Automorphisms of $\mathbb{C}^*$ Moduli Spaces Associated to a Riemann Surface

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Received August 27, 2015, in final form January 15, 2016; Published online January 20, 2016
http://dx.doi.org/10.3842/SIGMA.2016.007

Abstract. We compute the automorphism groups of the Dolbeault, de Rham and Betti moduli spaces for the multiplicative group $\mathbb{C}^*$ associated to a compact connected Riemann surface.

Key words: holomorphic connection; Higgs bundle; character variety; automorphism

2010 Mathematics Subject Classification: 14H60; 14J50; 53D30

1 Introduction

Let X be a compact connected Riemann surface of genus g, where $g \geq 1$. Let $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ be the multiplicative group. We are interested in studying the automorphism groups of certain $\mathbb{C}^*$-moduli spaces associated to X, arising from non-abelian Hodge theory. Namely these are the de Rham, Betti and Dolbeault moduli spaces $\mathcal{M}_C$, $\mathcal{M}_R$, $\mathcal{M}_H$ parametrizing holomorphic $\mathbb{C}^*$-connections, representations of the fundamental group into $\mathbb{C}^*$ and degree zero Higgs line bundles respectively. While these three moduli spaces are all homeomorphic, their algebraic structures are quite different ($\mathcal{M}_C$ and $\mathcal{M}_H$ are not even biholomorphic) and we find that their automorphism groups are also quite different.

In [2], a classification was obtained of the analytic automorphism groups of the moduli space of $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles, i.e., the $\mathrm{SL}(n,\mathbb{C})$ Dolbeault moduli space. It remains an open question to determine which of the analytic automorphisms found in [2] are algebraic and also to determine the corresponding automorphism groups for the $\mathrm{SL}(n,\mathbb{C})$ de Rham and Betti moduli spaces (note that de Rham and Betti moduli spaces are analytically but not algebraically isomorphic). As mentioned above, the goal of this paper is to address this classification problem for the corresponding $\mathbb{C}^*$-moduli spaces. We leave the task of extending our results to noncommutative reductive groups as an interesting and challenging open problem.

Motivation for studying the automorphisms of these moduli spaces arises from mirror symmetry, the geometric Langlands program and their relation to physics, as promoted in the celebrated work of Kapustin and Witten [9]. Namely, one is interested in the construction of examples of naturally defined subvarieties of these moduli spaces, known as *branes* in the language of physics. One way of constructing such subvarieties which has proved fruitful is as the fixed point set of an automorphism of the moduli space, as seen in [3, 4]. This has lead us to consider the problem of determining the automorphism groups of these moduli spaces in order to see how general our constructions are.

In what follows we shall describe the structure and results of this paper. We begin this paper by studying in Section 2 the structure of the de Rham moduli space $\mathcal{M}_C$ of holomorphic $\mathbb{C}^*$-connections on X up to gauge equivalence, i.e., pairs (L, D) where L is a holomorphic line bundle and D is a holomorphic connection on L. After recalling properties of the space, we give in Proposition 2.2 a gauge theoretic proof of the known result that every algebraic function on $\mathcal{M}_C$ is constant.

The moduli space $\mathcal{M}_C$ is a complex algebraic group with multiplication given by taking the tensor product of line bundles with connections, and thus $\mathcal{M}_C$ acts on itself by translations giving an injective homomorphism

$$\rho\colon\ \mathcal{M}_C \longrightarrow \mathrm{Aut}(\mathcal{M}_C),$$

where $\mathrm{Aut}(\mathcal{M}_C)$ denote the group of algebraic automorphisms of $\mathcal{M}_C$. This map is considered in Section 3, where we show the following (see Theorem 3.1):

Theorem 1.1. *The quotient* $\mathrm{Aut}(\mathcal{M}_C)/(\rho(\mathcal{M}_C))$ *is a countable group. In particular, the image of* ρ *is the connected component of* $\mathrm{Aut}(\mathcal{M}_C)$ *containing the identity element.*

Let $J(X)$ be the Jacobian of X and let $\rho_0\colon J(X) \longrightarrow \mathrm{Aut}(J(X))$ be the homomorphism given by letting $J(X)$ act on itself by translation. In Section 3, it is found that the quotient $\mathrm{Aut}(\mathcal{M}_C)/(\rho(\mathcal{M}_C))$ can be identified with a subgroup of $\mathrm{Aut}(J(X))/\rho_0(J(X))$.

From non-abelian Hodge theory it is seen that the moduli space $\mathcal{M}_C$ carries a naturally defined algebraic symplectic form [1, 7]. Let $\theta \in H^2(\mathcal{M}_C, \mathbb{C})$ denote the cohomology class of the symplectic form and let $\mathrm{Aut}_\theta(\mathcal{M}_C)$ be the subgroup of $\mathrm{Aut}(\mathcal{M}_C)$ preserving θ. In Section 3 we study this subgroup, and give its complete characterization in Theorem 3.2. For this, consider the homomorphism

$$\rho_C\colon\ \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(\mathcal{M}_C)$$

defined by sending $h \in \mathrm{Aut}(X)$ to the automorphism of $\mathcal{M}_C$ given by $(L, D) \longmapsto (h^*L, h^*D)$. We show in Section 3.1 that ρ_C is injective if $g \geq 2$. Let G denote the subgroup of $\mathrm{Aut}(\mathcal{M}_C)$ generated by $\rho_C(\mathrm{Aut}(X))$ together with the inversion $(L, D) \longmapsto (L^\vee, D^\vee)$ of the group $\mathcal{M}_C$; we denote the dual of a vector bundle, a vector space or a homomorphism by the superscript "$\vee$". Using the actions of G and $\rho_C(\mathrm{Aut}(X))$ on $\mathcal{M}_C$, consider the semi-direct products

$$\mathcal{G}_0 := \mathcal{M}_C \rtimes \rho_C(\mathrm{Aut}(X)) \qquad \text{and} \qquad \mathcal{G} := \mathcal{M}_C \rtimes G.$$

Through these groups we can characterize $\mathrm{Aut}_\theta(\mathcal{M}_C)$ (see Theorem 3.2):

Theorem 1.2. *The group* $\mathrm{Aut}_\theta(\mathcal{M}_C)$ *is given by*

1) $\mathrm{Aut}_\theta(\mathcal{M}_C) = \mathcal{G}$ *if* X *is not hyperelliptic;*
2) $\mathrm{Aut}_\theta(\mathcal{M}_C) = \mathcal{G}_0$ *if* X *is hyperelliptic.*

As a Corollary, we deduce that any automorphism of $\mathcal{M}_C$ preserving the cohomology class θ actually preserves the symplectic form and so the above theorem also gives the group of algebraic symplectomorphisms of $\mathcal{M}_C$.

In Section 4 we consider the Betti moduli space $\mathcal{M}_R$ of representations of $\pi_1(X)$ into the multiplicative group $\mathbb{C}^*$ (following [12]). The space $\mathcal{M}_R = \mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$, which is isomorphic to $(\mathbb{C}^*)^{2g}$. The group Γ of automorphisms of the $\mathbb{Z}$-module $H_1(X, \mathbb{Z})$ is isomorphic to $\mathrm{GL}(2g, \mathbb{Z})$, and thus there is a natural map

$$f\colon\ \mathrm{Aut}(\mathcal{M}_R) \longrightarrow \Gamma,$$

that sends an automorphism of $\mathcal{M}_R$ to its induced action on $H_1(X,\mathbb{Z})$. In Section 4, we show that f admits a right-splitting so that $\mathrm{Aut}(\mathcal{M}_R) = \mathrm{kernel}(f) \rtimes \Gamma$. Moreover, since the kernel of f is given by the natural action of $\mathcal{M}_R = (\mathbb{C}^*)^{2g}$ on itself by translations, we obtain that (see Theorem 4.2):

Theorem 1.3. *The automorphism group* $\mathrm{Aut}(\mathcal{M}_R)$ *is the semi-direct product* $\mathcal{M}_R \rtimes \Gamma$.

As with the de Rham moduli space, non-abelian Hodge theory determines a natural symplectic form on $\mathcal{M}_R$. We find that the subgroup of $\mathrm{Aut}(\mathcal{M}_R)$ preserving this form is given by $\mathcal{M}_R \rtimes \Gamma_{\mathrm{Sp}}$, where Γ_{Sp} is the subgroup of Γ preserving the cap product on $H_1(X,\mathbb{Z})$, so Γ_{Sp} is isomorphic to the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$.

Finally, in Section 5 we study the Dolbeault moduli space $\mathcal{M}_H$ of degree zero Higgs line bundles, that is pairs (L,Φ), where L is a degree zero line bundle on X and Φ is a holomorphic 1-form on X. This moduli space is the holomorphic cotangent bundle $T^\vee J(X)$ of the Jacobian $J(X)$. Considering the isomorphism $T^\vee J(X) = J(X) \times H^0(X,K_X)$, where K_X is the holomorphic cotangent bundle of X we obtain that (see Lemma 5.1):

Lemma 1.4. *Any* $f \in \mathrm{Aut}(\mathcal{M}_H)$ *is of the form*

$$f = f_1 \times f_2,$$

where $f_1 \in \mathrm{Aut}(J(X))$ *and* $f_2 \in \mathrm{Aut}(H^0(X,K_X))$.

Since the moduli space $\mathcal{M}_H$ is the cotangent bundle of $J(X)$, it carries a canonical symplectic form θ. We shall denote by $\mathrm{Aut}_\theta(\mathcal{M}_H)$ the subgroup of $\mathrm{Aut}(\mathcal{M}_H)$ preserving θ, and let $\Omega_{J(X)}$ denote the holomorphic cotangent bundle of $J(X)$. Recalling that there is an isomorphism $H^0(X,K_X) = H^0(J(X),\Omega_{J(X)})$, we conclude the paper showing that (see Theorem 5.2):

Theorem 1.5. *The group* $\mathrm{Aut}_\theta(\mathcal{M}_H)$ *is the semi-direct product*

$$H^0(J(X),\Omega_{J(X)}) \rtimes \mathrm{Aut}(J(X)),$$

where $\mathrm{Aut}(J(X))$ *acts on* $H^0(J(X),\Omega_{J(X)})$ *by* $f\cdot\alpha = (f^{-1})^*(\alpha)$, *for* $f \in \mathrm{Aut}(J(X))$, $\alpha \in H^0(J(X),\Omega_{J(X)})$.

2 Structure of the moduli space of $\mathbb{C}^*$-connections

Let X be a compact connected Riemann surface of genus $g \geq 1$, and K_X its holomorphic cotangent bundle. The Jacobian of X, which parametrizes all the isomorphism classes of holomorphic line bundles on X of degree zero, is denoted by $J(X)$. Let $\mathcal{M}_C$ be the moduli space of holomorphic connections on X of rank one. Therefore, $\mathcal{M}_C$ parametrizes the isomorphism classes of pairs of the form (L,D), where L is a holomorphic line bundle on X and D is a holomorphic connection on L. Since there are no nonzero $(2,0)$-forms on X, any holomorphic connection on X is automatically integrable.

The adjoint action of the algebraic group $\mathbb{C}^*$ on its Lie algebra $\mathrm{Lie}(\mathbb{C}^*) = \mathbb{C}$ is trivial. Consequently, for any $(L,D) \in \mathcal{M}_C$, the holomorphic tangent bundle to $\mathcal{M}_C$ at the point (L,D) is

$$T_{(L,D)}\mathcal{M}_C = H^1(X,\mathbb{C}). \tag{2.1}$$

Therefore, the real tangent bundle $T^{\mathbb{R}}_{(L,D)}\mathcal{M}_C$ is identified with $H^1(X,\mathbb{C})$, and the almost complex structure on $T^{\mathbb{R}}_{(L,D)}\mathcal{M}_C = H^1(X,\mathbb{C})$ is multiplication by $\sqrt{-1}$.

Since any holomorphic connection on X is flat, the degree of any holomorphic line bundle admitting a holomorphic connection is zero. Therefore, we have an algebraic morphism

$$\varphi\colon\ \mathcal{M}_C \longrightarrow J(X), \qquad (L, D) \longmapsto L. \tag{2.2}$$

This map φ is surjective because any holomorphic line bundle L on X of degree zero admits a holomorphic connection. More precisely, the space of all holomorphic connections on L is an affine space for the vector space $H^0(X, K_X)$. Therefore, φ makes $\mathcal{M}_C$ an algebraic principal $H^0(X, K_X)$-bundle over $J(X)$.

Let $\mathbb{V}$ denote the trivial holomorphic vector bundle $J(X) \times H^0(X, K_X)$ over $J(X)$ with fiber $H^0(X, K_X)$. The isomorphism classes of algebraic principal $H^0(X, K_X)$-bundles over $J(X)$ are parametrized by $H^1(J(X), \mathbb{V})$. We will calculate the cohomology class corresponding to $\mathcal{M}_C$. Note that φ does not admit any holomorphic section because $J(X)$ is compact and $\mathcal{M}_C$ is biholomorphic to $(\mathbb{C}^*)^{2g}$ thus ruling out the existence of any nonconstant holomorphic map from $J(X)$ to $\mathcal{M}_C$. Consequently, the class in $H^1(J(X), \mathbb{V})$ corresponding to $\mathcal{M}_C$ is nonzero.

We will briefly describe the Dolbeault type construction of cohomological invariants for principal $H^0(X, K_X)$-bundles.

Take an algebraic principal $H^0(X, K_X)$-bundle $q\colon E \longrightarrow J(X)$. Choose a C^∞ section

$$s\colon\ J(X) \longrightarrow E$$

for q; such a section exists because the fibers of the projection q are contractible. If s is holomorphic, then the holomorphic principal $H^0(X, K_X)$-bundle E is trivial. The invariant for E is a measure of the failure of s to be holomorphic. To explain this, let J_1 and J_2 denote the almost complex structures on $J(X)$ and E respectively. Let $ds\colon T^{\mathbb{R}}J(X) \longrightarrow T^{\mathbb{R}}E$ be the differential of the map s. For any $x \in J(X)$ and $y \in E_x$, consider the homomorphism

$$T^{\mathbb{R}}_x J(X) \longrightarrow T^{\mathbb{R}}_y E, \qquad v \longmapsto ds(J_1(v)) - J_2(ds(v)). \tag{2.3}$$

Since

- $q \circ s = \mathrm{Id}_{J(X)}$, and
- the map q is holomorphic,

it follows that the tangent vector $ds(J_1(v)) - J_2(ds(v))$ in (2.3) is vertical for q. Using the action of the group $H^0(X, K_X)$ on E, the vertical tangent bundle for q is the trivial vector bundle with fiber $H^0(X, K_X)$. Consequently, the homomorphism in (2.3) defines a section

$$c(E, s) \in C^\infty\big(J(X), \Omega^{0,1}_{J(X)} \otimes \mathbb{V}\big).$$

This $(0,1)$-form $c(E, s)$ is $\overline{\partial}$-closed because E is a holomorphic principal $H^0(X, K_X)$-bundle. Then, the Dolbeault cohomological class

$$c(E) \in H^1(J(X), \mathbb{V}) \tag{2.4}$$

defined by it is the invariant for E.

The Lie algebra $\mathrm{Lie}(J(X))$ of $J(X)$ is the abelian algebra $H^1(X, \mathcal{O}_X)$. The Serre duality theorem says that $H^1(X, \mathcal{O}_X) = H^0(X, K_X)^\vee$. Therefore, the vector bundle $\mathbb{V}$ is identified with the holomorphic cotangent bundle $\Omega_{J(X)}$. Consequently, we have

$$H^1(J(X), \mathbb{V}) = H^1(J(X), \Omega_{J(X)}).$$

Hence the isomorphism classes of holomorphic principal $H^0(X, K_X)$-bundles on $J(X)$ are parametrized by $H^1(J(X), \Omega_{J(X)})$. We note that every element of $H^1(J(X), \Omega_{J(X)})$ is the invariant (2.4) for some holomorphic principal $H^0(X, K_X)$-bundles on $J(X)$.

Let $\mathcal{M}_R := \mathrm{Hom}(\pi_1(X, x_0), \mathbb{C}^*) = \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}^*)$ be the space of 1-dimensional representations. Sending a flat connection to its monodromy representation, we get a holomorphic isomorphism

$$f\colon\ \mathcal{M}_C \xrightarrow{\sim} \mathcal{M}_R.$$

We have $\mathrm{Hom}(H_1(X, \mathbb{Z}), \mathrm{U}(1)) \hookrightarrow \mathcal{M}_R$ using the inclusion of $\mathrm{U}(1) = S^1$ in $\mathbb{C}^*$. From Hodge theory it follows that every $L \in J(X)$ admits a unique holomorphic connection such that the monodromy lies in $\mathrm{U}(1)$, and thus the composition

$$\mathrm{Hom}(H_1(X, \mathbb{Z}), \mathrm{U}(1)) \xrightarrow{f^{-1}} \mathcal{M}_C \xrightarrow{\varphi} J(X) \tag{2.5}$$

is a diffeomorphism, where φ is constructed in (2.2). We note that the above composition $\varphi \circ f^{-1}$ is a diffeomorphism because it is bijective and homomorphism of groups. We shall denote by

$$\xi\colon\ J(X) \longrightarrow \mathcal{M}_C \tag{2.6}$$

the C^∞ section of φ given by the inverse of the composition in (2.5).

Given any $L \in J(X)$, we consider $\nabla = \nabla^{1,0} + \nabla^{0,1}$ the unique unitary flat connection on L such that $(0,1)$-type component $\nabla^{0,1}$ is the Dolbeault operator on L. The real tangent space $T^{\mathbb{R}}_{\xi(L)}\mathcal{M}_C$ is $H^1(X, \mathbb{C})$, and the almost complex structure on $T^{\mathbb{R}}_{\xi(L)}\mathcal{M}_C$ coincides with the multiplication by $\sqrt{-1}$ on $H^1(X, \mathbb{C})$ (see (2.1) and the sentence following it). Therefore, the holomorphic tangent space to $\mathcal{M}_C$ is identified with $H^1(X, \mathbb{C})$. The inclusion of the Lie group $\mathrm{U}(1) \hookrightarrow \mathbb{C}^*$, identifies the Lie algebra $\mathrm{Lie}(\mathrm{U}(1))$ with the subspace

$$\sqrt{-1}\mathbb{R} \subset \mathrm{Lie}(\mathbb{C}^*) = \mathbb{C}.$$

Therefore, the subspace

$$T^{\mathbb{R}}_{\xi(L)} \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathrm{U}(1)) \subset T^{\mathbb{R}}_{\xi(L)}\mathcal{M}_C = H^1(X, \mathbb{C})$$

coincides with $H^1(X, \sqrt{-1}\mathbb{R})$ equipped with its natural inclusion

$$H^1(X, \sqrt{-1}\mathbb{R}) \hookrightarrow H^1(X, \mathbb{C}).$$

The anti-holomorphic tangent space $T^{0,1}_L J(X)$ is identified with $H^0(X, K_X)$ by sending any $\alpha \in H^0(X, K_X)$ to the flat unitary connection

$$\big(\nabla^{1,0} - \alpha\big) + \big(\nabla^{0,1} + \overline{\alpha}\big).$$

From the above, the complex structure on $T^{0,1}_L J(X)$ coincides with multiplication by $\sqrt{-1}$ on $H^0(X, K_X)$. If we identify $T^{0,1}_L J(X)$ with $T^{\mathbb{R}}_L J(X)$ by sending any $(0,1)$-tangent vector to its real part, then the isomorphism

$$T^{\mathbb{R}}_L J(X) \longrightarrow T^{\mathbb{R}}_{\xi(L)} \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathrm{U}(1))$$

given by the differential of the composition map in (2.5) sends any $\alpha \in H^0(X, K_X)$ to the element in

$$-2\sqrt{-1} \cdot \mathrm{Im}(\alpha) \in H^1\big(X, \sqrt{-1}\mathbb{R}\big).$$

The cup product

$$\bigwedge^2 H^1\big(X, \sqrt{-1}\mathbb{R}\big) \longrightarrow H^2(X, \mathbb{R}) = \mathbb{R}$$

produces a 2-form ω on $J(X)$. The form ω is closed because the translation action of $J(X)$ on itself preserves ω, and any translation invariant form on a torus is closed. In fact, ω is a Kähler form on $J(X)$. We shall let

$$\widehat{\omega} \in H^1(J(X), \Omega_{J(X)}) \tag{2.7}$$

be the Dolbeault cohomology class represented by ω.

From the above, the anti-holomorphic tangent space $T^{0,1}_L J(X)$ is identified with

$$T^{\mathbb{R}}_{\xi(L)} \operatorname{Hom}(H_1(X, \mathbb{Z}), \mathrm{U}(1)),$$

a subspace of $H^1(X, \mathbb{C})$ which in turn gives the holomorphic tangent space to $\mathcal{M}_C$. Hence, consider the almost complex structures obtained for $J(X)$ and $\mathcal{M}_C$, combined with the above description of the differential of ξ, one has that the class in $H^1(J(X), \mathbb{V})$ corresponding to the principal $H^0(X, K_X)$-bundle $\mathcal{M}_C$ coincides with $\widehat{\omega}$ in (2.7).

Let

$$0 \longrightarrow \Omega_{J(X)} \xrightarrow{\iota} E \xrightarrow{\sigma} \mathcal{O}_{J(X)} \longrightarrow 0 \tag{2.8}$$

be the extension of $\mathcal{O}_{J(X)}$ by $\Omega_{J(X)}$ associated to the extension class $\widehat{\omega}$ in (2.7). The section of $\mathcal{O}_{J(X)}$ given by the constant function 1 will be denoted by $1_{J(X)}$. We note that for the projection σ in (2.8), the inverse image $\sigma^{-1}(1_{J(X)}(J(X))) \subset E$ is a principal $H^0(X, K_X)$-bundle on $J(X)$ (recall that the dual vector space $\mathrm{Lie}(J(X))^\vee$ is identified with $H^0(X, K_X)$). Since the class in $H^1(J(X), \Omega_{J(X)})$ corresponding to the principal $H^0(X, K_X)$-bundle $\mathcal{M}_C$ coincides with $\widehat{\omega}$, we have the following:

Lemma 2.1. *The variety $\mathcal{M}_C$ is algebraically isomorphic to the inverse image*

$$\sigma^{-1}(1_{J(X)}(J(X))).$$

Through the above lemma, we can recover the following result, which from a different perspective can be deduced since the universal vector extension of the Jacobian parametrizes line bundles with connections [10, Chapter 1], and the universal vector extension of any abelian variety is anti-affine [5, Proposition 2.3(i)].

Proposition 2.2. *There are no nonconstant algebraic functions on $\mathcal{M}_C$.*

Proof. In view of Lemma 2.1 it suffices to show that the variety $\sigma^{-1}(1_{J(X)}(J(X)))$ does not admit any nonconstant algebraic function. We will first express $\sigma^{-1}(1_{J(X)}(J(X)))$ as a hyperplane complement $\mathcal{Y}$ in a projective bundle over $J(X)$ in Step 1. Then in Step 2 we shall study associated bundles, which in turn allow us to study $H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ in Step 3. From the description of the cohomology group that we obtain, we see that $\mathcal{Y}$ does not admit any nonconstant algebraic function if and only if certain natural inclusion is surjective. Hence, in Step 4 we study this inclusion, by taking the dual exact sequence to (2.8). Surjectivity of the inclusion can be then seen equivalent to injectivity of an associated map β. We conclude the proof of the proposition by showing in Step 5 that this map is indeed injective.

Step 1. Let $P(E) \longrightarrow J(X)$ and $P(\Omega_{J(X)}) \longrightarrow J(X)$ be the projective bundles parametrizing the lines in the fibers of E (constructed in (2.8)) and $\Omega_{J(X)}$ respectively. The homomorphism ι in (2.8) produces an embedding

$$\widehat{\iota}\colon\ P(\Omega_{J(X)}) \longrightarrow P(E).$$

The divisor $\widehat{\imath}(P(\Omega_{J(X)})) \subset P(E)$ will be denoted by $\mathcal{D}$. We have

$$\mathcal{Y} := P(E)\backslash\mathcal{D} \cong \sigma^{-1}(1_{J(X)}(J(X))) \tag{2.9}$$

by sending any $v \in \sigma^{-1}(1_{J(X)}(z))$ and $z \in J(X)$, to the line in the fiber E_z generated by v.

Step 2. Consider now the natural projection

$$p\colon\ P(E) \longrightarrow J(X).$$

For $\mathcal{L} \longrightarrow P(E)$ the dual of the tautological line bundle, the fiber of $\mathcal{L}$ over any $y \in P(E)$ is the dual of the line in $E_{p(y)}$ represented by y.

Note that for any point $z \in J(X)$, the two line bundles $\mathcal{L}|_{p^{-1}(z)}$ and $\mathcal{O}_{P(E)}(\mathcal{D})|_{p^{-1}(z)}$ on $p^{-1}(z)$ are isomorphic. Therefore, from the seesaw theorem (see [11, p. 51, Corollary 6]) it follows that there is a holomorphic line bundle L_0 on $J(X)$ such that

$$\mathcal{O}_{P(E)}(\mathcal{D}) = \mathcal{L} \otimes p^*L_0. \tag{2.10}$$

By the adjunction formula [8, p. 146], the restriction of $\mathcal{O}_{P(E)}(\mathcal{D})$ to $\mathcal{D}$ is the normal bundle $N_{\mathcal{D}}$ to the divisor $\mathcal{D} \subset P(E)$. This normal bundle $N_{\mathcal{D}}$ is identified with

$$\text{Hom}\left((\mathcal{L}|_{\mathcal{D}})^\vee, p_1^*(E/\Omega_{J(X)})\right) = p_1^*(E/\Omega_{J(X)}) \otimes (\mathcal{L}|_{\mathcal{D}}),$$

where

$$p_1 = p|_{\mathcal{D}}\colon\ \mathcal{D} \longrightarrow J(X)$$

is the restriction of p. Now, since the quotient $E/\Omega_{J(X)}$ is the trivial line bundle (see (2.8)), it follows that $N_{\mathcal{D}}$ is isomorphic to $\mathcal{L}|_{\mathcal{D}}$. Consequently from (2.10) it follows that the line bundle L_0 is trivial. This in turn implies that

$$\mathcal{O}_{P(E)}(\mathcal{D}) = \mathcal{L}. \tag{2.11}$$

Step 3. To calculate $H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, note that

$$H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = \varinjlim_{i\geq 0} H^0(P(E), \mathcal{O}_{P(E)}(i\mathcal{D})) = \varinjlim_{i\geq 0} H^0\left(P(E), \mathcal{L}^i\right) \tag{2.12}$$

(see (2.9) and (2.11)). Since $\mathcal{D}$ is an effective divisor, from (2.9) and (2.12) we conclude that $\sigma^{-1}(1_{J(X)}(J(X)))$ does not admit any nonconstant algebraic function if and only if the natural inclusion

$$\begin{aligned} H^0\left(P(E), \mathcal{L}^i\right) &= H^0(P(E), \mathcal{O}_{P(E)}(i\mathcal{D})) \\ &\hookrightarrow H^0(P(E), \mathcal{O}_{P(E)}((i+1)\mathcal{D})) = H^0\left(P(E), \mathcal{L}^{i+1}\right) \end{aligned} \tag{2.13}$$

is surjective for all $i \geq 0$. Note that

$$H^0(P(E), \mathcal{O}_{P(E)}(i\mathcal{D})) = H^0\left(J(X), \text{Sym}^i\left(E^\vee\right)\right).$$

Step 4. To prove that the homomorphism in (2.13) is indeed surjective, consider the dual of the exact sequence in (2.8):

$$0 \longrightarrow \mathcal{O}_{J(X)} \xrightarrow{\sigma^\vee} E^\vee \xrightarrow{\iota^\vee} TJ(X) \longrightarrow 0.$$

Taking its $(i+1)$-th symmetric power, we have

$$0 \longrightarrow \text{Sym}^i\left(E^\vee\right) \xrightarrow{\sigma'} \text{Sym}^{i+1}\left(E^\vee\right) \xrightarrow{\text{Sym}^{i+1}(\iota^\vee)} \text{Sym}^{i+1}(TJ(X)) \longrightarrow 0, \tag{2.14}$$

where $\mathrm{Sym}^{i+1}(\iota^\vee)$ is the homomorphism of symmetric products induced by the homomorphism $\iota^\vee$; the above homomorphism σ' is the symmetrization of the homomorphism

$$\otimes^i E^\vee = \mathcal{O}_{J(X)} \otimes \left(\otimes^i E^\vee\right) \stackrel{\sigma^\vee \otimes \mathrm{Id}}{\longrightarrow} \otimes^{i+1} E^\vee.$$

Let

$$\beta\colon\ H^0\big(J(X), \mathrm{Sym}^{i+1}(TJ(X))\big) \longrightarrow H^1\big(J(X), \mathrm{Sym}^i\left(E^\vee\right)\big)$$

be the connecting homomorphism in the long exact sequence of cohomologies associated to the short exact sequence in (2.14). Consider the homomorphisms

$$\begin{aligned} H^0\big(J(X), \mathrm{Sym}^{i+1}(TJ(X))\big) &\stackrel{\beta}{\longrightarrow} H^1\big(J(X), \mathrm{Sym}^i\left(E^\vee\right)\big) \\ &\stackrel{\gamma}{\longrightarrow} H^1\big(J(X), \mathrm{Sym}^i(TJ(X))\big), \end{aligned} \tag{2.15}$$

where γ is induced by the homomorphism $\mathrm{Sym}^i(\iota^\vee)$ (see (2.14)).

From the long exact sequence of cohomologies for (2.14) it follows immediately that the homomorphism in (2.13) is surjective if β in (2.15) is injective. To prove that β is injective, it is enough to show that the composition $\gamma \circ \beta$ in (2.15) is injective.

Step 5. Since the extension class for (2.8) is the cohomology class $\widehat{\omega}$, the extension class for (2.14) is $-(i+1)\widehat{\omega}$. Consequently, the homomorphism $\gamma \circ \beta$ sends any

$$\eta \in H^0\big(J(X), \mathrm{Sym}^{i+1}(TJ(X))\big)$$

to the Dolbeault cohomology class of the contraction

$$\omega \otimes' \eta \in C^\infty\big(J(X), \Omega^{0,1}_{J(X)} \otimes \mathrm{Sym}^{i+1}(TJ(X))\big)$$

of $\omega \otimes \eta$ of $\Omega^{1,0}_{J(X)}$ and $T(X)$; note that the tensor product $\omega \otimes \eta$ is a section of $\Omega^{0,1}_{J(X)} \otimes \Omega^{1,0}_{J(X)} \otimes \mathrm{Sym}^{i+1}(TJ(X))$ and hence its contraction $\omega \otimes' \eta$ is a section of $\Omega^{0,1}_{J(X)} \otimes \mathrm{Sym}^i(TJ(X)))$. Since both ω and η are invariant under translations of $J(X)$, it follows that $\omega \otimes' \eta$ is also invariant under translations of $J(X)$, and hence represents a nonzero cohomology class. The section $\omega \otimes' \eta$ is nonzero because ω is pointwise nondegenerate (recall that it is a Kähler form). Therefore, we conclude that the homomorphism $\gamma \circ \beta$ is injective. Hence the homomorphism in (2.13) is surjective, and the proof is complete. ■

3 Automorphisms of the moduli of $\mathbb{C}^*$-connections

The group of algebraic automorphisms of the variety $\mathcal{M}_C$ will be denoted by $\mathrm{Aut}(\mathcal{M}_C)$. The moduli space $\mathcal{M}_C$ is an algebraic group, with group operation

$$(L_1, D_1) \cdot (L_2, D_2) = \big(L_1 \otimes L_2, D_1 \otimes \mathrm{Id}_{L_2} + \mathrm{Id}_{L_1} \otimes D_2\big).$$

The algebraic map φ in (2.2) is a homomorphism of algebraic groups.

The translation action of $\mathcal{M}_C$ on itself produces an injective homomorphism

$$\rho\colon\ \mathcal{M}_C \longrightarrow \mathrm{Aut}(\mathcal{M}_C).$$

Theorem 3.1. *The quotient* $\mathrm{Aut}(\mathcal{M}_C)/(\rho(\mathcal{M}_C))$ *is a countable group. In particular, the image of* ρ *is the connected component of* $\mathrm{Aut}(\mathcal{M}_C)$ *containing the identity element.*

Proof. We will show that any automorphism of $\mathcal{M}_C$ descends to $J(X)$. For that, first note that there is no nonconstant algebraic map from $\mathbb{C}$ to an abelian variety. Indeed, such a map would extend to a nonconstant algebraic map from $\mathbb{CP}^1$, and therefore some holomorphic 1-form on the abelian variety would pull back to a nonzero holomorphic 1-form on $\mathbb{CP}^1$, but $\mathbb{CP}^1$ does not have any nonzero holomorphic 1-form. Since there is no nonconstant algebraic map from $\mathbb{C}$ to $J(X)$, there is no nonconstant algebraic map from a fiber of φ (see (2.2)) to the variety $J(X)$, because the fibers of φ are isomorphic to $\mathbb{C}^g$. This immediately implies that any automorphism of $\mathcal{M}_C$ descends to an automorphism of $J(X)$.

The group of all algebraic automorphisms of $J(X)$ will be denoted by $\mathrm{Aut}(J(X))$. The above observation produces a homomorphism

$$\delta\colon\ \mathrm{Aut}(\mathcal{M}_C) \longrightarrow \mathrm{Aut}(J(X)). \tag{3.1}$$

Recall that φ in (2.2) is a homomorphism of algebraic groups. Clearly, we have

$$\rho(\mathrm{kernel}(\varphi)) \subset \mathrm{kernel}(\delta).$$

We shall denote by

$$\rho_0\colon\ J(X) \longrightarrow \mathrm{Aut}(J(X))$$

the homomorphism given by the translation action of $J(X)$ on itself.

To prove the theorem, by the snake lemma it suffices to show the following two statements:

1) the quotient $\mathrm{Aut}(J(X))/(\rho_0(J(X)))$ is a countable group,
2) the inclusion $\rho(\mathrm{kernel}(\varphi)) \hookrightarrow \mathrm{kernel}(\delta)$ is surjective.

The first statement follows from the fact that $H^0(J(X), TJ(X)) = \mathrm{Lie}(J(X))$. In what follows we will prove the second statement.

Take any $\psi \in \mathrm{kernel}(\delta) \subset \mathrm{Aut}(\mathcal{M}_C)$, and for $\zeta \in H^0(X, K_X)^\vee$, define the function

$$F_{\psi,\zeta}\colon\ \mathcal{M}_C \longrightarrow \mathbb{C}, \qquad z \longmapsto \zeta(\psi(z) - z).$$

Note that $\varphi(\psi(z)) = \varphi(z)$ because $\psi \in \mathrm{kernel}(\delta)$, and hence $\psi(z) - z \in H^0(X, K_X)$. From Proposition 2.2 we know that $F_{\psi,\zeta}$ is a constant function. This implies that there is an element $v \in H^0(X, K_X)$ such that $\psi(z) = z + v$ for all $z \in \mathcal{M}_C$. So we have $\psi \in \rho(\mathrm{kernel}(\varphi))$, which completes the proof. ■

3.1 Automorphisms preserving cohomology class

As mentioned previously, the moduli space $\mathcal{M}_C$ is equipped with an algebraic symplectic form (see [1, 7]). The cohomology class in $H^2(\mathcal{M}_C, \mathbb{C})$ defined by the symplectic form will be denoted by θ. The pullback of the symplectic form on $\mathcal{M}_C$ by the section ξ in (2.6) coincides with the Kähler form on $J(X)$. Therefore, the cohomology class θ on $\mathcal{M}_C$ coincides with the pullback $\varphi^*\widehat{\omega}$ of the Kähler class on $J(X)$ (see (2.2) and (2.7)). Let $\mathrm{Aut}_\theta(\mathcal{M}_C)$ denote the group of all $\tau \in \mathrm{Aut}(\mathcal{M}_C)$ such that $\tau^*\theta = \theta$. Our aim in this subsection is to compute $\mathrm{Aut}_\theta(\mathcal{M}_C)$.

The group of all holomorphic automorphisms of X will be denoted by $\mathrm{Aut}(X)$. Let

$$\mathrm{Aut}^0(X) \subset \mathrm{Aut}(X)$$

be the connected component containing the identity element. If $g \geq 2$, then we have $\mathrm{Aut}^0(X)$=e. Let

$$\rho_C\colon\ \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(\mathcal{M}_C)$$

be the homomorphism that sends any $h \in \mathrm{Aut}(X)$ to the automorphism of $\mathcal{M}_C$ defined by $(L, D) \longmapsto (h^*L, h^*D)$. If $g \geq 2$, then ρ_C is injective. Indeed, the homomorphism $\mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(J(X))$ that sends any $h \in \mathrm{Aut}(X)$ to the automorphism $L \longmapsto h^*L$ is injective if $g \geq 2$. If $g = 1$ then X is an elliptic curve and $\mathrm{Aut}^0(X) = X$, acting on itself by translations. If $\tau\colon X \to X$ is any such translation then for any line bundle with holomorphic connection (L, D), we have $(\tau^*L, \tau^*D) \cong (L, D)$ since the corresponding flat connections have the same monodromy. Therefore the homomorphism $\rho_C|_{\mathrm{Aut}^0(X)}$ is trivial, and ρ_C produces an embedding of $\mathrm{Aut}(X)/\mathrm{Aut}^0(X)$ in $\mathrm{Aut}(J(X))$.

Let G denote the subgroup of $\mathrm{Aut}(\mathcal{M}_C)$ generated by $\rho_C(\mathrm{Aut}(X))$ together with the inversion $(L, D) \longmapsto (L^\vee, D^\vee)$ of the group $\mathcal{M}_C$. Using the actions of G and $\rho_C(\mathrm{Aut}(X))$ on $\mathcal{M}_C$, we construct the semi-direct products

$$\mathcal{G}_0 := \mathcal{M}_C \rtimes \rho_C(\mathrm{Aut}(X)) \qquad \text{and} \qquad \mathcal{G} := \mathcal{M}_C \rtimes G.$$

Note that using the action of $\rho_C(\mathrm{Aut}(X))$ (respectively, G) and the translation action of $\mathcal{M}_C$ on itself, the group $\mathcal{G}_0$ (respectively, $\mathcal{G}$) acts on $\mathcal{M}_C$.

Theorem 3.2. *The group* $\mathrm{Aut}_\theta(\mathcal{M}_C)$ *is given by*

1) $\mathrm{Aut}_\theta(\mathcal{M}_C) = \mathcal{G}$ *if* X *is not hyperelliptic;*
2) $\mathrm{Aut}_\theta(\mathcal{M}_C) = \mathcal{G}_0$ *if* X *is hyperelliptic.*

Proof. As mentioned before, we have $\theta = \varphi^*\widehat{\omega}$. From this it follows that for any element of $\mathcal{G}$, the corresponding automorphism of $\mathcal{M}_C$ preserves θ.

Let $\mathrm{Aut}_{\widehat{\omega}}(J(X))$ be the group of all automorphisms of the variety $J(X)$ that preserve the cohomology class $\widehat{\omega}$. From [13, Hauptsatz, p. 35] one has the following:

1. Assume that X is not hyperelliptic. Then $\mathrm{Aut}_{\widehat{\omega}}(J(X))$ is generated by translations of $J(X)$, $\mathrm{Aut}(X)$ and the inversion $L \longmapsto L^\vee$ of $J(X)$.
2. Assume that X is hyperelliptic. Then $\mathrm{Aut}_{\widehat{\omega}}(J(X))$ is generated by translations of $J(X)$ and $\mathrm{Aut}(X)$. (The hyperelliptic involution of X induces the inversion of $J(X)$.)

Consider the homomorphism δ in (3.1). In the proof of Theorem 3.1 it was shown that the inclusion

$$\rho(\mathrm{kernel}(\varphi)) \hookrightarrow \mathrm{kernel}(\delta) \tag{3.2}$$

is surjective. First assume that X is not hyperelliptic. Using φ in (2.2), we get a homomorphism

$$G \longrightarrow \mathrm{Aut}(J(X)).$$

From the above result of [13] we know that this homomorphism is injective, its image is a normal subgroup of $\mathrm{Aut}(J(X))$ and the composition

$$G \longrightarrow \mathrm{Aut}(J(X)) \longrightarrow \mathrm{Aut}(J(X))/J(X)$$

is an isomorphism. Therefore, from the surjectivity of the homomorphism in (3.2) we conclude that $\mathrm{Aut}_\theta(\mathcal{M}_C) = \mathcal{G}$.

If X is hyperelliptic, then $\mathrm{Aut}(J(X))/\mathrm{Aut}(X) = J(X)$ by the above theorem of [13]. Therefore, by the above argument it follows that $\mathrm{Aut}_\theta(\mathcal{M}_C) = \mathcal{G}_0$. ∎

From the definitions of $\mathcal{G}_0$ and $\mathcal{G}$, it is straightforward to verify that these groups preserve the algebraic symplectic form on $\mathcal{M}_C$. Therefore, Theorem 3.2 gives the following:

Corollary 3.3.

1. *Assume that X is not hyperelliptic. Then the group of algebraic automorphisms of $\mathcal{M}_C$ that preserve the symplectic form on $\mathcal{M}_C$ is $\mathcal{G}$.*
2. *Assume that X is hyperelliptic. Then the group of algebraic automorphisms of $\mathcal{M}_C$ that preserve the symplectic form on $\mathcal{M}_C$ is $\mathcal{G}_0$.*

4 Automorphisms of the representation space

The representation space

$$\mathcal{M}_R = \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{C}^*) = \mathrm{Hom}(\pi_1(X),\mathbb{C}^*)$$

is algebraically isomorphic to $(\mathbb{C}^*)^{2g}$. A choice of a basis of the $\mathbb{Z}$-module $H_1(X,\mathbb{Z})$ produces an isomorphism of $\mathcal{M}_R$ with $(\mathbb{C}^*)^{2g}$. The group structure of the multiplicative group $\mathbb{C}^*$ makes $\mathcal{M}_R$ a complex algebraic group.

The group $H_1(X,\mathbb{Z})$ is identified with $H^1(\mathcal{M}_R,\mathbb{Z})$ by the $(1,1)$-type Künneth component of the first Chern class of a Poincaré line bundle on $X\times\mathcal{M}_R$. It should be clarified that this $(1,1)$-type Künneth component is independent of the choice of the Poincaré line bundle. The group of all automorphisms of the $\mathbb{Z}$-module $H_1(X,\mathbb{Z})$ will be denoted by Γ. So Γ is isomorphic to $\mathrm{GL}(2g,\mathbb{Z})$.

Let $\mathrm{Aut}(\mathcal{M}_R)$ denote the group of all algebraic automorphisms of $\mathcal{M}_R$. Let

$$f\colon\ \mathrm{Aut}(\mathcal{M}_R)\longrightarrow\Gamma \tag{4.1}$$

be the homomorphism that sends any automorphism of $\mathcal{M}_R$ to the automorphism of

$$H^1(\mathcal{M}_R,\mathbb{Z}) = H_1(X,\mathbb{Z})$$

induced by it.

Lemma 4.1. *The homomorphism f in (4.1) is surjective.*

Proof. Given any $(a_{ij})_{i,j=1}^{2g}\in\mathrm{GL}(2g,\mathbb{Z})$, consider the automorphism T of $(\mathbb{C}^*)^{2g}$ defined as follows: the i-th coordinate of $T(z_1,\ldots,z_{2g})$, $(z_1,\ldots,z_{2g})\in(\mathbb{C}^*)^{2g}$, is

$$\prod_{j=1}^{2g} z_j^{a_{ij}}.$$

The automorphism of $H^1((\mathbb{C}^*)^{2g},\mathbb{Z}) = \mathbb{Z}^{2g}$ induced by T is given by the standard action of $(a_{ij})_{i,j=1}^{2g}$ on $\mathbb{Z}^{2g}$. ■

The map $(a_{ij})_{i,j=1}^{2g}\longmapsto\mathrm{Aut}((\mathbb{C}^*)^{2g})$ in the proof of Lemma 4.1 produces a canonical right-splitting of the homomorphism f in (4.1). Since f is surjective, this implies that the group $\mathrm{Aut}(\mathcal{M}_R)$ is the semi-direct product

$$\mathrm{Aut}(\mathcal{M}_R) = \mathrm{kernel}(f)\rtimes\Gamma.$$

The group of all algebraic automorphisms of $(\mathbb{C}^*)^{2g}$ that preserve every factor is $(\mathbb{C}^*)^{2g}$ acting on itself by translations. Therefore, we have the following:

Theorem 4.2. *The automorphism group $\mathrm{Aut}(\mathcal{M}_R)$ is the semi-direct product $\mathcal{M}_R\rtimes\Gamma$.*

We shall consider now $\Gamma_{\mathrm{Sp}} \subset \Gamma = \mathrm{Aut}(H_1(X,\mathbb{Z}))$ the group of automorphisms that preserve the cap product on $H_1(X,\mathbb{Z})$. So Γ_{Sp} is isomorphic to the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$. Using Theorem 4.2 it can be deduced that the group of automorphisms of $\mathcal{M}_R$ that preserve its symplectic form is $\mathcal{M}_R \rtimes \Gamma_{\mathrm{Sp}}$.

Although the holomorphic isomorphism between $\mathcal{M}_C$ and $\mathcal{M}_R$ is not algebraic, comparing Theorem 3.2 and Theorem 4.2 we obtain a relation between their automorphism groups. Let $h\colon \mathrm{Aut}(X)/\mathrm{Aut}^0(X) \times \mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{M}_R)$ be the homomorphism which sends $\mathrm{Aut}(X)/\mathrm{Aut}^0(X)$ to its image in $\mathrm{Aut}(\mathcal{M}_R)$ and maps the generator of $\mathbb{Z}_2$ to the inversion map $\phi \longmapsto \phi^{-1}$, sending a homomorphism $\phi\colon \pi_1(X) \to \mathbb{C}^*$ to its inverse ϕ^{-1}. Let $G_M \subset \mathrm{Aut}(\mathcal{M}_R)$ be the image of h. Then:

Corollary 4.3. *For any $\tau \in \mathrm{Aut}_\theta(\mathcal{M}_C)$, the holomorphic automorphism of $\mathcal{M}_R$ given by τ using the holomorphic identification between $\mathcal{M}_C$ and $\mathcal{M}_R$ is actually algebraic. More precisely, the set of automorphisms of $\mathcal{M}_R$ given by $\mathrm{Aut}_\theta(\mathcal{M}_C)$ is as follows:*

1) *it is $\mathcal{M}_R \rtimes G_M$ if X is not hyperelliptic,*
2) *it is $\mathcal{M}_R \rtimes h(\mathrm{Aut}(X)/\mathrm{Aut}^0(X))$ if X is hyperelliptic.*

Proof. Recall from Theorem 3.2 that $\mathrm{Aut}_\theta(\mathcal{M}_C)$ is generated by the translation action of $\mathcal{M}_C$ on itself together with the action of $\mathrm{Aut}(X)$ and the inversion $(L,D) \longmapsto (L^\vee, D^\vee)$ of the group $\mathcal{M}_C$. In the case that X is hyperelliptic the action of inversion coincides with the hyperelliptic involution, so may be omitted. As abstract groups, $\mathcal{M}_C$ and $\mathcal{M}_R$ are isomorphic, so the translation action of $\mathcal{M}_C$ coincides with the translation action of $\mathcal{M}_R$, hence is algebraic with respect to $\mathcal{M}_R$. It is also clear that the action of $\mathrm{Aut}(X)$ together with the inversion $(L,D) \longmapsto (L^\vee, D^\vee)$ act on $\mathcal{M}_R = \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{C}^*)$ as a subgroup of Γ_{Sp}, hence are also algebraic with respect to $\mathcal{M}_R$. This proves the claim that any $\tau \in \mathrm{Aut}_\theta(\mathcal{M}_C)$ acts as an algebraic automorphism of $\mathcal{M}_R$ and hence defines natural homomorphism $j\colon \mathrm{Aut}_\theta(\mathcal{M}_C) \to \mathrm{Aut}(\mathcal{M}_C)$.

We claim that j is injective. For this note that the restriction of f to G_M defines a homomorphism $f\colon G_M \to \Gamma_{\mathrm{Sp}}$ which sends an automorphism of X to its induced action on $H_1(X,\mathbb{Z})$ and sends the inversion map to $-\mathrm{Id}$. If X is not hyperelliptic, then the composition $f \circ h\colon \mathrm{Aut}(X)/\mathrm{Aut}^0(X) \times \mathbb{Z}_2 \mapsto \Gamma_{\mathrm{Sp}}$ is injective and if X is hyperelliptic then $f \circ h|_{\mathrm{Aut}(X)/\mathrm{Aut}^0(X)}\colon \mathrm{Aut}(X)/\mathrm{Aut}^0(X) \to \Gamma_{\mathrm{Sp}}$ is injective (for $g=1$ this is trivial, while for $g \geq 2$ this follows from, e.g., [6, Section V.2]). This proves the claim that j is injective and that the image of j is $\mathcal{M}_R \rtimes G_M$ if X is not hyperelliptic and $\mathcal{M}_R \rtimes h(\mathrm{Aut}(X)/\mathrm{Aut}^0(X))$ if X is hyperelliptic. ∎

5 Automorphisms of moduli space of Higgs line bundles

The moduli space of Higgs line bundles on X of degree zero is the Cartesian product

$$\mathcal{M}_H = J(X) \times H^0(X, K_X).$$

Let $\mathrm{Aut}(\mathcal{M}_H)$ denote the group of all algebraic automorphisms of the variety $\mathcal{M}_H$.

Lemma 5.1. *Any $f \in \mathrm{Aut}(\mathcal{M}_H)$ is of the form*

$$f = f_1 \times f_2,$$

where $f_1 \in \mathrm{Aut}(J(X))$ and $f_2 \in \mathrm{Aut}(H^0(X,K_X))$.

Proof. Let

$$\phi_1\colon \mathcal{M}_H = J(X) \times H^0(X,K_X) \longrightarrow J(X),$$
$$\phi_2\colon J(X) \times H^0(X,K_X) \longrightarrow H^0(X,K_X)$$

be the natural projections. As noted before, there are no nonconstant algebraic maps from $H^0(X, K_X)$ to $J(X)$. So given f, there is a unique automorphism

$$f_1 \in \mathrm{Aut}(J(X))$$

such that $f_1 \circ \phi_1 = \phi_1 \circ f$.

Note that given $v \in H^0(X, K_X)$, one can define the map

$$\begin{aligned} \widehat{v}\colon\ J(X) &\longrightarrow H^0(X, K_X), \\ z &\longmapsto \phi_2(f(z, v)), \end{aligned}$$

which is a constant map since it is holomorphic. Denoting by $v' \in H^0(X, K_X)$ the constant image of $\widehat{v}$, one can see that

$$\begin{aligned} f_2\colon\ H^0(X, K_X) &\longrightarrow H^0(X, K_X), \\ v &\longmapsto v' \end{aligned}$$

is an automorphism, and thus one has that $f = f_1 \times f_2$. ■

The moduli space $\mathcal{M}_H$ can be naturally identified with the cotangent bundle of $J(X)$, hence it carries a canonical symplectic form θ. Let $\mathrm{Aut}_\theta(\mathcal{M}_H)$ be the subgroup of $\mathrm{Aut}(\mathcal{M}_H)$ preserving θ. Recall that $\Omega_{J(X)}$ denotes the holomorphic cotangent bundle of $J(X)$ and that there is a naturally defined isomorphism $H^0(X, K_X) = H^0(J(X), \Omega_{J(X)})$.

Theorem 5.2. *The group* $\mathrm{Aut}_\theta(\mathcal{M}_H)$ *is the semi-direct product*

$$H^0(J(X), \Omega_{J(X)}) \rtimes \mathrm{Aut}(J(X)),$$

where $\mathrm{Aut}(J(X))$ *acts on* $H^0(J(X), \Omega_{J(X)})$ *by* $f \cdot \alpha = (f^{-1})^*(\alpha)$, *for* $f \in \mathrm{Aut}(J(X))$, $\alpha \in H^0(J(X), \Omega_{J(X)})$.

Proof. By Lemma 5.1, any automorphism of $T^\vee J(X) = J(X) \times H^0(J(X), \Omega_{J(X)})$ has the form $f(x, y) = (f_1(x), f_2(y))$ for $f_1 \in \mathrm{Aut}(J(X))$, $f_2 \in \mathrm{Aut}(H^0(J(X), \Omega_{J(X)})$. Since $f_1 \in \mathrm{Aut}(J(X))$, the derivative $(f_1)_*(x)\colon H^0(J(X), \Omega_{J(X)})^\vee \longrightarrow H^0(J(X), \Omega_{J(X)})^\vee$ is independent of x and will be denoted by A. Then, it is clear that $f_1 \times f_2$ preserves the symplectic form on $T^\vee J(X)$ if and only if $(f_2)_*(y) = (A^\vee)^{-1}$ for all y. Thus f_2 is an affine transformation of the form $f_2(y) = (A^\vee)^{-1} y + y_0$, for some $y_0 \in H^0(J(X), \Omega_{J(X)})$. So $f_1 \times f_2$ is the composition of $(f_1^*)^{-1}\colon T^\vee J(X) \longrightarrow T^\vee J(X)$ with a translation by y_0 in the fibers of $T^\vee J(X) \longrightarrow J(X)$. It follows easily that $\mathrm{Aut}_\theta(\mathcal{M}_H)$ is the semi-direct product $H^0(J(X), \Omega_{J(X)}) \rtimes \mathrm{Aut}(J(X))$. ■

Acknowledgements

We thank the referees for helpful comments. The first author is supported by a J.C. Bose Fellowship. The third author is partially supported by NSF grant DMS-1509693.

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