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RESEARCH ARTICLE

A geometric approach to orthogonal Higgs bundles

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Abstract We give a geometric characterisation of the topological invariants associated to $\mathrm{SO}(p+1,\,p)$-Higgs bundles through KO-theory and the Langlands correspondence between orthogonal and symplectic Hitchin systems. By defining the split orthogonal spectral data, we obtain geometric description of the intersection of the moduli space of those Higgs bundles with the $\mathrm{SO}(2p+1,\mathbb{C})$-Hitchin fibration in terms of a collection of compact abelian varieties, and provide a natural stratification of the moduli space of $\mathrm{SO}(p+1,\,p)$-Higgs bundles.

Keywords Higgs bundles · Hitchin fibration · (B, A, A)-branes

Mathematics Subject Classification 14D20 · 14D21 · 53C07 · 14H70 · 14P25 · 20C33

1 Introduction

Higgs bundles were first studied by Nigel Hitchin in 1987, and appeared as solutions of Yang–Mills self-duality equations on a Riemann surface [12].

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Definition 1.1 A *Higgs bundle* on a compact Riemann surface Σ of genus $g \geqslant 2$ is a pair (E, Φ) where E is a holomorphic vector bundle on Σ, and the *Higgs field* $\Phi\colon E \to E \otimes K$ is an endomorphism-valued holomorphic 1-form, for $K := T^*\Sigma$.

Higgs bundles can also be defined for complex semisimple groups $G_{\mathbb{C}}$ and their real forms, and through stability conditions one can construct their moduli spaces $\mathcal{M}_{G_{\mathbb{C}}}$ (e.g. see [13]).

A natural way of studying the moduli space of Higgs bundles is through the *Hitchin fibration*, sending the class of a Higgs bundle (E, Φ) to the coefficients of the characteristic polynomial $\det(\eta I - \Phi)$. The generic fibre is an abelian variety which can be seen through line bundles on an algebraic curve S, the *spectral curve* associated to the Higgs field as introduced in [13]. The *spectral data* is then given by the line bundle on S satisfying certain conditions.

Example 1.2 In the case of classical Higgs bundles, the smooth fibres are (non-canonically isomorphic to) Jacobian varieties of S [13].

The Hitchin fibration was defined for classical complex Lie groups in [13, Section 5], and following [14, Section 7] one may consider Higgs bundles with real structure group G as fixed point sets in the moduli space of Higgs bundles for the complexified group $G_{\mathbb{C}}$, therefore obtaining G-Higgs bundles as real points inside the Hitchin fibration (e.g. see [8,14,20] and references therein).

We dedicate this paper to the study of the geometry of the moduli space of $\mathrm{SO}(p+1, p)$-Higgs bundles inside $\mathcal{M}_{\mathrm{SO}(2p+1,\mathbb{C})}$. Recall from [13] the following.

Definition 1.3 An $\mathrm{SO}(2p+1, \mathbb{C})$-*Higgs bundle* is a pair (E, Φ) for E a holomorphic vector bundle of rank $2p+1$ with a nondegenerate symmetric bilinear form (v, w), and $\Phi \in H^0(\Sigma, \mathrm{End}_0(E) \otimes K)$ the Higgs field which satisfies $(\Phi v, w) = -(v, \Phi w)$.

In particular, whilst in this case there is no *Toledo invariant* when $m \neq 1, 3$ (see [16, Section 6]), one can consider the Langlands dual set-up of $\mathrm{Sp}(2p, \mathbb{C})$-Higgs bundles and Higgs bundles for its split real form to understand the role of the symplectic Toledo invariant from the orthogonal perspective, as well as to construct the spectral data.

Definition 1.4 An $\mathrm{Sp}(2p, \mathbb{C})$-*Higgs bundle* is a pair (E, Φ) where E is a rank $2p$ vector bundle with a symplectic form $\omega(\cdot, \cdot)$, and the Higgs field Φ is a section $H^0(\Sigma, \mathrm{End}(E) \otimes K)$ satisfying $\omega(\Phi v, w) = -\omega(v, \Phi w)$.

By considering real Higgs bundles as fixed points of an involution (e.g. see [20, Section 3.3.1]), we see the moduli space of $\mathrm{SO}(p+1, p)$-Higgs bundles inside the $\mathrm{SO}(2p+1, \mathbb{C})$-Hitchin fibration.

Definition 1.5 An $\mathrm{SO}(p+1, p)$-*Higgs bundle* inside the moduli space $\mathcal{M}_{\mathrm{SO}(2p+1,\mathbb{C})}$ is an $\mathrm{SO}(2p+1, \mathbb{C})$-Higgs bundle (E, Φ) where E decomposes into two orthogonal bundles $E = V_- \oplus V_+$ and the Higgs field $\Phi\colon E \to E \otimes K$ is given by

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \quad \text{for} \ \ \gamma = \beta^{\mathrm{T}},$$

where β^{T} is the orthogonal transpose of β, obtained using the orthogonal structures on V_+, V_-.

Definition 1.6 The coefficients a_i of the characteristic polynomial of an $\mathrm{SO}(p+1, p)$-Higgs field define a *spectral curve* $\pi : S \to \Sigma$ in the total space of the canonical bundle K whose equation is

$$\eta^{2p} + a_1\eta^{2p-2} + \cdots + a_{p-1}\eta^2 + a_p = 0, \tag{1}$$

for η the tautological section of π^*K and $a_i \in H^0(\Sigma, K^{2i})$.

Given an $\mathrm{Sp}(2p, \mathbb{C})$-Higgs bundle (E, Φ), the pair is an $\mathrm{Sp}(2p, \mathbb{R})$-Higgs bundle whenever E decomposes into $E = V \oplus V^*$ for V a rank p holomorphic vector bundle, and $\Phi = \left(\begin{smallmatrix} 0 & \beta \\ \gamma & 0 \end{smallmatrix}\right)$ for γ and β symmetric. Thus one can see that $\mathrm{Sp}(2p, \mathbb{R})$-Higgs bundles also define a curve as in Definition 1.6.

The curve S in Definition 1.6 is a $2p$-fold covering of the Riemann surface, generically smooth and ramified over $4p(g-1)$ distinct points, the zeros of a_p. In order to understand the topological invariants associated to $\mathrm{SO}(p+1, p)$-Higgs bundles, one has to consider a subdivisor D of the ramification divisor, over which a natural involution $\sigma : \eta \mapsto -\eta$ acts as -1, and whose degree we denote by M following the notation of [19]. The value of M is closely related to the Toledo invariant, and in particular one can deduce the following:

Proposition 3.3 *For each even invariant* $0 < M \leqslant 4p(g-1)$ *there is a component of the moduli space of* $\mathrm{Sp}(2p, \mathbb{R})$*-Higgs bundles which intersects the nonsingular fibres of the Hitchin fibration for* $\mathrm{Sp}(2p, \mathbb{C})$*-Higgs bundles. The component has a Zariski open set given by a fibration of a* $\mathbb{Z}_2$*-vector space over a Zariski open set in the total space of a vector bundle on the symmetric product* $S^M\Sigma$.

By taking into account the parity of p, one can obtain a geometric description of the intersection of the moduli space of $\mathrm{Sp}(2p, \mathbb{R})$-Higgs bundles with the generic fibres of the $\mathrm{Sp}(2p, \mathbb{C})$-Hitchin fibration:

Theorem 3.4 *When p is odd, for each fixed invariant M, the intersection of the moduli space* $\mathcal{M}_{\mathrm{Sp}(2p,\mathbb{R})}$ *with the smooth fibres of the* $\mathrm{Sp}(2p, \mathbb{C})$*-Hitchin fibration is given by* 2^{2g} *copies of*

$$\mathrm{Prym}(S/\sigma, \Sigma)[2].$$

In the case of orthogonal Higgs bundles, one has the following:

Theorem 4.2 *The intersection of the moduli space* $\mathcal{M}_{\mathrm{SO}(p+1,p)}$ *with the regular fibres of the* $\mathrm{SO}(2p+1, \mathbb{C})$*-Hitchin fibration is given by two copies of the space* $\mathrm{Prym}(S, S/\sigma)[2]/\rho^*H^1(\overline{S}, \mathbb{Z}_2)$ *where the* $\mathbb{Z}_2$ *space* $H^1(\overline{S}, \mathbb{Z}_2)$ *is given by*

$$\mathrm{Prym}(S/\sigma, \Sigma)[2] \oplus H^1(\Sigma, \mathbb{Z}_2),$$

for m odd*, and for m* even *it is given by*

$$H^1(\Sigma, \mathbb{Z}_2) \oplus \big(\mathrm{Prym}(S/\sigma, \Sigma)[2]/\pi^*H^1(\Sigma, \mathbb{Z}_2)\big) \oplus H^1(\Sigma, \mathbb{Z}_2).$$

Each of the two copies corresponds to whether the orthogonal bundle lifts to a spin bundle or not. Moreover, there is a decomposition of the torsion two points in the Prym variety

$$\mathrm{Prym}(S, S/\sigma)[2] \cong H^1(S/\sigma, \mathbb{Z}_2) \oplus \mathbb{Z}_2([a_p])^{\mathrm{ev}}/b_0,$$

where $\mathbb{Z}_2([a_p])^{\mathrm{ev}}$ denotes subdivisors of the effective divisor with simple zeros $[a_p]$ with even number of $+1$, and $b_0 := (1, \dots, 1)$. Thus one can make the following definition.

Definition 1.7 The *spectral data* of an $\mathrm{SO}(p+1, p)$-Higgs bundle is, up to equivalence, given by

- a line bundle $\mathcal{F} \in H^1(S/\sigma, \mathbb{Z}_2)$, and
- a divisor $D \in \mathbb{Z}_2([a_p])^{\mathrm{ev}}/b_0$ of degree M.

This description allows one to study the intersection of the space of real Higgs bundles with the complex Hitchin fibration. In particular, in the so-called *maximal Toledo invariant case* on the symplectic side, which corresponds to $M = 0$ in the orthogonal setting, one has a very neat description of the abelian intersection of $\mathcal{M}_{\mathrm{SO}(p+1,p)}$ with the generic fibres of the $\mathrm{SO}(2p+1, \mathbb{C})$-Hitchin fibration.

Since $\mathrm{SO}(p+1, p)$ retracts onto $\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(p+1))$, an $\mathrm{SO}(p+1, p)$-Higgs bundle $(V_+ \oplus V_-, \Phi)$ carries three topological invariants: the Stiefel–Whitney classes $\omega_1(V_+) = \det(V_+) \in H^1(\Sigma, \mathbb{Z}_2)$, and $\omega_2(V_\pm) \in H^2(\Sigma, \mathbb{Z}_2)$. Through a K-theoretic approach elaborating on the methods of [3,16], in Sect. 4 we can further classify these invariants in terms of their spectral data:

Theorem 4.6 *The Stiefel–Whitney classes of an* $\mathrm{SO}(p+1, p)$*-Higgs bundle* $(V = V_- \oplus V_+, \Phi)$ *with spectral data* $(S/\sigma, \mathcal{F}, D)$ *are given by*

$$\begin{aligned}
\omega_1(V_+) &= \mathrm{Nm}(\mathcal{F}) \in H^1(\Sigma, \mathbb{Z}_2);\\
\omega_2(V_+) &= \varphi_{S/\sigma}(\mathcal{F}) + \varphi_\Sigma(\mathrm{Nm}(\mathcal{F})) \in \mathbb{Z}_2;\\
\omega_2(V_-) &= \begin{cases} \varphi_{S/\sigma}(\mathcal{F}) + \varphi_\Sigma(\mathrm{Nm}(\mathcal{F})) & \text{if } \ \omega_2(V) = 0,\\ \varphi_{S/\sigma}(\mathcal{F}) + \varphi_\Sigma(\mathrm{Nm}(\mathcal{F})) + 1 & \text{if } \ \omega_2(V) = 1;\end{cases}
\end{aligned}$$

for φ_Σ *and* $\varphi_{S/\sigma}$ *the analytic mod* 2 *indices of the curves* (*see Definition 4.1*), *and* $\mathrm{Nm}(\mathcal{F})$ *the Norm on* Σ.

By considering $\mathcal{F} \otimes K_{\overline{S}}$ as a new spin structure, one can see $\omega_2(V_\pm)$ purely in terms of spin structures in Corollary 4.7. Moreover, by analysing spectral data through the induced 2-fold cover $\rho\colon S \to \overline{S} := S/\sigma$, and recalling that the orthogonal vector bundle $V_- \oplus V_+$ is recovered as an extension defined through the divisor D (see [15, Section 4.2]), one obtains the number of points in each of the regular fibres of the Hitchin fibration for a fixed invariant M.

Proposition 4.8 *The number of points in a regular fibre of the* $\mathrm{SO}(2p+1, \mathbb{C})$*-Hitchin fibration corresponding to* $\mathrm{SO}(p+1, p)$*-Higgs bundles with even invariant* M *is*

$$\binom{4p(g-1)}{M}.$$

By considering the parametrisation of the moduli space through spectral data, we obtain a natural stratification of the smooth loci of the moduli space of $\mathrm{SO}(p+1, p)$-Higgs bundles leading to a geometric description of Zariski dense open sets in each component:

Proposition 4.9 *For each fixed even invariant* $0 < M \leqslant 4p(g-1)$ *there is a component of the moduli space of* $\mathrm{SO}(p+1, p)$*-Higgs bundles which intersects the regular fibres of the* $\mathrm{SO}(2p+1, \mathbb{C})$*-Hitchin fibration. The component has a Zariski open set given by a covering of a Zariski open set in the total space of a vector space over the symmetric product* $S^M\Sigma$.

It is important to note that these components will possibly (and often do so) meet over the discriminant locus of the Hitchin fibration, and thus one needs to do further analysis to understand the connectivity of the moduli space. Whilst we shall not deepen into this matter in the current paper, an example of how to see the intersection of the components through the monodromy of the associated *Gauss–Manin connection* for $\mathrm{SO}(2, 3)$-Higgs bundles is discussed in [4, Section 6.3]. A geometric description of the covering of Proposition 4.9 is given in Sect. 4.5, recovering some of the results appearing in [7, Section 6.4].

The moduli space of $\mathrm{SO}(p+1, p)$-Higgs bundles considered in this paper is an example of what is known as (B, A, A)-brane in the moduli space $\mathcal{M}_{\mathrm{SO}(2p+1,\mathbb{C})}$ of complex Higgs bundles. As such, these branes have dual (B, B, B)-branes in the dual moduli space $\mathcal{M}_{\mathrm{Sp}(2p,\mathbb{C})}$ (see [17, Section 12]). In [3, Section 7] it was conjectured what the support of this dual brane should be the whole moduli space $\mathcal{M}_{\mathrm{Sp}(2p,\mathbb{C})}$ of symplectic Higgs bundles.

We conclude this note with some further comments on this duality in Sect. 5, as well as on the relation between the Hitchin components in both split symplectic and orthogonal (B, A, A)-branes in Langlands dual groups, and some implications of the geometric description of the spectral data given in this paper.

2 The Hitchin fibration

Recall from [13] that an $\mathrm{Sp}(2p, \mathbb{C})$-Higgs bundle is a pair (E, Φ') for E a rank $2p$ vector bundle with a symplectic form $\omega(\cdot,\cdot)$, and the Higgs field $\Phi' \in H^0(\Sigma, \mathrm{End}(E)\otimes K)$ satisfying $\omega(\Phi' v, w) = -\omega(v, \Phi' w)$. Similarly, an $\mathrm{SO}(2p+1, \mathbb{C})$-Higgs bundle is a pair (V, Φ) for V a holomorphic vector bundle of rank $2p+1$ with a nondegenerate symmetric bilinear form (v, w), and Φ a Higgs field in $H^0(\Sigma, \mathrm{End}_0(V)\otimes K)$ which satisfies $(\Phi v, w) = -(v, \Phi w)$.

The spectral curves defined by $\mathrm{SO}(2p+1, \mathbb{C})$-Higgs bundles and $\mathrm{Sp}(2p, \mathbb{C})$-Higgs bundles have similar equations (e.g. see [15, Sections 3–4]), and are given by a $2p$-fold cover $\pi : S \to \Sigma$ in the total space of K whose equation is $\det(\Phi' - \eta\,\mathrm{Id}) = \frac{1}{\eta}\det(\Phi - \eta\,\mathrm{Id}) = \eta^{2p} + a_1\eta^{2p-2} + \cdots + a_{p-1}\eta^2 + a_p = 0$ as in (1).

Definition 2.1 The curve S has an involution σ which acts as $\sigma(\eta) = -\eta$ and thus we define a *quotient curve* $\overline{S} := S/\sigma$ in the total space of K^2 for which $\rho : S \to \overline{S}$ is a double cover:

$$\begin{tikzcd} S \arrow[rr, "2:1", "\rho"'] \arrow[dr, "\pi", "2p:1"'] & & \overline{S} \arrow[dl, "\overline{\pi}"', "m:1"] \\ & \Sigma. & \end{tikzcd}$$

The covers S and $\overline{S}$ have, respectively, genus $g_S = 1 + 4p^2(g-1)$, and $g_{\overline{S}} = (2p^2 - p)(g-1) + 1$. Moreover, by the adjunction formula, their canonical bundles can be written, respectively, as $K_S = \pi^* K^{2p}$ and $K_{\overline{S}} = \overline{\pi}^* K^{2p-1}$. As shown in [13], the Hitchin fibration for both moduli spaces $\mathcal{M}_{\mathrm{SO}(2p+1,\mathbb{C})}$ and $\mathcal{M}_{\mathrm{Sp}(2p,\mathbb{C})}$ is given over $\mathcal{A} = \bigoplus_{i=1}^{p} H^0(\Sigma, K^{2i})$. From [15, Section 3], the generic fibres for $\mathcal{M}_{\mathrm{Sp}(2p,\mathbb{C})}$ are given by

$$\mathrm{Prym}(S, \overline{S}), \tag{2}$$

and from [15, Section 4], the generic fibres for $\mathcal{M}_{\mathrm{SO}(2p+1,\mathbb{C})}$ are given by (two copies of)

$$\mathrm{Prym}(S, \overline{S})/\rho^* H^1(\overline{S}, \mathbb{Z}_2). \tag{3}$$

In what follows we shall study the components of the moduli space of Higgs bundles for split real forms by considering, from [20, Theorem 4.12], points of order two in the generic fibres (2) and (3).

3 $\mathbf{Sp(2}\boldsymbol{p}\mathbf{,\mathbb{R})}$-Higgs bundles

We shall now consider $\mathrm{Sp}(2p,\mathbb{R})$-Higgs bundles, which from [20, Theorem 4.12] can be seen in the generic fibres of the Hitchin fibration as points of order two in (2), and are given by $\mathrm{Sp}(2p,\mathbb{C})$-Higgs bundles which decompose as $(E = W \oplus W^*, \Phi')$, for

$$\Phi = \begin{pmatrix} 0 & \beta' \\ \gamma' & 0 \end{pmatrix} \quad \text{where} \quad \begin{cases} \gamma' \colon W \to W^* \otimes K \text{ and } \gamma = \gamma'^t, \\ \beta' \colon W^* \to W \otimes K \text{ and } \beta = \beta'^t. \end{cases}$$

Fixing a choice of Θ characteristic $L_0 := K^{1/2}$, it is shown in [15] that the vector bundle E is recovered as $\pi_* U$ for $U := L \otimes K^{(2p-1)/2}$. Note that the condition $L \in \mathrm{Prym}(S, \overline{S})[2] := \{L \in \mathrm{Prym}(S, \overline{S}) : L^2 \cong \mathcal{O}\}$ is equivalent to requiring $U^2 \cong K_S \pi^* K^*$. Since points in $\mathrm{Prym}(S, \overline{S})[2]$ are given by line bundles L on S for which $\sigma^* L \cong L^* \cong L$, following [19, Theorem 3.5], they are classified by the action of the involution σ on L over its fixed point set (i.e., the ramification divisor of S). The involution σ acts as ± 1 over some subset of M points of the ramification divisor $[a_p]$, and the number of Higgs bundles appearing in each fibre for a fixed invariant M is described by Hitchin in [16].

Higgs bundles for the real symplectic group have associated a topological invariant, the Toledo invariant (e.g. see [8]), defined as

$$|\tau(W \oplus W^*, \Phi')| := |c_1(W)|,$$

and satisfy a Milnor–Wood type inequality $|c_1(W)| \leqslant p(g-1)$. Moreover, from [16, Section 6] the class can be expressed as

$$w_1(W) := c_1(W) = -\frac{M}{2} + p(g-1),$$

and its mod 2 value defines the invariant $c_1(W) \pmod 2$. Note that since the invariant M is even, within the moduli space of $\mathrm{Sp}(2p, \mathbb{R})$-Higgs bundles, the value of $c_1(W) \pmod 2$ differentiates components depending on the values of $M \pmod 4$.

Remark 3.1 For $\mathrm{Sp}(2p, \mathbb{C})$-Higgs bundles, the invariant M appears as n_- in [15, Section 4]. In the case of $\mathrm{Sp}(2p, \mathbb{R})$-Higgs bundles, it is the invariant l of [16, Section 6].

From [16, Section 6], the number of elements in $\mathrm{Prym}(S, \overline{S})[2]$ corresponding to M is

$$\binom{4p(g-1)}{M} \times 2^{2g_{\overline{S}}}.$$

In order to describe the geometry of the components given by these Higgs bundles, recall from [4, Proposition 4.15] that convenient splittings can be chosen for the short exact sequence

$$0 \to H^1(\overline{S}, \mathbb{Z}_2) \xrightarrow{\rho^*} \mathrm{Prym}(S, \overline{S})[2] \to \mathbb{Z}_2([a_p])^{\mathrm{ev}}/b_0 \to 0,$$

where b_0 is the divisor in Σ which has $+1$ for all zeros of a_p, by considering the action of σ as ± 1 on $\mathrm{Prym}(S, \overline{S})[2]$ over the zeros of $[a_p]$ up to switching σ by $-\sigma$, which is encoded by taking the quotient by b_0. Then, one can write

$$\mathrm{Prym}(S, \overline{S})[2] \cong H^1(\overline{S}, \mathbb{Z}_2) \oplus \mathbb{Z}_2([a_p])^{\mathrm{ev}}/b_0. \tag{4}$$

Hence, over each point in the base $\mathcal{A}$, one has the set of divisors D of degree M over which the involution acts as -1 (as in [19, Section 3]) given by a point in $\mathbb{Z}_2([a_p])^{\mathrm{ev}}/b_0$, together with the $\mathbb{Z}_2$ vector space $H^1(\overline{S}, \mathbb{Z}_2)$ of dimension $2g_{\overline{S}} = 2p(p-1)(g-1)+2$.

Proposition 3.2 *For p* odd *the space $H^1(\overline{S}, \mathbb{Z}_2)$ is given by*

$$\mathrm{Prym}(\overline{S}, \Sigma)[2] \oplus \overline{\pi}^* H^1(\Sigma, \mathbb{Z}_2),$$

and for p even *it is given by*

$$\overline{\pi}^* H^1(\Sigma, \mathbb{Z}_2) \oplus \left(\mathrm{Prym}(\overline{S}, \Sigma)[2]/\overline{\pi}^* H^1(\Sigma, \mathbb{Z}_2)\right) \oplus \overline{\pi}^* H^1(\Sigma, \mathbb{Z}_2).$$

Proof In order to understand the space $H^1(\overline{S}, \mathbb{Z}_2)$ one needs to take into account the parity of p, since $\overline{S}$ is a p-fold cover of the compact Riemann surface Σ. Considering the Norm map

$$0 \to \mathrm{Prym}(\overline{S}, \Sigma)[2] \to H^1(\overline{S}, \mathbb{Z}_2) \xrightarrow{\mathrm{Nm}} H^1(\Sigma, \mathbb{Z}_2) \to 0, \tag{5}$$

note that for a line bundle L on Σ one has that $\mathrm{Nm}(\overline{\pi}^* L) = pL$, and hence when p is odd the pullback $\overline{\pi}^*$ gives a splitting of the short exact sequence (5). Therefore, in this case one has

$$H^1(\overline{S}, \mathbb{Z}_2) \cong \mathrm{Prym}(\overline{S}, \Sigma)[2] \oplus \overline{\pi}^* H^1(\Sigma, \mathbb{Z}_2).$$

For p even, the image of $\overline{\pi}^* : H^1(\Sigma, \mathbb{Z}_2) \to H^1(\overline{S}, \mathbb{Z}_2)$ is contained in $\mathrm{Prym}(\overline{S}, \Sigma)[2]$, thus giving a filtration

$$H^1(\Sigma, \mathbb{Z}_2) \subset \mathrm{Prym}(\overline{S}, \Sigma)[2] \subset H^1(\overline{S}, \mathbb{Z}_2),$$

which induces the splitting (via isomorphism theorems for Nm)

$$H^1(\overline{S}, \mathbb{Z}_2) \cong \overline{\pi}^* H^1(\Sigma, \mathbb{Z}_2) \oplus \big(\mathrm{Prym}(\overline{S}, \Sigma)[2] / \overline{\pi}^* H^1(\Sigma, \mathbb{Z}_2)\big) \oplus \overline{\pi}^* H^1(\Sigma, \mathbb{Z}_2),$$

and thus the proposition follows. □

From the above, one has the following description of Zariski open sets in components of the moduli spaces of $\mathrm{Sp}(2p, \mathbb{R})$-Higgs bundles which intersect the smooth fibres of the $\mathrm{Sp}(2p, \mathbb{C})$-Hitchin fibration:

Proposition 3.3 *For each even invariant* $0 < M \leqslant 4p(g-1)$ *there is a component of the moduli space of* $\mathrm{Sp}(2p, \mathbb{R})$*-Higgs bundles which intersects the nonsingular fibres of the Hitchin fibration for* $\mathrm{Sp}(2p, \mathbb{C})$*-Higgs bundles. The component has a Zariski open set given by a fibration of a* $\mathbb{Z}_2$*-vector space over a Zariski open set in the total space of a vector bundle on the symmetric product* $S^M \Sigma$*. When* $M = 0$*, the intersection is given by a* $2^{2g_{\overline{S}}}$ *cover of a vector space over a point.*

Proof The overall idea of the proof is to follow the proof of [19, Theorem 4.2], taking into consideration the structure of the intersection of $\mathcal{M}_{\mathrm{Sp}(2p,\mathbb{R})}$ with the generic fibres of the Hitchin fibration given in (4) above.

The invariant M gives the degree of a divisor D which corresponds to the choice of an element in $\mathbb{Z}_2([a_p])^{\mathrm{ev}}/d_0$. In particular, whilst over generic points of the Hitchin base the divisor $[a_p]$ is an effective divisor with simple zeros, when considering Higgs bundles whose corresponding $2p$-differential has multiple zeros, one can deform those Higgs bundles to ones with simple zeros.

As in the case of $\mathrm{Sp}(2, \mathbb{R})$ analysed in [4, Proposition 10.2], every stable Higgs bundle is in a component which intersects the regular fibres of the Hitchin fibration. One should note that although this does not imply that all components intersect the regular fibres, it is what one needs in order to describe Zariski dense sets in the components.

Recall that the choice of a spectral curve is given by the choice of differentials a_i. Hence, given a point in $S^M \Sigma$, and the choice of the differential a_p is then given by a section in $H^0(\Sigma, K^{2p}(-D))$, or equivalently, a vector bundle $\mathcal{V}$ over the symmetric product. The remaining differentials are then parametrised by the bundle

$\bigoplus_{i=1}^{p-1} H^0(\Sigma, K^{2i})$ over the total space of $\mathcal{V}$ on $S^M\Sigma$. We shall denote by $\mathcal{W}$ the parameter space defining spectral curves as described above: $\bigoplus_{i=1}^{p-1} H^0(\Sigma, K^{2i})$ over the total space of $\mathcal{V}$ on $S^M\Sigma$.

When considering only generic spectral curves, one has to restrict to a Zariski open set inside the total space of $\mathcal{W}$. Finally, the Higgs bundles in the component are obtained by the remaining data in the fibre, which is a point in the $\mathbb{Z}_2$ vector space $H^1(\overline{S}, \mathbb{Z}_2)$.

Finally, from the above one can see that when $M = 0$, the space is given by a $2^{2g_{\overline{S}}}$ cover of a vector space over a point, and it is the monodromy action which needs to be considered from this perspective in order to deduce connectivity of this cover from this perspective. □

The following theorem follows from the above analysis.

Theorem 3.4 *When p is odd, for each fixed M the intersection of the moduli space $\mathcal{M}_{\mathrm{Sp}(2p,\mathbb{R})}$ with the smooth fibres of the* $\mathrm{Sp}(2p, \mathbb{C})$*-Hitchin fibration is given by* 2^{2g} *copies of* $\mathrm{Prym}(\overline{S}, \Sigma)[2]$.

Proof Recall that in terms of spectral data, the intersection of $\mathcal{M}_{\mathrm{Sp}(2p,\mathbb{R})}$ with the smooth fibres of the $\mathrm{Sp}(2p, \mathbb{C})$-Hitchin fibration is given $\mathrm{Prym}(S, \overline{S})[2]$, and by (4) this can be seen as

$$\mathrm{Prym}(S, \overline{S})[2] \cong H^1(\overline{S}, \mathbb{Z}_2) \oplus \mathbb{Z}_2([a_p])^{\mathrm{ev}}/b_0.$$

Moreover, when p is odd, the pullback $\overline{\pi}^*$ gives a splitting of the short exact sequence (5) leading to $H^1(\overline{S}, \mathbb{Z}_2) \cong \mathrm{Prym}(\overline{S}, \Sigma)[2] \oplus \overline{\pi}^* H^1(\Sigma, \mathbb{Z}_2)$. Therefore for p odd the intersection of $\mathcal{M}_{\mathrm{Sp}(2p,\mathbb{R})}$ with the smooth fibres of the $\mathrm{Sp}(2p, \mathbb{C})$-Hitchin fibration is given by

$$\mathrm{Prym}(S, \overline{S})[2] \cong \mathrm{Prym}(\overline{S}, \Sigma)[2] \oplus \overline{\pi}^* H^1(\Sigma, \mathbb{Z}_2) \oplus \mathbb{Z}_2([a_p])^{\mathrm{ev}}/b_0.$$

Then, once the topological invariants M have been fixed (giving a point in $\mathbb{Z}_2([a_p])^{\mathrm{ev}}/b_0$), the intersection with the generic fibres is given by the space $\mathrm{Prym}(\overline{S}, \Sigma)[2] \oplus \overline{\pi}^* H^1(\Sigma, \mathbb{Z}_2)$. Thus, over such curve, the fibre of the $\mathrm{Sp}(2p, \mathbb{C})$-Hitchin fibration intersects the moduli space of $\mathrm{Sp}(2p, \mathbb{R})$-Higgs bundles for fixed M in 2^{2g} copies of $\mathrm{Prym}(\overline{S}, \Sigma)[2]$. □

For $\mathrm{Sp}(4, \mathbb{R})$ it was shown by Gothen in [8] that $M = 0$ labels several connected components, and it is shown in [4, Section 6.3] how these components appear as orbits of the monodromy action of the corresponding Gauss–Manin connection on the $\mathrm{Sp}(4, \mathbb{C})$-Hitchin fibration. Further study of monodromy for low rank can be found in [4,18]. From a Morse theoretic perspective, connectivity has been studied under different restrictions by several authors—the reader should refer to [10] and references therein for further information in that direction.

As in the case of $\mathrm{U}(p, p)$-Higgs bundles of [19], the invariant and anti-invariant sections of $L \in \mathrm{Prym}(S, \overline{S})[2]$ decompose the direct image bundle into a direct sum of line bundles $\rho_* U := U_+ \oplus U_-$. Hence, the symplectic decomposition $E = W \oplus W^*$ can be seen as $W := \overline{\pi}_* U_+$ and $W^* := \overline{\pi}_* U_-$.

Proposition 3.5 *Let $D \in \mathbb{Z}_2[a_p]/b_0$ be the divisor of degree M on which the involution σ acts as -1. Then, there exists a line bundle $L_0 \in \mathrm{Prym}(\overline{S}, \Sigma)$ such that*

$$U_- = U_+ \otimes \mathcal{O}(D) \otimes K^* \otimes L_0.$$

Proof The symplectic structure of E is obtained though relative duality (e.g. see [16, Section 4]), and in particular it implies that $\overline{\pi}_* U_+ \cong (\overline{\pi}_* U_-)^*$. Hence, as in [19, Section 5], the line bundles $U_\pm$ are not independent. Indeed, from [19, Equation (15)] one has that

$$\mathrm{Nm}(U_+) = -\mathrm{Nm}(U_-) + 2p(p-1)K.$$

The above can be also written in terms of the divisor $D \in \mathbb{Z}_2[a_p]/b_0$ of degree M on which the involution σ acts as -1, as $D = \mathrm{Nm}(U_+^*) + \mathrm{Nm}(U_-) + pK$.

Therefore, viewing D as a divisor on $\overline{S}$ (since it is a subset of the ramification divisor of the p-fold cover $\overline{\pi}\colon \overline{S} \to \Sigma$ given by the divisor of $[a_p]$), one has that the line bundle U_- equals $U_+ \otimes \mathcal{O}(D) \otimes K^*$ up to a line bundle $L_0 \in \mathrm{Prym}(\overline{S}, \Sigma)$, and thus the result follows. □

From [19, Equations (9)–(10)] one can write the degrees of the line bundles $U_\pm$ in terms of M. In particular, recalling that $U = L \otimes K^{(2p-1)/2}$ one has that $\deg(U) = p(2p-1)(g-1)$, and therefore the degrees of the invariant and anti-invariant line bundles on $\overline{S}$ can be expressed as

$$\deg(U_+) = p(2p-1)(g-1) - \frac{M}{2},$$
$$\deg(U_-) = p(2p-3)(g-1) + \frac{M}{2}.$$

Equivalently, from [19, Equation (9)] one can write the degree of the rank p bundle W as

$$\deg(W) = \frac{\deg(U)}{2} - \frac{M}{2} - (2p^2 - 2p)(g-1) = -\frac{M}{2} + p(g-1),$$

recovering the result in [13, Equation (7)]. The case of $M = 0$ corresponds to the maximal Toledo invariant setting, for which it is known that within the covering space one has 2^{2g} connected components, the so-called *Hitchin components*, parametrising rich geometric structures. Moreover, when $m = 2$ the $\mathbb{Z}_2$ vector space $H^1(\overline{S}, \mathbb{Z}_2)$ can be understood in terms of line bundles of order 2 over the Riemann surface Σ, and we shall comment on this case in Sect. 5.

4 SO$(p+1, p)$-Higgs bundles

Recall from Definition 1.5 that an SO$(p+1, p)$-Higgs bundle is a pair (V, Φ) where $V = V_+ \oplus V_-$ for $V_\pm$ complex vector spaces with orthogonal structures of dimension p and $m+1$ respectively, and where

$$\Phi \in H^0\big(\Sigma, (\mathrm{Hom}(V_-, V_+)\oplus \mathrm{Hom}(V_+, V_-))\otimes K\big)$$

is given by

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \quad \text{for} \quad \gamma \equiv -\beta^{\mathrm{T}}, \tag{6}$$

where β^{T} is the orthogonal transpose of β. As mentioned previously, since the space $\mathrm{SO}(p+1, p)$ retracts onto $\mathrm{S}(\mathrm{O}(p)\times\mathrm{O}(p+1))$, the $\mathrm{SO}(p+1, p)$-Higgs bundles $(V_+\oplus V_-, \Phi)$ carry three topological invariants, the Stiefel–Whitney classes $\omega_1(V_+) \in H^1(\Sigma, \mathbb{Z}_2)$, and $\omega_2(V_\pm) \in H^2(\Sigma, \mathbb{Z}_2)$. Note that the class $\omega_1(V_-) = \omega_1(V_+)$ since $\det V_- = \det V_+^*$. By further requiring the Higgs bundle to be in the connected component of the identity, i.e. taking $\mathrm{SO}(p+1, p)_0$-Higgs bundles, one would obtain pairs with $\omega_1(V_\pm) = \mathcal{O}$ (as considered, for instance, in [1]). In what follows we shall give a geometric description of these topological invariants, relate them to the ones for $\mathrm{Sp}(2p, \mathbb{R})$-Higgs bundles obtained in [16, Section 6], and finally use this description to characterise Zariski dense open sets in each connected component of the moduli space of $\mathrm{SO}(p+1, p)$-Higgs bundles. On several occasions, it will be important to distinguish when p is even or odd, and we shall do so within this section.

4.1 KO-theory of Σ

In order to discuss the topology of orthogonal bundles on the surface Σ we use KO-theory. For this, we shall recall some results from [16, Section 6] and [2]. The Stiefel–Whitney classes of $V_\pm$ can be seen as classes $[V_\pm] \in \mathrm{KO}(\Sigma)$ where

$$\begin{aligned}[][V_\pm] \in \mathrm{KO}(\Sigma) &\simeq \mathbb{Z}\oplus H^1(\Sigma, \mathbb{Z}_2)\oplus\mathbb{Z}_2 \\ V_\pm &\mapsto (\mathrm{rk}(V_\pm), \omega_1(V_\pm), \omega_2(V_\pm)).\end{aligned}$$

Taking the map given by the total Stiefel–Whitney class $\omega = 1 + \omega_1 + \omega_2$ to the multiplicative group $\mathbb{Z}\oplus H^1(\Sigma, \mathbb{Z}_2)\oplus\mathbb{Z}_2$, we consider the generators given by holomorphic line bundles L such that $L^2 \simeq O$, and the class $\Omega = \mathcal{O}_p + \mathcal{O}_p^* - 2$ where $\mathcal{O}_p$ is the holomorphic line bundle given by a point $p \in \Sigma$. Then, for $\alpha(x)$ the class of a line bundle $x \in H^1(\Sigma, \mathbb{Z}_2)$ and (x, y) the intersection form, $\alpha(x+y) = \alpha(x) + \alpha(y) - 1 + (x, y)\Omega$. As in [16, Section 5], the isomorphism between the additive group $\widetilde{\mathrm{KO}}(\Sigma)$ and the multiplicative group $\mathrm{KO}(\Sigma)$ is determined by the relations

$$\begin{aligned}&\omega_1(\alpha(x)) = x, \quad \omega_1(\Omega) = 0, \quad \text{and} \\ &\omega_2(\Omega) = c_1(\mathcal{O}_p) \,(\mathrm{mod}\, 2) = [\Sigma] \in H^2(\Sigma, \mathbb{Z}_2).\end{aligned}$$

With this notation, the classes $[V_\pm]$ satisfy

$$[V_\pm] = \mathrm{rk}(V_\pm) - 1 + \alpha(\omega_1(V_\pm)) + \omega_2(V_\pm)\Omega.$$

Definition 4.1 Choosing a Θ characteristic $K^{1/2}$, the classes $[V_{\pm}]$ have associated an analytic mod 2 index

$$\varphi_{\Sigma}(V_{\pm}) = \dim H^0(\Sigma, V_{\pm} \otimes K^{1/2}) \pmod 2,$$

and the characteristic class ω_2 is independent of which spin structure $K^{1/2}$ is chosen.

It follows from [16, Theorem 1] that the classes $\omega_2(V_{\pm})$ satisfy

$$\omega_2(V_{\pm}) = \varphi_{\Sigma}(V_{\pm}) + \varphi_{\Sigma}(\det(V_{\pm})).$$

Moreover, $\varphi_{\Sigma}(\Omega) = 1$ and the map can be seen as the map to a point

$$\varphi_{\Sigma} : \mathrm{KO}(\Sigma) \to \mathrm{KO}^{-2}(\mathrm{pt}) \cong \mathbb{Z}_2.$$

Since we are interested in understanding Higgs bundles through their spectral data, we note that as in [16, Section 5], the spin structures together with the covers $\pi : S \to \Sigma$ and $\overline{\pi} : \overline{S} \to \Sigma$ define push forward maps $\mathrm{KO}(S) \to \mathrm{KO}(\Sigma)$ and $\mathrm{KO}(\overline{S}) \to \mathrm{KO}(\Sigma)$.

4.2 Spectral data for SO$(p+1, p)$-Higgs bundles

In order to give a geometric description of characteristic classes, we shall define here the spectral data associated to the $\mathrm{SO}(p+1, p)$-Higgs bundles. One should note that since $\mathrm{SO}(p+1, p)$-Higgs bundles lie completely inside the singular fibres of the $\mathrm{SL}(2p+1, \mathbb{C})$-Hitchin fibration, the analysis done in [16, Section 5] cannot be directly applied.

Theorem 4.2 *The intersection of the moduli space $\mathcal{M}_{\mathrm{SO}(p+1,p)}$ with the regular fibres of the* $\mathrm{SO}(2p+1, \mathbb{C})$*-Hitchin fibration is given by two copies of the space* $\mathrm{Prym}(S, \overline{S})[2]/\rho^* H^1(\overline{S}, \mathbb{Z}_2)$ *where the $\mathbb{Z}_2$ space $H^1(\overline{S}, \mathbb{Z}_2)$ is given by*

$$\mathrm{Prym}(\overline{S}, \Sigma)[2] \oplus H^1(\Sigma, \mathbb{Z}_2), \tag{7}$$

for m odd*, and for m* even *it is given by*

$$H^1(\Sigma, \mathbb{Z}_2) \oplus \big(\mathrm{Prym}(\overline{S}, \Sigma)[2]/\pi^* H^1(\Sigma, \mathbb{Z}_2)\big) \oplus H^1(\Sigma, \mathbb{Z}_2). \tag{8}$$

Proof From [20, Theorem 4.12] in this case we consider points of order 2 in the fibres (3) of the $\mathrm{SO}(2p+1, \mathbb{C})$-Hitchin fibration, which form two copies of the space

$$\mathrm{Prym}(S, \overline{S})[2]/\rho^* H^1(\overline{S}, \mathbb{Z}_2), \tag{9}$$

a $\mathbb{Z}_2$ vector space of dimension $4p(g-1)-2$. Moreover, the points in $\rho^* H^1(\overline{S}, \mathbb{Z}_2)$ are precisely those line bundles in $\mathrm{Prym}(S, \overline{S})[2]$ with trivial action of σ at all fixed points, i.e., with invariant $M = 0$. Together with the structure of $\rho^* H^1(\overline{S}, \mathbb{Z}_2)$ from Proposition 3.2, the description of the fibres in the theorem follows. ◻

From Theorem 4.2, the spectral data associated to an $\mathrm{SO}(p+1,p)$-Higgs bundle, up to equivalences by (7)–(8), is given by the intermediate spectral curve $\overline{S}$ together with a line bundle $\mathcal{F} \in H^1(\overline{S}, \mathbb{Z}_2)$, and a divisor $D \in \mathbb{Z}_2([a_p])^{\mathrm{ev}}/b_0$ of degree M.

Remark 4.3 It is interesting to note that when $m = 2$ the middle term in (8) gives in fact the spectral data for a K^2-twisted $\mathrm{PGL}(2,\mathbb{R})$-Higgs bundle. Moreover, the component $\mathrm{Prym}(\overline{S},\Sigma)[2]$ gives the spectral data for K^2-twisted $\mathrm{SL}(p,\mathbb{R})$-Higgs bundles.

In order to recover $\mathrm{SO}(p+1,p)$-Higgs bundles from the above spectral data, we shall recall the relation between symplectic and orthogonal Higgs bundles as described in [15, Section 4]. As mentioned in Sect. 3, the line bundle $L \in \mathrm{Prym}(S,\overline{S})[2]$ defines a symplectic vector bundle as $E := \pi_* U$, for $U = L \otimes K_S^{1/2} \otimes \pi^* K^{-1/2}$. Then, from [15, Equation (7)] the orthogonal bundle V is recovered as an extension

$$0 \to E \otimes K^{-1/2} \to V \to K^p \to 0, \tag{10}$$

and therefore near the divisor defined by the section a_p, the orthogonal bundle V of the $\mathrm{SO}(p+1,p)$-Higgs pair (V,Φ) is recovered as $V := (E \otimes K^{-1/2}) \oplus K^p$.

From [15, Section 4.1], the $2p+1$ vector bundle V has trivial determinant and a nondegenerate symmetric bilinear form $g(v,w)$ for which one has $g(\Phi v, w) + g(v, \Phi w) = 0$, related to the symplectic form on E. Indeed, by considering the Higgs field Φ on V/K^{-p}, one has a nondegenerate skew form on V/K^{-p}, and by choosing a square root $K^{1/2}$, one obtains a skew form on $E = V/K^{-p} \otimes K^{-1/2}$ which is generically nondegenerate: $\omega(v,w) = g(\Phi v, w)$. Moreover, the extension class in (10) can be seen as a choice of trivialization of the line bundle $L \in \mathrm{Prym}(S,\overline{S})$ which depends on the action of the involution σ, this is, on the divisor $D \in \mathbb{Z}_2[a_p]/b_0$ (see [15, Section 4.3]).

The orthogonal structure induced on the rank p and $p+1$ vector bundles $V_- \oplus V_+$ obtained through the spectral data in the fibre (9) can be understood in terms of a decomposition of the symplectic bundle $E := E_- \oplus E_+$, through which locally one has

$$V_- = E_- \otimes K^{-1/2} \oplus K^p, \tag{11}$$
$$V_+ = E_+ \otimes K^{-1/2}. \tag{12}$$

One should note that it is not the symplectic decomposition $E = W \oplus W^*$ which leads to the decomposition $E = E_- \oplus E_+$ on the orthogonal side. This becomes evident, for instance, by considering the Hitchin components for both groups described in Sect. 5.1. Furthermore, since $V_\pm$ form part of $\mathrm{GL}(p,\mathbb{C})$- and $\mathrm{GL}(p+1,\mathbb{C})$-Higgs bundles, from [13] and [6] there is a line bundle on $\overline{S}$ whose direct image gives V_+ on Σ. Adopting the notation of [16, Section 5] we define $\pi_!$ and $\overline{\pi}_!$ by

$$\pi_!(\mathcal{L}) = \pi_*(\mathcal{L} \otimes K_S^{1/2} \otimes \pi^* K^{-1/2}), \quad \text{and}$$
$$\overline{\pi}_!(\mathcal{L}) = \overline{\pi}_*(\overline{\mathcal{L}} \otimes K_{\overline{S}}^{1/2} \otimes \overline{\pi}^* K^{-1/2}),$$

for $\mathcal{L}$ and $\overline{\mathcal{L}}$ line bundles on S and $\overline{S}$. Then, as seen in Sect. 3, the symplectic vector bundle E is obtained as $E := \pi_!(\mathcal{L})$ for $\mathcal{L} \in \mathrm{Prym}(S, \overline{S})$. When $\mathcal{L}^2 \cong \mathcal{O}$ and $\overline{\mathcal{L}}^2 \cong \mathcal{O}$, the bundles $\pi_!(\mathcal{L})$ and $\overline{\pi}_!(\overline{\mathcal{L}})$ acquire orthogonal structures by relative duality, as shown in [16, Section 4]. Hence, since V_+ has an orthogonal structure, following [16, Section 4] and [6] for K^2-twisted Higgs bundles, the vector bundle V_+ is obtained, for some $\mathcal{F} \in H^1(\overline{S}, \mathbb{Z}_2)$, as

$$V_+ = \overline{\pi}_!(\mathcal{F}).$$

Remark 4.4 It is interesting to note that the vector bundles $V_\pm$ form part of K^2-twisted Higgs bundles obtained by taking the compositions of β and γ in (6). Moreover, one should note that the spectral curve associated to these compositions is in fact the quotient curve $\overline{S}$. The procedure to construct these Higgs bundles follows directly from the case of $\mathrm{SU}(p, p)$-Higgs bundles described in [19].

Proposition 4.5 *For $\mathcal{F} \in H^1(\overline{S}, \mathbb{Z}_2)$, one has $\det(\overline{\pi}_!(\mathcal{F})) = \mathrm{Nm}(\mathcal{F})$.*

Proof The determinant bundle of $\overline{\pi}_!(\mathcal{F})$ can be obtained through [6, Section 4], leading to $\det(\overline{\pi}_!(\mathcal{F})) = \mathrm{Nm}(\mathcal{F} \otimes K_{\overline{S}}^{1/2} \otimes \overline{\pi}^* K^{-(2p-1)/2}) = \mathrm{Nm}(\mathcal{F})$. □

In order to understand how the other orthogonal bundle V_- is reconstructed, we shall give now a construction of E_- via the spectral data $\mathcal{F}$ and D modulo (7)–(8) (which in particular implies modulo $\mathrm{Prym}(\overline{S}, \Sigma)$). Since $\det(E_+) \otimes \det(E_-) = \mathcal{O}$, from (11)–(12) one has that

$$\det(E_-) = \mathrm{Nm}(\mathcal{F}) \otimes K^{-p/2}.$$

Therefore, for some $L_0 \in \mathrm{Prym}(\overline{S}, \Sigma)$ one may write

$$V_- = \overline{\pi}_*\big(L_0 \otimes K_{\overline{S}}^{-1/2} \otimes \mathcal{F}\big) \otimes K^{-1/2}.$$

Note that the choice of L_0 is equivalent to the one done in Proposition 3.5, and the divisor D gives the extension class as in the complex case described in [15, Section 4.3].

4.3 Characteristic classes for SO$(p+1, p)$-Higgs bundles

In what follows we shall see that the three Stiefel–Whitney classes of $\mathrm{SO}(p+1, p)$-Higgs bundles $(V_- \oplus V_+, \Phi)$ can be described in terms of their spectral data, which from the previous sections is given modulo (7)–(8) by

$$(\mathcal{F}, D) \in H^1(\overline{S}, \mathbb{Z}_2) \oplus \mathbb{Z}_2([a_p])/b_0.$$

Theorem 4.6 *The Stiefel–Whitney classes of an $\mathrm{SO}(p+1, p)$-Higgs bundle $(V_- \oplus V_+, \Phi)$ with spectral data $(\mathcal{F}, D) \in \mathrm{Prym}(S, \overline{S})[2]/\rho^* H^1(\overline{S}, \mathbb{Z}_2)$ are given by*

$$\omega_1(V_+) = \mathrm{Nm}(\mathcal{F}) \in H^1(\Sigma, \mathbb{Z}_2);$$
$$\omega_2(V_+) = \varphi_{\overline{S}}(\mathcal{F}) + \varphi_\Sigma(\mathrm{Nm}(\mathcal{F})) \in \mathbb{Z}_2;$$

$$\omega_2(V_-) = \begin{cases} \omega_2(V_+) & \text{if } \omega_2(V) = 0, \\ \omega_2(V_+) + 1 & \text{if } \omega_2(V) = 1. \end{cases}$$

Proof Recall that $\varphi_\Sigma(\mathcal{L}) = \dim H^0(\Sigma, \mathcal{L} \otimes K^{1/2})\ (\mathrm{mod}\ 2)$, and from [16, Theorem 1] that for an even spin structure $K^{1/2}$, the orthogonal bundles $V_\pm$ satisfy

$$\omega_2(V_\pm) = \varphi_\Sigma(V_\pm) + \varphi_\Sigma(\det(V_\pm)). \tag{13}$$

Moreover, since $\deg(\det(V_\pm)) = 0$, one has that $\varphi_\Sigma(V_-) = \varphi_\Sigma(V_+)\ (\mathrm{mod}\ 2)$.

The above can also be seen in terms of the analytic mod 2 indices φ_S and $\varphi_{\overline{S}}$ of the spectral line bundles producing V_+ and V_-. The three mod 2 indices can be related by considering the definition of push forward of sheaves. Indeed, note that for $\mathcal{L}$ a torsion two line bundle on S, by definition of direct image sheaf

$$\begin{aligned} \varphi_S(\mathcal{L}) &= \dim H^0\big(S, \mathcal{L} \otimes K_S^{1/2}\big)\ (\mathrm{mod}\ 2) \\ &= \dim H^0\big(\Sigma, \pi_*\big(\mathcal{L} \otimes K_S^{1/2} \otimes \pi^* K^{-1/2}\big) \otimes K^{1/2}\big)\ (\mathrm{mod}\ 2), \end{aligned}$$

and hence $\varphi_S(\mathcal{L}) = \varphi_\Sigma(\pi_!(\mathcal{L}))$. An equivalent formula follows for $\overline{S}$, and therefore

$$\omega_1(V_+) = \mathrm{Nm}(\mathcal{F}) \in H^1(\Sigma, \mathbb{Z}_2).$$

Moreover, since $\varphi_{\overline{S}}(\mathcal{F}) = \varphi_\Sigma(\overline{\pi}_!(\mathcal{F}))$ it follows that

$$\omega_2(V_+) = \varphi_{\overline{S}}(\mathcal{F}) + \varphi_\Sigma(\mathrm{Nm}(\mathcal{F})).$$

In order to understand $\omega_2(V_-)$ through (13), we should recall that $\omega_2(V) = \omega_2(V_+) + \omega_2(V_-)$, and thus

$$\omega_2(V_-) = \varphi_{\overline{S}}(\mathcal{F}) + \varphi_\Sigma(\mathrm{Nm}(\mathcal{F})) + \varphi_\Sigma(V).$$

The value of $\varphi_\Sigma(V) = \omega_2(V)$ has been studied in [15, Section 4] and indicates whether V has a lift to a spin bundle or not. In particular, it is shown there that it is the identity component of the fibre that gives spin bundles, which is $\omega_2(V) = \varphi_\Sigma(V) = 0$, and the theorem follows. □

One should note that when further requiring V_+ to have trivial determinant, it becomes the vector bundle of an $\mathrm{SL}(p, \mathbb{R})$-Higgs pair and our result agrees with the description of $\omega_2(V_+)$ of [16, Theorem 1] for a fixed even spin structure. When considering $\mathrm{SO}(p+1, p)_0$-Higgs bundles, i.e. Higgs bundles in the component of the identity, both vector bundles $V_\pm$ satisfy $\det V_\pm = \mathcal{O}$, and thus they are obtained by choosing a point $L_+ \otimes \overline{\pi}^* K^{3/2}$ in the Prym variety $\mathrm{Prym}(\overline{S}, \Sigma)$, after fixing a choice of spin structure $K^{1/2}$. Moreover, in this case $M = 4p(g-1)$ and thus $L^* \in \mathrm{Prym}(S, \Sigma)[2]$ is the pullback of a line bundle on $\overline{S}$, hence determined by L_+.

Since the characteristic class ω_2 is independent of which spin structure $K^{1/2}$ is chosen, we may use this fact to further deduce the following from Theorem 4.6 by fixing $K^{1/2}$ such that $\varphi_\Sigma(\mathcal{O}) = 1$, along the lines of [16, Theorem 1] purely in terms of spin structures:

Corollary 4.7 *Let $\overline{S}$ be a smooth spectral curve in the total space of $K^2 \to \Sigma$ given by an equation*

$$\eta^p + a_1\eta^{p-1} + \cdots + a_{p-1}\eta + a_p = 0,$$

and let $\mathcal{F}$ be a line bundle on $\overline{S}$ such that $\mathcal{F}^2 \cong \mathcal{O}$. Define $V_+ := \pi_!(\mathcal{F})$ the image bundle with the orthogonal structure induced from relative duality. Let $K^{1/2}$ be an even spin structure on Σ, and for $K^{1/2}_{\overline{S}} = \overline{\pi}^* K^{p-1/2}$ the corresponding one on $\overline{S}$, consider the spin structure $\mathcal{F}_{\overline{S}} = \mathcal{F} \otimes K^{1/2}_{\overline{S}}$. Then, the characteristic classes of the corresponding $\mathrm{SO}(p+1, p)$*-Higgs pair are*

$$\omega_1(V_+) = \mathrm{Nm}(\mathcal{F}) \in H^1(\Sigma, \mathbb{Z}_2);$$
$$\omega_2(V_+) = \begin{cases} \varphi_\Sigma(\mathrm{Nm}(\mathcal{F})) & \text{if } \varphi_S(\mathcal{F}_{\overline{S}}) = 0, \\ 1 + \varphi_\Sigma(\mathrm{Nm}(\mathcal{F})) & \text{if } \varphi_S(\mathcal{F}_{\overline{S}}) = 1; \end{cases}$$
$$\omega_2(V_-) = \begin{cases} \varphi_\Sigma(\mathrm{Nm}(\mathcal{F})) & \text{if } \varphi_S(\mathcal{F}_{\overline{S}}) = \varphi_\Sigma(V), \\ 1 + \varphi_\Sigma(\mathrm{Nm}(\mathcal{F})) & \text{if } \varphi_S(\mathcal{F}_{\overline{S}}) \neq \varphi_\Sigma(V). \end{cases}$$

4.4 The divisor $D \in \mathbb{Z}([a_p])/b_0$

We shall finally consider the geometric implications of the divisor $D \in \mathbb{Z}([a_p])/b_0$ appearing in the spectral data of the Higgs bundles studied in this paper. As mentioned previously, the extension class giving the orthogonal bundle V is obtained through D. Moreover, its degree M appears both at the level of complex $\mathrm{SO}(2p+1, \mathbb{C})$-Higgs bundles (see [15, Remark 2, p. 14]) and real $\mathrm{SO}(p+1, p)$-Higgs bundles.

Proposition 4.8 *In each generic fibre of the* $\mathrm{SO}(2p+1, \mathbb{C})$*-Hitchin fibration there are*

$$\binom{4p(g-1)}{M}$$

points corresponding to $\mathrm{SO}(p+1, p)$*-Higgs bundles with even invariant M.*

Proof Recall from Theorem 4.2 that the intersection of the moduli space of $\mathrm{SO}(p+1, p)$-Higgs bundles with the complex Hitchin fibration is given by two copies of $\mathrm{Prym}(S, \overline{S})[2]/\rho^* H^1(\overline{S}, \mathbb{Z}_2)$, and thus we defined its spectral data, modulo (7)–(8), by

$$(\mathcal{F}, D) \in H^1(\overline{S}, \mathbb{Z}_2) \oplus \mathbb{Z}_2([a_p])/b_0.$$

In order to understand how many points in

$$\mathrm{Prym}(S, \overline{S})[2]/\rho^* H^1(\overline{S}, \mathbb{Z}_2)$$

correspond to $\mathrm{SO}(p+1,p)$-Higgs bundles with a fixed invariant M, real from [16, Section 6] (as recalled in Sect. 3) that the elements in $\mathrm{Prym}(S,\overline{S})[2]$ can be distinguished by their associated invariant M, and in each regular fibre there are

$$\binom{4p(g-1)}{M}\times 2^{2g_{\overline{S}}}$$

points with invariant M, where the genus of $\overline{S}$ is as before, $g_{\overline{S}}=(2m^2-m)\cdot(g-1)+1$.

Hence, in order to differentiate the characteristic classes for $\mathrm{SO}(p+1,p)$-Higgs bundles, one needs to understand the characteristic classes of elements in $\rho^*H^1(\overline{S},\mathbb{Z}_2)$. In particular, it should be noted that since the Prym variety $\mathrm{Prym}(S,\overline{S})$ is defined as the set of line bundles $L\in\mathrm{Jac}(S)$ for which $\sigma^*L\cong L^*$, the pulled-back line bundles in $\rho^*H^1(\overline{S},\mathbb{Z}_2)$ are acted on trivially by the involution σ and thus carry invariant $M=0$.

Therefore, recalling that the topological invariant M associated to $\mathrm{SO}(p+1,p)$-Higgs bundles can be seen from (4) as the degree of the subdivisor of $[a_p]$ giving an element in $\mathbb{Z}_2([a_p])^{\mathrm{ev}}$, the proposition follows. □

Since exchanging σ by $-\sigma$ exchanges the values of M and $4p(g-1)-M$, those two cases should be identified. Hence, the total number of points in each regular fibre is half of

$$\binom{4p(g-1)}{0}+\binom{4p(g-1)}{2}+\cdots+\binom{4p(g-1)}{4p(g-1)-2}+\binom{4p(g-1)}{4p(g-1)},$$

which is, as expected, $[2^{4p(g-1)-1}]/2$.

4.5 On the geometry of the moduli space

From the above analysis, one has a natural stratification of the moduli space leading to a geometric description of Zariski dense open sets in the moduli space of $\mathrm{SO}(p+1,p)$-Higgs bundles:

Proposition 4.9 *For each fixed even invariant $0<M\leqslant 4p(g-1)$ there is a component of the moduli space of $\mathrm{SO}(p+1,p)$-Higgs bundles which intersects the regular fibres of the $\mathrm{SO}(2p+1,\mathbb{C})$-Hitchin fibration. The component has a space given by a covering of a Zariski open set in the total space of a vector space over the symmetric product $S^M\Sigma$. When $M=0$ and p is odd the intersection with smooth fibres is given by 2^{2g} copies of $\mathrm{Prym}(\overline{S},\Sigma)$ over a vector space.*

Proof Recall from Theorem 4.2 that we defined its spectral data for generic $\mathrm{SO}(p+1,p)$-Higgs bundles, modulo (7)–(8), by

$$(\mathcal{F},D)\in H^1(\overline{S},\mathbb{Z}_2)\oplus\mathbb{Z}_2([a_p])/b_0.$$

Over a point in the regular locus of the Hitchin base defining a spectral curve (which is a point in $\mathcal{Z}$), one has $\mathbb{Z}_2([a_p])^{\mathrm{ev}}/d_0$. This is all choices of $4p(g-1)$ $\mathbb{Z}_2$-uples D

with an even number of $+1$, up to the element $(1, \dots, 1)$ and equivalence. From the construction of spectral data, we know that for each invariant M one has strictly stable Higgs bundles with that invariant: strictly stable since their characteristic polynomials are indecomposable and thus there are no Φ-invariant subbundles that could destabilise the Higgs bundle.

In order to show that for each M the component of $\mathrm{SO}(p+1, p)$-Higgs bundles has a space given by a covering of a Zariski open set in the total space of a vector space over the symmetric product $S^M\Sigma$, note that the choice of the divisor D (or equivalently, a point in $\mathbb{Z}_2([a_p])^{\mathrm{ev}}/d_0$) is given by a point in the symmetric product $S^M\Sigma$, for M the degree of D. Then, the choice of the differential a_p is given by the choice of a section $s \in H^0(\Sigma, K^{2p}(-D))$, leading to a vector bundle B over $S^M\Sigma$ of rank $(4p-1)(g-1) - M$. Finally, the choice of the spectral curve is completed by considering, as in the symplectic side, the space

$$\bigoplus_{i=1}^{2p-2} H^0(\Sigma, K^{2i}),$$

where the parametrisation is done up to $H^1(\overline{S}, \mathbb{Z}_2)$.

As in the proof of Proposition 3.3, in order to consider only the smooth loci of the Hitchin fibration, one should consider only a Zariski open set $\mathcal{Z}$ inside the space given by $\bigoplus_{i=1}^{2p-2} H^0(\Sigma, K^{2i})$ over the total space of $B \to S^M\Sigma$. Then, as in [16, Proposition 4], this agrees with the previous section, asserting that the intersection of the space with the fibre is a $\mathbb{Z}_2$ vector space of dimension $4p(g-1) - 2$.

Finally, from the above analysis it follows that when $M = 0$ and p is odd the intersection with smooth fibres is given by 2^{2g} copies of $\mathrm{Prym}(\overline{S}, \Sigma)$ over a vector space. □

One should keep in mind that the characteristic classes of the $\mathrm{SO}(p+1, p)$-Higgs bundles are topological invariants, and thus are constant within connected components. On the other hand, the invariant M labels components which often intersect over the singular locus of the Hitchin fibration. An interesting comparison can be made with [7, Section 6.4], where it is shown how the invariant M labels certain connected components of the moduli space. One should note also that the space $H^1(\overline{S}, \mathbb{Z}_2)$ is in fact the spectral data for K^2-twisted $\mathrm{GL}(p, \mathbb{R})$-Higgs bundles, and thus over each point in the Hitchin base one has the fibre giving the spectral data for a corresponding K^2-twisted $\mathrm{GL}(p, \mathbb{R})$-Higgs bundle, the Cayley partners.

5 Concluding remarks

In what follows, we shall describe some applications of the above methods in the context of understanding the moduli spaces for other real groups.

5.1 The $\mathrm{Sp}(2p,\mathbb{R})$- and $\mathrm{SO}(p+1,p)$-Hitchin components

When considering the Hitchin components for both split real forms $\mathrm{Sp}(2p,\mathbb{R})$ and $\mathrm{SO}(2p+1,\mathbb{R})$ as described in [14], one can see that the vector bundle for $\mathrm{Sp}(2p,\mathbb{R})$ is given by (e.g. see [16, p. 4])

$$E := \bigoplus_{i=1}^{2p} (K^{-p+i} \otimes K^{-1/2}).$$

Then, by considering $E \otimes K^{+1/2} \oplus K^{-p}$ one obtains, as expected, the orthogonal bundle for $\mathrm{SO}(p+1,p)$-Higgs bundles

$$V := \bigoplus_{i=0}^{2p} K^{-p+i}.$$

The pairing for the symplectic vector bundle $E = W \oplus W^*$ is obtained by considering the symplectic pairing between $K^{\pm a}$. On the other hand, the pairing for the orthogonal bundle $V = V_- \oplus V_+$ is obtained by taking the natural orthogonal structure for each $K^a \oplus K^{-a}$ and thus for p even one has

$$\begin{aligned} V_- &= \bigoplus_{i=0}^{p-1} K^{-p+2i+1}, \quad \text{and} \\ V_+ &= \bigoplus_{i=0}^{p} K^{-p+2i} = K^p \oplus \bigoplus_{i=0}^{p-1} K^{-p+2i}. \end{aligned} \tag{14}$$

Whenever p is odd, the roles of V_- and V_+ are interchanged. One should note that, in particular, separating the vector bundle $W = W_+ \oplus W_-$ into the odd $(-)$ and even $(+)$ values of i, one has

$$V_- = W_- \oplus W_-^* \quad \text{and} \quad V_+ = W_+ \oplus W_+^*,$$

and thus the relation between both decompositions of the symplectic bundle and orthogonal bundle become apparent. In the case of $\mathrm{SL}(p,\mathbb{R})$-Higgs bundles, the Hitchin component is given by Higgs bundles whose underlying vector $\widetilde{\mathcal{V}}$ bundle has the form (see [14, Section 3])

$$\widetilde{\mathcal{V}} = \bigoplus_{i=0}^{p-1} K^{(-p+1+2i)/2}.$$

This bundle is obtained from the origin in the fibre of the Hitchin fibration, and a similar construction leads to the Hitchin component for K^2 twisted $\mathrm{SL}(p,\mathbb{R})$-Higgs bundles, where

$$\mathcal{V} = \bigoplus_{i=0}^{p-1} K^{-p+1+2i}.$$

In particular, this rank p vector bundle coincides with V_- in (14), which is not surprising, as it follows from the proof of Theorem 4.6.

Remark 5.1 One should note the extension class involved when one goes from $\mathrm{Sp}(2p,\mathbb{C})$-Higgs bundles to $\mathrm{SO}(2p+1,\mathbb{C})$-Higgs bundles considered by Hitchin in [15] vanishes for Higgs bundles in the Hitchin component. This is because these Higgs bundles are of maximal Toledo invariant, which when seen through the action of the involution $\sigma:\eta\to-\eta$ on the spectral curve S, one can see that it implies that the action has the same sign over all the zeros of the divisor $[a_p]$, which implies that the extension class is the trivial one.

5.2 Maximal $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles and $\mathrm{SO}(2,3)$-Higgs bundles

Connectivity for $\mathrm{SO}(2,3)$-Higgs bundles has been studied in [4,11]. Moreover, from Gothen's work [8,9] on the so-called Gothen components, in the case of maximal Toledo invariant (i.e. $M=0$), the number of connected components for $\mathrm{Sp}(4,\mathbb{R})$ is $3\cdot 2^{2g}+2g-4$. As in the general case, the components are described by $H^1(\overline{S},\mathbb{Z}_2)$ over a vector bundle over the symmetric product $S^M\Sigma$. But in the case of $m=2$ one has one more correspondence to consider. Indeed, $H^1(\overline{S},\mathbb{Z}_2)$ becomes the spectral data for K^2-twisted $\mathrm{GL}(2,\mathbb{R})$-Higgs bundles, the Cayley partner of $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles. As in [4, Theorem 6.8], one has that

$$H^1(\overline{S},\mathbb{Z}_2)=\Lambda_\Sigma[2]\oplus\mathbb{Z}_2([a_p])^{\mathrm{ev}}/b_0\oplus\Lambda_\Sigma[2]$$

and as seen in [4, Corollary 6.9], one recovers the $3\cdot 2^{2g}+2g-4$ components as orbits of the monodromy action. Moreover, from the description in Sect. 3, these components appear as the components of K^2-twisted Higgs bundles over the vector space $\mathcal{A}$. The geometry of these components can be studied as in Sects. 3 and 4, by noting that a choice in $\mathbb{Z}_2([a_p])^{\mathrm{ev}}/b_0$ gives a point in a symmetric product labeled by the invariant M, and over that one has 2^{2g} covers coming from $H^1(\Sigma,\mathbb{Z}_2)$.

5.3 The dual (B,B,B)-branes

The smooth locus of the moduli space of $\mathrm{SO}(2p+1,\mathbb{C})$-Higgs bundles on Σ is a hyper-Kähler manifold, so there are natural complex structures I, J, K obeying the same relations as the imaginary quaternions (following the notation of [17]). Adopting physicists' language, a Lagrangian submanifold of a symplectic manifold is called an *A-brane* and a complex submanifold a *B-brane*. A submanifold of a hyper-Kähler manifold may be of type A or B with respect to each of the complex or symplectic structures, and thus choosing a triple of structures one may speak of branes of type (B,B,B), (B,A,A), (A,B,A) and (A,A,B). The moduli space of $\mathrm{SO}(p+1,p)$-Higgs bundles is a (B,A,A)-brane in the moduli space $\mathcal{M}_{\mathrm{SO}(2p+1,\mathbb{C})}$ of complex Higgs bundles. As such, it has a dual (B, B, B)-brane in the dual moduli space $\mathcal{M}_{\mathrm{Sp}(2p,\mathbb{C})}$ (see [17]).

It was conjectured by David Baraglia and the author, in [3, Section 7], that the support of the dual (B, B, B)-brane should be the whole moduli space of $\mathrm{Sp}(2p, \mathbb{C})$-Higgs bundles, which can now be understood through the spectral data description of the components of $\mathcal{M}_{\mathrm{SO}(p+1,p)}$ given in this paper. Indeed, the line bundles on the spectral curve S giving $\mathrm{SO}(p+1, p)$-Higgs bundles are given by $L \in \mathrm{Prym}(S, \overline{S})$, $L^2 \cong \mathcal{O}$, which is equivalent to requiring $\sigma^* L \cong L$. When one considers the complementary spectral data we recover the data for $\mathrm{Sp}(2p, \mathbb{C})$-Higgs bundles, given by $\sigma^* L \cong L^*$.

Since $\mathrm{SO}(p+1, p)$-Higgs bundles and $\mathrm{U}(p, p)$-Higgs bundles provide examples of (B, A, A)-branes whose dual (B, B, B)-brane should have the same support, the above can be compared to the hyperholomorphic (B, B, B)-brane constructed by Hitchin in [16, Section 7] dual to the $\mathrm{U}(p, p)$-Higgs bundles studied in [19], and one should be able to adapt the hyperholomorphic bundle in [16, Section 7] to the case of split orthogonal bundles. The study of this brane and the one appearing from a more generic setting for orthogonal Higgs bundles of any signature appears in [5].

Remark 5.2 The analogies between $\mathrm{SO}(p+1, p)$-Higgs bundles and $\mathrm{U}(p, p)$-Higgs bundles should not be surprising. Indeed, an $\mathrm{SO}(p+1, p)$-Higgs bundle $(V_- \oplus V_+, \Phi)$ defines the space $V_0 = \ker(\gamma) \subset V_-$. Then, the induced Higgs pair whose bundle is $V_-/V_0 \oplus V_+$ (with Higgs field defined through Φ') gives a canonical $\mathrm{U}(p, p)$-Higgs bundle with spectral data as in [19], and through [15] one can understand the extension data that relates these Higgs bundles.

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