

Higgs bundles and applications

LAURA P. SCHAPOSNIK

(joint work with David Baraglia, Nigel Hitchin)

Higgs bundles were first studied by Nigel Hitchin in 1987, and appeared as solutions of Yang-Mills self-duality equations on a Riemann surface [5]. Classically, a *Higgs bundle* on a compact Riemann surface Σ of genus $g \geq 2$ is a pair (E, Φ) where E is a holomorphic vector bundle on Σ , and Φ , the *Higgs field*, is a holomorphic 1-form in $H^0(\Sigma, \text{End}_0(E) \otimes K_\Sigma)$, for K_Σ the cotangent bundle of Σ and $\text{End}_0(E)$ the traceless endomorphisms of E . Higgs bundles can also be defined for complex groups G_c , and through stability conditions, one can construct their moduli spaces $\mathcal{M}_{G_c}$.

A natural way of studying the moduli space of Higgs bundles is through the *Hitchin fibration*, sending the class of a Higgs bundle (E, Φ) to the coefficients of the characteristic polynomial $\det(xI - \Phi)$. The generic fibre is an abelian variety, which can be seen through line bundles on an algebraic curve S , the *spectral curve* associated to the Higgs field. The *spectral data* is then given by a line bundle on S satisfying certain conditions, and it provides a geometric description of the fibres of the Hitchin fibration. For instance in the case of classical Higgs bundles, the smooth fibres can be seen through spectral data as Jacobian varieties of S .

Higgs bundles and branes. The smooth locus of the moduli space $\mathcal{M}_{G_c}$ of G_c -Higgs bundles on a compact Riemann surface Σ for a complex reductive Lie group G_c is a hyper-Kähler manifold, so there are natural complex structures I, J, K obeying the same relations as the imaginary quaternions (adopting the notation of [8]). Adopting physicists' language, a Lagrangian submanifold of a symplectic manifold is called an *A-brane* and a complex submanifold a *B-brane*. A submanifold of a hyper-Kähler manifold may be of type *A* or *B* with respect to each of the complex or symplectic structures, and thus choosing a triple of structures one may speak of branes of type (B, B, B) , (B, A, A) , (A, B, A) and (A, A, B) .

It is hence natural to construct different families of branes inside the moduli space $\mathcal{M}_{G_c}$, as was first done in [8] (see also [14]). Together with D. Baraglia we introduced a naturally defined triple of commuting real structures i_1, i_2, i_3 on $\mathcal{M}_{G_c}$, and through the *spectral data* gave a detailed picture of their fixed point sets as different types of branes. Given (E, Φ) a G_c -Higgs bundle on Σ , consider pairs $(\bar{\partial}_A, \Phi)$, where $\bar{\partial}_A$ denotes a $\bar{\partial}$ -connection on E defining a holomorphic structure, and Φ is a section of $\Omega^{1,0}(\Sigma, \text{ad}(E))$, for $\text{ad}(E)$ the adjoint bundle of E . Through the Cartan involution θ of a real form G of G_c one obtains

$$(1) \quad i_1(\bar{\partial}_A, \Phi) = (\theta(\bar{\partial}_A), -\theta(\Phi)).$$

Moreover, a real structure $f : \Sigma \rightarrow \Sigma$ on Σ induces an involution i_2 given by

$$(2) \quad i_2(\bar{\partial}_A, \Phi) = (f^*(\bar{\partial}_A), f^*(\Phi^*)) = (f^*(\rho(\bar{\partial}_A)), -f^*(\rho(\Phi))).$$

Lastly, by looking at $i_3 = i_1 \circ i_2$, one obtains $i_3(\bar{\partial}_A, \Phi) = (f^*\sigma(\bar{\partial}_A), f^*\sigma(\Phi))$. The fixed point sets of i_1, i_2, i_3 are branes of type (B, A, A) , (A, B, A) and (A, A, B) respectively. In [3], with D. Baraglia we studied these branes through the associated *spectral data* and described the topological invariants involved using KO , KR and equivariant K -theory. In particular, it was shown that amongst the fixed points of i_1 are solutions to the Hitchin equations with holonomy in G .

Higgs bundles and (A, B, A) -branes. Amongst the fixed points of the involution i_2 are representations of $\pi_1(\Sigma)$ that extend to certain 3-manifolds M whose boundary is Σ . Indeed, consider the space $\bar{\Sigma} = \Sigma \times [-1, 1]$ with involution $\tau(x, t) = (f(x), -t)$, for f the anti-holomorphic involution on Σ giving i_2 . The quotient $M = \bar{\Sigma}/\tau$ is a 3-manifold with boundary $\partial M = \Sigma$, and satisfies the following:

Theorem 1 (Baraglia, –). *Let (E, Φ) be a fixed point of i_2 with simple holonomy. Then the associated connection extends over M as a flat connection if and only if the class $[E] \in \tilde{K}_{\mathbb{Z}_2}^0(\Sigma)$ in reduced equivariant K -theory is trivial.*

Since Langlands duality can be seen in terms of Higgs bundles as a duality between the fibres of the Hitchin fibrations for $\mathcal{M}_{G_c}$ and $\mathcal{M}_{L_{G_c}}$, for ${}^L G_c$ the Langlands dual group of G_c (as was first seen in [4]), it is natural to ask what the duality between branes should be. In [3] we proposed the following:

Conjecture 1 (Baraglia, –). For ${}^L \rho$ the compact structure of ${}^L G_c$, the support of the dual brane of the fixed point set of i_2 is the fixed point set in $\mathcal{M}_{L_{G_c}}$ of

$$(3) \quad {}^L i_2(\bar{\partial}_A, \Phi) = (f^*({}^L \rho(\bar{\partial}_A)), -f^*({}^L \rho(\Phi))).$$

Higgs bundles and (B, A, A) -branes. Higgs bundles can be defined for complex Lie groups G_c , as well as for real forms G of G_c . Moreover, as mentioned in (A), the moduli spaces of real Higgs bundles $\mathcal{M}_G$ lie as (B, A, A) branes inside the moduli spaces $\mathcal{M}_{G_c}$. It is thus natural to ask how $\mathcal{M}_G$ intersects the Hitchin fibration for the complex moduli space $\mathcal{M}_{G_c}$, which is the main subject of [11, 12] (see [10]). Moreover, considering Langlands duality, in [3] we propose the following:

Conjecture 2 (Baraglia, –). The support of the dual brane to the fixed point set of i_1 is the moduli space $\mathcal{M}_{\check{H}} \subset \mathcal{M}_{L_{G_c}}$ of $\check{H}$ -Higgs bundles for $\check{H}$ the group associated to the Lie algebra $\check{\mathfrak{h}}$ in [9, Table 1].

On (B, A, A) -branes having finite intersection with smooth fibres. In the case of Higgs bundles for a split real form G of a complex reductive Lie group G_c , from [12] one has the following description of the intersection:

Theorem 2 (–). *The moduli space $\mathcal{M}_G$ as sitting inside $\mathcal{M}_{G_c}$ is given by points of order two in the smooth fibres of the Hitchin fibration $h : \mathcal{M}_{G_c} \rightarrow \mathcal{A}_{G_c}$.*

This result is used in [13] to study the moduli space of $SL(2, \mathbb{R})$ -Higgs bundles from a combinatorial point of view. The above theorem could also be used when studying L -twisted Higgs bundles (E, Φ) in [1], where the Higgs field is now twisted by any line bundle L obtaining $\Phi : E \rightarrow E \otimes L$.

Theorem 3 (Baraglia, –). *The monodromy action of an element $\tilde{s}_\gamma$ in the Hitchin base is the automorphism of $H^1(S, \mathbb{Z})$ induced by a Dehn twist of the spectral curve S around a loop l_γ . Let $c_\gamma \in H^1(S, \mathbb{Z})$ be the Poincaré dual of the homology class of l_γ . Then the monodromy of $\tilde{s}_\gamma$ acts on $H^1(S, \mathbb{Z})$ as a Picard-Lefschetz transformation: $x \mapsto x + \langle c_\gamma, x \rangle c_\gamma$.*

As an application, one can obtain information about some real character varieties $\text{Rep}(G)$. For instance one may prove that there are $3 \cdot 2^{2g} + g - 3$ connected components for $\text{Rep}(GL(2, \mathbb{R}))$, and recover the number of components of maximal representations $\text{Rep}_{2g-2}(Sp(4, \mathbb{R}))$, given by $3 \cdot 2^{2g} + 2g - 4$.

On (B, A, A) -branes having no intersection with smooth fibres. For other real forms the brane $\mathcal{M}_G$ may lie interlay inside the singular fibres of the Hitchin fibration for G_c -Higgs bundles. By extending the approach from [12], together with N. Hitchin we showed in [6] that this situation appears naturally when considering Higgs bundles corresponding to flat connections on Σ with holonomy in the real Lie groups $G = SL(m, \mathbb{H})$ and $SO(2n, \mathbb{H})$ (i.e., $SU^*(2m)$ and $SO^*(2n)$ respectively).

Theorem 4 (Hitchin, –). *The fibres of the (B, A, A) -brane in the Hitchin fibration for $SL(m, \mathbb{H})$, $SO(2n, \mathbb{H})$ and $Sp(2m, 2m)$ -Higgs bundles are not abelian varieties, but are instead moduli spaces of rank 2 bundles on a spectral curve, satisfying certain natural stability conditions.*

REFERENCES

- [1] D. Baraglia, L.P. Schaposnik, *Monodromy of rank 2 twisted Hitchin systems and real character varieties*, arXiv: 1506.00372 (2015)
- [2] D. Baraglia, L.P. Schaposnik, *Higgs bundles and (A, B, A) -branes*, Commun. Math. Phys. 331 3, (2014) 1271-1300.
- [3] D. Baraglia, L.P. Schaposnik, *Real structures on moduli spaces of Higgs bundles*, ArXiv:1309.1195 to appear in Advances in Theoretical and Mathematical Physics (2015).
- [4] T. Hausel, M. Thaddeus, *Mirror Symmetry, Langlands Duality, and the Hitchin System*, Invent. math., 153 1, (2003) 197–229.
- [5] N.J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. LMS, 55 3 (1987) 59–126.
- [6] N.J. Hitchin, L.P. Schaposnik, *Nonabelianization of Higgs bundles*, J. Differential Geom., 97 1 (2014), 79–89.
- [7] N.J. Hitchin, *Higgs bundles and characteristic classes*, arXiv:1308.4603 (2013).
- [8] A. Kapustin, E. Witten, *Electric-magnetic duality and the geometric Langlands program*, Commun. Number Theory Phys. 1 (2007) 1–236.
- [9] D. Nadler, *Perverse sheaves on real loop Grassmannians*, Invent. math., 159 1, (2005) 1–73.
- [10] L.P. Schaposnik, *An introduction to spectral data for Higgs bundles*, arXiv:1408.0333 (2014)
- [11] L.P. Schaposnik, *Spectral data for $U(m, m)$ -Higgs bundles*, Int Math Res Notices, 2015 11, (2015) 3486-3498. DOI: 10.1093/imrn/rnu029
- [12] L.P. Schaposnik, *Spectral data for G -Higgs bundles*, University of Oxford, (2013).
- [13] L.P. Schaposnik, *Monodromy of the SL_2 Hitchin fibration*, Internat. J. Math. 24 2 (2013), 1350013.
- [14] E. Witten, *More On Gauge Theory And Geometric Langlands*, ArXiv:1506.04293 (2015)