, we see that with probability at least $1 - 3k/\alpha$ we have $U_m = \overline{W}_1 \dots \overline{W}_m$. Also, from (7.2), we have that $\overline{W}_1 \dots \overline{W}_m = W_1 \dots W_m$ with probability at least $1 - 2km\nu$. Recall that $\mathbf{E}(\overline{W}_1) \dots \mathbf{E}(\overline{W}_m) = R_1 \dots R_m$. Therefore, with probability at least $4/5$,

$$|W_1 \dots W_m - R_1 \dots R_m| \leq \frac{\varepsilon}{2} R_1 \dots R_m,$$

which proves the lemma. \square

We can now prove the main theorem.

Proof of Theorem 1.1. We assume that $\varepsilon \geq 2^{-n}$, which only ignores cases which would take exponential time. Then, by Lemma 7.9, selecting $\sigma_0^2 = 1/(4n)$ implies that all but a negligible amount of volume of the starting Gaussian is contained in K .

Recall that our algorithm only has a bound on the expected number of steps. To account for this, we will run the algorithm $O(1)$ times to obtain a run which takes at most a constant factor of ball walk steps to proper steps, say with probability $1/20$. By Lemma 7.16, the answer returned by the algorithm will be within the target relative error with probability at least $4/5$. Thus the overall probability of failure is $3/4$. Note that we can boost this probability of failure to $1 - p$ by the standard trick of repeating the algorithm $\log 1/p$ times and returning the median.

We now analyze the runtime of the algorithm in Figure 2. Set $C = R \log(1/\varepsilon)/\sqrt{n}$. Assume that $C \geq 1$ (otherwise, arbitrarily increase C). When $\sigma^2 \leq 1$, using the value of k , the mixing time assigned to each phase, and the fact that there are $O(n \log n)$ phases, $O(n^{2.5} k \log n \log^2(n/\varepsilon)) = O(n^3 \log^2 n \log^2(n/\varepsilon)/\varepsilon^2) = O^*(n^3)$ is the total number of ball walk steps taken. When $\sigma^2 > 1$, the analysis is very similar if we note that the faster cooling rate and fewer number of samples cancel out the slower mixing time of $O^*(\sigma^2 n^2)$. Thus it follows that the total number of ball walk steps taken is

$$O\left(\frac{C^2 n^3 \log^2 n \log^2 \frac{n}{\varepsilon}}{\varepsilon^2}\right) = O\left(\frac{R^2 n^2}{\varepsilon^2} \cdot \log^2 n \log^2 \frac{1}{\varepsilon} \log^2 \frac{n}{\varepsilon}\right) = O^*(R^2 n^2). \quad \square$$

8. Conclusion.

We make a few concluding remarks:

1. In our algorithm, the complexity of volume computation for well-rounded bodies is essentially the same as the amortized complexity of generating a single uniform random sample—both are $O^*(n^3)$. This is in contrast to all previous volume algorithms where the amortized complexity of sampling is lower by at least a factor of n compared to the complexity of volume computation.

2. It would be interesting to extend our algorithm to integrating any well-rounded logconcave function; we expect this should be possible with essentially the same complexity. The variance of the ratio of integrals computed in each phase, as well as the isoperimetric inequality, are already proven in full generality for all logconcave functions.

3. The accelerated cooling schedule used in our algorithm can be seen as a worst-case analysis of the cooling schedule used in a practical algorithm for volume computation [8, 10]; in the latter, we used an adaptive schedule by empirically estimating the maximum tolerable change in the variance of the Gaussian that keeps the variance of the ratio estimator bounded by a constant.

4. An important open task is to find an $O^*(n^3)$ rounding algorithm for arbitrary convex bodies. The current best rounding complexity is $O^*(n^4)$ [24].

REFERENCES

- [1] D. APPLEGATE AND R. KANNAN, *Sampling and integration of near log-concave functions*, in STOC, 1991, pp. 156–163.
- [2] K. M. BALL, *Logarithmically concave functions and sections of convex sets in R^n* , Studia Math., 88 (1988), pp. 69–84.
- [3] I. BÁRÁNY AND Z. FÜREDI, *Computing the volume is difficult*, Discrete Comput. Geom., 2 (1987), pp. 319–326, <https://doi.org/10.1007/BF02187886>.
- [4] I. BÁRÁNY AND Z. FÜREDI, *Approximation of the sphere by polytopes having few vertices*, Proc. Amer. Math. Soc., 102 (1988), pp. 651–659.
- [5] H. J. BRASCAMP AND E. H. LIEB, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Functional Anal., 22 (1976), pp. 366–389.
- [6] S. BRAZITIKOS, A. GIANNOPoulos, P. VALETTAS, AND B. VRITSIOU, *Geometry of Isotropic Convex Bodies*, Math. Surveys Monogr. 196, AMS, Providence, RI, 2014.
- [7] R. BUBLEY, M. DYER, AND M. JERRUM, *An elementary analysis of a procedure for sampling points in a convex body*, Random Structures Algorithms, 12 (1998), pp. 213–235, [https://doi.org/10.1002/\(SICI\)1098-2418\(199805\)12:3<213::AID-RSA1>3.3.CO;2-R](https://doi.org/10.1002/(SICI)1098-2418(199805)12:3<213::AID-RSA1>3.3.CO;2-R).
- [8] B. COUSINS AND S. VEMPALA, *Volume computation of convex bodies*, MATLAB File Exchange, (2013), <http://www.mathworks.com/matlabcentral/fileexchange/43596-volume-computation-of-convex-bodies>.
- [9] B. COUSINS AND S. VEMPALA, *A cubic algorithm for computing Gaussian volume*, in SODA, 2014, pp. 1215–1228, <https://doi.org/10.1137/1.9781611973402.90>.
- [10] B. COUSINS AND S. VEMPALA, *A practical volume algorithm*, Math. Program. Comput., 8 (2016), pp. 133–160.
- [11] M. E. DYER AND A. M. FRIEZE, *On the complexity of computing the volume of a polyhedron*, SIAM J. Comput., 17 (1988), pp. 967–974, <https://doi.org/10.1137/0217060>.
- [12] M. E. DYER AND A. M. FRIEZE, *Computing the volume of a convex body: A case where randomness provably helps*, in Proceedings of the AMS Symposium on Probabilistic Combinatorics and Its Applications, 1991, pp. 123–170.
- [13] M. E. DYER, A. M. FRIEZE, AND R. KANNAN, *A random polynomial time algorithm for approximating the volume of convex bodies*, in STOC, 1989, pp. 375–381.
- [14] M. E. DYER, A. M. FRIEZE, AND R. KANNAN, *A random polynomial-time algorithm for approximating the volume of convex bodies*, J. Assoc. Comput. Mach., 38 (1991), pp. 1–17.
- [15] R. ELDAN, *Thin shell implies spectral gap up to polylog via a stochastic localization scheme*, Geom. Funct. Anal., 23 (2013), pp. 532–569.
- [16] R. ELDAN AND B. KLARTAG, *Approximately Gaussian marginals and the hyperplane conjecture*, in Concentration, Functional Inequalities and Isoperimetry, Contemp. Math. 545, AMS, Providence, RI, 2011, pp. 55–68.
- [17] B. FLEURY, *Concentration in a thin euclidean shell for log-concave measures*, J. Funct. Anal., 259 (2010), pp. 832–841.
- [18] R. KANNAN, L. LOVÁSZ, AND M. SIMONOVITS, *Isoperimetric problems for convex bodies and a localization lemma*, Discrete Comput. Geom., 13 (1995), pp. 541–559.
- [19] R. KANNAN, L. LOVÁSZ, AND M. SIMONOVITS, *Random walks and an $O^*(n^5)$ volume algorithm for convex bodies*, Random Structures Algorithms, 11 (1997), pp. 1–50.
- [20] L. LOVÁSZ AND M. SIMONOVITS, *Mixing rate of Markov chains, an isoperimetric inequality, and computing the volume*, in FOCS, 1990, pp. 482–491.
- [21] L. LOVÁSZ AND M. SIMONOVITS, *Random walks in a convex body and an improved volume algorithm*, Random Structures Algorithms, 4 (1993), pp. 359–412.
- [22] L. LOVÁSZ AND S. VEMPALA, *Fast algorithms for logconcave functions: Sampling, rounding, integration and optimization*, in FOCS, 2006, pp. 57–68.
- [23] L. LOVÁSZ AND S. VEMPALA, *Hit-and-run from a corner*, SIAM J. Comput., 35 (2006), pp. 985–1005, <https://doi.org/10.1137/S009753970544727X>.
- [24] L. LOVÁSZ AND S. VEMPALA, *Simulated annealing in convex bodies and an $O^*(n^4)$ volume algorithm*, J. Comput. System Sci., 72 (2006), pp. 392–417.
- [25] L. LOVÁSZ AND S. VEMPALA, *The geometry of logconcave functions and sampling algorithms*, Random Structures Algorithms, 30 (2007), pp. 307–358.