An Introduction to Spectral Data for Higgs Bundles

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These notes provide an introduction to Higgs bundles for complex and real Lie groups, as well as a description of their associated spectral data.

1. Introduction

The first lecture shall introduce classical Higgs bundles and the Hitchin fibration, and describe the associated spectral data in the case of principal Higgs bundles for classical complex Lie groups, following mainly Hitchin's papers [31, 32, 33, 34]. The second lecture is dedicated to the construction of Higgs bundles for real forms of classical complex Lie groups as fixed points of involutions, and the description of the corresponding spectral data, following mainly [35, 46, 47, 48]. Along the way, we shall mention different applications and open questions related to the methods introduced in both lectures.

Each lecture contains exercises of varying difficulty, whose solutions can be found in [47]. Open questions which might be tackled with methods similar to the ones introduced in the lectures appear indicated with **(*)** together with references which feature approaches that may be useful. Since it proves to be very difficult to give a comprehensive and exhaustive account of research in tangential areas, we shall restrain ourselves to mentioning related work only when it directly involves methods using *spectral data*. The reader should refer to references in the bibliography for further research in related topics (e.g., see references in [4, 42, 47]).

2. Spectral data for G_c-Higgs bundles

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

David Hilbert

Following [31, 32, 34] we dedicate this lecture to overview classical Higgs bundles as well as G_c-Higgs bundles for the groups $G_c = SL(n, \mathbb{C})$, $Sp(n, \mathbb{C})$, $SO(2n+1, \mathbb{C})$ and $SO(2n, \mathbb{C})$. In each case we introduce the Hitchin fibration and describe the generic fibres through *spectral data*, i.e., an associated spectral curve and a line bundle on it.

2.1. G_c-Higgs bundles

Consider Σ a compact Riemann surface of genus $g \geq 2$ with canonical bundle $K = T^*\Sigma$. Classically, a Higgs bundle on Σ is defined as follows:

Definition 2.1: A *Higgs bundle* is a pair (E, Φ) for E a holomorphic vector bundle on Σ, and Φ, the *Higgs field*, a holomorphic section in $H^0(\Sigma, \mathrm{End}(E) \otimes K)$.

In order to understand better what Higgs bundles are and how to generalise the definition, we shall first look at the moduli space of vector bundles and then study the moduli space of classical Higgs bundles and its associated spectral data. For more details the reader should refer to [31, 32, 23, 19, 51, 40, 52].

2.1.1. *Moduli space of vector bundles*

Holomorphic vector bundles E on a compact Riemann surface Σ of genus $g \geq 2$ are topologically classified by their rank $rk(E)$ and their degree $deg(E)$.

Definition 2.2: The *slope* of a holomorphic vector bundle E is defined as $\mu(E) := deg(E)/rk(E)$ and is used to define stability conditions: A vector bundle E is said to be *stable* (*semi-stable*) if for any proper, non-zero subbundle $F \subset E$ one has $\mu(F) < \mu(E)$ ($\mu(F) \leq \mu(E)$). It is *polystable* if it is a direct sum of stable bundles whose slope is the same as E.

It is known that the space of holomorphic bundles of fixed rank and fixed degree, up to isomorphism, is not a Hausdorff space. However, through Mumford's Geometric Invariant Theory one can construct the moduli space $\mathcal{N}(n, d)$ of stable bundles of fixed rank n and degree d, which has the natural structure of an algebraic variety.

Remark 2.3: For coprime n and d, the moduli space $\mathcal{N}(n, d)$ is a smooth projective algebraic variety of dimension $n^2(g-1)+1$.

Remark 2.4: All line bundles are stable, and thus $\mathcal{N}(1, d)$ contains all line bundles of degree d, and is isomorphic to $\mathrm{Jac}^d(\Sigma)$ of Σ, an abelian variety of dimension g.

Let G_c be a complex semisimple Lie group. Following [44] one can define stability for principal G_c-bundles as follows (see [4] for a comprehensive study):

Definition 2.5: A holomorphic principal G_c-bundle P is said to be *stable* (*semi-stable*) if for every reduction $\sigma : \Sigma \to P/Q$ to maximal parabolic subgroups Q of G_c one has $\deg \sigma^* T_{rel} > 0$ (≥ 0), where T_{rel} is the relative tangent bundle for the projection $P/Q \to \Sigma$.

The notion of polystability may be carried over to principal G_c-bundles, allowing one to construct the moduli space of isomorphism classes of polystable principal G_c-bundles of fixed topological type over the compact Riemann surface Σ.

2.1.2. *Moduli space of classical Higgs bundles*

In order to define the moduli space of Higgs bundles, the following stability condition is considered:

Definition 2.6: A vector subbundle F of E for which $\Phi(F) \subset F \otimes K$ is said to be a Φ-*invariant subbundle* of E. A Higgs bundle (E, Φ) is

- *stable* (*semi-stable*) if for each proper Φ-invariant $F \subset E$ one has $\mu(F) < \mu(E)$ (*equiv.* $\leq$);
- *polystable* if $(E, \Phi) = (E_1, \Phi_1) \oplus (E_2, \Phi_2) \oplus \cdots \oplus (E_r, \Phi_r)$, where (E_i, Φ_i) is stable with $\mu(E_i) = \mu(E)$ for all i.

Example 2.7: Choose a square root $K^{1/2}$ of the canonical bundle K, and a section ω of K^2. A family of classical Higgs bundles (E, Φ_ω) may be obtained by considering the vector bundle $E = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ and the Higgs bundle Φ_ω given by

$$\Phi_\omega = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix} \in H^0(\Sigma, \mathrm{End}(E) \otimes K).$$

Problem 2.8: *Show that the pairs (E, Φ_ω) from Example 2.7 are stable.*

Problem 2.9: *Prove that if a Higgs bundle (E, Φ) is stable, then for any $\lambda \in \mathbb{C}^*$ and α a holomorphic automorphism of E, the induced Higgs bundles $(E, \lambda\Phi)$ and $(E, \alpha^*\Phi)$ are stable.*

In order to define the moduli space of classical Higgs bundles, we shall first define an appropriate equivalence relation. For this, consider a strictly semi-stable Higgs bundle (E, Φ). As it is not stable, E admits a subbundle $F \subset E$ of the same slope which is preserved by Φ. If F is a subbundle of E of least rank and same slope which is preserved by Φ, it follows that F is stable and hence the induced pair (F, Φ) is stable. Then, by induction one obtains a flag of subbundles $F_0 = 0 \subset F_1 \subset \cdots \subset F_r = E$ where $\mu(F_i/F_{i-1}) = \mu(E)$ for $1 \leq i \leq r$, and where the induced Higgs bundles $(F_i/F_{i-1}, \Phi_i)$ are stable. This is the *Jordan-Hölder filtration* of E, and it is not unique. However, the graded object $\mathrm{Gr}(E, \Phi) := \bigoplus_{i=1}^{r} (F_i/F_{i=1}, \Phi_i)$ is unique up to isomorphism.

Definition 2.10: Two semi-stable Higgs bundles (E, Φ) and (E', Φ') are said to be *S-equivalent* if $\mathrm{Gr}(E, \Phi) \cong \mathrm{Gr}(E', \Phi')$.

Problem 2.11: *If a pair (E, Φ) is strictly stable, what is the induced Jordan-Hölder filtration?*

Following [40] we let $\mathcal{M}(n, d)$ be the moduli space of S-equivalence classes of semi-stable Higgs bundles of fixed degree d and fixed rank n. The moduli space $\mathcal{M}(n, d)$ is a quasi-projective scheme, and has an open subscheme $\mathcal{M}'(n, d)$ which is the moduli scheme of stable pairs. Thus, every point is represented by either a stable or a polystable Higgs bundle. When d and n are coprime, the moduli space $\mathcal{M}(n, d)$ is smooth.

The cotangent space of $\mathcal{N}(n, d)$ over the stable locus is contained in $\mathcal{M}(n, d)$ as a Zariski open subset. The moduli space $\mathcal{M}(n, d)$ is a non-compact variety which has complex dimension $2n^2(g-1) + 2$. Moreover, it is a hyperkähler manifold with natural symplectic form ω defined on the infinitesimal deformations $(\dot{A}, \dot{\Phi})$ of a Higgs bundle (E, Φ) by

$$\omega((\dot{A}_1, \dot{\Phi}_1), (\dot{A}_2, \dot{\Phi}_2)) = \int_\Sigma \mathrm{tr}(\dot{A}_1 \dot{\Phi}_2 - \dot{A}_2 \dot{\Phi}_1), \tag{2.1}$$

where $\dot{A} \in \Omega^{0,1}(\mathrm{End}_0 E)$ and $\dot{\Phi} \in \Omega^{1,0}(\mathrm{End}_0 E)$ (see [31, 32] for details). For simplicity, we shall fix n and d and write $\mathcal{M}$ for $\mathcal{M}(n, d)$.

2.1.3. *Moduli space of G_c-Higgs bundles*

The notion of Higgs bundle can be generalized to encompass principal G_c-bundles, for G_c a complex semi-simple Lie group. For more details, the reader should refer to [32].

Definition 2.12: A *G_c-Higgs bundle* is a pair (P, Φ) where P is a principal G_c-bundle over Σ, and the Higgs field Φ is a holomorphic section of the vector bundle $adP \otimes_{\mathbb{C}} K$, for adP the vector bundle associated to the adjoint representation.

When $G_c \subset GL(n, \mathbb{C})$, a G_c-Higgs bundle gives rise to a Higgs bundle in the classical sense, with some extra structure reflecting the definition of G_c. In particular, classical Higgs bundles are given by $GL(n, \mathbb{C})$-Higgs bundles.

Example 2.13: The Higgs bundles in Example 2.7 have traceless Higgs field, and the determinant $\Lambda^2 E$ is trivial. Hence, for each quadratic differential ω one has an $SL(2, \mathbb{C})$-Higgs bundle (E, Φ_ω).

By extending the stability definitions for principal G_c-bundles, one can define *stable*, *semi-stable* and *polystable* G_c-Higgs bundles. Since in these notes we shall be working with Higgs pairs which do not preserve any subbundle, they will be automatically stable and thus we shall not dedicate time to recall the main study of stability for principal Higgs bundles. For details about the corresponding constructions, the reader should refer for example to [12, 4]. We denote by $\mathcal{M}_{G_c}$ the moduli space of S-equivalence classes of polystable G_c-Higgs bundles.

In the remainder of this Section, following [32] and [34] we introduce the Hitchin fibration and describe the generic fibres for G_c-Higgs bundles where $G_c = GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $Sp(2n, \mathbb{C})$, $SO(2n+1, \mathbb{C})$ and $SO(2n, \mathbb{C})$. We shall cover with more detail the initial cases, and leave as an exercise to the reader some of the results for the latter groups.

2.2. *The Hitchin fibration*

A natural way of studying $\mathcal{M}_{G_c}$ is through the Hitchin fibration, as introduced in [32]. We shall denote by p_i, for $i = 1, \ldots, k$, a homogeneous basis for the algebra of invariant polynomials of the Lie algebra $\mathfrak{g}_c$ of G_c, and let d_i be their degrees. Then, the *Hitchin fibration* is given by

$$h \ : \ \mathcal{M}_{G_c} \longrightarrow \mathcal{A}_{G_c} := \bigoplus_{i=1}^{k} H^0(\Sigma, K^{d_i}), \tag{2.2}$$

$$(E,\Phi) \ \mapsto \ (p_1(\Phi), \ldots, p_k(\Phi)). \tag{2.3}$$

The map h is referred to as the *Hitchin map*, and is a proper map for any choice of basis [32]. Furthermore, $\dim \mathcal{A}_{G_c} = \dim \mathcal{M}_{G_c}/2$, making the Higgs bundle moduli space into an integrable system.

Remark 2.14: Note that, in a local frame, a Higgs field Φ has values in a Lie algebra, and thus since this is well defined up to conjugation, evaluating the invariant polynomials is globally well defined.

Remark 2.15: Let $\mathfrak{g}^c$ be one of the classical Lie algebras $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{sp}(2n,\mathbb{C})$, $\mathfrak{so}(2n+1,\mathbb{C})$. Then, for $\pi : \mathfrak{g}^c \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}^c$, the ring of invariant polynomials of $\mathfrak{g}^c$ is generated by $\mathrm{Tr}(\pi(X)^i)$, for $i \in \mathbb{N}$ and $X \in \mathfrak{g}^c$. Hence, a homogeneous basis of invariant polynomials for classical Higgs bundles (E,Φ) of rank n can be taken as $tr(\Phi^i)$ for $1 \leq i \leq n$.

Remark 2.16: Whilst a formal definition of the smooth locus of the Hitchin base can be given (e.g., see [20]) in these lectures we shall note that the generic fibres of the Hitchin fibration are smooth, and thus generic points in the Hitchin base are in the smooth locus.

In what follows we shall describe the spectral data associated to G_c-Higgs bundles as introduced in [32, 34].

2.2.1. $GL(n,\mathbb{C})$-*Higgs bundles*

As before, let K be the canonical bundle of Σ, and X its total space with projection $\pi : X \to \Sigma$. We shall denote by η the tautological section of the pull back π^*K on X. Abusing notation we denote with the same symbols the sections of powers K^i on Σ and their pull backs to X. The characteristic polynomial of a Higgs bundle (E,Φ) in a generic fibre $h^{-1}(a)$ defines a smooth curve $\pi : S_a \to \Sigma$ in X, the *spectral curve* of Φ, whose equation is

$$\det(\eta Id - \pi^*\Phi) = \eta^n + a_1\eta^{n-1} + a_2\eta^{n-2} + \cdots + a_{n-1}\eta + a_n = 0, \tag{2.4}$$

for $a_i \in H^0(\Sigma, K^i)$ (for simplicity, we shall write $\det(\eta - \Phi)$ for the characteristic polynomial of the Higgs field Φ, and drop the subscript a of S_a). By the adjunction formula on X (see e.g. [29]), since the canonical bundle

K has trivial cotangent bundle one has $K_S \cong \pi^* K^n$, and hence the genus of S is

$$g_S = 1 + n^2(g-1). \tag{2.5}$$

The spectral data *for classical Higgs bundles in a smooth fibre of the Hitchin fibration is given by a spectral curve S defined as in (2.4) and a line bundle $L \in$ Jac(S).*

In order to see that the smooth fibres of the Hitchin fibration are Jacobians, starting with a line bundle L on the smooth curve $\pi : S \to \Sigma$ with equation as in (2.4), we shall obtain a classical Higgs bundle by considering the direct image $\pi_* L$ of L. Recall that by definition of direct image, given an open set $\mathcal{U} \subset \Sigma$, one has $H^0(\pi^{-1}(\mathcal{U}), L) = H^0(\mathcal{U}, \pi_* L)$. Multiplication by the tautological section η induces the map

$$H^0(\pi^{-1}(\mathcal{U}), L) \xrightarrow{\eta} H^0(\pi^{-1}(\mathcal{U}), L \otimes \pi^* K),$$

which by definition of direct image can be pushed down to give

$$\Phi : \pi_* L \to \pi_* L \otimes K.$$

Then, one obtains a Higgs field $\Phi \in H^0(\Sigma, \mathrm{End}E \otimes K)$ for $E := \pi_* L$.

Problem 2.17: *Use Grothendieck-Riemann-Roch to show that the degree of E is $deg(E) = deg(L) + (n^2 - n)(1-g)$.*

Moreover, the Higgs field satisfies its characteristic equation, which by construction is given by $\eta^n + a_1\eta^{n-1} + a_2\eta^{n-2} + \cdots + a_{n-1}\eta + a_n = 0$. Furthermore, since S is irreducible, from Remark 2.21 there are no invariant subbundles of the Higgs field, making the induced Higgs bundle (E, Φ) stable.

Conversely, let (E, Φ) be a classical Higgs bundle. The characteristic polynomial is given by $\det(x - \Phi) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1}x + a_n$, and its coefficients define the *spectral curve* S in the total space X whose equation is (2.4).

From [11], there is a bijective correspondence between Higgs bundles (E, Φ) and the line bundles L on the spectral curve S described previously. This correspondence identifies the fibre of the Hitchin map with the Picard variety of line bundles of the appropriate degree. By tensoring the line bundles L with a chosen line bundle of degree $-\deg(L)$, one obtains a point in the Jacobian Jac(S), the abelian variety of line bundles of degree zero on S, which has dimension g_S as in (2.5). In particular, the Jacobian variety is the connected component of the identity in the Picard group $H^1(S, \mathcal{O}_S^*)$. Thus, the fibre of the classical Hitchin fibration $h : \mathcal{M} \to \mathcal{A}$ is isomorphic to the Jacobian of the spectral curve S. For more details, the reader should refer for example to [34].

Example 2.18: In the case of a classical rank 2 Higgs bundle (E, Φ), the characteristic polynomial of Φ defines a spectral curve $\pi : S \to \Sigma$. This is a 2-fold cover of Σ in the total space of K, and has equation $\eta^2 + a_2 = 0$, for a_2 a quadratic differential and η the tautological section of π^*K. By [11] the curve is smooth when a_2 has simple zeros, and in this case the ramification points are given by the divisor of a_2. For z a local coordinate near a ramification point, the covering is given by $z \mapsto z^2 := w$. In a neighbourhood of $z = 0$, a section of the line bundle M can be expressed as $f(w) = f_0(w) + zf_1(w)$. Since the Higgs field is obtained via multiplication by η, one has

$$\Phi(f_0(w) + zf_1(w)) = wf_1(w) + zf_0(w), \tag{2.6}$$

and thus a local form of the Higgs field Φ is given by

$$\Phi = \begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix}.$$

Remark 2.19: When $G_c \subset GL(n, \mathbb{C})$, for the groups G_c we are considering in these notes, the spectral data of a G_c-Higgs bundle is given by the spectral data of the pair as a classical Higgs bundle, satisfying extra conditions.

Remark 2.20: For general G_c, a similar description of the fibres can be obtained though a Lie theoretic approach, by means of what is known as *Cameral covers*. The reader should refer to [21] (see also [20]) for this generic description, and note that it is equivalent to the one given in the next sections for the groups considered in these lecture notes.

Remark 2.21: The characteristic polynomial of Φ restricted to an invariant subbundle divides the characteristic polynomial of Φ.

2.2.2. $SL(n, \mathbb{C})$-*Higgs bundles*

When $G_c = SL(n, \mathbb{C})$ we apply Definition 2.12 to obtain the following:

Definition 2.22: An $SL(n, \mathbb{C})$-Higgs bundle is a classical Higgs bundle (E, Φ) where the rank n vector bundle E has trivial determinant and the Higgs field has zero trace.

A basis for the invariant polynomials on the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ is given by the coefficients of the characteristic polynomial of a trace-free matrix $A \in \mathfrak{sl}(n, \mathbb{C})$. In this case, the spectral curve $\pi : S \to \Sigma$ associated to the Higgs bundle has equation

$$\eta^n + a_2\eta^{n-2} + \cdots + a_{n-1}\eta + a_n = 0, \tag{2.7}$$

where $a_i \in H^0(\Sigma, K^i)$ are the coefficients of the characteristic polynomial of the Higgs field Φ. In particular, one may consider $a_i = Tr(\Phi^i)$, from where it is clear that in this case $a_1 = Tr(\Phi) = 0$. Generically S is a smooth curve of genus $g_S = 1 + n^2(g - 1)$, and the coefficients define the corresponding Hitchin fibration

$$h \ : \ \mathcal{M}_{SL(n,\mathbb{C})} \longrightarrow \mathcal{A}_{SL(n,\mathbb{C})} := \bigoplus_{i=2}^{n} H^0(\Sigma, K^i). \tag{2.8}$$

In this case the generic fibres of the Hitchin fibration are given by the subset of $\mathrm{Jac}(S)$ of line bundles L on S for which $\pi_*L = E$ and $\Lambda^n\pi_*L$ is trivial. These conditions in terms of L lead to the following:

The generic fibre of the $SL(n, \mathbb{C})$ Hitchin fibration is biholomorphically equivalent to the Prym(S, Σ), for S the spectral curve defined as in (2.4).

In order to see why one has to take the Prym variety, recall that the Norm map

$$\mathrm{Nm} : \mathrm{Pic}(S) \to \mathrm{Pic}(\Sigma),$$

associated to π is defined on divisor classes by $\mathrm{Nm}(\sum n_i p_i) = \sum n_i \pi(p_i)$. In particular,

$$\mathrm{Nm}(\pi^{-1}(x)) = \pi(\pi^{-1}(x)) = nx.$$

The kernel of the Norm map is the *Prym variety*, and is denoted by $\mathrm{Prym}(S, \Sigma)$. From [11], the determinant bundle of L satisfies

$$\Lambda^n\pi_*L \cong \mathrm{Nm}(L) \otimes K^{-n(n-1)/2}.$$

Thus, $\Lambda^n\pi_*L$ is trivial if and only if

$$\mathrm{Nm}(L) \cong K^{n(n-1)/2}. \tag{2.9}$$

Equivalently, since $\mathrm{Nm}(\sum n_i\pi^{-1}(p_i)) = n\sum n_i p_i$, the determinant bundle $\Lambda^n\pi_*L$ is trivial if the line bundle $M := L \otimes \pi^*K^{-(n-1)/2}$ is in the Prym variety.

Remark 2.23: In the case of even rank, equation (2.9) implies a choice of a square root of K (see [31] and [34] for more details).

2.2.3. $Sp(2n,\mathbb{C})$-*Higgs bundles*

Let $G_c = Sp(2n,\mathbb{C})$, and let V be a $2n$-dimensional vector space with a non-degenerate skew-symmetric form $<,>$. For v_i, v_j eigenvectors of $A \in \mathfrak{sp}(2n,\mathbb{C})$ with eigenvalues λ_i and λ_j,

$$\begin{aligned}
\lambda_i < v_i, v_j > &= < \lambda_i v_i, v_j > && (2.10)\\
&= < Av_i, v_j > && (2.11)\\
&= - < v_i, Av_j > && (2.12)\\
&= - < v_i, \lambda_j v_j > = -\lambda_j < v_i, v_j > .
\end{aligned}$$

From the above, $< v_i, v_j >= 0$ unless $\lambda_i = -\lambda_j$. Since $< v_j, v_j >= 0$, from the non-degeneracy of the symplectic inner product it follows that if λ_i is an eigenvalue so is $-\lambda_i$. Thus, distinct eigenvalues of A must occur in $\pm\lambda_i$ pairs, and the corresponding eigenspaces are paired by the symplectic form. The characteristic polynomial of A must therefore be of the form

$$\det(x - A) = x^{2n} + a_1 x^{2n-2} + \cdots + a_{n-1} x^2 + a_n,$$

and a basis for the invariant polynomials on the Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$ is given by $a_1, \ldots, a_n$.

Definition 2.24: An $Sp(2n,\mathbb{C})$-Higgs bundle is a pair (E, Φ) for E a rank $2n$ vector bundle with a symplectic form $\omega(\ ,\ )$, and the Higgs field $\Phi \in H^0(\Sigma, \mathrm{End}(E) \otimes K)$ satisfying

$$\omega(\Phi v, w) = -\omega(v, \Phi w).$$

The volume form ω^n trivialises the determinant bundle $\Lambda^{2n}E^*$. The characteristic polynomial $\det(\eta - \Phi)$ defines a spectral curve $\pi : S \to \Sigma$ in X with equation

$$\eta^{2n} + a_1 \eta^{2n-1} + \cdots + a_{n-1}\eta^2 + a_n = 0, \tag{2.13}$$

whose genus is $g_S := 1 + 4n^2(g-1)$. The curve S has a natural involution $\sigma(\eta) = -\eta$ and thus one can define the quotient curve $\bar{\pi} : \overline{S} = S/\sigma \to \Sigma$, of which S is a 2-fold cover $p : S \to \overline{S}$. Note that the Norm map associated to p satisfies $p^*\mathrm{Nm}(x) = x + \sigma x$, and thus the Prym variety $\mathrm{Prym}(S, \overline{S})$ is given by the line bundles $M \in \mathrm{Jac}(S)$ for which $\sigma^* M \cong M^*$.

As in the case of classical Higgs bundles, the characteristic polynomial of a Higgs field Φ gives the Hitchin fibration

$$h \ : \ \mathcal{M}_{Sp(2n,\mathbb{C})} \longrightarrow \mathcal{A}_{Sp(2n,\mathbb{C})} := \bigoplus_{i=1}^{n} H^0(\Sigma, K^{2i}), \tag{2.14}$$

and one has the following:

The generic fibres $h^{-1}(a)$ of the Hitchin fibration for $Sp(2n,\mathbb{C})$-Higgs bundles is given by Prym varieties Prym$(S,\overline{S})$, where S and its quotient $\bar{S}$ are the curves defined by a as above.

The spectral data described above for an $Sp(2n,\mathbb{C})$-Higgs bundle (E,Φ) can be obtained by looking at the extra conditions needed on $L \in \mathrm{Jac}(S)$ associated to the corresponding classical Higgs pair for which $\pi_* L = E$. In order to understand this, note that for $\mathcal{V} \subset S$ an open set, we have $\mathcal{V} \subset \pi^{-1}(\pi(\mathcal{V}))$ and hence a natural restriction map $H^0(\pi^{-1}(\pi(\mathcal{V})), L) \to H^0(\mathcal{V}, L)$, which gives the evaluation map $ev: \ \pi^*\pi_* L \to L$. Multiplication by η commutes with this linear map and so the action of $\pi^*\Phi$ on the dual of the vector bundle $\pi^*\pi_* L$ preserves a one-dimensional subspace. Hence L^* is an eigenspace of $\pi^*\Phi^t$, with eigenvalue η. Equivalently, L is the cokernel of $\pi^*\Phi - \eta$ acting on $\pi^*E \otimes \pi^*K^*$. By means of the Norm map for π, this correspondence can be seen on the curve S via the exact sequence

$$0 \to L \otimes \pi^*K^{1-2n} \to \pi^*E \xrightarrow{\pi^*\Phi - \eta} \pi^*(E \otimes K^*) \xrightarrow{ev} L \otimes \pi^*K \to 0, \tag{2.15}$$

and its dualised sequence

$$0 \to L^* \otimes \pi^*K^* \to \pi^*(E^* \otimes K^*) \to \pi^*E^* \to L^* \otimes \pi^*K^{2n-1} \to 0. \tag{2.16}$$

In particular, from the relative duality theorem one has that

$$\pi_*(L)^* \cong \pi_*(K_S \otimes \pi^*K^{-1} \otimes L^*), \tag{2.17}$$

and thus E^* is the direct image sheaf $\pi_*(L^* \otimes \pi^*K^{2n-1})$.

Given an $Sp(2n,\mathbb{C})$-Higgs bundle (E,Φ), one has $\Phi^t = -\Phi$ and an eigenspace L of Φ with eigenvalue η is transformed to σ^*L for the eigenvalue $-\eta$. Moreover, since the line bundle L is the cokernel of $\pi^*\Phi - \eta$ acting on $\pi^*(E \otimes K^*)$, one can consider the corresponding exact sequences (2.15) and its dualised sequence, which identify L^* with $L \otimes \pi^*K^{1-2n}$, or equivalently, $L^2 = \pi^*K^{2n-1}$. By choosing a square root $K^{1/2}$ one has a line bundle $M := L \otimes \pi^*K^{-n+1/2}$ for which $\sigma^*M \cong M^*$, i.e., which is in the Prym variety $\mathrm{Prym}(S,\overline{S})$.

Conversely, an $Sp(2p,\mathbb{C})$-Higgs bundle can be recovered from a line bundle $M \in \mathrm{Prym}(S,\overline{S})$, for S a smooth curve with equation (2.13) and $\bar{S}$ its quotient curve. Indeed, by Bertini's theorem, such a smooth curve S with equation (2.13) always exists. Letting $E := \pi_* L$ for $L = M \otimes \pi^* K^{n-1/2}$, one has the exact sequences (2.15) and its dualised on the curve S. Moreover, since $L^2 \cong \pi^* K^{2n-1}$, there is an isomorphism $E \cong E^*$ which induces the symplectic structure on E. Hence, the generic fibres of the corresponding Hitchin fibration can be identified with the Prym variety $\mathrm{Prym}(S, \overline{S})$.

2.2.4. *$SO(2n+1,\mathbb{C})$-Higgs bundles*

We shall now consider the special orthogonal group $G_c = SO(2n+1,\mathbb{C})$ and the corresponding Higgs bundles. Following a similar analysis as in the previous case, one can see that for a generic matrix $A \in \mathfrak{so}(2n+1,\mathbb{C})$, its distinct eigenvalues occur in $\pm\lambda_i$ pairs, and necessarily A has a zero eigenvalue. Thus, the characteristic polynomial of A must be of the form

$$\det(x - A) = x(x^{2n} + a_1 x^{2n-2} + \cdots + a_{n-1} x^2 + a_n), \qquad (2.18)$$

where the coefficients $a_1, \ldots, a_n$ give a basis for the invariant polynomials on $\mathfrak{so}(2n+1,\mathbb{C})$.

Definition 2.25: An $SO(2n+1,\mathbb{C})$-Higgs bundle is a pair (E, Φ) for E a holomorphic vector bundle of rank $2n+1$ with a non-degenerate symmetric bilinear form (v, w), and Φ a Higgs field in $H^0(\Sigma, \mathrm{End}_0(E) \otimes K)$ which satisfies $(\Phi v, w) = -(v, \Phi w)$.

The moduli space $\mathcal{M}_{SO(2n+1,\mathbb{C})}$ has two connected components, characterised by a class $w_2 \in H^2(\Sigma, \mathbb{Z}_2) \cong \mathbb{Z}_2$, depending on whether E has a lift to a spin bundle or not. The spectral curve induced by the characteristic polynomial in (2.18) is a reducible curve: an $SO(2n+1,\mathbb{C})$-Higgs field Φ always has a zero eigenvalue, and from [34] the zero eigenspace E_0 is given by $E_0 \cong K^{-n}$.

From (2.18), the characteristic polynomial $\det(\eta - \Phi)$ defines a component of the spectral curve, which we shall denote by $\pi : S \to \Sigma$, and whose equation is $\eta^{2n} + a_1 \eta^{2n-2} + \cdots + a_{n-1}\eta^2 + a_n = 0$, where $a_i \in H^0(\Sigma, K^{2i})$. This is a $2n$-fold cover of Σ, with genus $g_S = 1 + 4n^2(g-1)$. The Hitchin fibration in this case is given by the map

$$h \;:\; \mathcal{M}_{SO(2n+1,\mathbb{C})} \longrightarrow \mathcal{A}_{SO(2n+1,\mathbb{C})} := \bigoplus_{i=1}^{n} H^0(\Sigma, K^{2i}), \qquad (2.19)$$

which sends each pair (E, Φ) to the coefficients of $\det(\eta - \Phi)$. As in the case of $Sp(2n,\mathbb{C})$, the curve S has an involution σ which acts as $\sigma(\eta) = -\eta$.

Thus, we may consider the quotient curve $\overline{S} = S/\sigma$ in the total space of K^2, for which S is a double cover $p : S \to \overline{S}$. In this case the regular fibres can be described as follows:

The regular fibres $h^{-1}(a)$ of the $SO(2n+1,\mathbb{C})$ Hitchin fibration are given by Prym varieties $Prym(S,\bar{S})$ together with a trivialization of each $M \in Prym(S,\bar{S})$ over the zeros of a_n defining S as in (2.18).

Following [34], the symmetric bilinear form (v,w) canonically defines a skew form $(\Phi v, w)$ on E/E_0 with values in K. Moreover, choosing a square root $K^{1/2}$ one can define

$$V = E/E_0 \otimes K^{-1/2},$$

on which the corresponding skew form is non-degenerate. The Higgs field Φ induces a transformation Φ' on V which has characteristic polynomial

$$\det(x - \Phi') = x^{2n} + a_1 x^{2n-2} + \cdots + a_{n-1}x^2 + a_n.$$

Note that this is exactly the case of $Sp(2n,\mathbb{C})$ described in Section 2.2.3, and thus we may describe the above with a choice of a line bundle M_0 in the Prym variety $\mathrm{Prym}(S,\overline{S})$. In particular, S corresponds to the smooth spectral curve of an $Sp(2n,\mathbb{C})$-Higgs bundle.

When reconstructing the vector bundle E with an $SO(2n+1,\mathbb{C})$ structure from an $Sp(2n,\mathbb{C})$-Higgs bundle (V,Φ') as in [34], there is a mod 2 invariant associated to each zero of the coefficient a_n of the characteristic polynomial $\det(\eta - \Phi')$. This data comes from choosing a trivialisation of $M \in \mathrm{Prym}(S,\overline{S})$ over the zeros of a_n, and defines a covering P' of the Prym variety $\mathrm{Prym}(S,\overline{S})$. The covering has two components corresponding to the spin and non-spin lifts of the vector bundle. The identity component of P', which corresponds to the spin case, is isomorphic to the dual of the symplectic Prym variety, and this is the generic fibre of the $SO(2n+1,\mathbb{C})$ Hitchin map - the reader should refer to Hitchin's work [32] and [33] for a thorough explanation of how the above description is obtained.

2.2.5. $SO(2n,\mathbb{C})$-*Higgs bundles*

Lastly, we consider $G_c = SO(2n,\mathbb{C})$. As in previous cases, the distinct eigenvalues of a matrix $A \in \mathfrak{so}(2n,\mathbb{C})$ occur in pairs $\pm\lambda_i$, and thus the characteristic polynomial of A is of the form $\det(x - A) = x^{2n} + a_1 x^{2n-2} + \cdots + a_{n-1}x^2 + a_n$. In this case the coefficient a_n is the square of a polynomial

p_n, the Pfaffian, of degree n. A basis for the invariant polynomials on the Lie algebra $\mathfrak{so}(2n, \mathbb{C})$ is $a_1, a_2, \ldots, a_{n-1}, p_n$, (the reader should refer, for example, to [5] and references therein for further details).

Definition 2.26: An $SO(2n, \mathbb{C})$-Higgs bundle is a pair (E, Φ), for E a holomorphic vector bundle of rank $2n$ with a non-degenerate symmetric bilinear form $(\ ,\ )$, and the Higgs field $\Phi \in H^0(\Sigma, \mathrm{End}_0(E) \otimes K)$ satisfying $(\Phi v, w) = -(v, \Phi w)$.

Considering the characteristic polynomial $\det(\eta - \Phi)$ of a Higgs bundle (E, Φ) one obtains a $2n$-fold cover $\pi : S \to \Sigma$ whose equation is given by

$$\det(\eta - \Phi) = \eta^{2n} + a_1 \eta^{2n-2} + \cdots + a_{n-1}\eta^2 + p_n^2,$$

for $a_i \in H^0(\Sigma, K^{2i})$ and $p_n \in H^0(\Sigma, K^n)$. Note that this curve has always singularities, which are given by $\eta = 0$. The curve S has a natural involution $\sigma(\eta) = -\eta$, whose fixed points in this case are the singularities of S. The *virtual* genus of S can be obtained via the adjunction formula, giving $g_S = 1 + 4n^2(g-1)$.

In order to define the spectral data, one may consider its non-singular model $\hat{\pi} : \hat{S} \to \Sigma$, whose genus is given by

$$\begin{aligned} g_{\hat{S}} &= g_S - \#\text{singularities} \\ &= 1 + 4n^2(g-1) - 2n(g-1) \\ &= 1 + 2n(2n-1)(g-1). \end{aligned}$$

As the fixed points of σ are double points, the involution extends to an involution $\hat{\sigma}$ on $\hat{S}$ which does not have fixed points. Considering the associated basis of invariant polynomials for each Higgs field Φ, one may define the Hitchin fibration

$$h \ : \ \mathcal{M}_{SO(2n,\mathbb{C})} \longrightarrow \mathcal{A}_{SO(2n,\mathbb{C})} := H^0(\Sigma, K^n) \oplus \bigoplus_{i=1}^{n-1} H^0(\Sigma, K^{2i}). \quad (2.20)$$

In this case the line bundle associated to an $SO(2n, \mathbb{C})$-Higgs bundle is defined on the desingularisation $\hat{S}$ of S:

The smooth fibres of the $SO(2n, \mathbb{C})$ Hitchin fibration are given by Prym$(\hat{S}, \hat{S}/\hat{\sigma})$, for $\hat{S}$ the desingularisation of the curve S associated to the regular base point a.

Starting with an $SO(2n, \mathbb{C})$-Higgs bundle, since $\hat{S}$ is smooth we obtain an eigenspace bundle $L \subset \ker(\eta - \Phi)$ inside the vector bundle E pulled back to $\hat{S}$. In particular, this line bundle satisfies $\hat{\sigma}^*L \cong L^* \otimes (K_{\hat{S}} \otimes \pi^*K^*)^{-1}$, thus defining a point in $\mathrm{Prym}(\hat{S}, \hat{S}/\hat{\sigma})$ given by

$$M := L \otimes (K_{\hat{S}} \otimes \pi^*K^*)^{1/2}.$$

Conversely, a Higgs bundle (E, Φ) may be recovered from a curve S with has equation $\eta^{2n} + a_1\eta^{2n-2} + \cdots + a_{n-1}\eta^2 + p_n^2 = 0$, and a line bundle L on its desingularisation $\hat{S}$. Note that given the sections

$$s = \eta^{2n} + a_1\eta^{2n-2} + \cdots + a_{n-1}\eta^2 + p_n^2$$

for fixed p_n with simple zeros, one has a linear system whose only base points are when $\eta = 0$ and $p_n = 0$. Hence, by Bertini's theorem the generic divisor of the linear system defined by the sections s has those base points as its only singularities. Moreover, as p_n is a section of K^n, in general there are $2n(g-1)$ singularities which are generically ordinary double points. A generic divisor of the above linear system defines a curve S which has an involution $\sigma(\eta) = -\eta$ whose only fixed points are the base points.

The involution σ induces an involution $\hat{\sigma}$ on the desingularisation $\hat{S}$ of S which has no fixed points, and thus we may consider the quotient $\hat{S}/\hat{\sigma}$ and the corresponding Prym variety $\mathrm{Prym}(\hat{S}, \hat{S}/\hat{\sigma})$. Following a similar procedure as before, a line bundle $M \in \mathrm{Prym}(\hat{S}, \hat{S}/\hat{\sigma})$ induces a Higgs bundle (E, Φ) where E is the direct image sheaf of $L = M \otimes (K_{\hat{S}} \otimes \pi^*K^*)^{-1/2}$. It is thus the Prym variety of $\hat{S}$ which is a generic fibre of the corresponding Hitchin fibration.

Problem 2.27: *Show that the genus $g_{\hat{S}/\hat{\sigma}}$ of $\hat{S}/\hat{\sigma}$ is $n(2n-1)(g-1)$.*

2.3. *Spectral data for complex Higgs bundles*

Considering S a spectral curve, $\hat{S}$ a normalized spectral curve, and $\overline{S}$ and $\overline{\hat{S}}$ the quotients of S and $\hat{S}$ by the involution $\eta \mapsto -\eta$. Moreover, let D be a sub-divisor of $[a_n]$. Then, the spectral data described in this lecture can be summarised as follows:

Table 1. Spectral data for complex Higgs bundles.

Group	Spectral curve	Generic fibre	Ref.
$GL(n,\mathbb{C})$	$\eta^n + a_1\eta^{n-1} + \cdots + a_{n-1}\eta + a_n$	$\mathrm{Jac}(S)$	[32], [34]
$SL(n,\mathbb{C})$	$\eta^n + a_2\eta^{n-2} + \cdots + a_{n-1}\eta + a_n$	$\mathrm{Prym}(S/\Sigma)$	[32], [34]
$Sp(n,\mathbb{C})$	$\eta^{2n} + a_1\eta^{2n-2} + \cdots + a_{n-1}\eta^2 + a_n$	$\mathrm{Prym}(S/\overline{S})$	[32], [34]
$SO(2n+1,\mathbb{C})$	$\eta^{2n} + a_1\eta^{2n-2} + \cdots + a_{n-1}\eta^2 + a_n$	$\mathrm{Prym}(S/\overline{S})$ + D	[32], [34]
$SO(2n,\mathbb{C})$	$\eta^{2n} + a_1\eta^{2n-2} + \cdots + a_{n-1}\eta^2 + p_n^2$	$\mathrm{Prym}(\hat{S}/\overline{\hat{S}})$	[32]

3. Spectral data for G-Higgs bundles

But most of all a good example is a thing of beauty. It shines and convinces. It gives insight and understanding. It provides the bedrock of belief.

Sir Michael Atiyah

Higgs bundles for real forms were first studied by N. Hitchin in [31], and the results for $SL(2,\mathbb{R})$ were generalised in [33], where Hitchin studied the case of $G = SL(n,\mathbb{R})$. Using Higgs bundles he counted the number of connected components and, in the case of split real forms, he identified a component homeomorphic to $\mathbb{R}^{\dim G(2g-2)}$ and which naturally contains a copy of a Teichmüller space. The aim of this Lecture is to introduce principal Higgs bundles for real forms and their corresponding spectral data as studied in [47] and further developed in [35, 36].

3.1. *G-Higgs bundles*

We shall begin by reviewing definitions and properties related to real forms of Lie algebras and Lie groups (see e.g., [24, 37, 41, 39, 45]), and then define G-Higgs bundles for real forms G of classical semisimple complex Lie groups G_c, or of $GL(n,\mathbb{C})$. Through the approach of [33], we describe these Higgs bundles as the fixed points of a certain involution on the moduli space of G_c-Higgs bundles. In later sections we study G-Higgs bundles for non-compact real forms G and in each case give an overview of the corresponding spectral data when available.

3.1.1. *Real forms*

Let $\mathfrak{g}_c$ be a complex Lie algebra with complex structure i, whose Lie group is G_c.

Definition 3.1: A *real form of* $\mathfrak{g}^c$ *is a real Lie algebra which satisfies* $\mathfrak{g}^c = \mathfrak{g} \oplus i\mathfrak{g}$.

Given a real form $\mathfrak{g}$ of $\mathfrak{g}^c$, an element $Z \in \mathfrak{g}^c$ in the Lie algebra may be written as $Z = X + iY$ for $X, Y \in \mathfrak{g}$. The mapping $X + iY \mapsto X - iY$ is called the *conjugation* with respect to $\mathfrak{g}$.

Remark 3.2: Any real form $\mathfrak{g}$ of $\mathfrak{g}^c$ is given by the fixed points set of an antilinear involution τ on $\mathfrak{g}^c$. In particular the conjugation with respect to $\mathfrak{g}$ satisfies these properties.

Definition 3.3: A real form of a complex Lie group G_c is an antiholomorphic Lie group automorphism $\tau : G_c \to G_c$ of order two, i.e., $\tau^2 = 1$.

Every $X \in \mathfrak{g}^c$ defines an endomorphism $\mathrm{ad}X$ of the Lie algebra $\mathfrak{g}^c$ given by $\mathrm{ad}X(Y) = [X, Y]$ for $Y \in \mathfrak{g}^c$. For Tr the trace of a vector space endomorphism, $B(X, Y) = \mathrm{Tr}(\mathrm{ad}X\mathrm{ad}Y)$ is a the bilinear form on $\mathfrak{g}^c \times \mathfrak{g}^c$ called the *Killing form* of $\mathfrak{g}^c$.

Definition 3.4: A real Lie algebra $\mathfrak{g}$ is called *compact* if the Killing form is negative definite on it. The corresponding Lie group G is a compact Lie group.

Definition 3.5: Let $\mathfrak{g}$ be a real form of a complex simple Lie algebra $\mathfrak{g}^c$, given by the fixed points of an antilinear involution τ. Then, if there is a Cartan subalgebra invariant under τ on which the Killing form is negative definite, the real form $\mathfrak{g}$ is called a *compact real form*. Such a compact real form of $\mathfrak{g}^c$ corresponds to a compact real form G of G_c; if there is an invariant Cartan subalgebra on which the Killing form is positive definite, the form is called a *split (or normal) real form*. The corresponding Lie group G is the split real form of G_c.

Any complex semisimple Lie algebra $\mathfrak{g}^c$ has a compact and a split real form which are unique up to conjugation via $\mathrm{Aut}_{\mathbb{C}}\mathfrak{g}^c$ (e.g., for $\mathfrak{sl}(\mathfrak{n}, \mathbb{C})$ these are $\mathfrak{su}(\mathfrak{n})$ and $\mathfrak{sl}(n, \mathbb{R})$ respectively).

Remark 3.6: Recall that all Cartan subalgebras $\mathfrak{h}$ of a finite dimensional Lie algebra $\mathfrak{g}$ have the same dimension. The rank of $\mathfrak{g}$ is defined to be this dimension, and a real form $\mathfrak{g}$ of a complex Lie algebra $\mathfrak{g}^c$ is split if and only if the real rank of $\mathfrak{g}$ equals the complex rank of $\mathfrak{g}^c$.

An involution θ of a real semisimple Lie algebra $\mathfrak{g}$ such that the symmetric bilinear form $B_\theta(X, Y) = -B(X, \theta Y)$ is positive definite is called a *Cartan involution*. Any real semisimple Lie algebra has a Cartan involution, and any two Cartan involutions θ_1, θ_2 of $\mathfrak{g}$ are conjugate via an automorphism of $\mathfrak{g}$, i.e., there is a map φ in $\mathrm{Aut}\mathfrak{g}$ such that $\varphi\theta_1\varphi^{-1} = \theta_2$. The decomposition of $\mathfrak{g}$ into eigenspaces of a Cartan involution θ is called the *Cartan decomposition* of $\mathfrak{g}$.

Proposition 3.7: [39] *Let* $\mathfrak{g}^c$ *be a complex semisimple Lie algebra, and* ρ *the conjugation with respect to a compact real form* $\mathfrak{u}$ *of* $\mathfrak{g}^c$*. Then,* ρ *is a Cartan involution.*

Proposition 3.8: [37] *Any non-compact real form* $\mathfrak{g}$ *of a complex simple Lie algebra* $\mathfrak{g}^c$ *can be obtained from a pair* $(\mathfrak{u}, \theta)$*, for* $\mathfrak{u}$ *the compact real form of* $\mathfrak{g}^c$ *and* θ *an involution on* $\mathfrak{u}$*.*

For completion, we shall recall here the construction of real forms from [37]. Let $\mathfrak{h}$ be the $+1$-eigenspaces of θ and $i\mathfrak{m}$ the -1-eigenspace of θ acting on $\mathfrak{u}$, thus having

$$\mathfrak{u} = \mathfrak{h} \oplus i\mathfrak{m}. \tag{3.1}$$

Since $\mathfrak{g}^c = \mathfrak{h} \oplus \mathfrak{m} \oplus i(\mathfrak{h} \oplus \mathfrak{m})$, there is a natural non-compact real form $\mathfrak{g}$ of $\mathfrak{g}^c$ given by

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}. \tag{3.2}$$

Moreover, if a linear isomorphism θ_0 induces the decomposition as in (3.2), then θ_0 is a Cartan involution of $\mathfrak{g}$ and $\mathfrak{h}$ is the maximal compact subalgebra of $\mathfrak{g}$.

Following the notation of Proposition 3.8, let ρ be the antilinear involution defining the compact form $\mathfrak{u}$ of a complex simple Lie algebra $\mathfrak{g}^c$ whose decomposition via an involution θ is given by equation (3.1). Moreover, let τ be an antilinear involution which defines the corresponding non-compact real form $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of $\mathfrak{g}^c$. Considering the action of the two antilinear involutions ρ and τ on $\mathfrak{g}^c$, we may decompose the Lie algebra $\mathfrak{g}^c$ into eigenspaces

$$\mathfrak{g}^c = \mathfrak{h}^{(+,+)} \oplus \mathfrak{m}^{(-,+)} \oplus (i\mathfrak{m})^{(+,-)} \oplus (i\mathfrak{h})^{(-,-)}, \tag{3.3}$$

where the upper index $(\cdot, \cdot)$ represents the $\pm$-eigenvalue of ρ and τ respectively. From the decomposition (3.3), the involution θ on the compact real form $\mathfrak{u}$ giving a non-compact real form $\mathfrak{g}$ of $\mathfrak{g}^c$ can be seen as acting on $\mathfrak{g}^c$ as $\sigma := \rho\tau$. Moreover, this induces an involution on the corresponding Lie group $\sigma := G_c \to G_c$.

Remark 3.9: The fixed point set $\mathfrak{g}^\sigma$ of σ is given by $\mathfrak{g}^\sigma = \mathfrak{h} \oplus i\mathfrak{h}$, and thus it is the complexification of the maximal compact subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Equivalently, the anti-invariant set under the involution σ is given by $\mathfrak{m}^{\mathbb{C}}$.

3.1.2. *G-Higgs bundles through involutions*

As mentioned previously, non-abelian Hodge theory on the compact Riemann surface Σ gives a correspondence between the moduli space of reductive representations of $\pi_1(\Sigma)$ in a complex Lie group G_c and the moduli space of G_c-Higgs bundles. The anti-holomorphic operation of conjugating by a real form τ of G_c in the moduli space of representations can be seen via this correspondence as a holomorphic involution Θ of the moduli space of G_c-Higgs bundles.

Following [32], in order to obtain a G-Higgs bundle, for A the connection which solves Hitchin equations, one requires the flat $GL(n,\mathbb{C})$ connection

$$\nabla = \nabla_A + \Phi + \Phi^* \tag{3.4}$$

to have holonomy in a non-compact real form G of $GL(n,\mathbb{C})$, whose real structure is τ and Lie algebra is $\mathfrak{g}$. More generally, for a complex Lie group G_c with non-compact real form G and real structure τ, one requires

$$\nabla = \nabla_A + \Phi - \rho(\Phi) \tag{3.5}$$

to have holonomy in G, where ρ is the compact real structure of G_c. Since A has holonomy in the compact real form of G_c, we have $\rho(\nabla_A) = \nabla_A$. Hence, requiring $\nabla = \tau(\nabla)$ is equivalent to requiring $\nabla_A = \tau(\nabla_A)$ and $\Phi - \rho(\Phi) = \tau(\Phi - \rho(\Phi))$. In terms of $\sigma = \rho\tau$, these two equalities are given by $\sigma(\nabla_A) = \nabla_A$ and $\Phi - \rho(\Phi) = \tau(\Phi - \rho(\Phi)) = \tau(\Phi) - \sigma(\Phi) = \sigma(\rho(\Phi) - \Phi)$. Hence, ∇ has holonomy in the real form G if ∇_A is invariant under σ, and Φ anti-invariant. In terms of a G_c-Higgs bundle (P, Φ), one has that for $\mathcal{U}$ and $\mathcal{V}$ two trivialising open sets in the compact Riemann surface Σ, the involution σ induces an action on the transition functions $g_{uv} : \mathcal{U} \cap \mathcal{V} \to G_c$ given by $g_{uv} \mapsto \sigma(g_{uv})$, and on the Higgs field by sending $\Phi \mapsto -\sigma(\Phi)$.

Concretely, for G a real form of a complex semisimple lie group G_c, we may construct G-Higgs bundles as follows. For H the maximal compact subgroup of G, we have seen that the Cartan decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, for $\mathfrak{h}$ the Lie algebra of H, and $\mathfrak{m}$ its orthogonal complement. This induces the following decomposition of the Lie algebra $\mathfrak{g}^c$ of G_c in terms of the eigenspaces of the corresponding involution σ as defined before: $\mathfrak{g}^c = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$. Note that the Lie algebras satisfy $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Hence there is an induced isotropy representation given by $\mathrm{Ad}|_{H^{\mathbb{C}}} : H^{\mathbb{C}} \to GL(\mathfrak{m}^{\mathbb{C}})$. Then, Definition 2.12 generalises to the following (see e.g. [27]):

Definition 3.10: A principal *G-Higgs bundle* is a pair (P, Φ) where P is a holomorphic principal $H^{\mathbb{C}}$-bundle on Σ, and Φ is a holomorphic section of $P \times_{Ad} \mathfrak{m}^{\mathbb{C}} \otimes K$.

Example 3.11: For a compact real form G, one has $G = H$ and $\mathfrak{m} = \{0\}$, and thus σ is the identity and the Higgs field must vanish: a G-Higgs bundle becomes a principal G_c-bundle.

In terms of involutions, following [33] and recalling the previous analysis leading to Remark 3.9, we have the following:

Proposition 3.12: *Let G be a real form of a complex semi-simple Lie group G_c, whose real structure is τ. Then,* G-Higgs bundles *are given by the fixed points in $\mathcal{M}_{G_c}$ of the involution Θ_G acting by*

$$\Theta_G : \ (P, \Phi) \mapsto (\sigma(P), -\sigma(\Phi)),$$

where $\sigma = \rho\tau$, for ρ the compact real form of G_c.

Similarly to the case of G_c-Higgs bundles, there is a notion of stability, semi stability and polystability for G-Higgs bundles. Following [14] and [15], one can see that the polystability of a G-Higgs bundle for $G \subset GL(n, \mathbb{C})$ is equivalent to the polystability of the corresponding $GL(n, \mathbb{C})$-Higgs bundle. However, a G-Higgs bundle can be stable as a G-Higgs bundle but not as a $GL(n, \mathbb{C})$-Higgs bundle. We shall denote by $\mathcal{M}_G$ the moduli space of polystable G-Higgs bundles on Σ.

Problem 3.13: *(*) Considering the notion of "strong real form" from* [2], *describe the corresponding Higgs bundles and give a definition of Θ_G for which one does not have the problem described in the above paragraph. The reader might find useful the notes in* [1, 2] *for a concise definition.*

One should note that a fixed point of Θ_G in $\mathcal{M}_{G^c}$ gives a representation of $\pi_1(\Sigma)$ into the real form G up to the equivalence of conjugation by the normalizer of G in G^c. This may be bigger than G itself, and thus two distinct classes in $\mathcal{M}_G$ could be isomorphic in $\mathcal{M}_{G^c}$ via a complex map. Hence, although there is a map from $\mathcal{M}_G$ to the fixed point subvarieties in $\mathcal{M}_{G^c}$, this might not be an embedding. The reader may refer to [25] for the Hitchin-Kobayashi type correspondence for real forms.

Remark 3.14: A description of the above phenomena in the case of rank 2 Higgs bundles is given in [46], where one can see how the $SL(2, \mathbb{R})$-Higgs bundles which have different topological invariants lie in the same connected component as $SL(2, \mathbb{C})$-Higgs bundles.

Remark 3.15: As mentioned previously, the study of real Higgs bundles as fixed point sets of involutions was initiated by Hitchin in [33] in the case of split real forms, and developed for other real forms in [47]. Moreover, this approach has been taken in several papers recently (see, among others, [7, 8, 10, 13, 18, 49]), and continues to be used (see, among others, [9]).

Remark 3.16: The point of view of Proposition 3.12, which is considered throughout [47], fits into a more global picture where Θ_G is one of three natural involutions acting on the moduli space of Higgs bundles [7, 8], giving three families of (B, A, A), (A, B, A) and (A, A, B) branes in $\mathcal{M}_{G_c}$ as the fixed point sets. One should note that the fixed point sets of these involutions are of great importance when studying the relation of Langlands duality with Higgs bundles, as initiated in [30, 38] and [34].

3.2. *Spectral data for G-Higgs bundles*

As mentioned in the first Lecture, the moduli spaces $\mathcal{M}_{G_c}$ have a natural symplectic structure, which we denoted by ω. Moreover, following [32], the involutions Θ_G send $\omega \mapsto -\omega$. Thus, at a smooth point, the fixed point set must be Lagrangian and so the expected dimension of $\mathcal{M}_G$ is half the dimension of $\mathcal{M}_{G_c}$. In order to describe the spectral data for real G-Higgs bundles, one considers the moduli space $\mathcal{M}_G$ sitting inside $\mathcal{M}_{G_c}$ as fixed points of Θ_G in the Hitchin base $\mathcal{A}_{G_c}$ and the corresponding preserved fibres.

By considering Cartan's classification of classical semisimple Lie algebras, we shall now describe G-Higgs bundles and their spectral data for non-compact real forms of a classical semisimple complex Lie algebra $\mathfrak{g}^c$. For I_n the unit matrix of order n, we denote by $I_{p,q}$, J_n and $K_{p,q}$ the matrices

$$I_{p,q} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}, \quad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad K_{p,q} = \begin{pmatrix} -I_p & 0 & 0 & 0 \\ 0 & I_q & 0 & 0 \\ 0 & 0 & -I_p & 0 \\ 0 & 0 & 0 & I_q \end{pmatrix}.$$

Following Proposition 3.8, we study each complex Lie algebra $\mathfrak{g}_c$ and compact form $\mathfrak{u}$ with different involutions θ which give decompositions $\mathfrak{u} = \mathfrak{h} \oplus i\mathfrak{m}$. Then the corresponding natural non-compact real form is $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, and to make sense of Proposition 3.12 we consider the following Lie algebras, Lie groups, real forms, and holomorphic and anti holomorphic involutions:

Table 2. Compact forms $\mathfrak{u}$ of classical Lie algebras.

$\mathfrak{g}_c$	Lie group G_c	Split form	Compact form $\mathfrak{u}$	ρ fixing $\mathfrak{u}$
$\mathfrak{a}_n$	$SL(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{su}(n)$	$\rho(X)=-\overline{X}^t$
$\mathfrak{b}_n$	$SO(2n+1,\mathbb{C})$	$\mathfrak{so}(n,n+1)$	$\mathfrak{so}(2n+1)$	$\rho(X)=\overline{X}$
$\mathfrak{c}_n$	$Sp(2n,\mathbb{C})$	$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{sp}(n)$	$\rho(X)=J_n\overline{X}J_n^{-1}$
$\mathfrak{d}_n$	$SO(2n,\mathbb{C})$	$\mathfrak{so}(n,n)$	$\mathfrak{so}(2n)$	$\rho(X)=\overline{X}$

Table 3. Non-compact forms G of classical Lie algebras G_c.

$\mathfrak{g}_c$	Real form G	τ fixing G	Involution θ on $\mathfrak{u}$
$\mathfrak{a}_n$	$SL(n,\mathbb{R})$	$\rho(X)=-\overline{X}^t$	$\theta(X)=\overline{X}$.
	$SU^*(2m)$	$\tau(X)=J_m\overline{X}J_m^{-1}$	$\theta(X)=J_m\overline{X}J_m^{-1}$
	$SU(p,q)$	$\tau(X)=-I_{p,q}\overline{X}^tI_{p,q}$	$\theta(X)=I_{p,q}XI_{p,q}$
$\mathfrak{b}_n$	$SO(p,q)$	$\tau(X)=I_{p,q}\overline{X}I_{p,q}$.	$\theta(X)=I_{p,q}XI_{p,q}$
$\mathfrak{c}_n$	$Sp(2n,\mathbb{R})$	$\tau(X)=\overline{X}$	$\theta(X)=\overline{X}$
	$Sp(2p,2q)$	$\tau(X)=-K_{p,q}X^*K_{p,q}$.	$\theta(X)=K_{p,q}XK_{p,q}$,
$\mathfrak{d}_n$	$SO(p,q)$	$\tau(X)=I_{p,q}\overline{X}I_{p,q}$.	$\theta(X)=I_{p,q}XI_{p,q}$
	$SO^*(2m)$	$\tau(X)=J_m\overline{X}J_m^{-1}$.	$\theta(X)=J_m\overline{X}J_m^{-1}$

In the case of split real forms, following the methods of [33] one obtains a description of real Higgs bundles which we shall use in subsequent sections:

Theorem 3.17: [47] *For G the split real form of G_c, the fixed points of Θ_G in the smooth fibres of the Hitchin fibration for G_c-Higgs bundles are given by points of order two.*

3.2.1. $SL(n,\mathbb{R})$-*Higgs bundles*

Higgs bundles for $SL(n,\mathbb{R})$ were first considered in [33], where Hitchin studied a copy of Teichmüller space inside the moduli space of Higgs bundles for split real forms. Following Definition 3.10, an $SL(n,\mathbb{R})$-Higgs bundle is a pair (E,Φ) where E is a rank n orthogonal vector bundle and the Higgs field $\Phi: E\to E\otimes K$ is a symmetric and traceless holomorphic map.

Proposition 3.18: *$SL(n,\mathbb{R})$-Higgs bundles are given by the fixed points of*

$$\Theta_{SL(n,\mathbb{R})}:(E,\Phi)\mapsto(E^*,\Phi^t)$$

in $\mathcal{M}_{SL(n,\mathbb{C})}$ corresponding to automorphisms $f: E\to E^*$ giving a symmetric form on E.

Problem 3.19: *Find the decomposition of $\mathfrak{u}=\mathfrak{su}(n)$ induced by the corresponding θ in Table 5, and use this to deduce Proposition 3.18.*

Recalling that the trace is invariant under transposition, one has that the ring of invariant polynomials of $\mathfrak{g}^c = \mathfrak{sl}(n, \mathbb{C})$ is acted on trivially by the involution $-\sigma$, and thus the Hitchin base is preserved by $\Theta_{SL(n,\mathbb{R})}$. In order to find the spectral data for $SL(n, \mathbb{R})$-Higgs bundles, following Theorem 3.17 we look at elements of order two in the fibres of the Hitchin fibration for $SL(n, \mathbb{C})$-Higgs bundles:

Over a smooth point in the Hitchin base $\mathcal{A}_{SL(n,\mathbb{C})}$, Higgs bundles for $SL(n, \mathbb{R})$ correspond to line bundles $L \in Prym(S, \Sigma)$ such that $L^2 \cong \mathcal{O}$.

In the case of $n = 2$, the $SL(2, \mathbb{C})$-spectral curve S given as in (2.7) has a natural involution $\sigma : \eta \mapsto -\eta$ and $\mathrm{Prym}(S, \Sigma) = \{L \in \mathrm{Jac}(S) : \sigma^* L \cong L^*\}$. Hence, points in the smooth fibres corresponding to $SL(2, \mathbb{R})$-Higgs bundles are given by line bundles $L \in \mathrm{Jac}(S)$ such that $\sigma^* L \cong L$.

Problem 3.20: *Let $L \in Prym(S, \Sigma)$ be a line bundle of order two. Then, its direct image is a rank 2 bundle on Σ which decomposes into the sum of two line bundles $V \oplus V^*$. How can the Lefschetz fixed point formula (which relates the action of an involution on a line bundle, and the dimension of the spaces of invariant and anti-invariant sections of a line bundle) from* [6] *be used to relate the degree of V and the action of σ on L in the spirit of* [48]*?*

The topological invariant associated to $SL(n, \mathbb{R})$-Higgs bundles is the characteristic class $\omega_2 \in \mathbb{Z}_2$ which is the obstruction to lifting the orthogonal bundle to a spin bundle, and its study was carried through in [36].

Problem 3.21: *For $n = 2$, use the approach of* [36] *to relate ω_2 to the invariants in Problem 3.20.*

The spectral data of $SL(n, \mathbb{R})$-Higgs bundles gives a finite cover of the smooth locus of the Hitchin fibration. For $n = 2$, an explicit description of the monodromy action whose orbits are the connected components of $\mathcal{M}_{SL(2,\mathbb{R})}$ is given in [46].

Problem 3.22: *(*) How can the methods in* [46] *be extended to study monodromy for $SL(n, \mathbb{R})$-Higgs bundles for $n \geq 3$?*

3.2.2. *$SU^*(2m)$-Higgs bundles*

The group $SU^*(2m)$ is the subgroup of $SL(2m, \mathbb{C})$ which commutes with an antilinear automorphism J of $\mathbb{C}^{2m}$ such that $J^2 = -1$. At the level of the Lie algebras we have that the involution θ decomposes $\mathfrak{u} = \mathfrak{h} \oplus i\mathfrak{m}$ where $\mathfrak{h} = \mathfrak{sp}(m)$. The induced non-compact real form $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is

$$\mathfrak{g} = \mathfrak{su}^*(2m) = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\overline{Z}_2 & \overline{Z}_1 \end{pmatrix} \middle| \begin{array}{c} Z_1, Z_2 \ m \times m \quad \text{complex matrices,} \\ \mathrm{Tr} Z_1 + \mathrm{Tr}\overline{Z}_1 = 0 \end{array} \right\}.$$

Definition 3.23: An $SU^*(2m)$ Higgs bundle on Σ is a pair (E, Φ) for E a rank $2m$ vector bundle with a symplectic form ω, and the Higgs field $\Phi \in H^0(\Sigma, \mathrm{End}(E) \otimes K)$ traceless and symmetric with respect to ω.

These Higgs bundles are the first example considered in this notes for which one has nonabelian spectral data. It was first studied in [35], providing what one may call the nonabelianization of the Hitching fibration. In what follows we shall describe the nonabelian spectral data, and also do so for other cases which behave similarly, which have been studied in [35, 47].

Proposition 3.24: *Isomorphism classes of $SU^*(2m)$-Higgs bundles are given by fixed points of the involution*

$$\Theta_{SU^*} : (E, \Phi) \mapsto (E^*, \Phi^t)$$

on $SL(2m, \mathbb{C})$-Higgs bundles corresponding to pairs which have an automorphism $f : E \to E^*$ endowing it with a symplectic structure, and which trivialises its determinant bundle.

As the trace is invariant under conjugation and transposition, one has that the involution $-\sigma(X) = J_m X^t J_m^{-1}$ acts trivially on the ring of invariant polynomials of $\mathfrak{sl}(2m, \mathbb{C})$, and thus preserves the Hitchin base. The spectral data associated to $SU^*(2m)$-Higgs bundles (E, Φ) was studied in [35], and we shall describe here its main features.

The characteristic polynomial of an $SU^*(2m)$-Higgs bundle (E, Φ) can be seen to be the square of a Pfaffian, $\det(\eta - \Phi) = p(\eta)^2$ and thus all fixed points of Θ_{SU^*} lie over singular points of the $SL(2m, \mathbb{C})$ Hitchin fibration. With a slight abuse of notation, we denote by S the spectral curve in the total space of K defined by

$$p(\eta) = \eta^m + a_2 \eta^{m-2} + \cdots + a_m = 0$$

where the coefficients $a_i \in H^0(\Sigma, K^i)$. It is a ramified m-fold cover of Σ whose ramification points are the zeros of a_m. As in the case of complex groups, we interpret $p(\eta) = 0$ as the vanishing of a section of $\pi^* K^m$ over the total space of the canonical bundle $\pi : K \to \Sigma$, where η is the tautological section of $\pi^* K$, and Bertini's theorem assures us that for generic a_i the curve is nonsingular.

Problem 3.25: *What is the genus g_S of S?*

On the spectral curve S, the cokernel of $(\eta - \Phi)$ is a rank two holomorphic vector bundle V on S. Then, following [11] (and using $p(\Phi) = 0$ instead of the Cayley-Hamilton theorem), we can identify E with the direct image $\pi_* V$ and Φ as the direct image of $\eta : V \to V \otimes \pi^* K$. From [35] one has a description of the spectral data:

The fixed point set of $\Theta_{SU^*(2m)}$ in a smooth fibre of the $SL(2m, \mathbb{C})$-Hitchin fibration is the moduli space of semi-stable rank 2 vector bundles on S with fixed determinant $\pi^* K^{m-1}$.

Problem 3.26: *Use Remark 2.21 together with Grothendieck-Riemann-Roch to show that semi-stability of V implies semi-stability of (E, Φ).*

Problem 3.27: *Follow the approach of $SL(n, \mathbb{C})$-Higgs bundles to note that by fixing the determinant of V one obtains a trivialization of the determinant of $\pi_* V$ on Σ.*

3.2.3. *$SU(p, q)$-Higgs bundles*

Definition 3.28: An $SU(p, q)$-Higgs bundle over Σ is a pair (E, Φ) where $E = W_1 \oplus W_2$ for W_1, W_2 vector bundles over Σ of rank p and q such that $\Lambda^p W_1 \cong \Lambda^q W_2^*$, and the Higgs field Φ is given by $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$, for $\beta : W_2 \to W_1 \otimes K$ and $\gamma : W_1 \to W_2 \otimes K$.

Problem 3.29: *Find the decomposition $\mathfrak{u} = \mathfrak{h} \oplus i\mathfrak{m}$ via the action of θ in Table 2 and deduce that $\theta\rho$ is the anti-holomorphic involution fixing the non-compact real form $\mathfrak{u}(p, q)$.*

Proposition 3.30: *$SU(p, q)$-Higgs bundles are fixed points of $\Theta_{SU(p,q)}$: $(E, \Phi) \mapsto (E, -\Phi)$ on $SL(p + q, \mathbb{C})$-Higgs bundles corresponding to bundles E which have an automorphism conjugate to $I_{p,q}$ sending Φ to $-\Phi$, and whose ± 1 eigenspaces have dimensions p and q.*

The involution $-\sigma$ acts trivially on the polynomials of even degree. Whilst the spectral data is not known for $p \neq q$, in the case of $p = q$ it has been described in [47] and [48] by looking at $U(p, p)$-Higgs bundles $(W_1 \oplus W_2, \Phi)$, which when satisfying $\Lambda^p W_1 \cong \Lambda^q W_2^*$ correspond to $SU(p, p)$-Higgs bundles. In this case, the characteristic polynomial defines a spectral curve $\pi : S \to \Sigma$ through the equation $\det(\eta - \Phi) = \eta^{2p} + a_2 \eta^{2p-2} +$

$\cdots + a_{2p-2}\eta^2 + a_{2p} = 0$, where η is the tautological section of π^*K and $a_i \in H^0(\Sigma, K^i)$. This is a $2p$-fold cover of Σ, ramified over the $4p(g-1)$ zeros of a_{2p}, and has a natural involution $\eta \mapsto -\eta$ which has as fixed points the ramification points of the cover, and which by abuse of notation, we shall call σ.

The involution σ plays an important role when constructing the spectral data as described in [48]. A line bundle L on S which defines a classical Higgs bundle induces a $U(p,p)$-Higgs bundle if and only if $\sigma^*L \cong L$. In this case, at a fixed point $x \in S$ of the involution, there is a linear action of σ on the fibre L_x given by scalar multiplication of ± 1. This description of the spectral data can be then seen in terms of Jacobians through [48]:

The fixed point set of $\Theta_{U(p,p)}$ in a smooth fibre of the classical Hitchin fibration can be seen in terms of pull backs of $Jac(S/\sigma)$ on a symmetric product of Σ to a point of the Hitching base.

As described in [48], the topological invariants associated to a $U(p,p)$-Higgs bundle $(W_1 \oplus W_2, \Phi)$ are the degrees $\deg(W_1)$ and $\deg(W_2)$, and can be seen in terms of the degree of the line bundle L on S and the number of ramification points of S over which the linear action of σ on the fibre of L is -1.

Problem 3.31: *Use the Lefschetz fixed point formula in* [6] *to see that the parity of the degree of L and the number of points over which σ acts as -1 needs to be the same.*

Problem 3.32: *Following* [14], *a $U(p,p)$-Higgs bundles has an associated invariant $\tau(deg(W_1), deg(W_2)) := deg(W_1) - deg(W_2)$, known as the Toledo invariant. Use Problem 3.31 to express the invariant in terms of fixed points of σ and obtain natural bounds.*

In the case of $SU(p,p)$-Higgs bundles, for maximal Toledo invariant (i.e., when the invariant in the above problem achieves the bounds), the fixed point set of $\Theta_{SU(p,p)}$ in a smooth fibre of the $SL(2p,\mathbb{C})$-Hitchin fibration is given by a covering of $\mathrm{Prym}(S/\sigma, \Sigma)$, the Prym variety of the quotient curve S/σ. For $SU(p,p+1)$-Higgs bundles, the methods and arguments of [20] can be adapted and used to obtain the spectral data as seen in [42].

Problem 3.33: *(*) How can the methods from* [48] *together with the approach of* [53] *be used to obtain the spectral data for $SU(p,1)$-Higgs bundles?*

Remark 3.34: The moduli space of real Higgs bundles is a brane in the moduli space of complex Higgs bundles, and as such it has a dual space, a brane in the moduli space of complex Higgs bundles for the Langlands dual group. Properties of this dual space have been conjectured in [8], and it is interesting to note that the spectral data for $SU(p,p)$-Higgs bundles from [48] is used to conjecture a dual space to $\mathcal{M}_{SU(p,p)}$ through Langlands duality in [36].

3.2.4. *SO(p, q)-Higgs bundles*

In this case, if $p+q$ is even, $\mathfrak{g}$ is a split real form if and only if $p = q$; if $p+q$ is odd, $\mathfrak{g}$ is a split real form if and only if $p = q+1$. Whilst we shall give some details on the construction of $SO(p,q)$-Higgs bundles, for a more detailed description of the approach needed to understand groups with signature the reader should refer to the following section on $Sp(2p, 2q)$-Higgs bundles.

The vector space V associated to the standard representation of $\mathfrak{h}^{\mathbb{C}}$ can be decomposed into $V = V_p \oplus V_q$, for V_p and V_q complex vector spaces of dimension p and q respectively, with orthogonal structures. The maximal compact subalgebra of $\mathfrak{so}(p,q)$ is $\mathfrak{h} = \mathfrak{so}(p) \times \mathfrak{so}(q)$ and the Cartan decomposition of $\mathfrak{so}(p+q,\mathbb{C})$ is given by $(\mathfrak{so}(p,\mathbb{C}) \oplus \mathfrak{so}(q,\mathbb{C})) \oplus \mathfrak{m}^{\mathbb{C}}$, for

$$\mathfrak{m} = \left\{ \left(\begin{matrix} 0 & X_2 \\ X_2^t & 0 \end{matrix} \right) \middle| X_2 \ \text{ real } p \times q \ \text{ matrix} \right\}.$$

Definition 3.35: An $SO(p,q)$ Higgs bundle is a pair (E, Φ) where the vector bundle is $E = V_p \oplus V_q$ for V_p and V_q complex vector spaces of dimension p and q respectively, with orthogonal structures, and the Higgs field is a section in $H^0(\Sigma, (\mathrm{Hom}(V_q, V_p) \oplus \mathrm{Hom}(V_p, V_q)) \otimes K)$ given by

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \quad \text{for} \ \ \gamma \equiv -\beta^{\mathrm{T}}, \ \text{and} \ \beta^{\mathrm{T}} \text{the orthogonal transpose of } \beta.$$

Proposition 3.36: *$SO(p,q)$ Higgs bundles are fixed points of*

$$\Theta_{SO(p,q)} : \ (E, \Phi) \mapsto (E, -\Phi)$$

on the moduli space of $SO(p+q,\mathbb{C})$ corresponding to vector bundles E which have an automorphism f conjugate to $I_{p,q}$ sending Φ to $-\Phi$ and whose ± 1 eigenspaces have dimensions p and q.

Problem 3.37: *The involution θ from the Table 2 decomposes $\mathfrak{u} = \mathfrak{h} \oplus i\mathfrak{m}$. Give an explicit description of $\mathfrak{m}$ and $\mathfrak{h}$ and of the real form $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.*

Since the ring of invariant polynomials of $\mathfrak{g}^c = \mathfrak{so}(2m+1, \mathbb{C})$ is generated by $\mathrm{Tr}(X^i)$ for $X \in \mathfrak{g}^c$, for $p+q = 2m+1$ one has that the induced action of the involution $\Theta_{SO(p,q)}$ is trivial on the ring of invariant polynomials of the Lie algebra $\mathfrak{so}(2m+1, \mathbb{C})$, i.e., when p and q have different parity.

Problem 3.38: *In the case of $\mathfrak{so}(2m, \mathbb{C})$, for $2m = p+q$, the ring of invariant polynomials is generated by $Tr(X^i)$ for $X \in \mathfrak{g}^c$ and $i < 2m$, together with the Pfaffian p_m, which is of degree m. Under which conditions on p and q is the induced action of $\Theta_{SO(p,q)}$ trivial on the ring of invariant polynomials?*

The spectral data for $SO(p,q)$-Higgs bundles when $p = q$ or $p = q+1$ can be seen through Theorem 3.17 from [47] *as points of order two in the smooth fibres of the $SO(p+q, \mathbb{C})$-Hitchin fibration.*

In both cases a key ingredient is the double cover $p : S \to S/\sigma$ given by the spectral curve (the desingularised curve in the case of $SO(2n, \mathbb{C})$) over the quotient curve, which through K-theoretic methods allows one to express the topological invariants involved in terms of the action of σ [9].

3.2.5. *$SO^*(2m)$-Higgs bundles*

The action of θ in Table 2 decomposes the compact form $\mathfrak{u} = \mathfrak{h} \oplus i\mathfrak{m}$ for $\mathfrak{h} = \mathfrak{u}(m) \cong \mathfrak{so}(2m) \cap \mathfrak{sp}(m)$, and

$$i\mathfrak{m} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \middle| X_1,\ X_2 \in \mathfrak{so}(m) \right\}, \tag{3.6}$$

and the induced non-compact real form $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is

$$\mathfrak{g} = \mathfrak{so}^*(2m) = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\overline{Z}_2 & \overline{Z}_1 \end{pmatrix} \middle| \begin{matrix} Z_1, Z_2 \ m \times m \quad \text{complex matrices} \\ Z_1 \ \text{ skew symmetric, } Z_2 \text{ Hermitian} \end{matrix} \right\}.$$

The vector space associated to the standard representation of $\mathfrak{h}^{\mathbb{C}}$ has an orthogonal and symplectic structure J. Since $J^{-1} = J^t$ and $J^2 = -1$, the vector space may be expressed in terms of the $\pm i$ eigenspaces of J as $V \oplus V^*$, for V a rank m vector space. Thus, we have the following definition:

Definition 3.39: An $SO^*(2m)$-Higgs bundle is given by a pair (E, Φ) where $E = V \oplus V^*$ for V a rank m holomorphic vector bundle, and where the Higgs field Φ is given by

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \quad \text{for} \quad \begin{cases} \gamma : \ V \to V^* \otimes K \ \text{ satisfying } \gamma = -\gamma^t \\ \beta : \ V^* \to V \otimes K \ \text{ satisfying } \beta = -\beta^t \end{cases}.$$

In terms of involutions, these Higgs bundles may be seen as follows:

Proposition 3.40: *$SO^*(2m)$-Higgs bundles are fixed points of the involution*

$$\Theta_{SO^*(2m)} : \ (E, \Phi) \mapsto (E, -\Phi)$$

on the moduli space of $SO(2m,\mathbb{C})$-Higgs bundles corresponding to vector bundles E which have an orthogonal automorphism f conjugate to J_m, sending Φ to $-\Phi$ and which squares to -1, equipping E with a symplectic structure.

As in the previous case, the involution induced action of $\Theta_{SO^*(2m)}$ is trivial on the ring of invariant polynomials of $\mathfrak{g}^c$. The spectral data for these Higgs bundles is studied in [35], and we shall give a short description bellow.

In order to understand the associated spectral data, one notes that $SO^*(2m)$-Higgs bundles (E,Φ) may be regarded as $SU^*(2m)$-Higgs bundle with extra conditions. Hence, one may define a natural m cover of the Riemann surface $\pi : S \to \Sigma$ by taking

$$\sqrt{\mathrm{char}(\Phi)} = \eta^m + a_2\eta^{2m-2} + \cdots + a_m,$$

and a rank 2 vector bundle V on S whose direct image on Σ gives E. Since in this case the equation of the spectral curve only has even coefficients, there is a natural involution $\sigma : \eta \to -\eta$ and one may consider the induced action of σ on V and on its determinant bundle. In particular, from [35] the vector bundle V gives an $SO^*(2m)$-Higgs bundle if and only if it is preserved by the involution and the induced action on it satisfies some conditions:

The fixed point set of $\Theta_{SO^*(2m)}$ in a smooth fibre of the $SO(2m,\mathbb{C})$-Hitchin fibration is given by the moduli space of semi-stable rank 2 vector bundles V on S with fixed determinant π^*K^{2m-1}, whose induced action by σ on the determinant bundle is trivial.

Problem 3.41: *The relative duality theorem gives*

$$(\pi_*(V))^* \cong \pi_*(V^* \otimes K_S) \otimes K^*.$$

Use this to see that in order to have $E \cong E^*$ through a skew form, the action of σ needs to be trivial on the determinant bundle of V for $\pi_*V = E$.

Problem 3.42: *(*) Describe how the vector bundles of rank 2 in* [35] *appear in the description of the connected components of $\mathcal{M}_{SO^*(2m)}$ in* [17].

3.2.6. *$Sp(2n,\mathbb{R})$-Higgs bundles*

In this section and the one which follows we consider the non-compact real forms of the complex Lie group $Sp(2n,\mathbb{C})$. For this, recall that the symplectic Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$ is given by the set of $2n\times 2n$ complex matrices X that satisfy $J_nX+X^tJ_n=0$ or equivalently, $X=-J_n^{-1}X^tJ_n$.

Let $\mathfrak{u}$ be the compact real form $\mathfrak{u}=\mathfrak{sp}(n)$ and $\theta(X)=\overline{X}=J_n\overline{X}J_n^{-1}$. The Lie algebra $\mathfrak{sp}(n)$ is given by the quaternionic skew-Hermitian matrices; that is, the set of $n\times n$ quaternionic matrices X which satisfy $X=-\overline{X}^t$. The compact form is $\mathfrak{u}=\mathfrak{h}\oplus i\mathfrak{m}$, for $\mathfrak{h}=\mathfrak{u}(n)\cong\mathfrak{so}(2n)\cap\mathfrak{sp}(n)$, which leads to the split real form $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ defined by

$$\mathfrak{g}=\mathfrak{sp}(2n,\mathbb{R})=\left\{\begin{pmatrix}X_1 & X_2\\ X_3 & -X_1^t\end{pmatrix}\;\middle|\;\begin{matrix}X_i\ \text{ real } n\times n \quad \text{matrices}\\ X_2,X_3\ \text{ symmetric}\end{matrix}\right\}.$$

Definition 3.43: An $Sp(2n,\mathbb{R})$-Higgs bundle is given by a pair (E,Φ) where $E=V\oplus V^*$ for V a rank n holomorphic vector bundle, and for Φ the Higgs field given by

$$\Phi=\begin{pmatrix}0 & \beta\\ \gamma & 0\end{pmatrix}\quad \text{for}\quad \begin{cases}\gamma:\ V\to V^*\otimes K & \text{satisfying } \gamma=\gamma^t\\ \beta:\ V^*\to V\otimes K & \text{satisfying } \beta=\beta^t\end{cases}.$$

Proposition 3.44: *$Sp(2n,\mathbb{C})$ Higgs bundles, and $Sp(2n,\mathbb{R})$-Higgs bundles are given by the fixed points of*

$$\Theta_{Sp(2n,\mathbb{R})}:\ (E,\Phi)\mapsto(E,-\Phi)$$

on $Sp(2n,\mathbb{C})$-Higgs bundles corresponding to vector bundles E which have a symplectic isomorphism sending Φ to $-\Phi$, and whose square is the identity, endowing E with an orthogonal structure.

The invariant polynomials of $\mathfrak{g}^c$ are of even degree, and hence the involution $-\sigma$ acts trivially on them, making $\Theta_{Sp(2n,\mathbb{R})}$ preserve the whole Hitchin base $\mathcal{A}_{Sp(2n,\mathbb{C})}$. In the case of rank 4 Higgs bundles, the spectral data was first consider in P. Gothen's thesis [27, 28], and through Theorem 3.17 and the spectral data for complex Higgs bundles one has the following:

The fixed points of $\Theta_{Sp(2n,\mathbb{R})}$ in the smooth fibres of the $Sp(2n,\mathbb{C})$-Hitchin fibration are given by line bundles $L\in Prym(S,S/\sigma)$ such that $L^2\cong\mathcal{O}$.

In particular, since S is a ramified double cover of S/σ, one has that $L \in \mathrm{Prym}(S, S/\sigma)$ if and only if $\sigma^* L \cong L^*$. Hence, by considering points of order two one has that $\sigma^* L \cong L$ and thus there is a natural induced action of σ on the line bundle L. The topological invariants associated to these Higgs bundles were studied in [36] through the natural action of σ.

Problem 3.45: *Compare the calculations in* [27] *which lead to Milnor-Wood type inequalities for* $Sp(2n, \mathbb{R})$*-Higgs bundles, with the inequalities one obtains by using the involution* σ *as in* [36].

Problem 3.46: *(*) Express the invariants from* [27] *in terms of different choices of the natural involution* σ *on* S *as well as in terms of the action of a second natural involution appearing in some situations on* S/σ.

3.2.7. $Sp(2p, 2q)$*-Higgs bundles*

The induced non-compact real form $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is

$$\mathfrak{sp}(2p, 2q) = \left\{ \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ \overline{Z}_{12}^t & Z_{22} & Z_{14}^t & Z_{24} \\ -\overline{Z}_{13} & \overline{Z}_{14} & \overline{Z}_{11} & -\overline{Z}_{12} \\ \overline{Z}_{14}^t & -\overline{Z}_{24} & -Z_{12}^t & \overline{Z}_{22} \end{pmatrix} \middle| \begin{array}{c} Z_{i,j} \quad \text{complex matrices,} \\ Z_{11},\ Z_{13} \ \text{ order } \ p, \\ Z_{12},\ Z_{14}\ p \times q \ \text{ matrices,} \\ Z_{11},\ Z_{22} \ \text{ skew Hermitian,} \\ Z_{13},\ Z_{24} \ \text{ symmetric.} \end{array} \right\}.$$

Problem 3.47: *Show that* $\mathfrak{m}^{\mathbb{C}}$ *can be expressed as subset of certain off-diagonal matrices.*

Definition 3.48: An $Sp(2p, 2q)$-Higgs bundle is given by a pair (E, Φ) for $E = V_{2p} \oplus V_{2q}$ is a direct sum of symplectic vector spaces of rank $2p$ and $2q$, and

$$\Phi = \begin{pmatrix} 0 & -\gamma^{\mathrm{T}} \\ \gamma & 0 \end{pmatrix} \quad \text{for} \quad \left\{ \begin{array}{l} \gamma \quad : V_{2p} \to V_{2q} \otimes K \\ -\gamma^{\mathrm{T}} : V_{2q} \to V_{2p} \otimes K \end{array} \right. ,$$

for γ^{T} the symplectic transpose of γ.

Proposition 3.49: $Sp(2p, 2q)$*-Higgs bundles are the fixed points of*

$$\Theta_{Sp(2p,2q)} : \ (E, \Phi) \mapsto (E, -\Phi^T)$$

on the moduli space of $Sp(2p + 2q, \mathbb{C})$*-Higgs bundles corresponding to symplectic vector bundles* E *which have an endomorphism* $f : E \to E$ *conjugate to* $\tilde{K}_{p,q}$*, sending* Φ *to the symplectic transpose* $-\Phi^T$*, and whose* ± 1 *eigenspaces are of dimension* $2p$ *and* $2q$.

As the trace is invariant under conjugation and transposition, the induced action of $\Theta_{Sp(2p,2q)}$ is trivial on the ring of invariant polynomials of $\mathfrak{g}^c = \mathfrak{sp}(2(p+q), \mathbb{C})$. In the case of $p = q$, one can see that $Sp(2p, 2p)$-Higgs bundles are a particular case of $SU^*(2p)$-Higgs bundles, and thus one needs to understand which extra conditions to the spectral data for $SU^*(2p)$-Higgs bundles needs to be added in order to have the Higgs bundles for the symplectic real form.

From the previous section, when $p = q$ the corresponding spectral curve is a $2p$-fold cover of the Riemann surface Σ whose equation is given by the square root of the characteristic polynomial of the Higgs field. Moreover, it has a natural involution σ whose action determines the associated spectral data. More precisely, the following is shown in [47] and [35]:

The fixed point set of $\Theta_{Sp(2p,2p)}$ in a smooth fibre of the $Sp(4p,\mathbb{C})$-Hitchin fibration is given by the moduli space of semi-stable rank 2 vector bundles V on S with fixed determinant π^*K^{2p-1}, whose induced action by σ on $\Lambda^2 V$ is -1.

Since the action on $\Lambda^2 V$ is -1, the involution $\sigma : S \to S$ acts with different eigenvalues ± 1 on the fibres of V over the ramification points of S, and thus through [3], the spectral data relates to the moduli space of admissible parabolic rank 2 bundles on S/σ as seen in [47].

Problem 3.50: *(*) Nonabelianization can also be seen through Cameral covers* [42]*. Realise the action of σ in terms of Cameral covers.*

4. Spectral data for real Higgs bundles

From the above sections, we have seen that spectral data can be been defined for G-Higgs bundles, and this has been done for several groups G. A summary of the state of the art in this direction is given as follows (to the best of the author's knowledge), where the notation is as in the previous sections (for the precise objects in each case, the reader should refer to the previous sections)[a]:

[a]For the groups missing, no spectral data has yet been defined.

Table 4. Spectral data for real Higgs bundles.

Group	Spectral curve	generic fibre	Ref.
$GL(n,\mathbb{R})$	$\eta^n + a_1\eta^{n-1} + \cdots + a_{n-1}\eta + a_n$	Jac(S)[2]	[47]
$U(p,p)$	$\eta^{2p} + a_1\eta^{2p-2} + \cdots + a_{p-1}\eta^2 + a_p$	Jac($\overline{S}$) + D + m	[47, 48]
$SL(n,\mathbb{R})$	$\eta^n + a_2\eta^{n-2} + \cdots + a_{n-1}\eta + a_n$	Prym(S,Σ)[2]	[47, 46]
$SU^*(2m)$	$\eta^{2p} + a_1\eta^{2p-2} + \cdots + a_{p-1}\eta^2 + a_p$	Subspace of $\mathcal{N}_2$(S)	[35]
$SU(p,p)$	$\eta^{2p} + a_1\eta^{2p-2} + \cdots + a_{p-1}\eta^2 + a_p$	Prym($S,\overline{S}$) + $D + m$	[47, 48]
$SU(p,p+1)$	$\eta^{2p} + a_1\eta^{2p-2} + \cdots + a_{p-1}\eta^2 + a_p$	Jac($\overline{S}$) + f	[43, 47]
$SO(p,p+1)$	$\eta^{2p} + a_1\eta^{2p-2} + \cdots + a_{p-1}\eta^2 + a_p$	Prym($S,\overline{S}$)[2] +D	[47, 50, 9]
$Sp(2p,\mathbb{R})$	$\eta^{2p} + a_1\eta^{2p-2} + \cdots + a_{p-1}\eta^2 + a_p$	Prym($S,\overline{S}$)[2]	[36, 47]
$Sp(2p,2p)$	$\eta^{2p} + a_1\eta^{2p-2} + \cdots + a_{p-1}\eta^2 + a_p$	Subspace of $\mathcal{N}_2$(S)	[35]
$Sp(2p,2p+2q)$	$\eta^{2q}(\eta^{4p} + a_1\eta^{4p-2} + \cdots + a_{2p-1}\eta^2 + a_{2p})$	Abelian & non-abelian	[9]
$SO(p,p)$	$\eta^{2p} + a_1\eta^{2p-2} + \cdots + a_{p-1}\eta^2 + a_p^2$	Prym($\hat{S},\overline{\hat{S}}$)[2]	[47]
$SO(p,p+q)$	$\eta^q(\eta^{2p} + a_1\eta^{2p-2} + \cdots + a_{p-1}\eta^2 + a_p)$	Abelian & non-abelian	[9]
$SO^*(2p)$	$\eta^{2p} + a_1\eta^{2p-2} + \cdots + a_{p-1}\eta^2 + a_p$	Subspace of $\mathcal{N}_2$(S)	[35, 47]

In the above table, we have considered the following notation, following the previous sections:

- S stands for a spectral curve and $\hat{S}$ for the normalized curve;
- $\overline{S}$ and $\overline{\hat{S}}$ denote the quotients of S and $\hat{S}$ by the involution $\eta \mapsto -\eta$;
- D denotes a positive divisor and f an extension class;
- $\mathcal{N}_2(S)$ is the moduli space of semi-stable rank 2 vector bundles on the spectral curve S.

Although Morse theoretic approaches (following [31], for a partial list, see [14, 15, 17] and references therein) are usually considered to study connectivity of the moduli space of G-Higgs bundles, spectral data may also be used to calculate the number of connected components of the moduli spaces of G-Higgs bundles. This approach was taken for the following groups[b]:

[b]In the table we give references for work done through spectral data, and in Remark 4 we mention the original sources of those results, when done previously with other methods.

Table 5. Connectivity for real Higgs bundles.

Group	restriction	components	Ref.
$PGL_i(n,\mathbb{R})$	$n=2$ and $i=0$	$2^{2g}+g-1$	[10]
	$n=2$ and $i=1$	$2^{2g}+g-2$	[10]
$PSL_i(n,\mathbb{R})$	$n=2$ and $i=0$	$2g-1$	[10]
	$n=2$ and $i=1$	$2g-2$	[10]
$GL(n,\mathbb{R})$	$n=2$	$3\cdot 2^{2g}+g-3$	[10]
$SL(n,\mathbb{R})$	$n=2$	$2\cdot 2^{2g}+2g-3$	[10, 46, 47]
$SU^*(2m)$	-	1	[35]
$SU(p,q)$	$p=q$, maximal, over generic loci	2^{2g}	[46, 47]
$Sp(2n,\mathbb{R})$	$n=2$, maximal	$3\cdot 2^{2g}+2g-4$	[10]
$Sp(2p,2q)$	$p=q$	1	[35, 47]
$SO(p,p)$	$p=2$	$3\cdot 6^{2g}+4g^2-6g-3$	[10]
$SO^*(2m)$	-	1	[35]

Some of the above connectivity results have been obtained before with other methods, some of which do not require the restrictions in the table:

- The number of components for $Sp(4,\mathbb{R})$ was obtained originally by Gothen in [27].
- The number of components $2.2^{2g}+2g-3$ for $SL(2,\mathbb{R})$ and $4g-3$ for $PSL(2,\mathbb{R})$ were shown by Goldman in [26]. Xia [54, 55] showed that the number of components of the space of homomorphisms $Hom(\pi_1(\Sigma), PSL(2,\mathbb{R}))$ is $2.2^{2g}+4g-5$. This number is different to the number $2.2^{2g}+2g-3$ of components of $PGL(2,\mathbb{R})$ because upon taking the quotient of the conjugation action of $PGL(2,\mathbb{R})$, certain pairs of components are identified.
- Connectivity for $SU(p,p)$-Higgs bundles was first studied in [14], and for $SU^*(2m)$ in [16] and for $SO^*(2m)$ in [17].

The remaining results mentioned in the above table have not, as far as we are aware, been obtained elsewhere.

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