

Higgs Bundles and (A, B, A) -Branes

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Abstract: Through the action of anti-holomorphic involutions on a compact Riemann surface Σ we construct families of (A, B, A) -branes \mathcal{L}_{G_c} in the moduli spaces \mathcal{M}_{G_c} of G_c -Higgs bundles on Σ . We study the geometry of these (A, B, A) -branes in terms of spectral data and show they have the structure of real integrable systems.

1. Introduction

Since Higgs bundles were introduced in 1987 [Hit87], they have found applications in many areas of mathematics and mathematical physics. In particular, Hitchin showed in [Hit87] that their moduli spaces carry a natural hyperkähler structure and furthermore are algebraically completely integrable systems [Hit87a]. More recently Hausel and Thaddeus [HT03] related Higgs bundles to mirror symmetry, and in the work of Kapustin and Witten [KW07] Higgs bundles were used to give a physical derivation of the geometric Langlands correspondence.

Classically a Higgs bundle (E, Φ) on a compact Riemann surface Σ of genus $g \geq 2$ is given by a holomorphic vector bundle E on Σ together with a holomorphic section $\Phi : E \rightarrow E \otimes K$, for K the canonical bundle of the surface. Moreover, given a complex semisimple Lie group G_c one can define G_c -Higgs bundles [Hit87a] as pairs (P, Φ) on Σ , where P is a principal G_c -bundle and the Higgs field Φ is a holomorphic section of $\text{ad}(P) \otimes K$, where $\text{ad}(P)$ is the adjoint bundle of P . By considering parabolic stability one can construct the moduli space \mathcal{M}_{G_c} of G_c -Higgs bundles (for details see [BiGo08, Section 3] and references therein) which carries a hyperkähler structure.

As a hyperkähler manifold the moduli space \mathcal{M}_{G_c} carries a natural triple of complex structures I, J, K satisfying the familiar quaternionic relations and a triple of associated Kähler forms $\omega_I, \omega_J, \omega_K$, following the notation in [Hit87]. It is natural to consider submanifolds of \mathcal{M}_{G_c} which are Lagrangian or complex with respect to each of these three

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structures. From the perspective of physics such submanifolds are referred to as A-branes and B-branes respectively. We dedicate this paper to the study of branes of type (A, B, A) arising naturally from anti-holomorphic involutions on the Riemann surface Σ . These (A, B, A) -branes are submanifolds of \mathcal{M}_{G_c} which are Lagrangian with respect to the two symplectic structures ω_I, ω_K and complex with respect to J . Such branes are of considerable interest due to the connections believed to exist between Langlands duality and homological mirror symmetry. Additionally the (A, B, A) -branes we construct turn out to be integrable systems making them interesting structures to consider in their own right.

Given an anti-holomorphic involution f on a compact Riemann surface Σ of genus $g \geq 2$, one has a natural induced action on the moduli space of G_c -Higgs bundles as well as on the moduli space of representations of $\pi_1(\Sigma)$ into G_c , which we study in Sects. 2–4. By looking at the fixed point set \mathcal{L}_{G_c} of this involution, in Sect. 5 we obtain a natural (A, B, A) -brane in the sense of Kapustin and Witten [KW07]:

Theorem 14. *For each choice of anti-holomorphic involution f on a compact Riemann surface there is a natural (A, B, A) -brane \mathcal{L}_{G_c} defined in the moduli space of G_c -Higgs bundles.*

In order to study these branes we consider in Sects. 6 and 7 the spectral data associated to G_c -Higgs bundles introduced in [Hit87a], for G_c a complex classical Lie group. Looking at \mathcal{L}_{G_c} as sitting inside the corresponding Hitchin fibration, we obtain new examples of real integrable systems over a real subspace L of the base of the Hitchin fibration:

Theorem 17. *If \mathcal{L}_{G_c} contains smooth points then the restriction of the Hitchin fibration $h|_{\mathcal{L}_{G_c}} : \mathcal{L}_{G_c} \rightarrow L$ to \mathcal{L}_{G_c} is a Lagrangian fibration with singularities. The generic fibre is smooth and consists of a finite number of tori.*

Through the study of the spectral data associated to such (A, B, A) -branes, we prove in Sect. 7 that \mathcal{L}_{G_c} always contains smooth points for the classical Lie groups. Hence Theorem 17 applies and we obtain many families of real integrable systems through this construction.

In Sect. 8 we study the connectivity of the fibres of \mathcal{L}_{G_c} . In the case of $G_c = GL(n, \mathbb{C})$, the number of components can be expressed in terms of the number of components of the fixed point set of an induced involution on the spectral curves (Proposition 32). Moreover, in the rank 2 case we are able to reduce this to an exact formula in terms of the associated quadratic differential (Theorem 36), and also obtain exact formulas for $G_c = SL(2, \mathbb{C})$ (Theorem 38).

For certain groups G_c we find that the Higgs bundles fixed by the induced involution may have a real or quaternionic structure. In particular, we consider this for $SL(2, \mathbb{C})$ -Higgs bundles in Sect. 9. We find that if (E, Φ) is a stable $SL(2, \mathbb{C})$ -Higgs bundle then E carries either a real or quaternionic structure. Moreover, we determine which of the two possibilities occur in terms of the spectral data associated to (E, Φ) .

Under Langlands duality, it is known ([KW07, Section 12.4]) that (A, B, A) -branes in \mathcal{M}_{G_c} map to (A, B, A) -branes in the moduli space $\mathcal{M}_{L G_c}$ of $L G_c$ -Higgs bundles, for $L G_c$ the Langlands dual group of G_c . We propose that the dual of the (A, B, A) -brane \mathcal{L}_{G_c} is the corresponding (A, B, A) -brane $\mathcal{L}_{L G_c}$ defined using the same involution f on the Riemann surface. In Sect. 10 we present some preliminary evidence to support this proposal.

Following the ideas of Kapustin–Witten [KW07] and Gukov [Gu07], the constructions given in this paper can be shown to be closely related to representations of

3-manifolds whose boundary is the Riemann surface Σ . In Sect. 11 we give an outline of this relation, which are developed in the companion paper [BaSch].

2. Anti-holomorphic Involutions on a Riemann Surface

Throughout the paper we consider anti-holomorphic involutions $f : \Sigma \rightarrow \Sigma$ on a compact Riemann surface Σ of genus $g > 1$ always taken to be connected. Such an involution f induces involutions on the moduli space of representations $\pi_1(\Sigma) \rightarrow G_c$ and the moduli space of G_c -Higgs bundles on Σ , for a given complex semisimple Lie group G_c . In this section we consider the topological properties of such involutions and the induced actions on the fundamental group $\pi_1(\Sigma)$.

2.1. Topological classification. The classification of anti-holomorphic involutions $f : \Sigma \rightarrow \Sigma$ of a compact Riemann surface is a classical result of Klein. The reader should refer to [GH81] for a thorough study of this situation. All such involutions on Σ may be characterised by two integer invariants (n, a) as follows. The fixed point set of f is known to be a disjoint union of copies of the circle embedded in the surface. Let n denote the number of components of the fixed point set which by a classical theorem of Harnack can be at most $g + 1$. The complement of the fixed point set has one or two components. Set $a = 0$ if the complement is disconnected and $a = 1$ otherwise.

Example 1. For Σ of genus 2 and $n(\Sigma) = 1$, one may have (Fig. 1):

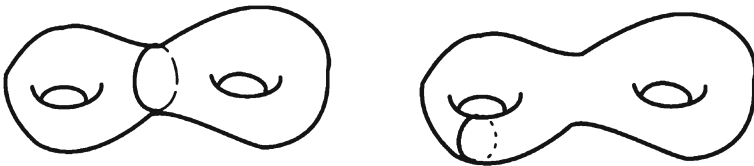


Fig. 1. *Left invariant $a(\Sigma) = 0$; right invariant $a(\Sigma) = 1$*

Remark 2. One should note that an anti-involution f as considered in this paper can also be found in the literature as an anti-conformal map of Σ , whose species $Sp_i(f)$ is $+k$ or $-k$ according to whether $\Sigma - \text{Fix}(f)$ is connected or not (e.g., see [BCGG] and references therein).

Proposition 3 ([GH81]). *Let g be the genus of Σ . The invariants (n, a) associated to an orientation reversing involution on Σ satisfy the following conditions*

- $0 \leq n \leq g + 1$.
- If $n = 0$ then $a = 1$. If $n = g + 1$ then $a = 0$.
- If $a = 0$ then $n = g + 1 \pmod{2}$.

Conversely any pair (n, a) satisfying these conditions determines such an orientation reversing involution on Σ , unique up to homeomorphism.

Orientation reversing involutions on a compact Riemann surface Σ may be constructed as follows. Given a compact surface S (not necessarily orientable) with boundary let \tilde{S} be its orientation double cover, which is an oriented surface with boundary. The involution f which interchanges the sheets of the double cover reverses the orientation

of \tilde{S} . From each boundary component of S we obtain a corresponding pair of boundary components in \tilde{S} which are exchanged by f . We then define Σ by gluing together the boundary components of \tilde{S} through the restriction of f to the boundary. Moreover such a surface Σ has a natural orientation reversing involution induced by f , and the conditions of Proposition 3 are satisfied by this construction.

In order to see that the above construction produces all such anti-holomorphic involutions on a compact Riemann surface, consider Σ a compact Riemann surface with an anti-holomorphic involution f which has fixed points. In a neighbourhood of a fixed point, one can find a local coordinate z such that the involution f is given by the complex conjugation $f : z \mapsto \bar{z}$ (e.g., see [Se91] or [GH81]). Hence the fixed point set can be seen as a union of copies of the unit circle embedded in the surface. By making cuts in Σ around these circles we obtain a Riemann surface Σ' with two boundary components for each cut and a fixed point free anti-holomorphic involution, which permutes each pair of components. Since the induced involution on Σ' does not fix any boundary component, it pairs off components and thus Σ' is the orientation double cover of a Riemann surface with boundary.

2.2. Action on spin structures. Recall that a spin structure on an oriented Riemannian n -manifold M with $SO(n)$ -frame bundle P may be defined as a class $\xi \in H^1(P, \mathbb{Z}_2)$ which restricted to any fibre of $P \rightarrow M$ agrees with the class in $H^1(SO(n), \mathbb{Z}_2)$ corresponding to the double cover $Spin(n) \rightarrow SO(n)$. In the case that $M = \Sigma$ is a Riemann surface we may identify the frame bundle P with the bundle $U\Sigma$ of unit tangent vectors, since any unit vector can be uniquely extended to an oriented frame.

Throughout the paper we let K denote the canonical bundle of the Riemann surface Σ . Given a compatible Riemannian metric on Σ , the spin structures correspond to theta characteristics, holomorphic line bundles L for which there is an isomorphism $L^2 \simeq K$. The hermitian structure on K determines a hermitian structure on L such that the bundle of unit vectors in L is a double cover of the unit vectors of K . Then since the unit vectors of K can be naturally identified with the unit tangent bundle $U\Sigma$ we obtain a double cover of $U\Sigma$, which is then a spin structure. The spin structures on Σ for different choices of metric may be canonically identified and the spin structure associated to L does not depend on the choice of metric, for different choices of metrics compatible with the complex structure on Σ .

The induced map $f^* : H^1(U\Sigma, \mathbb{Z}_2) \rightarrow H^1(U\Sigma, \mathbb{Z}_2)$ of an anti-holomorphic involution f on Σ preserves the subset of classes in $H^1(U\Sigma, \mathbb{Z}_2)$ which define spin structures. Therefore the anti-holomorphic involution f has a natural action on the set of spin structures. From [Ati71, GH81] we have the following.

Proposition 4. *Given Σ a compact oriented surface and $f : \Sigma \rightarrow \Sigma$ an orientation reversing involution, there exists a spin structure preserved by f .*

There is also a natural action of f on the theta characteristics. Indeed as f is anti-holomorphic there is a natural isomorphism $f^*(\bar{K}) \simeq K$. It follows that if L is a theta characteristic, then so is $f^*(\bar{L})$.

Proposition 5. *The action of f on the set of spin structures of the Riemann surface Σ interpreted as theta characteristics agrees with the action of f on the set of spin structures considered as a subset of $H^1(U\Sigma, \mathbb{Z}_2)$, where the action of f is induced by the differential $f_* : U\Sigma \rightarrow U\Sigma$.*

Proof. Let L be a holomorphic line bundle such that $L^2 \simeq K$. In particular, L inherits a hermitian metric h from the hermitian metric on K . Let $Q \rightarrow U\Sigma$ be the corresponding double cover of $U\Sigma$ and $\xi \in H^1(U\Sigma, \mathbb{Z}_2)$ the class defined by Q . The hermitian metric on L determines a metric on $f^*(\bar{L})$ and it is clear that the induced double cover of $U\Sigma$ is isomorphic to $f^*(Q)$, which corresponds to $f^*(\xi) \in H^1(U\Sigma, \mathbb{Z}_2)$.

From the above propositions we conclude the existence of theta characteristics $K^{1/2}$ such that $f^*(\bar{K}^{1/2}) \simeq K^{1/2}$.

2.3. Action on the fundamental group. Since an anti-holomorphic involution of the Riemann surface $f : \Sigma \rightarrow \Sigma$ is a homeomorphism, it induces an isomorphism $f_* : \pi_1(\Sigma, x_0) \rightarrow \pi_1(\Sigma, f(x_0))$ for $x_0 \in \Sigma$. Let \cdot denote the operation of joining paths. Given a path γ joining x_0 to $f(x_0)$ we obtain an isomorphism

$$\phi_\gamma : \pi_1(\Sigma, f(x_0)) \rightarrow \pi_1(\Sigma, (x_0)),$$

which sends a loop u based at $f(x_0)$ to the loop $\gamma.u.\gamma^{-1}$ based at x_0 . Hence the composition

$$\hat{f} = \phi_\gamma \circ f_* : \pi_1(\Sigma, x_0) \rightarrow \pi_1(\Sigma, x_0)$$

is an automorphism of $\pi_1(\Sigma, x_0)$. Different choices of γ change \hat{f} by composition with an inner automorphism. Observe that $f(\gamma)$ is a path from $f(x_0)$ to x_0 so the composition $h = \gamma.f(\gamma)$ is a loop based at x_0 . One sees that \hat{f}^2 is the inner automorphism $u \mapsto huh^{-1}$.

If the map f has fixed points we can choose x_0 to be a fixed point and γ to be the constant path. Then h is the trivial loop and \hat{f} is an involutive automorphism of $\pi_1(\Sigma)$. On the contrary if f is fixed point free then Σ is a double cover $\pi : \Sigma \rightarrow \Sigma'$ of a non-orientable surface Σ' and thus we get an exact sequence

$$1 \rightarrow \pi_1(\Sigma) \rightarrow \pi_1(\Sigma') \rightarrow \mathbb{Z}_2 \rightarrow 1.$$

The image $\gamma' = \pi(\gamma)$ of γ is a class in $\pi_1(\Sigma')$ lifting the generator of \mathbb{Z}_2 . The automorphism \hat{f} is conjugation by γ' in the sense that $\pi_*\hat{f}(v) = \gamma'\pi_*(v)(\gamma')^{-1}$. In particular, \hat{f}^2 is conjugation by $(\gamma')^2$.

2.4. Action on principal bundles. Let $P \rightarrow \Sigma$ be a principal G_c -bundle, for G_c a complex connected Lie group and $f^*(P)$ be the pullback principal bundle. Since G_c is connected the principal bundle P can be trivialized over the 1-skeleton of Σ . The obstruction to extending the trivialization to the 2-skeleton is a cohomology class in $H^2(\Sigma, \pi_1(G_c)) \simeq \pi_1(G_c)$. Moreover from [Ra75, Proposition 5.1] one has the following:

Proposition 6. *Isomorphism classes of G_c -bundles are in bijection with the cohomology group $H^2(\Sigma, \pi_1(G_c))$.*

From the above proposition the action of f^* can be seen to be the pullback in cohomology

$$f^* : H^2(\Sigma, \pi_1(G_c)) \rightarrow H^2(\Sigma, \pi_1(G_c)).$$

Moreover since $H^2(\Sigma, \pi_1(G_c)) \simeq H^2(\Sigma, \mathbb{Z}) \otimes \pi_1(G_c)$, the induced action f^* is multiplication by -1 . Therefore a principal G_c -bundle $P \rightarrow \Sigma$ is fixed by f if and only if its topological class $x \in \pi(G_c)$ is a 2-torsion element.

3. Action on Representations and the Fixed Point Set

In this section we shall denote by G_c a complex connected Lie group with Lie algebra \mathfrak{g} , and assume that \mathfrak{g} admits an invariant symmetric non-degenerate bilinear form B .

3.1. Moduli space of representations. Let Σ be a compact orientable surface of genus $g > 1$ and let $\pi = \pi_1(\Sigma, x_0)$ be its fundamental group. We denote by $\text{Hom}(\pi, G_c)$ the set of homomorphisms $\rho : \pi \rightarrow G_c$ with the compact-open topology. Given a representation $\rho : \pi \rightarrow G_c$ composition with the adjoint representation of G_c defines a representation of π on \mathfrak{g} . We denote by \mathfrak{g}_ρ the Lie algebra \mathfrak{g} equipped with this representation.

The space $\text{Hom}(\pi, G_c)$ has a natural action of G_c by conjugation. To obtain a good quotient one restricts to the subspace $\text{Hom}^+(\pi, G_c)$ of reductive representations. Recall that a representation $\rho : \pi \rightarrow G_c$ is called reductive if \mathfrak{g}_ρ splits into a direct sum of irreducible representations. Restricted to reductive representations the action of G_c on $\text{Hom}^+(\pi, G_c)$ is proper [Go84], and thus the quotient is a Hausdorff space.

Let $\text{Rep}^+(\pi, G_c)$ be the quotient space $\text{Hom}^+(\pi, G_c)/G_c$, the moduli space of reductive G_c -representations of $\pi_1(\Sigma)$. In general $\text{Rep}^+(\pi, G_c)$ has singularities, but there is a dense open subset of smooth points over which $\text{Rep}^+(\pi, G_c)$ naturally has the structure of a complex manifold. For a representation ρ , let $G_\rho \subseteq G_c$ be the stabilizer of ρ , and let $Z(G_c)$ be the centre of G_c . If ρ is a reductive representation such that $\dim(G_\rho) = \dim(Z(G_c))$, then ρ is a smooth point of $\text{Hom}^+(\pi, G_c)$, whose dimension is $(2g - 1)\dim(G_c) + \dim(Z(G_c))$ [Go84]. This condition also ensures that the stabilizer $G_\rho/Z(G_c)$ of ρ in $G_c/Z(G_c)$ is discrete, and hence finite since the action is proper. It follows from general theory (see [MM03]) that around such points the quotient space $\text{Rep}^+(\pi, G_c)$ has the structure of an orbifold. Moreover if ρ is a reductive representation with $G_\rho = Z(G_c)$, then the corresponding point in $\text{Rep}^+(\pi, G_c)$ is smooth of dimension $(2g - 2)\dim(G_c) + 2\dim(Z(G_c))$.

Definition 7. *We say that a reductive representation ρ is simple if $G_\rho = Z(G_c)$. In particular if the representation of π on \mathfrak{g} induced by ρ is irreducible, then ρ is simple.*

If $\rho \in \text{Rep}^+(\pi, G_c)$ is a simple point then the tangent space at ρ is naturally given in terms of group cohomology by $H^1(\pi, \mathfrak{g}_\rho)$. The pairing $B : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{C}$ determines a symplectic pairing $H^1(\pi, \mathfrak{g}_\rho) \otimes H^1(\pi, \mathfrak{g}_\rho) \rightarrow H^2(\pi, \mathbb{C}) \simeq \mathbb{C}$. This defines a closed complex symplectic form Ω on the smooth part of $\text{Rep}^+(\pi, G_c)$ [Go84].

3.2. Induced action on space of representations. We have seen in Sect. 2.3 that given an orientation reversing involution $f : \Sigma \rightarrow \Sigma$, there is an induced map $\hat{f} : \pi \rightarrow \pi$ defined up to an inner automorphism. Accordingly there is an induced action on $\text{Hom}(\pi, G_c)$ sending a representation ρ to $\rho \circ \hat{f}$. This action preserves the subspace of reductive representations and descends through the quotient to an action $f : \text{Rep}^+(\pi, G_c) \rightarrow \text{Rep}^+(\pi, G_c)$. We may identify $\text{Rep}^+(\pi, G_c)$ with the moduli space of gauge equivalence classes of flat G_c -connections on Σ with reductive holonomy. Then $f : \text{Rep}^+(\pi, G_c) \rightarrow \text{Rep}^+(\pi, G_c)$ corresponds to the pullback of connections by $f : \Sigma \rightarrow \Sigma$.

Proposition 8. *The induced map $f : \text{Rep}^+(\pi, G_c) \rightarrow \text{Rep}^+(\pi, G_c)$ is an involution which preserves the subspace of simple points. On the simple points f is holomorphic and anti-symplectic, that is $f^*\Omega = -\Omega$.*

Proof. Recall that the automorphism $\hat{f} : \pi \rightarrow \pi$ is such that \hat{f}^2 is an inner automorphism. It follows that the induced map $f : \text{Rep}^+(\pi, G_c) \rightarrow \text{Rep}^+(\pi, G_c)$ is an involution which preserves the space of simple points. It is clear that f acts smoothly on the space of simple points, and its differential is the induced pullback in group cohomology $f^* : H^1(\pi, \mathfrak{g}_\rho) \rightarrow H^1(\pi, \mathfrak{g}_{\rho \circ \hat{f}})$. Since this is a complex linear map, $f : \text{Rep}^+(\pi, G_c) \rightarrow \text{Rep}^+(\pi, G_c)$ is holomorphic on the simple points. Moreover f is orientation reversing from which the anti-symplectic condition $f^*\Omega = -\Omega$ follows.

Lemma 9. *Let X be a complex manifold of dimension n and $f : X \rightarrow X$ an anti-holomorphic involution. If the fixed point set of f is non-empty, then it is a real analytic submanifold of real dimension n .*

Proof. Let p be a fixed point of f , and let g be a Riemannian metric on X . Replacing g by $g + f^*(g)$ we may assume that g is f -invariant. We can also assume that g is analytic near p . Let $e : T_p X \rightarrow X$ be the exponential mapping at p , and $C = f_*(p) : T_p X \rightarrow T_p X$ the differential of f at p . Then C is an anti-linear involution on $T_p X$, and since g is f -invariant, for any $v \in T_p X$ we have that $e(C(v)) = f(e(v))$. Thus a neighborhood of the origin in $T_p X$ defines local analytic coordinates near p such that f corresponds to the anti-linear involution C . Therefore the fixed point set of f near p is a real analytic submanifold of real dimension n .

We shall denote by \mathcal{L}_{G_c} the fixed point set of the induced involution $f : \text{Rep}^+(\pi, G_c) \rightarrow \text{Rep}^+(\pi, G_c)$. From Proposition 8 and Lemma 9 we obtain:

Proposition 10. *If non-empty, the set of simple points of \mathcal{L}_{G_c} is a smooth complex Lagrangian submanifold of the simple points of $\text{Rep}^+(\pi, G_c)$.*

Through the theory of spectral curves, we show in Sect. 7 that \mathcal{L}_{G_c} does indeed contain smooth points for $G_c = GL(n, \mathbb{C})$, and for G_c a classical complex semi-simple Lie group.

4. Moduli Space of G_c -Higgs Bundles

A Higgs bundle on a Riemann surface Σ is a pair (E, Φ) , where E is a holomorphic vector bundle on Σ and Φ is a holomorphic section of $\text{End}(E) \otimes K$. More generally for a complex Lie group G_c we define a G_c -Higgs bundle to be a pair (P, Φ) where P is a holomorphic principal G_c -bundle with adjoint bundle $\text{ad}(P)$ and Φ is a holomorphic section of $\text{ad}(P) \otimes K$. In the following subsections we shall study the geometry of the moduli space of G_c -Higgs bundles, which is closely related to the moduli space of surface group representations into G_c .

4.1. Higgs bundles and the Hitchin equations. For simplicity assume G_c is connected and semisimple. Let G denote the compact real form of G_c . Further, we use \mathfrak{g}_c to denote the Lie algebra of G_c , and \mathfrak{g} for the Lie algebra of G . Given a choice of principal G_c -bundle P , the Killing form $k(x, y)$ on \mathfrak{g}_c naturally defines a bilinear form on the adjoint bundle $\text{ad}(P)$ which we will also denote by k . Suppose P is given a reduction of structure to G . This amounts to equipping the adjoint bundle $\text{ad}(P)$ with an anti-linear involution $\rho : \text{ad}(P) \rightarrow \text{ad}(P)$ such that taking the hermitian adjoint x^* of a

section x of $\text{ad}(P)$ is given by $x^* = -\rho(x)$. The associated hermitian form h is given by $h(x, y) = k(x^*, y) = -k(\rho(x), y)$. Note that $k(x, y) = k(x^*, y^*)$.

Consider pairs $(\bar{\partial}_A, \Phi)$, where $\bar{\partial}_A$ denotes a $\bar{\partial}$ -connection on P defining a holomorphic structure, and Φ is a section of $\Omega^{1,0}(\Sigma, \text{ad}(P))$. Note that if $\bar{\partial}_A \Phi = 0$ then $(\bar{\partial}_A, \Phi)$ defines a G_c -Higgs bundle on P . From [Hit87], the Hitchin equations for a pair $(\bar{\partial}_A, \Phi)$ are

$$\begin{aligned} \bar{\partial}_A \Phi &= 0 \\ F_A + [\Phi, \Phi^*] &= 0, \end{aligned} \tag{1}$$

where F_A is the curvature of the unitary connection $\nabla_A = \partial_A + \bar{\partial}_A$ associated to $\bar{\partial}_A$. Two solutions to the Hitchin equations on P are considered equivalent if they are related by a G -valued gauge transform. We let $\mathcal{M}_{G_c}(P)$ denote the moduli space of gauge equivalence classes of solutions to the Hitchin equations on P , and \mathcal{M}_{G_c} the union of the $\mathcal{M}_{G_c}(P)$ as P ranges over the set of isomorphism classes of principal G_c -bundles.

A solution $(\bar{\partial}_A, \Phi)$ to the Hitchin equations determines a flat G_c -connection $\nabla = \nabla_A + \Phi + \Phi^*$. Using results of Donaldson [D87] and Corlette [Cor88], the assignment $(\bar{\partial}_A, \Phi) \mapsto \nabla$ gives an isomorphism between the Higgs bundle moduli space \mathcal{M}_{G_c} of equivalence classes of solutions to the Hitchin equations and the moduli space $\text{Rep}^+(\pi_1(\Sigma), G_c)$ of reductive representations of $\pi_1(\Sigma)$ in G_c . In particular every flat G_c -connection ∇ with reductive holonomy representation admits a decomposition $\nabla = \nabla_A + \Phi + \Phi^*$ associated to a solution $(\bar{\partial}_A, \Phi)$ of the Hitchin equations.

The slope of a holomorphic vector bundle $E \rightarrow \Sigma$ is defined as the number $\mu(E) := \text{deg}(E)/\text{rank}(E)$. We say that a Higgs bundle (E, Φ) is semi-stable if for each proper, non-zero subbundle $F \subset E$ which is Φ -invariant we have $\mu(F) \leq \mu(E)$. If this inequality is always strict then (E, Φ) is said to be stable. Finally the Higgs bundle (E, Φ) is poly-stable if it is a sum of stable Higgs bundles of the same slope. It is possible to adapt these definitions to the case of principal G_c -Higgs bundles for a complex semisimple Lie group G_c (e.g., see [BiGo08, Section 3]).

One may define a moduli space of semi-stable G_c -Higgs bundles $\mathcal{M}_{G_c}^{\text{Higgs}}$, and by a fundamental result of Hitchin [Hit87] and Simpson [S88] there is an isomorphism between $\mathcal{M}_{G_c}^{\text{Higgs}}$ and the moduli space \mathcal{M}_{G_c} of solutions to the Hitchin equations, when G_c is semi-simple. The key result used to establish this is that a G_c -Higgs bundle $(\bar{\partial}_A, \Phi)$ is gauge equivalent to a solution of the Hitchin equations if and only if it is poly-stable. A similar isomorphism exists when the group is $G_c = GL(n, \mathbb{C})$, except that in this case $\mathcal{M}_{GL(n, \mathbb{C})}$ corresponds to the subspace $\mathcal{M}_{GL(n, \mathbb{C})}^{\text{Higgs}, 0} \subset \mathcal{M}_{G_c}^{\text{Higgs}}$ of Higgs bundles of degree 0.

4.2. Geometry of the Higgs bundle moduli space. The moduli space \mathcal{M}_{G_c} of G_c -Higgs bundles, for G_c a complex semisimple Lie group is a hyperkähler manifold with singularities, obtained by taking a hyperkähler quotient of the infinite dimensional space of pairs of complex structures and Higgs fields [Hit87, S88].

Recall that a pair $(\bar{\partial}_A, \Phi)$ consists of a holomorphic structure $\bar{\partial}_A$ on the principal bundle P , and a section $\Phi \in \Omega^{1,0}(\Sigma, \text{ad}(P))$. The space of all pairs $(\bar{\partial}_A, \Phi)$ on P is an infinite dimensional manifold and an affine space modelled on $\Omega^{0,1}(\Sigma, \text{ad}(P)) \oplus \Omega^{1,0}(\Sigma, \text{ad}(P))$. We write $(\Psi_1, \Phi_1), (\Psi_2, \Phi_2), \dots$ for tangent vectors to this space. Furthermore given a pair $(\bar{\partial}_A, \Phi)$ we write ∇_A for the unitary connection corresponding to $\bar{\partial}_A$, and F_A its curvature. Explicitly one has $\nabla_A = \bar{\partial}_A + \partial_A$ where $\partial_A = \rho \circ \bar{\partial}_A \circ \rho$. The metric on this infinite dimensional space is given by

$$g((\Psi_1, \Phi_1), (\Psi_1, \Phi_1)) = 2i \int_{\Sigma} k(\Psi_1^*, \Psi_1) - k(\Phi_1^*, \Phi_1). \tag{2}$$

There are compatible complex structures I, J, K satisfying the quaternionic relations given by

$$\begin{aligned} I(\Psi_1, \Phi_1) &= (i\Psi_1, i\Phi_1) \\ J(\Psi_1, \Phi_1) &= (i\Phi_1^*, -i\Psi_1^*) \\ K(\Psi_1, \Phi_1) &= (-\Phi_1^*, \Psi_1^*). \end{aligned} \tag{3}$$

We shall denote by $\omega_I, \omega_J, \omega_K$ the corresponding Kähler forms defined by

$$\omega_I(X, Y) := g(IX, Y), \quad \omega_J(X, Y) := g(JX, Y), \quad \omega_K(X, Y) := g(KX, Y).$$

The induced complex symplectic forms $\Omega_I, \Omega_J, \Omega_K$ are given by

$$\Omega_I = \omega_J + i\omega_K, \quad \Omega_J = \omega_K + i\omega_I, \quad \Omega_K = \omega_I + i\omega_J.$$

Example 11. For example Ω_J is given by:

$$\Omega_J((\Psi_1, \Phi_1), (\Psi_2, \Phi_2)) = -i \int_{\Sigma} k(\Psi_1 - \Psi_1^* + \Phi_1 + \Phi_1^*, \Psi_2 - \Psi_2^* + \Phi_2 + \Phi_2^*).$$

After taking the hyperkähler quotient this agrees up to the factor of $-i$ with the symplectic form Ω defined in Sect. 3.1, that is $\Omega_J = -i\Omega$ on \mathcal{M}_{G_c} .

To get the moduli space \mathcal{M}_{G_c} , as seen in [Hit87] one interprets the Hitchin Eq. (1) as the moment map equations for the action of the unitary gauge group and takes the hyperkähler quotient. Given $(\bar{\partial}_A, \Phi)$ a solution of the Hitchin Eq. (1), the tangent space $T_{(\bar{\partial}_A, \Phi)}\mathcal{M}_{G_c}$ of the moduli space at a smooth point $(\bar{\partial}_A, \Phi)$ can be described as equivalence classes of deformations (Ψ_1, Φ_1) in $\Omega^{0,1}(\Sigma, \text{ad}(P)) \oplus \Omega^{1,0}(\Sigma, \text{ad}(P))$ satisfying:

$$\begin{aligned} \bar{\partial}_A \Phi_1 + [\Psi_1, \Phi] &= 0 \\ \partial_A \Psi_1 - \bar{\partial}_A \Psi_1^* + [\Phi, \Phi_1^*] + [\Phi_1, \Phi^*] &= 0. \end{aligned}$$

Two deformations are equivalent if they are related by an infinitesimal unitary gauge transformation. Thus (Ψ_2, Φ_2) is equivalent to (Ψ_1, Φ_1) if there is a skew-adjoint section $\psi \in \Omega^0(\Sigma, \text{ad}(P))$ of the adjoint bundle such that

$$\begin{aligned} \Psi_2 &= \Psi_1 + \bar{\partial}_A \psi \\ \Phi_2 &= \Phi_1 + [\Phi, \psi]. \end{aligned}$$

5. Higgs Bundles and the (A, B, A) -Brane \mathcal{L}_{G_c}

Given $f : \Sigma \rightarrow \Sigma$ an anti-holomorphic involution on Σ , we saw in Sect. 3 that f induces an involution on $\text{Rep}^+(\pi_1(\Sigma), G_c)$ with fixed point set \mathcal{L}_{G_c} . By identifying $\text{Rep}^+(\pi_1(\Sigma), G_c)$ with \mathcal{M}_{G_c} we obtain an involution on the moduli space of G_c -Higgs bundles. From this perspective we can interpret the fixed point set \mathcal{L}_{G_c} in terms of A- and B-branes following [KW07].

Let P be a principal G_c -bundle over Σ and fix a reduction of structure of P to G . For $x \in \mathcal{M}_{G_c}$ a point in the moduli space represented by a G_c -Higgs bundle pair $(\bar{\partial}_A, \Phi)$ on P , applying a gauge transform we may assume that $(\bar{\partial}_A, \Phi)$ satisfies the Hitchin equations. In particular the flat connection ∇ corresponding to $(\bar{\partial}_A, \Phi)$ is given by $\nabla = \partial_A + \bar{\partial}_A + \Phi + \Phi^*$. The involution f acts on this flat connection by pullback, so

we obtain $f^*\nabla = f^*(\partial_A) + f^*(\bar{\partial}_A) + f^*(\Phi) + f^*(\Phi^*)$. Therefore $(f^*(\partial_A), f^*(\Phi^*))$ is a G_c -Higgs bundle on $f^*(P)$ which satisfies the Hitchin equations. The flat connection associated to the pair $(f^*(\partial_A), f^*(\Phi^*))$ is $f^*\nabla$, so we have found a Higgs bundle pair $(f^*(\partial_A), f^*(\Phi^*))$ representing $f(x) \in \mathcal{M}_{G_c}$.

Proposition 12. *The induced involution $f : \mathcal{M}_{G_c} \rightarrow \mathcal{M}_{G_c}$ on the moduli space of solutions $(\bar{\partial}_A, \Phi)$ to the G_c -Hitchin equations (1) is holomorphic with respect to the complex structure J and anti-holomorphic with respect to the complex structures I and K . Moreover f is an isometry with respect to the hyperkähler metric.*

Proof. As seen above, the action of f on \mathcal{M}_{G_c} is given by sending a solution $(\bar{\partial}_A, \Phi)$ of the Hitchin equations to the pair $(f^*(\partial_A), f^*(\Phi^*))$. The differential of f at $(\bar{\partial}_A, \Phi)$ sends a deformation (Ψ_1, Φ_1) of $(\bar{\partial}_A, \Phi)$ to a corresponding deformation of $(f^*(\partial_A), f^*(\Phi^*))$ given by $(-f^*(\Psi_1^*), f^*(\Phi_1^*))$. From this and (3) we see that f is anti-holomorphic with respect to I, K and holomorphic with respect to J . Similarly it follows from (2) that f is an isometry.

Proposition 13. *The fixed point set \mathcal{L}_{G_c} of f on the moduli space of solutions $(\bar{\partial}_A, \Phi)$ to the G_c -Hitchin equations (1) meets the smooth points in a complex Lagrangian submanifold with respect to J, Ω_J .*

Proof. Since f is anti-holomorphic in I , we know immediately that its fixed point set \mathcal{L}_{G_c} is a mid-dimensional submanifold of \mathcal{M}_{G_c} . Clearly \mathcal{L}_{G_c} is a complex submanifold with respect to J , and the symplectic forms ω_I, ω_K must vanish on \mathcal{L}_{G_c} . In particular this means the complex symplectic form $\Omega_J = \omega_K + i\omega_I$ vanishes on \mathcal{L}_{G_c} . Thus \mathcal{L}_{G_c} is a complex Lagrangian submanifold with respect to J .

Following [KW07], we say that \mathcal{L}_{G_c} is an (A, B, A) -brane with respect to the complex structures I, J and K , and thus have the following:

Theorem 14. *For each choice of anti-holomorphic involution f on a compact Riemann surface, there is a natural (A, B, A) -brane \mathcal{L}_{G_c} defined in the moduli space of G_c -Higgs bundles.*

In subsequent sections we shall describe how the brane \mathcal{L}_{G_c} lies with respect to the Hitchin fibration for the moduli space \mathcal{M}_{G_c} (Sect. 6), consider the spectral data for G_c -Higgs bundle to show that \mathcal{L}_{G_c} is non-empty (Sect. 7), and study connectivity of \mathcal{L}_{G_c} (Sect. 8).

6. The Hitchin Fibration and \mathcal{L}_{G_c}

Let $R^* = \mathbb{C}[\mathfrak{g}_c^*]^{G_c}$ be the graded \mathbb{C} -algebra of invariant polynomials on \mathfrak{g}_c and Sym^* the graded \mathbb{C} -algebra with $\text{Sym}^j = H^0(\Sigma, K^j)$. We shall denote by \mathcal{A}_{G_c} the space $\text{Hom}(R^*, \text{Sym}^*)$ of graded \mathbb{C} -algebra homomorphisms from R^* to Sym^* . Choosing a homogeneous basis of generators p_1, p_2, \dots, p_l for R^* , we have that $R^* \simeq \mathbb{C}[p_1, \dots, p_l]$ and hence we obtain a non-canonical isomorphism

$$\mathcal{A}_{G_c} = \bigoplus_{i=1}^l H^0(\Sigma, K^{d_i}),$$

where d_i is the degree of p_i . From [Hit92], we consider the Hitchin fibration $h : \mathcal{M}_{G_c} \rightarrow \mathcal{A}_{G_c}$, which assigns to a Higgs bundle (P, Φ) the homomorphism $h(P, \Phi) : R^* \rightarrow \text{Sym}^*$ sending an invariant polynomial p to its evaluation $p(\Phi)$ on Φ .

The anti-holomorphic involution f on Σ induces a natural anti-holomorphic involution on the spaces $H^0(\Sigma, K^j)$ given by sending a holomorphic differential q to $\overline{f^*(q)}$. The induced map $f : \text{Sym}^* \rightarrow \text{Sym}^*$ is an anti-linear graded ring involution. We may also define an anti-linear graded ring involution $f : R^* \rightarrow R^*$ as follows. Given $p \in R^*$ and $x \in \mathfrak{g}_c$, set $(fp)(x) = -\overline{p(\rho(x))}$, for ρ the compact anti-involution defined as in Sect. 4. Combining these actions on R^* and Sym^* we obtain an anti-linear involution $f : \mathcal{A}_{G_c} \rightarrow \mathcal{A}_{G_c}$. It is important to note that whilst other anti involutions may be used instead of ρ , the choice ρ is natural, and this will be seen in following sections as well as in [BaSch].

Let (P, Φ) be a Higgs bundle pair which satisfies the Hitchin equations. Then $f(P, \Phi)$ has Higgs field $f^*(\Phi^*) = -f^*(\rho\Phi)$. Given any invariant polynomial p , we find that

$$\begin{aligned} h(f(P, \Phi))(p) &= p(-f^*(\rho\Phi)) \\ &= -f^*(p\rho\Phi) \\ &= f^*((fp)(\Phi)). \end{aligned}$$

Therefore the Hitchin map commutes with the involutions:

$$\begin{array}{ccc} \mathcal{M}_{G_c} & \xrightarrow{f} & \mathcal{M}_{G_c} \\ \downarrow h & & \downarrow h \\ \mathcal{A}_{G_c} & \xrightarrow{f} & \mathcal{A}_{G_c} \end{array}$$

Remark 15. In the case of $G_c = GL(n, \mathbb{C})$, a generating basis of invariant polynomials is given by the traces of powers $p_j(x) = \text{tr}(x^j)$, and

$$(fp_j)(x) = -\overline{\text{tr}(\rho(x)^j)} = \text{tr}((x^j)^t) = p_j(x),$$

where we have used $\rho x = -x^* = -\overline{x^t}$. Hence for $GL(n, \mathbb{C})$ the above basis of generators for the invariant polynomials are fixed by the involution $f : R^* \rightarrow R^*$.

Let $L \subset \mathcal{A}_{G_c}$ denote the fixed point set of f on \mathcal{A}_{G_c} . It follows that the Hitchin fibration restricts to a map $h|_{\mathcal{L}_{G_c}} : \mathcal{L}_{G_c} \rightarrow L$. For simplicity, we shall drop the subscript of \mathcal{L}_{G_c} , and refer to the (A, B, A) -brane $\mathcal{L} \subset \mathcal{M}_{G_c}$.

Proposition 16. *Away from the singular fibres of the Hitchin map, the restriction $h|_{\mathcal{L}} : \mathcal{L} \rightarrow L$ is a submersion.*

Proof. Recall from Proposition 13 that the restriction of \mathcal{L} to the smooth points of \mathcal{M}_{G_c} is a complex submanifold of the smooth part of \mathcal{M}_{G_c} with respect to J . Thus it is naturally a complex Kähler manifold with complex structure $J|_{\mathcal{L}}$ and Kähler form $\omega_J|_{\mathcal{L}}$. From [Hit92] the non-singular fibres of the Hitchin map are Lagrangian with respect to the complex symplectic form $\Omega_I = \omega_J + i\omega_K$. In particular, ω_J vanishes on the fibres of the Hitchin map.

For any smooth point $x \in \mathcal{L}$ the kernel K_x of $(h|_{\mathcal{L}})_*$ is an isotropic subspace of $T_x\mathcal{L}$ with respect to $\omega_J|_{\mathcal{L}}$, and thus $\dim(K_x) \leq \frac{1}{2}\dim(\mathcal{L})$. But on the other hand the image I_x of $(h|_{\mathcal{L}})_*$ lies in L , so we also have $\dim(I_x) \leq \dim(L) = \frac{1}{2}\dim(\mathcal{L})$. So for any $x \in \mathcal{L}$ the dimensions satisfy

$$\dim(\mathcal{L}) = \dim(K_x) + \dim(I_x) \leq \frac{1}{2}\dim(\mathcal{L}) + \frac{1}{2}\dim(\mathcal{L}) = \dim(\mathcal{L}).$$

Hence we must have $\dim(I_x) = \dim(L)$ and the restriction $h|_{\mathcal{L}}$ is a submersion.

It follows from the above proof that away from the singular fibres of the Hitchin fibration, the fibres of $h|_{\mathcal{L}} : \mathcal{L} \rightarrow L$ are Lagrangian submanifolds of \mathcal{L} , where the symplectic structure on \mathcal{L} is given by $\omega_J|_{\mathcal{L}}$.

Theorem 17. *If \mathcal{L} contains smooth points then the restriction of the Hitchin fibration*

$$h|_{\mathcal{L}} : \mathcal{L} \rightarrow L$$

to \mathcal{L} is a Lagrangian fibration with singularities. The generic fibre is smooth and consists of a finite number of tori.

Proof. Assume that \mathcal{L} contains smooth points. Then \mathcal{L} meets the smooth points of the Higgs bundle moduli space \mathcal{M}_{G_c} in a smooth real analytic submanifold. At a smooth fixed point p the tangent space to \mathcal{L} defines a real structure on $T_p\mathcal{M}_{G_c}$. On the other hand the smooth points where the Hitchin map is not a submersion form a complex analytic subvariety V . By considering power series one sees that the smooth points of \mathcal{L} can not lie entirely in V , so that the Hitchin map is a submersion at a generic smooth point of \mathcal{L} . This also shows that the image $h(U)$ of any non-empty open subset U of \mathcal{L} does not lie entirely in the space of critical values of h . Hence the generic fibre of the restricted Hitchin map $h|_{\mathcal{L}} : \mathcal{L} \rightarrow L$ is smooth, giving a Lagrangian fibration with singularities. On the generic smooth fibres of $h|_{\mathcal{L}}$ we obtain $\dim(\mathcal{L})/2$ commuting vector fields, which are complete since h is a proper map (e.g.: [Hit87] for the rank 2 case). It follows that the generic fibres of $h|_{\mathcal{L}}$ consist of finitely many copies of a torus.

Remark 18. Let $\Delta \subset L$ denote the points of L over which the Hitchin map is not a submersion. Each fibre of \mathcal{L} over $L \setminus \Delta$ consists of a finite number of tori. In general $L \setminus \Delta$ may be disconnected and the number of tori in a given fibre varies as one moves between the components of $L \setminus \Delta$.

The above analysis establishes that the fixed point set \mathcal{L} has the structure of a real integrable system with singularities.

7. Spectral Data for \mathcal{L}_{G_c}

As seen in Proposition 12, the map f induces an involution on the moduli space \mathcal{M}_{G_c} of polystable G_c -Higgs bundles

$$f : \mathcal{M}_{G_c} \rightarrow \mathcal{M}_{G_c}$$

$$(\bar{\partial}_A, \Phi) \mapsto (f^*(\partial_A), f^*(\Phi^*)).$$

In this section we shall describe how f acts in terms of the spectral data associated to G_c -Higgs bundles, for G_c a classical complex Lie group and see that the fixed point set \mathcal{L} is non-empty (see e.g., [Hit07] for the construction of the spectral data).

Remark 19. In the case of $G_c = GL(n, \mathbb{C})$ the action $(\bar{\partial}_A, \Phi) \mapsto (f^*(\partial_A), f^*(\Phi^*))$ makes sense for Higgs bundles of any degree, so that the action of f can be extended to the full moduli space of $GL(n, \mathbb{C})$ -Higgs bundles of arbitrary degree. However we shall see that there can only be fixed points in degree 0. This extended action can be interpreted as the induced action of f on representations of a central extension of $\pi_1(\Sigma)$, following [AB82].

7.1. *Spectral data for $GL(n, \mathbb{C})$.* From [Hit87a] the fibre of the Hitchin fibration for $GL(n, \mathbb{C})$ -Higgs bundles is isomorphic to the Jacobian of a curve. To see this let $p : K \rightarrow \Sigma$ denote the projection from the total space of K to Σ and let η denote the tautological section of p^*K . A $GL(n, \mathbb{C})$ -Higgs bundle (E, Φ) has associated an n -fold cover $p : S \rightarrow \Sigma$ in the total space of K , with equation $\det(\eta - \Phi) = 0$, together with a line bundle U on S satisfying $p_*U = E$. Explicitly, S has equation

$$\eta^n + a_1\eta^{n-1} + a_2\eta^{n-2} + a_3\eta^{n-3} + \dots + a_n = 0, \tag{4}$$

where $a_m \in H^0(\Sigma, K^m)$. The line bundle U associated to (E, Φ) corresponds to the eigenspaces of Φ and may be defined by the exact sequence [BNR89]

$$0 \rightarrow U \otimes p^*K^{1-n} \rightarrow p^*E \xrightarrow{\eta - p^*(\Phi)} p^*(E \otimes K) \rightarrow U \otimes p^*K \rightarrow 0. \tag{5}$$

Conversely given sections $a_i \in H^0(\Sigma, K^i)$ such that the associated curve S defined by Eq. (4) is smooth and a line bundle U on S , one has a corresponding rank n stable Higgs bundle (E, Φ) by defining $E = p_*(U)$ and taking Φ to be the map obtained by pushing forward the tautological section $\eta : U \rightarrow U \otimes p^*K$.

As seen in Sect. 4, we may view a Higgs bundle $(\bar{\partial}_A, \Phi)$ as consisting of a holomorphic structure $\bar{\partial}_A$ and a Higgs field Φ , where $\Phi \in \Omega^{1,0}(\Sigma, \mathfrak{g}_c)$ is holomorphic with respect to $\bar{\partial}_A$. From Proposition 12, the action of the anti-holomorphic involution f sends the pair $(\bar{\partial}_A, \Phi)$ to $f(\bar{\partial}_A, \Phi) = (f^*(\bar{\partial}_A), f^*(\Phi^*))$. Hence the spectral curve for $f(\bar{\partial}_A, \Phi)$ has equation

$$\det(\eta - f^*(\Phi^*)) = 0. \tag{6}$$

The anti-involution f naturally lifts to an anti-holomorphic involution on the total space $\tilde{f} : K \rightarrow K$ by sending $y \in K_x$ to $f^*(\bar{y}) \in K_{f(x)}$.

Definition 20. We denote by \tilde{f} the natural lift of f to all powers K^m of K .

Let $p_K : K \rightarrow \Sigma$ be the projection from the total space of K . Then the map \tilde{f} can be lifted to a natural action $\tilde{f} : p_K^*(K^m) \rightarrow p_K^*(K^m)$ on the total space of $p_K^*(K^m)$. Moreover since $\tilde{f}^*(\eta) = \eta$, one has

$$\tilde{f}(\det(\eta - \Phi)) = \det(\tilde{f}(\eta) - f^*(\Phi^*)) = \det(\eta - f^*(\Phi^*)),$$

and thus the action of \tilde{f} on K restricts to a bijection between the spectral curves for $(\bar{\partial}_A, \Phi)$ and $f(\bar{\partial}_A, \Phi)$. Let $(\bar{\partial}_A, \Phi)$ be a fixed point in the moduli space, so $f(\bar{\partial}_A, \Phi)$ is gauge equivalent to $(\bar{\partial}_A, \Phi)$ under a complex gauge transformation. In particular, since $(\bar{\partial}_A, \Phi)$ and $f(\bar{\partial}_A, \Phi)$ are in the same isomorphism class, this requires the spectral curves to coincide.

Proposition 21. The spectral curves for $(\bar{\partial}_A, \Phi)$ and $f(\bar{\partial}_A, \Phi)$ coincide if and only if the coefficients a_m in (4) satisfy the reality conditions

$$a_m = f^*(\bar{a}_m). \tag{7}$$

Proof. Since Φ and Φ^* have conjugate eigenvalues we find from (6) that the coefficient of λ^{n-m} in the spectral curve for $f(A, \Phi)$ is given by $f^*(\bar{a}_m)$. Hence from (6), the spectral curves for $(\bar{\partial}_A, \Phi)$ and $f(\bar{\partial}_A, \Phi)$ coincide if and only if $a_m = f^*(\bar{a}_m)$ for all m .

Remark 22. When condition (7) holds the map \tilde{f} restricts to an anti-holomorphic involution on the spectral curve S covering the action of f on Σ .

Proposition 23. *For U the eigen-line bundle associated to a classical Higgs pair $(\bar{\partial}_A, \Phi)$, the spectral data on the curve S for $f(\bar{\partial}_A, \Phi)$ is given by the line bundle*

$$\tilde{f}^*(\bar{U}^* \otimes p^*K^{n-1}) = \tilde{f}^*(\bar{U}^*) \otimes p^*K^{n-1}.$$

Proof. Let U' denote the spectral line associated to $f(\bar{\partial}_A, \Phi) = (f^*(\partial_A), f^*(\Phi^*))$. If $\bar{\partial}_A$ is a $\bar{\partial}$ -operator on E then $f^*(\partial_A)$ is the $\bar{\partial}$ -operator naturally associated to $f^*(\bar{E}^*)$. To see this we can locally write $\bar{\partial}_A$ in the form $\bar{\partial}_A = \bar{\partial} + A$ with respect to a unitary frame. Then the corresponding complex structure on $f^*(\bar{E})$ is $\bar{\partial} + f^*(\bar{A})$, and the associated complex structure on $f^*(\bar{E}^*)$ is $\bar{\partial} - f^*(\bar{A}^t)$. The claim follows since $\partial_A = \partial + \rho(A) = \partial - \bar{A}^t$. Similarly the induced Higgs field on $f^*(\bar{E}^*)$ is $f^*(\bar{\Phi}^t) = f^*(\Phi^*)$. To determine the spectral line associated to $f^*(\Phi^*)$ consider the exact sequence (5) defining U . Dualising, pulling back by \tilde{f} , conjugating and tensoring by p^*K , we obtain the exact sequence

$$0 \rightarrow \tilde{f}^*(\bar{U}^*) \rightarrow p^*f^*(\bar{E}^*) \xrightarrow{\eta - p^*(\Phi^*)} p^*(f^*(\bar{E}^*) \otimes K) \rightarrow \tilde{f}^*(\bar{U}^*) \otimes p^*K^n \rightarrow 0.$$

Therefore the spectral line U' for $f(\bar{\partial}_A, \Phi)$ is given by $U' = \tilde{f}^*(\bar{U}^*) \otimes p^*K^{n-1}$.

Definition 24. *We denote by $\iota : Pic(S) \rightarrow Pic(S)$ the natural involution*

$$U \mapsto \tilde{f}^*(\bar{U}^*) \otimes p^*K^{n-1}.$$

It is clear that ι is anti-holomorphic with respect to the natural complex structure on $Pic(S)$. We have thus established the following:

Proposition 25. *Let $(\bar{\partial}_A, \Phi)$ be a stable classical Higgs bundle with smooth spectral curve $S \rightarrow \Sigma$ and eigen-line $U \in Pic(S)$. Then $(\bar{\partial}_A, \Phi)$ is gauge equivalent to $f(\bar{\partial}_A, \Phi)$ if and only if*

- S is carried to itself under the natural lift $\tilde{f} : K \rightarrow K$ of f ;
- U is fixed by the natural involution ι on $Pic(S)$.

Definition 26. *Define $\iota_0 : Jac(S) \rightarrow Jac(S)$ as the involution*

$$\iota_0 : U \mapsto \tilde{f}^*(\bar{U}^*).$$

The involution ι on $Jac(S)$ in Definition 24 can be expressed as

$$U \mapsto \iota_0(U) \otimes p^*K^{n-1},$$

and changes degree according to

$$\deg(\iota U) = -\deg(U) + n(n - 1)(2g - 2).$$

Therefore only line bundles of degree $n(n - 1)(g - 1)$ can be fixed by ι . Let $Jac_d(S)$ denote the component of $Pic(S)$ consisting of line bundles of degree d and write $Jac(S)$ for the Jacobian $Jac_0(S)$. From the above analysis, any fixed point of ι must lie in $Jac_d(S)$ for $d = n(n - 1)(g - 1)$, but we have not yet established the existence of any fixed points. For this let $K^{1/2}$ be a theta characteristic corresponding to an f -invariant

spin structure on Σ . Such spin structures exist by Proposition 4, and by Proposition 5 this corresponds to the existence of a theta characteristic $K^{1/2}$ such that $f^*(\overline{K}^{1/2}) \simeq K^{1/2}$. Now since $p^*K^{(n-1)/2}$ has degree $d = n(n - 1)(g - 1)$, tensoring by $p^*K^{(n-1)/2}$ defines a bijection $Jac(S) \rightarrow Jac_d(S)$. For M a line bundle in $Jac(S)$, the action of ι on $M \otimes p^*K^{(n-1)/2}$ is

$$\iota(M \otimes p^*K^{(n-1)/2}) = \iota_0(M) \otimes p^*K^{(1-n)/2} \otimes p^*K^{n-1} = \iota_0(M) \otimes p^*K^{(n-1)/2}.$$

Thus $M \otimes p^*K^{(n-1)/2}$ is fixed by ι if and only if M is fixed by ι_0 .

Lemma 27. *Let A be a complex torus of dimension m and $\theta : A \rightarrow A$ an anti-holomorphic group automorphism. Then the fixed point set A^θ of A is isomorphic to a product $T \times (\mathbb{Z}_2)^d$ of a real torus T of dimension m and a discrete group $(\mathbb{Z}_2)^d$, for some $0 \leq d \leq m$. Every component of A^θ contains a point of order 2 and there are exactly 2^{m+d} points of order 2 fixed by θ .*

Proof. Since θ is an automorphism the fixed point set A^θ is a closed subgroup, hence a compact abelian subgroup. The identity component A_0^θ is therefore a torus which must have real dimension m since θ is an anti-holomorphic involution. Let $H = A^\theta/A_0^\theta$ be the finite abelian group of components of A^θ . The exact sequence $0 \rightarrow A_0^\theta \rightarrow A^\theta \rightarrow H \rightarrow 0$ splits because A_0^θ is a divisible group, and every element of H is 2-torsion. Indeed if $x \in A^\theta$ then $x + x = x + \theta(x)$ lies in the identity component A_0^θ because it lies in the image of the connected group A under the map $A \rightarrow A^\theta$ which sends a point $y \in A$ to $y + \theta(y)$. It remains only to show that every component of A^θ contains a point of order 2. This follows since A_0^θ is a divisible group. The inequality $d \leq m$ holds since A has 2^{2m} points of order 2.

Proposition 28. *The fixed point set $\mathcal{L}_{GL(n, \mathbb{C})}$ of the involution f on $\mathcal{M}_{GL(n, \mathbb{C})}$ is non-empty and contains smooth points of $\mathcal{M}_{GL(n, \mathbb{C})}$. A generic fibre of the restricted Hitchin fibration $\mathcal{L}_{GL(n, \mathbb{C})} \rightarrow L$ is diffeomorphic to a product $T \times (\mathbb{Z}_2)^d$ of a torus of real dimension $1 + n^2(g - 1)$ and a discrete group $(\mathbb{Z}_2)^d$ for some $d \geq 0$.*

Proof. A generic point in the base L of the restricted Hitchin map $\mathcal{L}_{GL(n, \mathbb{C})} \rightarrow L$ defines a smooth spectral curve $S \rightarrow \Sigma$ and the corresponding fibre is diffeomorphic to the fixed point set of the involution $\iota_0 : Jac(S) \rightarrow Jac(S)$, where $\iota_0(L) = \tilde{f}^*(\overline{L}^*)$. Thus ι_0 is an anti-holomorphic group automorphism of $Jac(s)$ and we may apply Lemma 27.

Similar methods can be used to show that \mathcal{L}_{G_c} is non-empty for classical complex Lie groups G_c . For completion, we shall show here how the method applies in the case of $G_c = SL(n, \mathbb{C})$.

7.2. Spectral data for $SL(n, \mathbb{C})$. An $SL(n, \mathbb{C})$ -Higgs bundle for $n \geq 2$ is a pair (E, Φ) where E is a rank n vector bundle with trivial determinant and Φ is a trace-free Higgs field. As usual we let ∂_A denote the $\bar{\partial}$ -operator defining the holomorphic structure on E .

The spectral curve $p : S \rightarrow \Sigma$ associated to an $SL(n, \mathbb{C})$ -Higgs pair (∂_A, Φ) has equation

$$\eta^n + a_2\eta^{n-2} + a_3\eta^{n-3} + \dots + a_n = 0, \tag{8}$$

where a_m is a holomorphic section in $H^0(\Sigma, K^m)$. From Proposition 12 the action of f maps $(\bar{\partial}_A, \Phi) \mapsto f(\bar{\partial}_A, \Phi) = (f^*(\bar{\partial}_A), f^*(\Phi^*))$. Thus one can define the spectral curve for $f(\bar{\partial}_A, \Phi)$ which has equation $\det(\eta - f^*(\Phi^*)) = 0$.

From Proposition 21, the spectral curves for $(\bar{\partial}_A, \Phi)$ and $f(\bar{\partial}_A, \Phi)$ coincide if and only if the coefficients a_m in (8) satisfy the reality conditions $a_m = f^*(\overline{a_m})$. In such a case we have that \tilde{f} restricts to an anti-holomorphic involution on the spectral curve S covering f .

From [Hit07, Section 2.2], the spectral data in this case is given by an eigen-line bundle U in the Jacobian of S for which $U \otimes p^*K^{(n-1)/2}$ lies in the Prym variety $\text{Prym}(S, \Sigma)$. Recall that $\text{Prym}(S, \Sigma)$ is defined by the exact sequence

$$1 \longrightarrow \text{Prym}(S, \Sigma) \longrightarrow \text{Jac}(S) \xrightarrow{Nm} \text{Jac}(\Sigma) \longrightarrow 1,$$

where the norm map $Nm : \text{Jac}(S) \rightarrow \text{Jac}(\Sigma)$ is defined by sending a divisor $D = \sum_i s_i$ on S to the divisor $Nm(D) = \sum_i p(s_i)$ on Σ . Note that $\text{Prym}(S, \Sigma)$ in this situation is connected, hence a complex torus which is in fact an abelian variety.

Through Proposition 23, the spectral data associated to the pair $f(\bar{\partial}_A, \Phi)$ is given by the line bundle

$$\tilde{f}^*(\overline{U}^* \otimes p^*\overline{K}^{n-1}) = \tilde{f}^*(\overline{U}^*) \otimes p^*K^{n-1}.$$

To proceed to $SL(n, \mathbb{C})$ -bundles we must understand how the fixed point set of ι_0 meets the Prym variety.

Proposition 29. *The action of ι_0 on $\text{Jac}(S)$ preserves the Prym variety.*

Proof. Recall that the norm map Nm may also be defined as the dual map of the pullback $p^* : \text{Jac}(\Sigma) \rightarrow \text{Jac}(S)$. Let $\iota'_0 : \text{Jac}(\Sigma) \rightarrow \text{Jac}(\Sigma)$ be defined by $\iota'_0(L) = f^*(\overline{L}^*)$, then clearly $p^*\iota'_0 = \iota_0 p^* : \text{Jac}(\Sigma) \rightarrow \text{Jac}(S)$. It follows that $\iota'_0 \circ Nm = Nm \circ \iota_0$, and in particular ι_0 preserves the Prym variety, which is given by the kernel of Nm .

Applying Lemma 27 we obtain the following:

Proposition 30. *The generic fibres of the restriction of the Hitchin fibration $\mathcal{L}_{SL(n, \mathbb{C})} \rightarrow L$ are products $T \times (\mathbb{Z}_2)^d$ of a real torus T of dimension $(n^2 - 1)(g - 1)$ and a discrete group $(\mathbb{Z}_2)^d$, where the number of points of order 2 in $\text{Prym}(S, \Sigma)$ fixed by ι_0 is $2^{(n^2-1)(g-1)+d}$.*

The above study establishes the existence of fixed points for the action of f on the moduli space of $SL(n, \mathbb{C})$ -Higgs bundles. Moreover, it confirms the picture given in Sect. 6, where the fixed point set is shown to be a real integrable system.

Remark 31. One should note that the moduli space of $SU(p, p)$ -Higgs bundles can be seen inside the moduli space of $SL(2p, \mathbb{C})$ -Higgs bundles. In particular from [Sch, Chapter 6], the spectral data for this subspace corresponds to line bundles L of fixed determinant which are preserved by the natural involution $\sigma : \eta \mapsto -\eta$ on the corresponding spectral curve. Hence, by looking at the relation between σ and \tilde{f} one can understand the fixed point set of f in the moduli space $\mathcal{M}_{SU(p,p)}$ (equivalently, in $\mathcal{M}_{U(p,p)}$).

Applying the same methods for the case of $G_c = PGL(n, \mathbb{C}), Sp(2n, \mathbb{C}), SO(2n + 1, \mathbb{C})$, and $SO(2n, \mathbb{C})$, one can see through the spectral data introduced in [Hit07] that the fixed point set \mathcal{L}_{G_c} is non-empty and contains smooth points. Moreover the generic fibre of the restricted Hitchin map is the set of real points in an abelian variety, diffeomorphic to 2^d torus components for some d . In the following section we shall study the topology of the (A, B, A) -branes \mathcal{L}_{G_c} .

8. Connectivity of the Brane \mathcal{L}_{G_c}

In order to study connectivity of the (A, B, A) -brane \mathcal{L}_{G_c} in the moduli space \mathcal{M}_{G_c} of G_c -Higgs bundles, we consider the fibres of the restricted Hitchin fibration

$$h|_{\mathcal{L}_{G_c}} : \mathcal{L}_{G_c} \rightarrow L.$$

In particular we shall see that the number of connected components of the fibres depends on the invariants (n, a) associated to the Riemann surface and the spectral curves, as introduced in Sect. 2.1 following [GH81].

8.1. Connectivity for classical Higgs bundles.

Proposition 32. *Let $S \rightarrow \Sigma$ be a smooth $GL(n, \mathbb{C})$ spectral curve, n_S the number of fixed point components of the lifted involution $\tilde{f} : S \rightarrow S$, and $g_S = 1 + n^2(g - 1)$ the genus of S . Then the number of connected components of the corresponding fibre of*

$$h|_{\mathcal{L}_{GL(n, \mathbb{C})}} : \mathcal{L}_{GL(n, \mathbb{C})} \rightarrow L$$

is 2^{n_S-1} if $n_S > 0$. If $n_S = 0$ the number of components is 1 if g_S is even, and 2 if g_S is odd.

Proof. Recall that $\iota_0 : Jac(S) \rightarrow Jac(S)$ is given by $\iota_0(L) = \tilde{f}^*(\overline{L}^*)$ and that the fibre of the Hitchin map corresponding to the spectral curve S is given by the fixed point set of ι_0 . Thus we must determine the number of components of the fixed point set of ι_0 . From Lemma 27 we see that the number of components of this fixed point set is the same as the number of components of the fixed point set of the involution $\theta : Jac(S) \rightarrow Jac(S)$, where $\theta(L) = \tilde{f}^*(\overline{L})$. The involution θ is studied in [GH81, Propositions 3.2–3.3], where it is determined that the number of components of the fixed point set is 2^{n_S-1} if $n_S > 0$. In the case that $n_S = 0$ the number of components is shown to be 1 or 2 depending on whether the genus of the curve is even or odd.

It remains to determine how n_S depends on the pair (Σ, f) and the coefficients $a_i \in H^0(\Sigma, K^i)$ defining the spectral curve S . As the rank n increases this quickly becomes difficult so we restrict attention to the rank 2 case in Sect. 8.2.

Remark 33. In the case of Accola–Maclachlan and Kulkarni surfaces, the invariants (n, a) are determined in [BBCGG] for any genus g . Thus, the study of the invariants for the ramified covering S in this case gets simplified, and connectivity of $\mathcal{L}_{GL(n, \mathbb{C})}$ for these Riemann surfaces can be determined through [GH81]. For hyperelliptic surfaces, a study of the invariants associated to f has been done in [BCGG], and extended to our case in Appendix A.1.

8.2. *Rank 2-case.* Consider rank 2 Higgs bundles. In this case the Hitchin fibration is given by

$$h : (E, \Phi) \mapsto (a_1, a_2) \in H^0(\Sigma, K) \oplus H^0(\Sigma, K^2),$$

and the corresponding twofold cover $p : S \rightarrow \Sigma$ has genus $g_S = 1 + 4(g - 1)$ and equation $\eta^2 + a_1\eta + a_2 = 0$. Without loss of generality we assume that $a_1 = 0$, since we can re-write the previous equation as $(\eta + a_1/2)^2 + (a_2 - a_1^2/4) = 0$. The spectral curve S is therefore a double cover given by

$$S = \{(\eta, z) \in K \mid \eta^2 = q(z)\}$$

for $q = -a_2 \in H^0(\Sigma, K^2)$. Generically q has $4g - 4$ different zeros, which give the ramification points of the smooth curve S . The reality condition on the defining equation for S is simply that $f^*(\bar{q}) = q$, and in particular f must act on the zero set of q .

The anti-linear involution f on the Riemann surface Σ induces an involution $\tilde{f} : S \rightarrow S$, which from Sect. 6 acts by

$$(\eta, z) \mapsto (\overline{f^*(\eta)}, f(z)) \tag{9}$$

The study of the invariant n_S depends on how f acts on the zero set of q . First note that since \tilde{f} covers f , any fixed component of \tilde{f} must lie over a fixed component of f . Therefore to determine n_S we need only determine how many fixed components of \tilde{f} lie over each fixed component of f . Given a fixed component of f there are two cases to consider depending on whether or not q has zeros along the component.

Proposition 34. *If S^1 is a fixed circle component of f in Σ and the differential q does not have any zeros on S^1 , then $p^{-1}(S^1)$ is a disjoint union of two circles. Let $x \in S^1$ be any point of the fixed component. With respect to a local coordinate z centred on x such that $f(z) = \bar{z}$, q may be written as $q(z) = q_1(z)(dz)^2$, where $q_1(\bar{z}) = \overline{q_1(z)}$. Then*

- the two S^1 factors in S are fixed by \tilde{f} if $q_1(0) > 0$ on S^1 ,
- the two S^1 factors in S are exchanged by \tilde{f} if $q_1(0) < 0$ on S^1 .

Proof. Let x be a fixed point of f such that $q(x) \neq 0$. By choosing an appropriate local holomorphic coordinate z , we can write $f(z) = \bar{z}$ in a neighbourhood of x , corresponding to $z = 0$. In local coordinates the differential can be expressed as

$$q(z) = q_1(z)(dz)^2 \quad \text{for } q_1(0) \neq 0,$$

and by rescaling z by a real factor if necessary, we may assume that $q_1(0) = \pm 1$. The curve S is thus given by

$$\eta^2 = q_1(z)(dz)^2,$$

and so at $z = 0$ we have

$$\eta = \begin{cases} \pm dz & \text{for } q_1(0) = 1; \\ \pm i dz & \text{for } q_1(0) = -1. \end{cases}$$

In the case of $q_1(0) = 1$ the points lying over x are $(\pm dz, 0)$, and by (9) they are both fixed by \tilde{f} . In the case of $q_1(0) = -1$ the points over x are $(\pm i dz, 0)$, which are interchanged by \tilde{f} .

Consider now $S^1 \subset \Sigma$ a component of the fixed points of f in Σ which does not contain zeros of q . As above, for $x \in S^1$ we can choose a holomorphic local coordinate z such that $f(z) = \bar{z}$, and such that the point x corresponds to $z = 0$. For any other such coordinate w , one can write

$$w = a_0z + a_1z^2 + \dots \text{ for } a_0 \neq 0, a_i \in \mathbb{R}.$$

Then $dw|_x = a_0dz|_x$, and this defines a real subbundle $K|_{S^1}(\mathbb{R})$ of the restriction $K|_{S^1}$ of K to the fixed circle. Note that since we can find a non-vanishing vector field $X \in T\Sigma|_{S^1}$ tangent to S^1 , the subbundle $K|_{S^1}(\mathbb{R})$ is trivial. Therefore since $K|_{S^1}(\mathbb{R})$ is a trivial real bundle and $\pm\eta$ are non-vanishing along the circle, the double cover $p^{-1}(S^1) \rightarrow S^1$ defined by $\eta^2 = q$ must be a trivial double cover $p^{-1}(S^1) = S^1 \cup S^1$.

We shall now consider the case of a fixed circle $S^1 \subset \Sigma$ which contains one or more zeros of the quadratic differential q . Let $x \in S^1$ be such that $q(x) = 0$. Choose a local holomorphic coordinate z centred at x satisfying $f(z) = \bar{z}$. As in the above proof we can write $q(z) = q_1(z)(dz)^2$, where $q_1(\bar{z}) = \overline{q_1(z)}$ and now $q_1(0) = 0$. Since we are assuming that q has only simple zeros we have $dq_1(z) \neq 0$ at $z = 0$. Hence we can take $q_1(z)$ as a local coordinate $w = q_1(z)$. In terms of this coordinate, the double cover $p : S \rightarrow \Sigma$ given by $(\eta dz, z) \mapsto z$ can be written as

$$\eta \mapsto \eta^2 = q_1(z) = w.$$

Recall that $\tilde{f}(\eta) = \bar{\eta}$ and $f(w) = \bar{w}$. If η is real one has $\tilde{f}(\eta) = \eta$ and if η is imaginary we get $\tilde{f}(\eta) = -\eta$. Locally the fixed points of f are the points where w is real and so

- for $q_1 = w < 0$, we have that $\pm\eta \in i\mathbb{R}$ and \tilde{f} exchanges η and $-\eta$,
- for $q_1 = w > 0$, we have that $\pm\eta \in \mathbb{R}$ and \tilde{f} fixes $\pm\eta$.

From the above analysis one has the following proposition:

Proposition 35. *Let $S^1 \subset \Sigma$ be a fixed component of f on which q has at least one zero. Then q has an even number $2k$ of zeros which divides the circle into $2k$ segments over which we have $q \geq 0$ and $q \leq 0$ alternately. For each of the k segments where $q \geq 0$, the inverse image of the segment in S is a circle fixed by \tilde{f} . For each of the k segments where $q \leq 0$, the inverse image of the segment is a circle such that \tilde{f} acts as a reflection. In particular the inverse image $p^{-1}(S)$ contains exactly k circles fixed by \tilde{f} .*

It is possible to strengthen this result to a precise description of S in a neighborhood of the inverse image $p^{-1}(S^1)$. The inverse image $p^{-1}(S^1)$ is the graph obtained by taking $2k$ circles and joining them in a chain which has Euler characteristic $2k - 4k = -2k$. We can enlarge each circle of $p^{-1}(S^1)$ to a tubular neighbourhood in S such that their union is a neighbourhood of $p^{-1}(S^1)$ diffeomorphic to a surface of genus $k - 1$ with 4 points removed. Combining Propositions 34 and 35 we obtain:

Theorem 36. *Let $q \in H^0(\Sigma, K^2)$ be a quadratic differential which has only simple zeros and such that $f^*(\bar{q}) = q$, let S be the spectral curve given by $\eta^2 = q$. Let n_+ be the number of fixed components of f on which q is non-vanishing and positive, and u the number of zeros of q which are fixed by f . Then the number of components of the induced involution $\iota_0 : Jac(S) \rightarrow Jac(S)$ is 2^d , where $d = 2n_+ + u/2 - 1$ if $2n_+ + u/2 > 0$, and $d = 1$ otherwise.*

It is natural to ask which pairs $(n_+, u/2)$ can occur for a given pair (Σ, f) with associated invariants (n, a) . Note that in particular one has $0 \leq n_+ \leq n$ and $0 \leq u/2 \leq 2g - 2$. Moreover if $u > 0$ then $n_+ < n$, and if $n = 0$ then $u = 0$. Even with these constraints there are a large number of possibilities that can occur. For example in the genus $g = 2$ case the invariants $(n, a, n_+, u/2)$ subject to the above constraints together with the constraints on (n, a) from Proposition 3 give rise to a total of 26 possible cases. To determine which of these possible cases actually occur one can use the explicit description of the space of quadratic differentials on a genus 2 hyperelliptic curve given in Appendix A. Remarkably it turns out that all but one of the 26 cases can actually be realized, the exception being the case $(n, a, n_+, u/2) = (1, 0, 0, 1)$.

8.3. *Connectivity for $SL(2, \mathbb{C})$ -Higgs bundles.* For $SL(2, \mathbb{C})$ -Higgs bundles we have seen that the fibres of the Hitchin fibration are Prym varieties $\text{Prym}(S, \Sigma)$, for S the spectral curve defined by the point in the Hitchin base. Hence in this case we are interested in the number of connected components of the intersection of $\text{Prym}(S, \Sigma)$ with the real points in the Jacobian of S .

Let $q \in H^0(\Sigma, K^2)$ be a quadratic differential with simple zeros and satisfying $f^*(q) = \bar{q}$. In particular f preserves the set of $4g - 4$ branch points of q . For $p : S \rightarrow \Sigma$ the branched double cover of degree 2 defined by the equation $\eta^2 = q$, its genus is $g_S = 1 + 4(g - 1)$. The natural action of f on K defines a lift $\tilde{f} : S \rightarrow S$ of f . We shall denote by $\sigma : S \rightarrow S$ the involution which exchanges sheets, which commutes with \tilde{f} .

Since $S \rightarrow \Sigma$ is a double cover the Prym variety $\text{Prym}(S, \Sigma)$ may be defined as those line bundles L satisfying $\sigma^*L \simeq L^*$ and is a complex torus of dimension $3g - 3$. We are interested in the set of elements $L \in \text{Prym}(S, \Sigma)$ which satisfy the reality condition $\tilde{f}^*(\bar{L}^*) = L$. Since this set is the product of a real torus of dimension $3g - 3$ and a discrete group $(\mathbb{Z}_2)^d$, the fixed point set has 2^d components for some d . The strategy we will employ to find d is to look for points of order 2 in $\text{Prym}(S, \Sigma)$ satisfying the reality condition.¹ Let P denote the set of such points, which is a subgroup of $\text{Prym}(S, \Sigma)$. From Lemma 27, the set P is isomorphic to $(\mathbb{Z}_2)^{3g-3+d}$, and thus it will suffice to determine the order of P .

By construction, elements of P are line bundles L on S satisfying the following three conditions: L^2 is trivial, $\sigma^*(L) = L$ and $\tilde{f}^*(\bar{L}) = L$. Since L has order 2, it is equivalent to view L as a flat line bundle with transition functions in \mathbb{Z}_2 . Then as flat line bundles $L \simeq \bar{L} \simeq L^*$. Thus the set P corresponds to elements of $H^1(S, \mathbb{Z}_2)$ which are invariant under the actions of σ and \tilde{f} . Let $H^1(S, \mathbb{Z}_2)^\sigma$ denote the subgroup of elements of $H^1(S, \mathbb{Z}_2)$ fixed by σ . We will first determine this group and then find the subgroup of elements fixed by \tilde{f} .

Let $b = 4g - 4$ be the number of branch points of p , and $B = \{x_1, \dots, x_b\}$ the set of branch points and $\tilde{x}_i = p^{-1}(x_i)$. For $\Sigma' = \Sigma \setminus B$ and $S' = S \setminus p^{-1}(B)$, there is a commutative diagram

$$\begin{array}{ccc}
 S' & \xrightarrow{i_S} & S \\
 \downarrow p & & \downarrow p \\
 \Sigma' & \xrightarrow{i_\Sigma} & \Sigma
 \end{array}$$

¹ One should note that from [Sch11] these points correspond to real points given by $SL(2, \mathbb{R})$ -Higgs bundles.

Moreover $p : S' \rightarrow \Sigma'$ is a degree 2 covering space, so we obtain an exact sequence

$$1 \longrightarrow \pi_1(S', \tilde{x}) \longrightarrow \pi_1(\Sigma', x) \xrightarrow{\omega} \mathbb{Z}_2 \longrightarrow 1, \tag{10}$$

where $\tilde{x} \in S'$ and $x = p(\tilde{x}) \in \Sigma'$. The map $\omega : \pi_1(\Sigma', x) \rightarrow \mathbb{Z}_2$ is the cohomology class $\omega \in H^1(\Sigma', \mathbb{Z}_2)$ corresponding to the double cover $p : S' \rightarrow \Sigma'$.

Proposition 37. *There is an exact sequence*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow H^1(\Sigma', \mathbb{Z}_2) \xrightarrow{p^*} i_S^*(H^1(S, \mathbb{Z}_2)^\sigma) \longrightarrow 0,$$

where the kernel of p^* is generated by ω . Moreover, $i_S^* : H^1(S, \mathbb{Z}_2) \rightarrow H^1(S', \mathbb{Z}_2)$ is injective and thus one has an isomorphism $H^1(S, \mathbb{Z}_2)^\sigma \simeq H^1(\Sigma', \mathbb{Z}_2)/\langle \omega \rangle$.

Proof. The Mayer–Vietoris sequence applied to Σ and S gives a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(S, \mathbb{Z}_2) & \xrightarrow{i_S^*} & H^1(S', \mathbb{Z}_2) & \longrightarrow & \mathbb{Z}_2^b & \xrightarrow{\delta} & H^2(S, \mathbb{Z}_2) & \longrightarrow & 0 \\ & & \uparrow p^* & & \uparrow p^* & & \uparrow 0 & & \uparrow 0 & & \\ 0 & \longrightarrow & H^1(\Sigma, \mathbb{Z}_2) & \xrightarrow{i_\Sigma^*} & H^1(\Sigma', \mathbb{Z}_2) & \longrightarrow & \mathbb{Z}_2^b & \xrightarrow{\delta} & H^2(\Sigma, \mathbb{Z}_2) & \longrightarrow & 0 \end{array}$$

This shows that $i_S^* : H^1(S, \mathbb{Z}_2) \rightarrow H^1(S', \mathbb{Z}_2)$ is injective and that the image of the map $p^* : H^1(\Sigma', \mathbb{Z}_2) \rightarrow H^1(S', \mathbb{Z}_2)$ is contained in the image of i_S^* . Clearly also the image of p^* is σ -invariant since $p\sigma = p$, so one has the inclusion $p^*(H^1(\Sigma', \mathbb{Z}_2)) \subseteq i_S^*(H^1(S, \mathbb{Z}_2)^\sigma)$. From the exact sequence (10) we see that the kernel of $p^* : H^1(\Sigma', \mathbb{Z}_2) \rightarrow i_S^*(H^1(S, \mathbb{Z}_2)^\sigma)$ is $\langle \omega \rangle = \mathbb{Z}_2$, so it remains to show surjectivity of p^* .

Let \tilde{x} be in a small σ -invariant disc $D \subset S$ containing a single branch point \tilde{x}_1 , such that $\tilde{x} \neq \tilde{x}_1$. Let γ be a path in $D \setminus \{\tilde{x}_1\}$ from \tilde{x} to $\sigma(\tilde{x})$. We may also choose D small enough so that the restriction $p|_D$ has the form $z \mapsto z^2$. Define a map $\hat{\sigma} : \pi_1(S, \tilde{x}) \rightarrow \pi_1(S, \tilde{x})$ by $\hat{\sigma}(\alpha) = \gamma \cdot \sigma(\alpha) \cdot \gamma^{-1}$. Then $\hat{\sigma}$ is an automorphism of $\pi_1(S, \tilde{x})$ whose pullback induces an action on cohomology groups $\sigma^* : H^1(S, \mathbb{Z}_2) \rightarrow H^1(S, \mathbb{Z}_2)$ under the identification $H^1(S, \mathbb{Z}_2) = \text{Hom}(\pi_1(S, \tilde{x}), \mathbb{Z}_2)$.

Let $\rho : \pi_1(S) \rightarrow \mathbb{Z}_2$ be a homomorphism which is σ^* -invariant, so $\rho \circ \hat{\sigma} = \rho$. We may then define a representation $\tau : \pi_1(\Sigma') \rightarrow \mathbb{Z}_2$ such that $p^*(\tau) = i_S^*(\rho)$. Indeed consider a loop α in Σ' based at x . Then exactly one of the loops α , $p(\gamma) \cdot \alpha$ lifts to a loop in S' based at \tilde{x} . Let $\hat{\alpha}$ be this lift and set $\tau(\alpha) = \rho(\hat{\alpha})$. We claim that τ is a homomorphism. This follows from the fact that ρ is $\hat{\sigma}$ -invariant, and since $\rho(\gamma \cdot \sigma(\gamma)) = 1$ because $\gamma \cdot \sigma(\gamma)$ is null-homotopic in S . By construction it is immediate that $p^*(\tau) = i_S^*(\rho)$.

To find the group of \tilde{f} -invariant elements of $H^1(S, \mathbb{Z}_2)^\sigma$ it is equivalent to find the f -invariant elements of $H^1(\Sigma', \mathbb{Z}_2)/\langle \omega \rangle$, since we have the commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}_2 & \longrightarrow & H^1(\Sigma', \mathbb{Z}_2) & \xrightarrow{p^*} & i_S^*(H^1(S, \mathbb{Z}_2)^\sigma) \\ \uparrow id & & \uparrow f^* & & \uparrow \tilde{f}^* \\ \mathbb{Z}_2 & \longrightarrow & H^1(\Sigma', \mathbb{Z}_2) & \xrightarrow{p^*} & i_S^*(H^1(S, \mathbb{Z}_2)^\sigma). \end{array}$$

For $A = H^1(\Sigma', \mathbb{Z}_2)/\langle \omega \rangle$, the involution f acts on A , and we are after the group A^f of f -invariant elements of A . Let $A^* = \text{Hom}(A, \mathbb{Z}_2)$. Since A is a \mathbb{Z}_2 -vector space, A and A^* are isomorphic, and the fixed point subspace A^f is isomorphic to the subspace $(A^*)^f$ of A^* fixed by the dual action of f on A^* . The action $f_* : A^* \rightarrow A^*$ fits into a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & A^* & \longrightarrow & H_1(\Sigma', \mathbb{Z}_2) & \xrightarrow{\omega} & \mathbb{Z}_2 & \longrightarrow & 0 \\
 & & \uparrow f_* & & \uparrow f_* & & \uparrow id & & \\
 0 & \longrightarrow & A^* & \longrightarrow & H_1(\Sigma', \mathbb{Z}_2) & \xrightarrow{\omega} & \mathbb{Z}_2 & \longrightarrow & 0.
 \end{array}$$

Thus, to determine $(A^*)^f$ it is sufficient to determine the group $H_1(\Sigma', \mathbb{Z}_2)$, the action of the map $f_* : H_1(\Sigma', \mathbb{Z}_2) \rightarrow H_1(\Sigma', \mathbb{Z}_2)$ and the homomorphism $\omega : H_1(\Sigma', \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$. Note that the Mayer-Vietoris sequence applied to Σ gives an exact sequence

$$0 \longrightarrow H_2(\Sigma, \mathbb{Z}_2) \xrightarrow{\partial} \mathbb{Z}_2^b \longrightarrow H_1(\Sigma', \mathbb{Z}_2) \xrightarrow{(i_\Sigma)_*} H_1(\Sigma, \mathbb{Z}_2) \longrightarrow 0.$$

The above \mathbb{Z}_2^b group corresponds to cycles around each of the b branch points, and the boundary $\partial : H^2(\Sigma, \mathbb{Z}_2) = \mathbb{Z}_2 \rightarrow \mathbb{Z}_2^b$ is the diagonal $\mathbb{Z}_2 \rightarrow \mathbb{Z}_2^b$. In particular, $H_1(\Sigma', \mathbb{Z}_2)$ is a \mathbb{Z}_2 -vector space of dimension $2g + b - 1 = 6g - 5$.

Theorem 38. *Suppose that at least one branch point of $p : S \rightarrow \Sigma$ is fixed by f . Then the fixed point set of the action of \tilde{f} on the Prym variety $\text{Prym}(S, \Sigma)$ has $2^{n_0+u/2-1}$ connected components, where n_0 is the number of fixed components of $f : \Sigma \rightarrow \Sigma$ which do not contain branch points, and u is the number of branch points which are fixed by f .*

Proof. Let A^* be the kernel of $\omega : H_1(\Sigma', \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ and $(A^*)^f$ the subspace of A^* fixed by f_* . We must show that $(A^*)^f$ has dimension $3g - 3 + n_0 + u/2 - 1$. Consider a cycle $D \in H_1(\Sigma', \mathbb{Z}_2)$ around a branch point which is fixed by f . Then $f_*(D) = D$ and $\omega(D) = 1$. Therefore it suffices to show that the kernel of $\theta = f_* - 1 : H_1(\Sigma', \mathbb{Z}_2) \rightarrow H_1(\Sigma', \mathbb{Z}_2)$ has dimension $3g - 3 + n_0 + u/2$.

Recall that there are $b = 4g - 4$ branch points and write $b = 2t + u$, where t is the number of pairs of branch points exchanged by f and u is the number of fixed branch points. For $i = 1, \dots, t$ let C_i, C'_i be cycles around a pair of branch points exchanged by f . Further, for $j = 1, \dots, u$ denote by D_j cycles around the fixed branch points. From the Mayer-Vietoris sequence we have a single relation between these cycles $\sum_{i=1}^t C_i + C'_i + \sum_{j=1}^u D_j = 0$. These cycles span a $b - 1$ -dimensional subspace of $H_1(\Sigma', \mathbb{Z}_2)$ and we have $\theta(C_i) = \theta(C'_i) = C_i + C'_i$, and $\theta(D_j) = 0$.

Let $a_1, \dots, a_{2g} \in H_1(\Sigma, \mathbb{Z}_2)$ be a basis of cycles in Σ . We may assume that the cycles do not touch the branch points so that a_1, \dots, a_{2g} are also cycles in Σ' . Together with the cycles C_i, C'_i, D_j we have a generating set for $H_1(\Sigma', \mathbb{Z}_2)$ satisfying $\sum_{i=1}^t C_i + C'_i + \sum_{j=1}^u D_j = 0$. To proceed we consider three subcases depending on the topology of (Σ, f) . As in previous sections, we let (n, a) be the topological invariants associated to the anti-holomorphic involution f .

Case 1: $a = 0$. Write $g = 2s + r, n = r + 1$. We have a generating set for $H_1(\Sigma', \mathbb{Z}_2)$ consisting of $A_i, B_i, A'_i, B'_i, A''_j, B''_j, C_k, C'_k, D_l$ for $1 \leq i \leq s$, for $1 \leq j \leq r$, for $0 \leq k \leq t$, and for $1 \leq l \leq u$ with one relation given by $\sum_{k=1}^t C_k + C'_k + \sum_{l=1}^u D_l = 0$.

The cycles A_i, A'_i are interchanged by f_* and similarly for B_i, B'_i . The cycles A''_i correspond to a choice of $r = n - 1$ of the fixed components of f , perturbed slightly so as to avoid the branch points. Let $\sum_{h \in A''_j} D_h$ denote the sum of cycles D_h where D_h is a cycle around a branch point which lies in the fixed component of f corresponding to A''_j . Finally the B''_j are cycles that cross two of the fixed components of f . We can choose the A''_j, B''_j so that $\theta(A''_j) = \sum_{h \in A''_j} D_h$ and $\theta(B''_j) = 0$. It follows that $\text{Ker}(\theta)$ has dimension $2s + r + (n_0 - 1) + t + u = 3g - 3 + n_0 + u/2$ as required.

Case 2: $a = 1$, and $g - n$ even. Write $g = 2s + r, n = r$. We have a generating set for $H_1(\Sigma', \mathbb{Z}_2)$ consisting of $A_i, B_i, A'_i, B'_i, A''_j, B''_j, C_k, C'_k, D_l$ with one relation as before. As previously, f_* exchanges the pairs A_i, A'_i and B_i, B'_i , and the A''_j correspond to the fixed components of f . We again have $\theta(A''_j) = \sum_{h \in A''_j} D_h$, but now $\theta(B''_j) = \sum_{j=1}^r A''_j + \sum_{k=1}^t C_k$. As before, we find that $\text{Ker}(\theta)$ has dimension $3g - 3 + n_0 + u/2$.

Case 3: $a = 1$, and $g - n$ odd. Write $g = 2s + r + 1, n = r$. We have a generating set for $H_1(\Sigma', \mathbb{Z}_2)$ consisting of X, Y , and $A_i, B_i, A'_i, B'_i, A''_j, B''_j, C_k, C'_k, D_l$, with one relation as before. Moreover, $f_*(X) = X$ and $f_*(Y) = Y + \sum_{j=1}^r A''_j$. The pairs A_i, A'_i and B_i, B'_i are exchanged by f_* . As before A''_j correspond to fixed components of the involution f and satisfy $\theta(A''_j) = \sum_{h \in A''_j} D_h$. Finally we have $\theta(B''_j) = Y$. In this case $\text{Ker}(\theta)$ has dimension $3g - 3 + n_0 + u/2$.

9. Real and Quaternionic Bundles

In this section we consider the relation between Higgs bundles fixed by the induced action of an anti-holomorphic involution $f : \Sigma \rightarrow \Sigma$ and bundles with real or quaternionic structure. In particular we focus on the case of $SL(2, \mathbb{C})$ -Higgs bundles.

Let E, F be rank n holomorphic vector bundles on Σ and $\phi : E \rightarrow F$ an anti-linear bundle isomorphism covering f . The map ϕ can be extended to a map $\phi : \Omega^{(p,q)}(E) \rightarrow \Omega^{(p,q)}(F)$ of form-valued sections as follows. For any point $x \in \Sigma$ choose a local trivialisation of E near x , and a trivialisation of F near $f(x)$. A local section s of E in this trivialisation is a \mathbb{C}^n -valued function defined near x . The corresponding local section of F near $f(x)$ is of the form $\phi(s) = g f^*(\bar{s})$, for some locally defined $GL(n, \mathbb{C})$ -valued function g . The extension of ϕ to form-valued sections is given by setting $\phi(\omega \otimes s) = f^*(\bar{\omega}) \otimes g f^*(\bar{s})$, where ω is a form on Σ defined near x . We say that such a map $\phi : E \rightarrow F$ is holomorphic if it sends holomorphic sections to holomorphic sections, or letting $\bar{\partial}_E, \bar{\partial}_F$ denote the corresponding $\bar{\partial}$ -operators on E, F , we have $\phi \circ \bar{\partial}_E = \bar{\partial}_F \circ \phi$.

Let (E, Φ) be a rank n Higgs bundle satisfying the Hitchin equations and whose isomorphism class is fixed by f . Since E must have degree 0 we can take it to be the trivial rank n bundle equipped with the constant Hermitian structure. Let $\bar{\partial}_A$ be the $\bar{\partial}$ -operator defining the holomorphic structure, so $\bar{\partial}_A = \bar{\partial} + A$ for some $A \in \Omega^{0,1}(\mathfrak{gl}(n, \mathbb{C}))$. Since (E, Φ) satisfies the Hitchin equations we have that $f(E, \Phi) = (f^*(\partial_A), f^*(\bar{\Phi}^t))$, where $\partial_A = \partial - \bar{A}^t$. Moreover $f^*(\partial_A) = \bar{\partial} - f^*(\bar{A}^t)$ and thus supposing that $f(E, \Phi)$ is isomorphic to (E, Φ) is equivalent to the existence of an anti-linear isomorphism $\phi : E \rightarrow E^*$ covering f , holomorphic in the sense described above and such that $\phi \circ \Phi^t = \Phi \circ \phi$. Note that this gives a purely holomorphic interpretation of the condition for a Higgs bundle (E, Φ) to be fixed by f .

Suppose now that (E, Φ) is an $SL(2, \mathbb{C})$ -Higgs bundle. Then we have an isomorphism $(E^*, \Phi^t) \simeq (E, -\Phi)$. In this case if the isomorphism class is fixed by f we have a holomorphic anti-linear isomorphism $\phi : E \rightarrow E$ covering f and such that $\phi^{-1} \circ \Phi \circ \phi = -\Phi$. Note then that $\phi^2 : E \rightarrow E$ is a linear isomorphism covering the identity, preserving the holomorphic structure and Φ . As shown in [Hit87], it follows that ϕ^2 preserves the corresponding flat connection $\nabla = \nabla_A + \Phi + \Phi^*$. Assuming ∇ is stable then $\phi^2 = \lambda \in \mathbb{C}^*$ is constant. Rescaling by a positive constant we may assume λ has norm 1.

Proposition 39. *The constant λ satisfies $\lambda = \pm 1$.*

Proof. Taking determinants we have a holomorphic anti-linear isomorphism covering f given by $\det(\phi) : \mathbb{C} \rightarrow \mathbb{C}$. Thus $\det(\phi)(s) = \alpha f^*(\bar{s})$, for some constant $\alpha \in \mathbb{C}^*$. However we also have that $\det(\phi) \circ \det(\phi) = \lambda^2$, hence $\lambda^2 = \alpha\bar{\alpha}$ is real and positive. But λ has norm 1, so $\lambda^2 = 1$.

From the above analysis, if we assume (E, Φ) is stable, the bundle E has either a real or quaternionic structure according to whether $\lambda = 1$ or $\lambda = -1$ respectively. We will now examine how this distinction is reflected in the spectral data for (E, Φ) .

Suppose that (E, Φ) is a stable $SL(2, \mathbb{C})$ -Higgs bundle with smooth spectral curve $p : S \rightarrow \Sigma$ and is fixed by the action of f . Recall from Section 8.2 that S is the double cover $S = \{(\eta, z) \in K \mid \eta^2 = q(z)\}$ associated to a quadratic differential $q \in H^0(\Sigma, K^2)$ such that $f^*(\bar{q}) = q$, and q has only simple zeros. Let $K^{1/2}$ be an f -invariant theta characteristic. Then we have $E = p_*(L \otimes p^*(K^{1/2}))$ for some $L \in \text{Prym}(S, \Sigma)$, and Φ is obtained by pushing forward the tautological section $\eta : L \otimes p^*(K^{1/2}) \rightarrow L \otimes p^*(K^{3/2})$. Let $\tilde{f} : S \rightarrow S$ be the natural lift of f to S and $\sigma : S \rightarrow S$ the automorphism which exchanges sheets of the covering. Since (E, Φ) is fixed by the action of f we have $\tilde{f}^*(\bar{L}) = L^*$ and since $L \in \text{Prym}(S, \Sigma)$ we have $\sigma^*(L) = L^*$. Recall that $\tilde{f} : K \rightarrow K$ is defined by $\tilde{f}(\omega) = f^*(\bar{\omega})$.

Lemma 40. *There exists a holomorphic anti-linear isomorphism*

$$\gamma : K^{1/2} \rightarrow K^{1/2}$$

covering f such that $\gamma \circ \gamma = \epsilon_1 = \pm 1$ and $\gamma \otimes \gamma = \tilde{f}$. If f has fixed points then $\epsilon_1 = 1$.

Proof. Since $f^*(\bar{K}^{1/2}) \simeq K^{1/2}$, we have that there exists a holomorphic anti-linear isomorphism $\gamma : K^{1/2} \rightarrow K^{1/2}$ covering f . Then $\gamma \circ \gamma = \beta \in \mathbb{C}^*$ for some constant β . Rescaling by a positive constant we can assume β has norm 1. We also have $(\gamma \otimes \gamma)(\omega) = \alpha f^*(\bar{\omega})$ for some constant $\alpha \in \mathbb{C}^*$. Then $\alpha\bar{\alpha} = \beta^2$, so that $\alpha\bar{\alpha} = \beta^2 = 1$. Choose $u \in \mathbb{C}^*$ with $u^2 = \alpha$ and replace γ by $u\gamma$. Then $\gamma \otimes \gamma = \tilde{f}$ and $\gamma \circ \gamma = \beta = \epsilon_1$, where $\beta^2 = 1$. Finally if f has fixed points then we must have $\beta = 1$ since otherwise we would obtain a quaternionic structure on a rank one complex vector space.

Let $\eta \in \Gamma(S, p^*(K))$ denote the tautological section. We have $\tilde{f}^*(\eta) = \eta$ and $\sigma^*(\eta) = -\eta$. Setting $\tau = \tilde{f}\sigma$, the map $\tau : S \rightarrow S$ is an anti-holomorphic involution and $\tau^*(\eta) = -\eta$. Moreover we also have $\tau^*(\bar{L}) = L$. Suppose now that $\tilde{\tau} : L \rightarrow L$ is a lift of τ to a holomorphic anti-linear map satisfying $\tilde{\tau} \circ \tilde{\tau} = \epsilon_2$, where $\epsilon_2 = \pm 1$. Let $\gamma : K^{1/2} \rightarrow K^{1/2}$ be as in Lemma 40. Then since $E = p_*(L \otimes p^*(K^{1/2}))$, it is clear that $\tilde{\tau} \otimes p^*(\gamma)$ pushes-forward to a holomorphic anti-linear involution $\phi : E \rightarrow E$ covering f , satisfying $\phi^{-1} \circ \Phi \circ \phi = -\Phi$ and $\phi^2 = \epsilon$, where $\epsilon = \epsilon_1\epsilon_2$. Thus E is real or quaternionic according to whether $\epsilon = 1$ or $\epsilon = -1$.

Proposition 41. *If q has a zero fixed by f , then there exists a lift $\tilde{\tau} : L \rightarrow L$ of τ with $\tilde{\tau} \circ \tilde{\tau} = 1$. In this case E has a real structure.*

Proof. Since $\tau = \tilde{f}\sigma$ it follows that τ has fixed points. Let $x \in S$ be a fixed point of τ . Since L is a holomorphic line bundle, then there is associated to L a corresponding unitary homomorphism $\rho : \pi_1(S, x) \rightarrow U(1)$. Let $\tilde{S} \rightarrow S$ be the universal cover viewed as a principal $\pi_1(S, x)$ -bundle. Then L may be identified as the quotient of $\tilde{S} \times \mathbb{C}$ by the relation $(pg, s) \sim (p, \rho(g)s)$, for $g \in \pi_1(S, x)$. Let $\hat{\tau} : \pi_1(\tilde{S}, x) \rightarrow \pi_1(S, x)$ be the automorphism of $\pi_1(S, x)$ constructed as in Sect. 2.3. Then since x is a fixed point, $\hat{\tau}$ is an involution. Moreover since $\tau^*(\bar{L}) = L$ we have $\rho \circ \hat{\tau} = \bar{\rho}$. We can lift τ to an involution $\tau' : \tilde{S} \rightarrow \tilde{S}$ which satisfies $\tau'(pg) = \tau'(p)\hat{\tau}(g)$. Now define $\tilde{\tau} : L \rightarrow L$ by $\tilde{\tau}(p, s) = (\tau'(p), \bar{s})$. This is well-defined since $\rho \circ \hat{\tau} = \bar{\rho}$, and is an involution since τ' is an involution. Finally note that ϵ_1 in Lemma 40 equals 1 since f has fixed points.

Proposition 42. *Suppose q has no zeros fixed by f . Let $x \in S$ and choose a path l from x to $\tau(x)$. Let μ be the loop $\mu = l.\tau(l)$ and let $\rho : \pi_1(S, x) \rightarrow U(1)$ be the flat unitary structure associated to the holomorphic line bundle L . Then $\epsilon := \rho(\mu) = \pm 1$ and there exists a lift $\tilde{\tau} : L \rightarrow L$ of τ with $\tilde{\tau} \circ \tilde{\tau} = \epsilon$.*

Proof. The proof is nearly identical to that of Proposition 41, except now the lift $\tau' : \tilde{S} \rightarrow \tilde{S}$ satisfies $\tau'(\tau'(p)) = p\mu^{-1}$, hence the lift $\tilde{\tau} : L \rightarrow L$ given by $\tilde{\tau}(p, s) = (\tau(p), \bar{s})$ squares to $\rho(\mu)^{-1}$. Note also that since $\rho \circ \hat{\tau} = \bar{\rho}$ and $\hat{\tau}(\mu) = \mu$, we have that $\rho(\mu) = \rho(\mu)^{-1} = \pm 1$.

10. Langlands Duality for \mathcal{L}_{G_c}

Let ${}^L G_c$ denote the Langlands dual group of G_c . The bases $\mathcal{A}_{G_c}, \mathcal{A}_{L_{G_c}}$ of the Hitchin fibrations $h : \mathcal{M}_{G_c} \rightarrow \mathcal{A}_{G_c}$ and $h : \mathcal{M}_{L_{G_c}} \rightarrow \mathcal{A}_{L_{G_c}}$ can naturally be identified so that $\mathcal{M}_{G_c}, \mathcal{M}_{L_{G_c}}$ are torus fibrations over a common base. Langlands duality is then interpreted as the statement that the moduli spaces $\mathcal{M}_{G_c}, \mathcal{M}_{L_{G_c}}$ are duals in the sense of mirror symmetry. One aspect of this duality is that the connected components of the non-singular fibres of \mathcal{M}_{G_c} and $\mathcal{M}_{L_{G_c}}$ should be dual abelian varieties.

From [KW07, Section 12.4], under Langlands duality we know that (A, B, A) -branes in \mathcal{M}_{G_c} map to (A, B, A) -branes in the moduli space $\mathcal{M}_{L_{G_c}}$ of ${}^L G_c$ -Higgs bundles for ${}^L G_c$. Informally the mapping of branes can be thought of as a fibrewise Fourier–Mukai transform. Hence we know that $\mathcal{L}_{G_c} \subset \mathcal{M}_{G_c}$ maps under the duality to an (A, B, A) -brane ${}^L \mathcal{L}_{G_c} \subset \mathcal{M}_{L_{G_c}}$. We claim that the dual (A, B, A) -brane ${}^L \mathcal{L}_{G_c} \subset \mathcal{M}_{L_{G_c}}$ coincides with the (A, B, A) -brane $\mathcal{L}_{L_{G_c}}$.

Consider first the case $G_c = {}^L G_c = GL(n, \mathbb{C})$ and restrict attention to the moduli space of degree 0 Higgs bundles $\mathcal{M}_{GL(n, \mathbb{C})} = \mathcal{M}_{GL(n, \mathbb{C})}^{\text{Higgs}, 0}$. Let $p : S \rightarrow \Sigma$ be the spectral curve corresponding to a generic point $a \in \mathcal{A}_{GL(n, \mathbb{C})}$. After fixing a choice of spin structure the fibre of $\mathcal{M}_{GL(n, \mathbb{C})}$ is given by the Jacobian $Jac(S)$, and the self-duality of $\mathcal{M}_{GL(n, \mathbb{C})}$ is reflected in the self-duality of $Jac(S)$. Now, with a slight abuse of notation, let $L \subset \mathcal{A}_{GL(n, \mathbb{C})}$ be the subspace of $\mathcal{A}_{GL(n, \mathbb{C})}$ fixed by the induced action of f , so that if $a \in L$ then f lifts to an involution $\tilde{f} : S \rightarrow S$. Then $\mathcal{L}_{GL(n, \mathbb{C})}$ fibres over L with fibre corresponding to the subspace of $Jac(S)$ fixed by the map $\iota_0 : Jac(S) \rightarrow Jac(S)$ as in Definition 26. Let $B \subset Jac(S)$ be the fixed point set of ι_0 .

The self-duality of $\mathcal{L}_{GL(n, \mathbb{C})}$ is reflected in the fact that B is a Lagrangian submanifold of $Jac(S)$, where the symplectic structure on $Jac(S)$ is given by the standard principal

polarization. This follows since the map $H^1(S, \mathbb{R}) \rightarrow H^1(S, \mathbb{R})$ corresponding to ι_0 is given by $x \mapsto -\tilde{f}^*(x)$, which is anti-symplectic.

To understand the above claim we need a brief digression into Fourier-Mukai duality. Let A, \hat{A} be dual abelian varieties. There is a natural dual pairing of $H_1(A, \mathbb{R})$ and $H_1(\hat{A}, \mathbb{R})$. Given a subspace $V \subseteq H_1(A, \mathbb{R})$ there is a corresponding subspace $\hat{V} \subseteq H_1(\hat{A}, \mathbb{R})$, namely the annihilator $\hat{V} = V^\perp$ of V . Suppose V is the subspace determined by a subtorus $B \subseteq A$. Then V^\perp defines a corresponding subtorus $\hat{B} \subseteq \hat{A}$. Under the cohomological Fourier-Mukai transform [Huy] $H^*(A, \mathbb{R}) \rightarrow H^*(\hat{A}, \mathbb{R})$, the Poincaré dual cohomology classes $\eta_B \in H^*(A, \mathbb{R})$ and $\eta_{\hat{B}} \in H^*(\hat{A}, \mathbb{R})$ correspond to one another (up to a sign factor). Based on this we say that closed subtori $B \subset A$ and $\hat{B} \subset \hat{A}$ are Fourier-Mukai dual if the corresponding subspaces $V \subset H_1(A, \mathbb{R})$ and $\hat{V} \subset H_1(\hat{A}, \mathbb{R})$ are annihilators of one another. When A is self-dual we can use a principal polarization to identify $H_1(A, \mathbb{R})$ with $H_1(\hat{A}, \mathbb{R})$ and we find that a subspace $V \subseteq H_1(A, \mathbb{R})$ is self-dual if and only if it is a Lagrangian subspace. This explains our claim that $\mathcal{L}_{GL(n, \mathbb{C})}$ is a self-dual brane.

Consider now the case $G_c = SL(n, \mathbb{C})$, for which ${}^L G_c = PGL(n, \mathbb{C})$. Clearly we have an identification of the bases $\mathcal{A}_{SL(n, \mathbb{C})} \simeq \mathcal{A}_{PGL(n, \mathbb{C})}$. Let a be a generic point of the base with corresponding spectral curve $p : S \rightarrow \Sigma$. To see the duality of the fibres we note that the exact sequence

$$1 \longrightarrow \text{Prym}(S, \Sigma) \longrightarrow \text{Jac}(S) \xrightarrow{Nm} \text{Jac}(\Sigma) \longrightarrow 1$$

dualises to

$$1 \longrightarrow \text{Jac}(\Sigma) \xrightarrow{p^*} \text{Jac}(S) \longrightarrow \widehat{\text{Prym}}(S, \Sigma) \longrightarrow 1.$$

In particular this shows that the dual of $\text{Prym}(S, \Sigma)$ is $\text{Jac}(S)/p^*(\text{Jac}(\Sigma))$ which is especially seen to describe the spectral data for $PGL(n, \mathbb{C})$ -Higgs bundles.

Let a be a point in the base which is fixed by the action of f , so that f lifts to $\tilde{f} : S \rightarrow S$. As before, denote by $B \subset \text{Jac}(S)$ the fixed point set of ι_0 . Then the fibre of $\mathcal{L}_{SL(n, \mathbb{C})}$ over a is $B \cap \text{Prym}(S, \Sigma)$ and the fibre of $\mathcal{L}_{PGL(n, \mathbb{C})}$ is the image of B in $\text{Jac}(S)/p^*(\text{Jac}(\Sigma))$. The components of these fibres are then seen to be dual in the sense described above.

11. Higgs Bundles and 3-Manifolds

Given an anti-holomorphic involution $f : \Sigma \rightarrow \Sigma$ on a compact Riemann surface, one may construct an associated 3-manifold M with boundary Σ . Moreover we show that there is a close relationship between representations of the fundamental group of M and the (A, B, A) -brane \mathcal{L}_{G_c} defined by f . To define M consider the product $\bar{\Sigma} = \Sigma \times [-1, 1]$. On $\bar{\Sigma}$ there is a natural involution

$$\sigma : (x, t) \mapsto (f(x), -t),$$

which is orientation preserving and a product action (e.g., [KMCC96, Section 1]). The quotient space $M = \bar{\Sigma}/\sigma$ is a 3-dimensional manifold with boundary $\partial M = \Sigma$. From M we obtain a distinguished subspace of representations of $\pi_1(\Sigma)$, namely those representations which viewed as flat connections on Σ extend to flat connections over M .

Proposition 43. *The representations of Σ into G_c which extend to M belong to the (A, B, A) -brane \mathcal{L}_{G_c} .*

Proof. Let $i : \Sigma \rightarrow \overline{\Sigma}$ be the inclusion $i(x) = (x, 0)$. Then clearly $i \circ f = \sigma \circ i$. Fix a point $x_0 \in \Sigma$ and choose a path γ in Σ from x_0 to $f(x_0)$. Recall from Sect. 2.3 that the automorphism $\hat{f} : \pi_1(\Sigma, x_0) \rightarrow \pi_1(\Sigma, x_0)$ is given by $\hat{f}(u) = \gamma.f(u).\gamma^{-1}$. Let $m_0 \in M$ be the point of M corresponding to $(x_0, 0)$ and τ the image of $i(\gamma)$ in M , which is a loop at m_0 . Let $j : \Sigma \rightarrow M$ be the composition of i with the projection $\overline{\Sigma} \rightarrow M$. We then have a commutative diagram

$$\begin{CD} \pi_1(\Sigma, x_0) @>j_*>> \pi_1(M, m_0) \\ @V\hat{f}VV @VV\text{Ad}_\tau V \\ \pi_1(\Sigma, x_0) @>j_*>> \pi_1(M, m_0) \end{CD}$$

where Ad_τ denotes conjugation by τ . For $\rho : \pi_1(M, m_0) \rightarrow G_c$ a representation of M , we have $\rho \circ j_* \circ \hat{f} = \text{Ad}_{\rho(\tau)} \circ \rho \circ j_*$. Hence $\rho \circ j_*$ defines a fixed point of the action of f on $\text{Rep}^+(\pi_1(\Sigma), G_c)$ as required.

The study of this correspondence, as well as the further characterisation of the (A, B, A) -branes \mathcal{L}_{G_c} appears in the companion paper [BaSch].

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A. Anti-holomorphic Involutions on Hyperelliptic Surfaces

As seen in [GH81, Section 6], in the case of hyperelliptic curves more information can be deduced concerning the topological invariants (n, a) . We shall see here some examples.

A.1. Classification. We shall begin by classifying pairs (Σ, f) where Σ is a hyperelliptic curve of genus $g \geq 2$ and $f : \Sigma \rightarrow \Sigma$ an anti-holomorphic involution.

Definition 44. *Let $\tau : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the anti-holomorphic involution which in projective coordinates $[z_1, z_2]$ is given by $\tau([z_1, z_2]) = [\bar{z}_1, \bar{z}_2]$. Let $\alpha : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be given by $\alpha([z_1, z_2]) = [z_2, -z_1]$ so that $\alpha\tau = \tau\alpha$ is the antipodal map.*

Proposition 45. *There exists a non-constant meromorphic function $z : \Sigma \rightarrow \mathbb{P}^1$ with polar divisor $z^{-1}(\infty)$ of degree 2 such that either $z(f(w)) = \tau(z(w))$ or $z(f(w)) = \alpha\tau(z(w))$. In addition, we may choose z such that $\infty = [0, 1] \in \mathbb{P}^1$ is not a branch point.*

Proof. Since Σ is hyperelliptic there exists a meromorphic function $z' : \Sigma \rightarrow \mathbb{P}^1$ which has two simple poles, or a single pole of order 2. Any other meromorphic function with this property is obtained from z' by a Möbius transformation. In particular this applies to $\tau \circ z' \circ f$, so there exists a matrix $M \in GL(2, \mathbb{C})$ such that $\tau \circ z' \circ f = M \circ z'$, where M acts on \mathbb{P}^1 as a Möbius transformation. Note that since τ and f are involutions we have $\overline{MM} = \pm I$. Choose a matrix $A \in GL(2, \mathbb{C})$ and set $z = A \circ z'$. We have $\tau \circ z \circ f = \overline{AMA}^{-1}z$.

If $\overline{MM} = I$ then we can choose $A \in GL(2, \mathbb{C})$ such that $\overline{A}^{-1}A = M$. Thus $z \circ f = \tau \circ z$ as required. By composing with a transformation in $GL(2, \mathbb{R})$ we can

ensure that ∞ is not a branch point of z . If $\overline{M}M = -I$ then we can choose $A \in GL(2, \mathbb{C})$ such that $\overline{A}^{-1}JA = M$, where J is the linear transformation $J(z_1, z_2) = (z_2, -z_1)$. Thus $z \circ f = \tau \alpha \circ z$ as required. The linear transformations in $GL(2, \mathbb{C})$ commuting with the antipodal map $(z_1, z_2) \rightarrow (\overline{z_2}, -\overline{z_1})$ form the group $SU(2)$. Composing with an element of $SU(2)$ we can ensure that ∞ is not a branch point of z .

We have established that the anti-holomorphic involution f on Σ covers an anti-holomorphic involution on \mathbb{P}^1 which is either the conjugation map τ or the antipodal map $\alpha\tau$. Let $z : \Sigma \rightarrow \mathbb{P}^1$ be the meromorphic function as above. Then z exhibits Σ as a branched double cover of \mathbb{P}^1 . We may choose z so that ∞ is not a branch point, and let $P_1, \dots, P_{2g+2} \in \Sigma$ be the branch points with $z(P_1), \dots, z(P_{2g+2}) \in \mathbb{C} \subset \mathbb{P}^1$ their images in \mathbb{P}^1 . On Σ there is a meromorphic function w satisfying

$$w^2 = \prod_{j=1}^{2g+2} (z - z(P_j)) := p(z),$$

and a holomorphic involution $\sigma : \Sigma \rightarrow \Sigma$, the hyperelliptic involution, defined by exchanging the sheets of the branched covering $\Sigma \rightarrow \mathbb{P}^1$. Thus $z \circ \sigma = z$ and $w \circ \sigma = -w$.

Proposition 46. *If the anti-holomorphic involution f covers the conjugation map τ from Definition 44, then it is given by either $f' : (w, z) \mapsto (\overline{w}, \overline{z})$ or by $\sigma \circ f'$.*

Proof. For simplicity we shall write \overline{z} in place of $\tau(z)$. Since f sends branch points of z to branch points, the set of images $z(P_1), \dots, z(P_{2g+2})$ of branch points is left invariant under the conjugation map τ . Therefore each $z(P_j)$ is either real or occurs in a conjugate pair. The polynomial $p(z) = \prod_{j=1}^{2g+2} (z - z(P_j))$ thus satisfies $p(\overline{z}) = \overline{p(z)}$. In particular we obtain an anti-holomorphic involution $f' : \Sigma \rightarrow \Sigma$ by sending a pair (w, z) such that $w^2 = p(z)$ to the corresponding pair $(\overline{w}, \overline{z})$. Clearly f' covers τ . The only other anti-holomorphic involution covering τ is given by sending a pair (w, z) to $(-\overline{w}, \overline{z})$ and this is just $\sigma \circ f'$. Thus f is one of f' or $\sigma \circ f'$.

Proposition 47. *Let $2k$ be the number of real roots of $p(z)$. Consider the topological invariants (n_Σ, a_Σ) associated to the pair (Σ, f') . If $k = 0$ then $(n_\Sigma, a_\Sigma) = (1, 0)$ if g is even, and $n_\Sigma = 2$ if g is odd. If $0 < k < g + 1$ then $(n_\Sigma, a_\Sigma) = (k, 1)$, and if $k = g + 1$ then $(n_\Sigma, a_\Sigma) = (g + 1, 0)$.*

Proof. Consider the zeros $z(P_1), \dots, z(P_{2g+2})$ of $p(z)$ in the complex plane. Draw branch cuts between conjugate pairs of roots and joining adjacent pairs of real roots. From this data the invariants (n_Σ, a_Σ) can easily be determined.

Similarly, the topological invariants of the involution $\sigma \circ f'$ can be found.

Proposition 48. *Let $2k$ be the number of real roots of $p(z)$. Consider the topological invariants (n_Σ, a_Σ) associated to the pair $(\Sigma, \sigma \circ f')$. If $0 \leq k < g + 1$ then $(n_\Sigma, a_\Sigma) = (k, 1)$, and if $k = g + 1$ then $(n_\Sigma, a_\Sigma) = (g + 1, 0)$.*

We shall now consider involutions covering the antipodal map $\alpha\tau$ as in Definition 44. In this case the antipodal map must permute the zeros of $p(z)$, so the zeros occur in antipodal pairs. For $\lambda := \prod_{j=1}^{2g+2} \overline{z(P_j)}$, it follows that λ is of the form $\lambda = \mu(-1/\overline{\mu}) =$

$-\mu^2/|\mu|^2$, for some non-zero $\mu \in \mathbb{C}$. Define $c = i\mu/|\mu|$ so that $c^2 = \lambda$ and $c\bar{c} = 1$. It is clear that if $w^2 = p(z)$ then $\left(\frac{\bar{w}}{c\bar{z}^{g+1}}\right)^2 = p(-1/\bar{z})$, so we obtain an anti-holomorphic map $h : \Sigma \rightarrow \Sigma$ such that $z \circ h = z$ by setting $h(w, z) = \left(\frac{\bar{w}}{c\bar{z}^{g+1}}\right), -1/\bar{z}$. Note that h is an involution if and only if g is odd. We have thus established:

Proposition 49. *If g is even, then there are no anti-holomorphic involutions on Σ covering the antipodal map on \mathbb{P}^1 . If g is odd then there are precisely two such involutions, h and $\sigma \circ h$. The topological invariants of (Σ, h) and $(\Sigma, \sigma \circ h)$ are $(n_\Sigma, a_\Sigma) = (0, 1)$.*

A.2. Genus 2 case. Let z, w be the meromorphic functions on Σ as in Sect. A.1. Then $H^0(\Sigma, K)$ has dimension g and is spanned by the elements $z^j dz/w$ for $j = 0, 1, \dots, g-1$. A divisor is in the linear system $\mathbb{P}(H^0(\Sigma, K))$ if and only if it is of the form $D = z^{-1}(a_1) + \dots + z^{-1}(a_{g-1})$, where a_1, \dots, a_{g-1} are any $(g-1)$ points on \mathbb{P}^1 .

For a hyperelliptic surface of genus $g = 2$ the anti-holomorphic involution f must cover the conjugation map on \mathbb{P}^1 . Moreover since the natural map $H^0(\Sigma, K) \rightarrow H^0(\Sigma, K^2)$ is surjective, in this case we can describe the linear system of quadratic differentials. These all have the form $D = z^{-1}(a_1) + z^{-1}(a_2)$ for two points $a_1, a_2 \in \mathbb{P}^1$. The quadratic differentials q with divisor $(q) = D$ have simple zeros provided that $a_1 \neq a_2$ and that a_1, a_2 are not zeros of $p(z)$.

Let $a_1, a_2 \in \mathbb{P}^1$ be distinct points that are not zeros of $p(z)$, and let q be any non-zero quadratic differential with $(q) = D = z^{-1}(a_1) + z^{-1}(a_2)$. Then $f^*(\bar{q})$ has divisor $z^{-1}(\bar{a}_1) + z^{-1}(\bar{a}_2)$. Thus to obtain a quadratic differential q with $f^*(\bar{q}) = q$, the points a_1, a_2 must be real or conjugates. For such a pair of points $a_1, a_2 \in \mathbb{P}^1$, the corresponding quadratic differentials with divisor $D = z^{-1}(a_1) + z^{-1}(a_2)$ determine a complex 1-dimensional space of $H^0(\Sigma, K^2)$ invariant under the induced action of f . Thus we can always find a non-zero quadratic differential q with $(q) = D$ and $f^*(\bar{q}) = q$. Moreover q is unique up to the action of \mathbb{R}^* .

Consider the spectral curve $S \rightarrow \Sigma$ given by the characteristic equation $\lambda^2 = q$ and with induced involution $\tilde{f}(\lambda, x) = (f^*(\bar{\lambda}), f(x))$, where $x \in \Sigma$ and $\lambda^2 = q(x)$. Replacing q by any positive multiple of itself will simply require a corresponding rescaling of λ , so the pair (S, \tilde{f}) is essentially independent of such rescalings. On the other hand replacing q by $-q$ will leave S unchanged but replace \tilde{f} by $\pi \circ \tilde{f}$, where $\pi : S \rightarrow S$ is the involution which exchanges sheets of $S \rightarrow \Sigma$.

To summarise, we choose distinct points $a_1, a_2 \in \mathbb{P}^1$ which are either both real or complex conjugates. Up to positive rescalings there are two non-zero quadratic differentials $q, -q$ with $(q) = (-q) = z^{-1}(a_1) + z^{-1}(a_2)$. In fact, we may take q to be $q = (z - a_1)(z - a_2)(dz)^2/w^2$, if a_1, a_2 are finite. If say $a_2 = \infty$ then we omit the factor $(z - a_2)$. From here it is possible to calculate the invariants $(n_+, u/2)$ associated to q as in Sect. 8.2, where n_+ is the number of fixed components of f on which q is non-vanishing and positive and u is the number of zeros of q which are fixed by f .

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