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Spectral Data for $U(m, m)$-Higgs Bundles

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We define and study spectral data associated to $U(m, m)$-Higgs bundles through the $\mathrm{GL}(2m, \mathbb{C})$ Hitchin fibration. We give a new interpretation of the topological invariants involved as well as a geometric description of the moduli space.

1 Introduction

Examples of G-Higgs bundles on a Riemann surface Σ of genus $g \geq 2$, for G a real form of a complex Lie group G_c, were initially considered in [7, 9]. Whilst the particular case of G-Higgs bundles for the unitary group with signature has received much attention in the past decade (e.g., see [5] and references therein), the work found in the literature is mostly on the counting of connected components via Morse theory, where the critical points lie in the most singular fiber of the corresponding Hitchin fibration. In contrast, this paper is dedicated to the study of the geometry of the moduli space of $U(m, m)$-Higgs bundles via the most generic fibers of the $\mathrm{GL}(2m, \mathbb{C})$ Hitchin fibration.

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Definition 1.1. A $U(m,m)$-Higgs bundle over Σ is a pair (V,Φ), where $V = W_1 \oplus W_2$ for W_i rank m vector bundles over Σ, and the Higgs field Φ is given by

$$\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}, \tag{1}$$

where $\beta : W_2 \to W_1 \otimes K$ and $\gamma : W_1 \to W_2 \otimes K$ are holomorphic sections for K the canonical bundle of Σ. □

We define the spectral data for $U(m,m)$-Higgs bundles in Section 2, and in Section 3 we describe the topological invariants (Proposition 3.1) and the Milnor–Wood type inequalities they satisfy, leading to our main result:

Theorem 3.2. There is a 1-to-1 correspondence between isomorphism classes of $U(m,m)$-Higgs bundles $(W_1 \oplus W_2, \Phi)$ for which char(Φ) defines a smooth curve and $\deg W_1 > \deg W_2$, and triples $(\bar{S}, E_+, D)$ where

(1) $\bar{\pi} : \bar{S} \to \Sigma$ is a smooth m-fold cover of Σ in the total space of K^2 defined by

$$\xi^m + a_1 \xi^{m-1} + \cdots + a_{m-1}\xi + a_m = 0,$$

for $a_i \in H^0(\Sigma, K^{2i})$ and ξ the tautological section of the pullback of K^2;

(2) E_+ is a line bundle on $\bar{S}$ whose degree is $\deg E_+ = \deg W_1 + (2m^2 - 2m)(g-1)$;

(3) D is a positive subdivisor of the divisor of a_m of degree $M = \deg W_2 - \deg W_1 + 2m(g-1)$. □

Interchanging W_1 and W_2, the above correspondence holds for any $U(m,m)$-Higgs bundle for which $\deg W_1 \neq \deg W_2$, and in the case of $W_1 = W_2$, we show that the construction fails to be 1-to-1. For each choice of invariants $\deg E_+$, M, in Section 4 we give a geometric description of a distinguished component of the moduli space $\mathcal{M}_{U(m,m)}$ of $U(m,m)$-Higgs bundles that intersects a generic fiber of the classical Hitchin fibration (Theorem 4.2), studying all the components which are known to exist (see [5, p. 116]).

Finally, we specialize the above results to study the stable locus of the moduli space of $\mathrm{SU}(m,m)$-Higgs bundles as sitting inside the smooth fibers of the $\mathrm{SL}(2m,\mathbb{C})$ Hitchin fibration in Section 5.

2 Spectral Data for $U(m, m)$-Higgs Bundles

As shown in [8], there is a natural fibration of the moduli space $\mathcal{M}$ of classical Higgs bundles of rank $2m$ and fixed degree, the so-called Hitchin fibration, which maps isomorphism classes of Higgs bundles to the coefficients of the characteristic polynomial of the Higgs field

$$h : \mathcal{M} \to \mathcal{A} := \bigoplus_{i=1}^{2m} H^0(\Sigma, K^i). \tag{2}$$

The regular fiber of the Hitchin fibration is isomorphic to the Jacobian of a curve S in the total space of K defined by the characteristic polynomial of Φ. In particular, given a line bundle E on $\pi : S \to \Sigma$, one can obtain a classical Higgs pair (V, Φ) by taking the direct image $V := \pi_* E$, and letting the Higgs field Φ be the map obtained through the direct image of the tautological section $E \xrightarrow{\eta} E \otimes \pi^* K$.

Remark 2.1. From Definition 1.1, it follows that $U(m, m)$-Higgs bundles can be seen as fixed points in $\mathcal{M}$ of the involution $\Theta : (V, \Phi) \mapsto (V, -\Phi)$ on $\mathcal{M}$ corresponding to vector bundles V, which have an automorphism sending Φ to $-\Phi$ conjugate to

$$I_{m,m} = \begin{pmatrix} -I_m & 0 \\ 0 & I_m \end{pmatrix}.$$ □

The characteristic polynomial of a $U(m, m)$-Higgs field Φ, which from the above remark is invariant under $\Phi \mapsto -\Phi$, defines the spectral curve $\pi : S \to \Sigma$ in the total space of K with equation

$$\eta^{2m} + a_1 \eta^{2m-2} + \cdots + a_{m-1} \eta^2 + a_m = 0, \tag{3}$$

for $a_i \in H^0(\Sigma, K^{2i})$. Throughout the paper, we shall assume the curve S to be smooth.

The $2m$-fold cover S has a natural involution $\sigma : \eta \mapsto -\eta$, and so we may consider the quotient curve $\bar{\pi} : \bar{S} = S/\sigma \to \Sigma$ in the total space of K^2. This is an m-fold cover of Σ which is smooth since S is assumed to be smooth, and has equation

$$\xi^m + a_1 \xi^{m-1} + \cdots + a_{m-1} \xi + a_m = 0, \tag{4}$$

for $\xi = \eta^2$ the tautological section of $\bar{\pi}^* K^2$. Let g_S and $g_{\bar{S}}$ be the genus of S and $\bar{S}$, respectively, and K_S and $K_{\bar{S}}$ their canonical bundles. By the adjunction formula, we have $K_S \cong \pi^* K^{2m}$ and $K_{\bar{S}} \cong \bar{\pi}^* K^{2m} \otimes \bar{\pi}^* K^{-1}$, hence $g_S = 4m^2(g-1) + 1$, and $g_{\bar{S}} = (2m^2 - m)(g-1) + 1$.

Under certain conditions, the line bundle $E \in \mathrm{Jac}(S)$ associated to a $\mathrm{GL}(2m, \mathbb{C})$-Higgs bundle defines a $U(m, m)$-Higgs bundle:

Proposition 2.2. Let S be a smooth curve in the total space of K as in (3) with a natural involution $\sigma : \eta \mapsto -\eta$, and E a line bundle on it. Then, E defines a $U(m, m)$-Higgs bundle if and only if $\sigma^* E \cong E$. □

Proof. For S, E as above, there are two possible lifts of the action of σ to E which differ by ± 1. By abuse of notation, we shall denote the choice of lifted action on E also by σ. On an invariant open set $\pi^{-1}(\mathcal{U}) \subset S$, we may decompose the sections of E into the invariant and anti-invariant parts, labeled by the upper indices $\pm$ as follows:

$$H^0(\pi^{-1}(\mathcal{U}), E) = H^0(\pi^{-1}(\mathcal{U}), E)^+ \oplus H^0(\pi^{-1}(\mathcal{U}), E)^-.$$

From the definition of direct image, there is an equivalent decomposition of the cohomology $H^0(\mathcal{U}, \pi_* E)$ into $H^0(\mathcal{U}, \pi_* E) = H^0(\mathcal{U}, \pi_* E)^+ \oplus H^0(\mathcal{U}, \pi_* E)^-$. Hence $\pi_* E = W_+ \oplus W_-$, where $W_\pm$ are vector bundles on Σ. At a point x such that $a_m(x) \neq 0$, the involution σ has no fixed points on $\pi^{-1}(x)$. Moreover, if x is not a branch point, $\pi^{-1}(x)$ consists of $2m$ points $e_1, \ldots, e_m, \sigma e_1, \ldots \sigma e_m$. The fiber of the direct image is then isomorphic to $\mathbb{C}^m \oplus \mathbb{C}^m$, and the involution is $\sigma : (v, w) \mapsto (w, v)$. Then, the fiber of W_+ is given by the invariant points (v, v), and the one of W_- is given by the anti-invariant points $(v, -v)$, and so $\mathrm{rk} W_+ = \mathrm{rk} W_- = m$.

The Higgs field associated to E is defined as in the case of classical Higgs bundles through the multiplication map $E \xrightarrow{\eta} E \otimes \pi^* K$. Since $\sigma(\eta) = -\eta$, the Higgs field Φ maps $W_+ \mapsto W_- \otimes K$ and $W_- \mapsto W_+ \otimes K$ and thus it may be written as in (1).

Conversely, as a classical Higgs bundle, a $U(m, m)$-Higgs bundle (V, Φ) with smooth spectral curve S as in (3) has associated a line bundle E on S defined as the cokernel of $\eta I - \pi^* \Phi$ in $\pi^* V \otimes \pi^* K$. The involution σ transforms the line bundle E for eigenvalue η to $\sigma^* E$ for eigenvalue $-\eta$. Furthermore, as noted before the isomorphism classes of $U(m, m)$-Higgs bundles are fixed by $\Theta : (V, \Phi) \mapsto (V, -\Phi)$ and thus $\sigma^* E \cong E$. ■

Following [5], we say that a $U(m, m)$-Higgs bundle (V, Φ) as in Definition 1.1 is stable if for every subbundle $V' = W_1' \oplus W_2'$ with $W_1' \subset W_1$ and $W_2' \subset W_2$, satisfying $\Phi(V') \subset V' \otimes K$, one has that $\deg V'/\mathrm{rk} V' < \deg V/\mathrm{rk} V$. Since in Proposition 2.2, we have considered an irreducible curve S, there is no proper subbundle of V which is preserved by Φ, and hence the $U(m, m)$-Higgs pair constructed through E is stable.

By means of the two-fold cover $\rho: S \to \bar{S}$, the line bundle E associated to a $U(m, m)$-Higgs bundle $(W_1 \oplus W_2, \Phi)$ can be seen in terms of line bundles on $\bar{S}$. The invariant and anti-invariant sections of E give a decomposition $H^0(\rho^{-1}(\mathcal{V}), E) = H^0(\rho^{-1}(\mathcal{V}), E)^+ \oplus H^0(\rho^{-1}(\mathcal{V}), E)^-$, for $\mathcal{V} \subset \bar{S}$ an open set. Then, by definition of direct image, there are two line bundles E_+ and E_- on the quotient curve $\bar{S}$ such that $H^0(\rho^{-1}(\mathcal{V}), E)^\pm \cong H^0(\mathcal{V}, E_\pm)$. Moreover, by similar arguments as above, one has $\rho_* E = E_+ \oplus E_-$ on $\bar{S}$ for which $\bar{\pi}_* E_+ = W_1$ and $\bar{\pi}_* E_- = W_2$.

Remark 2.3. Considering the maps $\beta\gamma$ and $\gamma\beta$, one obtains K^2-twisted Higgs bundles whose associated spectral curve is $\bar{S}$ (e.g., see [4]), with corresponding line bundles $E_\pm$ on $\bar{S}$. □

3 The Associated Invariant

The topological invariants associated to $U(m, m)$-Higgs bundles as in Definition 1.1 are the degrees $\deg W_1$ and $\deg W_2$. These invariants arise in a natural way in terms of the spectral data from the isomorphism $\sigma^* E \cong E$ of the line bundle obtained through Proposition 2.2 and the quotient curve $\bar{S}$. At a fixed point $a \in S$ of the involution, there is a linear action of σ on the fiber E_a given by scalar multiplication of ± 1. We shall denote by M the number of points at which the action of σ is -1 on E.

Proposition 3.1. The topological invariants of a $U(m, m)$-Higgs bundle $(W_1 \oplus W_2, \Phi)$ with corresponding line bundle E can be expressed in terms of the spectral data as follows:

$$\deg W_1 = \frac{\deg E - M}{2} + (2m - 2m^2)(g - 1), \tag{5}$$

$$\deg W_2 = \frac{\deg E + M}{2} - 2m^2(g - 1). \tag{6}$$

□

Proof. From Proposition 2.2, the involution σ preserves the line bundle E and over its $4m(g - 1)$ fixed points, which are the zeros of a_m in (3), it acts as ± 1. Furthermore, over the fixed points σ acts as $+1$ on $\rho^* L$, for any line bundle L on $\bar{S}$. Hence, the involution acts as -1 on $E \otimes \rho^* L$ over exactly M points. Choosing L of sufficiently large degree such that $H^1(S, E \otimes \rho^* L)$ vanishes, by Riemann–Roch and Serre duality, we have that

$$\dim H^0(S, E \otimes \rho^* L)^+ + \dim H^0(S, E \otimes \rho^* L)^- = \deg E + \deg L - 4m^2(g - 1), \tag{7}$$

where, as before the $\pm$ upper script labels the ± 1 eigenspaces of σ in $H^0(S, E \otimes \rho^* L)$. From [1, Theorem 4.12], the holomorphic Lefschetz theorem gives us

$$\dim H^0(S, E \otimes \rho^* L)^+ - \dim H^0(S, E \otimes \rho^* L)^- = \frac{(-M) + (4m(g-1) - M)}{2}. \tag{8}$$

Hence, from (7) we have

$$\dim H^0(S, E \otimes \rho^* L)^+ = \frac{\deg E + \deg L - M}{2} + (m - 2m^2)(g-1),$$
$$\dim H^0(S, E \otimes \rho^* L)^- = \frac{\deg E + \deg L + M}{2} - (m + 2m^2)(g-1).$$

Finally, recall that $\rho_* E = E_+ \oplus E_-$ satisfying $\bar{\pi}_* E_+ = W_1$ and $\bar{\pi}_* E_- = W_2$. Since by construction $\dim H^0(S, E \otimes \rho^* L)^{\pm} = \dim H^0(\bar{S}, E_{\pm} \otimes L)$, applying Riemann–Roch and Serre duality, we obtain (5) and (6) as required, and also note that $M = \deg W_2 - \deg W_1 + 2m(g-1)$. ■

Remark 3.2. Throughout this section, we have assumed that W_1 and W_2 are obtained via the invariant and anti-invariant sections of E, respectively. Interchanging the role of W_1 and W_2 corresponds to considering the involution $-\sigma$ acting on S, and the associated invariants $\deg E$ and $\bar{M} = 4m(g-1) - M$. □

For $m > 1$, the degrees of the bundles E_+ and E_- are given by

$$\deg E_+ = \deg W_1 + (2m^2 - 2m)(g-1) = \frac{\deg E}{2} - \frac{M}{2}, \tag{9}$$

$$\deg E_- = \deg W_2 + (2m^2 - 2m)(g-1) = \frac{\deg E}{2} + \frac{M}{2} - 2m(g-1). \tag{10}$$

Remark 3.3. The parity of the $\deg E$ and M need to be the same. □

Given a stable $U(m, m)$-Higgs field $\Phi = (\beta, \gamma)$ as in Definition 1.1, if $\deg W_1 \geq \deg W_2$ (else one can interchange W_1 and W_2), then $\gamma \neq 0$ because otherwise the invariant subbundle W_1 would contradict stability. Moreover, the section γ induces a nonzero map $c \in H^0(\bar{S}, E_+^* \otimes E_- \otimes \bar{\pi}^* K)$ obtained through the multiplication map $H^0(S, E)^+ \xrightarrow{\eta} H^0(S, E \otimes \pi^* K)^-$. Hence, since c is odd, it must vanish at the fixed points over which σ acts as -1 on E.

Since $\bar{\pi}$ is a degree m cover, from (9) and (10) we have that $\deg(E_+^* \otimes E_- \otimes \bar{\pi}^*K) = M$. Therefore, the section c vanishes only at the M points over which σ acts as -1 on E, and defines a positive subdivisor D of the divisor of $a_m = \det(\beta\gamma)$ giving the whole set of fixed points. For $[D]$, the line bundle on $\bar{S}$ associated to the divisor D, it follows that

$$[D] = E_+^* \otimes E_- \otimes \bar{\pi}^*K. \tag{11}$$

Remark 3.4. For $m = 1$, the surface Σ and the curve $\bar{S}$ coincide, and the spectral data are studied in [7]. □

From the study of the spectral data for $U(m, m)$-Higgs bundles, we have the following theorem.

Theorem 3.5. There is a 1-to-1 correspondence between isomorphism classes of $U(m, m)$-Higgs bundles $(W_1 \oplus W_2, \Phi)$ on Σ with nonsingular spectral curve for which $\deg W_1 > \deg W_2$, and triples $(\bar{S}, E_+, D)$ up to isomorphism, where

(1) $\bar{\pi} : \bar{S} \to \Sigma$ is a smooth m-fold cover of Σ in the total space of K^2 defined by

$$\xi^m + a_1\xi^{m-1} + \cdots + a_{m-1}\xi + a_m = 0, \tag{12}$$

for $a_i \in H^0(\Sigma, K^{2i})$ and ξ the tautological section of the pullback of K^2;

(3) E_+ is a line bundle on $\bar{S}$ whose degree is $\deg E_+ = \deg W_1 + (2m^2 - 2m)(g-1)$;

(4) D is a positive subdivisor of the divisor of a_m of degree $M = \deg W_2 - \deg W_1 + 2m(g-1)$. □

Proof. Starting with a $U(m, m)$-Higgs bundle, consider the line bundle E as in Proposition 2.2, and let D be the positive subdivisor of the divisor of a_m over which σ acts as -1 on E as in Proposition 3.1. As seen previously, the direct image ρ_*E on $\bar{S} = S/\sigma$ decomposes into $\rho_*E = E_+ \oplus E_-$, for $E_\pm$ line bundles on $\bar{S}$ satisfying $[D] = E_+^* \otimes E_- \otimes \bar{\rho}^*K$. Note that from Equations (9) and (10) since $\deg W_1 > \deg W_2$ one has that $\deg E_+ > \deg E_-$. Hence, considering the line bundle E_+ of biggest degree, and identifying $\mathrm{Pic}^{\deg E_+}(\bar{S})$ with $\mathrm{Jac}(\bar{S})$, one can construct the triple $(\bar{S}, E_+, D)$.

Conversely, given a triple $(\bar{S}, E_+, D)$, since the curve $\bar{S}$ is smooth, the section a_m has simple zeros [4], and we may write $[a_m] = D + \bar{D}$, for $\bar{D}$ a positive divisor on Σ. Furthermore, following (11) we define the line bundle E_- on $\bar{S}$ by $E_- = [D] \otimes E_+ \otimes \bar{\pi}^*K^*$.

One should note that in the case of $\deg W_1 = \deg W_2$ the number of points in D and $\bar{D}$ is the same, and thus the line bundle E_- could be constructed through either of the two divisors, making the correspondence fail to be 1-to-1 in this case.

On the curve $\bar{S}$, we may consider the sections associated to the divisors D and $\bar{D}$, which induce the natural maps $\bar{\beta} : E_- \to [\bar{D}] \otimes E_- = E_+ \otimes \bar{\pi}^* K$ and $\bar{\gamma} : E_+ \to [D] \otimes E_+ = E_- \otimes \bar{\pi}^* K$. Then, there is a natural rank 2 Higgs bundle $(E_+ \oplus E_-, \bar{\Phi})$ whose Higgs field $\bar{\Phi}$ has off diagonal entries $\bar{\beta}$ and $\bar{\gamma}$. Moreover, $\bar{\beta}\bar{\gamma}$ is given by a_m up to scalar multiplication, and the spectral curve of this $\bar{\pi}^* K$-twisted Higgs bundle $\bar{\Phi}$ is a curve S whose quotient under $\sigma : \eta \mapsto -\eta$ gives $\bar{S}$.

Following the methods for classical Higgs bundles, $(E_+ \oplus E_-, \bar{\Phi})$ has an associated line bundle E on S which is preserved by the involution σ on S (e.g., [4]) and such that $\rho_* E = E_+ \oplus E_-$ via the invariant and anti-invariant sections and by similar arguments as the ones leading to (11), the involution σ acts as -1 on E over the divisor D, hence proving the theorem. ■

Remark 3.6. By interchanging the involutions σ and $-\sigma$, one can show that an equivalent correspondence exists in the case of $U(m, m)$-Higgs bundles for which $\deg W_1 < \deg W_2$. □

Following [5], the Toledo invariant $\tau(\deg W_1, \deg W_2)$ associated to $U(m, m)$-Higgs bundles is defined as $\tau(\deg W_1, \deg W_2) := \deg W_1 - \deg W_2$, and so from the previous calculations, this invariant may be expressed as $\tau(\deg W_1, \deg W_2) = -M + 2m(g-1)$. By definition, M satisfies $0 \le M \le 4m(g-1)$, and thus

$$0 \le |\tau(\deg W_1, \deg W_2)| \le 2m(g-1), \tag{13}$$

which agrees with the bounds given in [5] for the Toledo invariant.

Remark 3.7. The maximal Toledo invariant corresponds to $M = 0$, and thus, from Section 3 in this case, the nonzero map $c \in H^0(\bar{S}, E_+^* \otimes E_- \otimes \bar{\pi}^* K)$ associated to a $U(m, m)$-Higgs bundle is an isomorphism. Then, the section γ is nowhere vanishing, giving an isomorphism $W_1 \cong W_2 \otimes K$. For L_0 a choice of square root of K one can construct a Cayley pair $(\tilde{W}_2, \tilde{\theta})$, a K^2-twisted Higgs bundle on Σ naturally associated to the $U(m, m)$-Higgs bundle [5]. This is done by considering $\tilde{\theta} = [(\gamma \otimes I_K) \circ \beta] \otimes I_{L_0} \ : \ W_2 \otimes L_0 \to W_2 \otimes L_0 \otimes K^2$, for I_K and I_{L_0}, the identities on K and L_0. Moreover, the spectral curve of the Cayley pair is $\bar{S}$, the corresponding line bundle is $E_- \otimes \bar{\pi}^* L_0$, and Theorem 3.5 provides a realization of the Cayley correspondence. □

Remark 3.8. The methods developed here to identify the topological invariants associated to $U(m,m)$-Higgs bundles in terms of an action on fixed points can be extended to the study of Higgs bundles for other real forms, and we do this in [11, 13]. □

4 The Moduli Space of $U(m,m)$-Higgs Bundles

Through dimensional arguments, one can show that the space of isomorphism classes of $U(m,m)$-Higgs bundles satisfying Theorem 3.5 is included in the moduli of stable $U(m,m)$-Higgs bundles $\mathcal{M}_{U(m,m)}$ as a Zariski open set. We shall denote by $\mathcal{A}' \subset \mathcal{A}$ the points in the Hitchin base of the classical Hitchin fibration over which $\mathcal{M}_{U(m,m)}$ lies, this is,

$$\mathcal{A}' = \bigoplus_{i=1}^{m} H^0(\Sigma, K^{2i}). \tag{14}$$

Proposition 4.1. The dimension of the parameter space of the triples $(\bar{S}, E_+, D)$ associated to $U(m,m)$-Higgs bundles as in Theorem 3.5 is, as expected, $4m^2(g-1)+1$. □

Proof. Since by Bertini's theorem a generic point in $\mathcal{A}'$ gives a smooth curve $\bar{S}$, the parameter space of $\bar{S}$ has dimension $\dim \mathcal{A}' = (2m^2+m)(g-1)$. The choice of the divisor D gives a partition of the zeros of a_m, and from Theorem 3.5, the choice of the line bundle E_+ on $\bar{S}$ is given by an element in $\mathrm{Jac}(\bar{S})$, which has dimension $g_{\bar{S}} = 1 + (2m^2-m)(g-1)$. Therefore, the parameter space of the triples $(\bar{S}, E_+, D)$ has dimension $(2m^2+m)(g-1)+1+(2m^2-m)(g-1) = 4m^2(g-1)+1$. Moreover, we have that $\dim \mathcal{M}_{\mathrm{GL}(2m,\mathbb{C})} = \dim \mathrm{GL}(2m,\mathbb{C})(g-1) = 8m^2(g-1)+2$, and the expected dimension of the moduli space of $U(m,m)$-Higgs bundles is $4m^2(g-1)+1$.

Finally, since the discriminant locus in the base $\mathcal{A}'$ which guarantees a smooth spectral curve is given by algebraic equations, the space of isomorphism classes of $U(m,m)$-Higgs bundles satisfying Theorem 3.5 is included in the moduli space $\mathcal{M}_{U(m,m)}$ as a Zariski open set. ■

From the construction of the spectral data in Theorem 3.5, we have the following theorem.

Theorem 4.2. Each pair of invariants $\deg E$, M as in Proposition 3.1 labels exactly one connected component of the moduli space of $U(m,m)$-Higgs bundles which intersects the nonsingular fibers of the Hitchin fibration $\mathcal{M}_{\mathrm{GL}(2m,\mathbb{C})} \to \mathcal{A}_{\mathrm{GL}(2m,\mathbb{C})}$. This component is given by a fibration of Jacobians over the total space of a vector bundle on the symmetric product $S^M\Sigma$. □

Proof. In order to describe the connected component of $\mathcal{M}_{U(m,m)}$ which intersects the regular fibers of the classical Hitchin fibration for fixed invariants $\deg E$ and M, we shall analyze the parameter space for the choices of $(\bar{S}, D, E_+)$. From its construction, the choice of the curve $\bar{S}$ is given by the space $\mathcal{A}'$ as in (14). Then, since the degree $\deg E_+$ is fixed, the choice of E_+ is given by a fibration $\mathcal{J}$ whose fibers are the Jacobians $\mathrm{Jac}(S_a)$ of the smooth curves S_a defined by each $a \in \mathcal{A}'$. We shall see now that the choice of the divisor D of degree M is equivalent to replacing $H^0(\Sigma, K^{2m})$ in (14) by a vector bundle over the symmetric product $S^M\Sigma$.

Let a_m be a fixed differential corresponding to some $a = (1_1, \dots, a_m) \in \mathcal{A}'$ for $a_i \in H^0(\Sigma, K^{2i})$. In order to construct the divisor D, one has to select a subdivisor of a_m consisting of M points, such that $[a_m] = D + \bar{D}$ for a positive divisor $\bar{D}$, and hence choose $x \in S^M\Sigma$. In particular, since it is a ramification subdivisor, D and $\bar{D}$ can be also thought of as divisors on S and $\bar{S}$.

Following the proof of Theorem 3.5 and the ideas in [7, Proposition 10.2], the divisor D defines a natural map $\bar{\gamma} : E_+ \to [D] \otimes E_+$ up to scalar multiple. The fiber of this point $x \in S^M\Sigma$ is the choice of $\bar{\beta}$ up to scalar multiple, which corresponds to $\bar{D}$, and gives us a point in a vector bundle B of rank $(4m-1)(g-1) - M$ over $S^M\Sigma$ whose fibers are $H^0(\Sigma, K^{2m}\mathcal{O}(-x))$, for $x \in S^M\Sigma$.

Starting with an effective divisor D given by M points in the Riemann surface Σ, consider the exact sequence

$$0 \to K^{2m}(-D) \to K^{2m} \to K^{2m}|_D \to 0,$$

and the corresponding sequence of cohomology groups

$$0 \to H^0(\Sigma, K^{2m}(-D)) \to H^0(\Sigma, K^{2m}) \to H^0(D, K^{2m}) \to H^1(\Sigma, K^{2m}(-D)) \to \cdots .$$

Note that since $\deg D = M < 2m(g-1)$ and $m > 1$, by Riemann Roch, and Serre duality $\dim H^1(\Sigma, K^{2m}(-D)) = 0$, leading to the short exact sequence

$$0 \to H^0(\Sigma, K^{2m}(-D)) \xrightarrow{f_1} H^0(\Sigma, K^{2m}) \xrightarrow{f_2} H^0(D, K^{2m}) \to 0.$$

Then, for a section $s \in H^0(\Sigma, K^{2m}(-D))$ in the $(4m-1)(g-1) - M$-dimensional fiber over D, the injective map f_1 defines a corresponding differential $a_m := f_1(s) \in H^0(\Sigma, K^{2m})$.

From the above analysis, the connected parameter space of the triples $(\bar{S}, D, E_+)$ for fixed topological invariants can be obtained by replacing $H^0(\Sigma, K^{2m})$ in the base (14) by the vector bundle B over $S^M\Sigma$, and looking at the Zariski open set there which gives smooth curves $\bar{S}$. ■

5 Spectral Data for SU(m, m)

An SU(m, m)-Higgs bundle is given by a $U(m, m)$-Higgs bundle $(W_1 \oplus W_2, \Phi)$ for which $\Lambda^m W_1 \cong \Lambda^m W_2^*$. In particular, since we have that $\deg W_1 = -\deg W_2$ and thus one can adapt Theorem 3.5 to obtain the condition $\deg E = (4m^2 - 2m)(g-1)$, and $M = 2\deg W_2 + 2m(g-1)$. Moreover, $\Lambda^m \bar{\pi}_* E_+ \cong \Lambda^m \bar{\pi}_* E_-$. Considering the norm map Nm: $\mathrm{Pic}(\bar{S}) \to \mathrm{Pic}(\Sigma)$, from [4, Section 4], we have that $\Lambda^m \bar{\pi}_* E_\pm = \mathrm{Nm}(E_\pm) \otimes K^{-m(m-1)}$, where we are identifying divisors of Σ and their corresponding line bundles. Hence, one has that the norms satisfy $\mathrm{Nm}(E_+) = -\mathrm{Nm}(E_-) + 2m(m-1)K$. In terms of divisors on Σ, one has $D = \mathrm{Nm}\ E_+^* + \mathrm{Nm}\ E_- + mK$. Hence, the condition for the spectral data of Theorem 3.5 to give an SU(m, m)-Higgs bundle can be expressed as

$$2\mathrm{Nm}\ E_+ = m(2m-1)K - D, \tag{15}$$

or equivalently $\mathrm{Nm}\ ([D] \otimes E_+^2 \otimes \bar{\rho}^* K^{1-2m}) = 0$. The choice of E_+ is thus determined by the choice of an element in ker(Nm), that is, in $\mathrm{Prym}(\bar{S}, \Sigma)$. As in the case of $U(m, m)$-Higgs bundles, in this case the choice of D is given by a point in a symmetric product of Σ, and the section a_m is given by the divisor D together with a section of $K^{2m}[-D]$ (whose divisor is $\bar{D}$).

Since for maximal Toledo invariant $\tau(\deg W_1, \deg W_2) = \deg W_2 - \deg W_1 = 2m(g-1)$, in this case the divisor D is empty and thus (15) reduces to $2\mathrm{Nm}E_+ = m(2m-1)K$. Hence, for each choice of square root of $m(2m-1)K$, the line bundle E_+ is determined by an element in the Prym variety $\mathrm{Prym}(\bar{S}, \Sigma)$. From Section 4, in this case the choice of $\bar{S}$ and D is given by a point in a Zariski open subset of $\mathcal{A}'$ as in (14). Therefore, for each of the 2^{2g} choices, one has a copy of $\mathrm{Prym}(\bar{S}, \Sigma)$, giving a fibration over the Zariski open in $\mathcal{A}'$ whose fibers are the disjoint union 2^{2g} Prym varieties. Note that if m is even, there is a distinguished choice of square root. For $m = 1$, the study of these components is done through the monodromy action in [14]. For a review of the theory of Higgs bundles for groups of Hermitian type (which, in particular, includes the case of SU(m, m)-Higgs bundles for arbitrary m), the reader should refer, for example, to [13] or [6].

Remark 5.1. One should note that the fibers of the Hitchin fibration for real groups are abelian varieties only for some groups, and this has been investigated during the last year from different points of view. In [12], it was recently shown that the fibers are abelian if and only if the real form is quasi-split, and in [11] the author together with N. Hitchin considered the nonabelian cases of $\mathrm{Sp}(m, m)$, $\mathrm{SL}(2m, \mathbb{H})$, and $\mathrm{SO}(2m, \mathbb{H})$-Higgs bundles. □

Remark 5.2. As noted in [3], the spectral data defined in previous sections can be used, in physics terminology, to give a geometric realization of (B, A, A)-branes and understand their intersection with the (A, B, A)-branes defined in the above paper. Moreover, the spectral data can be studied through KO-theory as in [2]. More recently, the spectral data for $U(m, m)$-Higgs bundles appearing in this paper was used in [10] to study Langland's duality for real forms. □

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References

[1] Atiyah, M. F. and R. Bott. "A Lefschetz fixed point formula for elliptic complexes: II. Applications." *Annals of Mathematics* 88, no. 3 (1968): 451–91.

[2] Baraglia, D. and L. P. Schaposnik. "Real structures on moduli spaces of Higgs bundles." Advances in Theoretical and Mathematical Physics, (2014).

[3] Baraglia, D. and L. P. Schaposnik. "Higgs Bundles and (A, B, A)-Branes." *Communications in Mathematical Physics* (2014): preprint ArXiv:1305.4638.

[4] Beauville, A., M. S. Narasimhan, and S. Ramanan. "Spectral curves and the generalised theta divisor." *Journal für die Reine und Angewandte Mathematik* 398 (1989): 169–79.

[5] Bradlow, S., O. Garc´a-Prada, and P. G. Gothen. "Surface group representations and $U(p, q)$-bundles." *Journal of Differential Geometry* 64 (2003): 111–70.

[6] Bradlow, S. B., O. Garcia-Prada, and P. B. Gothen. "Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces." *Geometriae Dedicata* 122 (2006): 185–213.

[7] Hitchin, N. J. "The self-duality equations on a Riemann surface." *Proceedings of the London Mathematical Society* 55, no. 3 (1987): 59–126.

[8] Hitchin, N. J. "Stable bundles and integrable systems." *Duke Mathematical Journal* 54, no. 1 (1987): 91–114.

[9] Hitchin, N. J. "Lie groups and Teichmüller space." *Topology* 31, no. 3 (1992): 449–73.

[10] Hitchin, N. J. *Higgs bundles and characteristic classes*. Mathematische Arbeitstagung, 2013.

[11] Hitchin, N. J. and L. P. Schaposnik. "Nonabelianization of Higgs bundles." *Journal of Differential Geometry*, (2014).

[12] Peon, A. "Higgs bundles, real forms and the Hitchin fibration." PhD Thesis, UAM-ICMAT, Madrid (2013).

[13] Schaposnik, L. P. "Spectral data for G-Higgs bundles." DPhil. Thesis, Oxford (2013).

[14] Schaposnik, L. P. "Monodromy of the SL_2 Hitchin fibration." *International Journal of Mathematics* 24, no. 2 (2013): 1–21.