Management Science

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informs
http://pubsonline.informs.org/journal/mnsc/

MANAGEMENT SCIENCE
Vol. 65, No. 5, May 2019, pp. 2142–2160
ISSN 0025-1909 (print), ISSN 1526-5501 (online)

Population Monotonicity in Newsvendor Games

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Received: April 5, 2016
Revised: April 19, 2017; September 18, 2017
Accepted: January 10, 2018
Published Online in Articles in Advance: January 17, 2019

https://doi.org/10.1287/mnsc.2018.3053

Abstract. A newsvendor game allows the players to collaborate on inventory pooling and share the resulting total cost. There are several possible ways to allocate the cost. Previous studies have focused on the core of the game. It is known that the core of the newsvendor game is nonempty, and one can use duality theory in stochastic programming to construct an allocation—referred to as the dual-based allocation—belonging to the core. Yet, an allocation that lies in the core does not necessarily guarantee the unhindered formation of a coalition, as some existing members' allocated costs may increase when new members are added in the process. In this work, we use the concept of population monotonic allocation scheme (PMAS), which requires the cost allocated to every member of a coalition to decrease as the coalition grows, to study allocation schemes in a growing population. We show that when the demands faced by the newsvendors are independent, log-concavity of their distributions is sufficient to guarantee the existence of a PMAS. Specifically, for continuous demands, log-concavity ensures that the game is convex, which in turn implies a PMAS exists. We also show that under the same condition the dual-based allocation scheme is a PMAS. For discrete and log-concave demands, however, the game may no longer be convex, but we manage to show that, even so, the dual-based allocation scheme is a PMAS. When the demands are dependent, the game is in general not convex. We derive a sufficient condition based on the dependence structure, measured by the copula, to ensure that the dual-based allocation scheme is still a PMAS. We also include an example of a game with no PMAS.

History: Accepted by Yinyu Ye, optimization.

Funding: This research of the first author is supported by National Science Foundation (NSF) [Grants CMMI-1363261, CMMI-1538451, and CMMI-1635160] and the National Science Foundation of China (NSFC) [Grant 71520107001]. The third author is partially supported by a research grant at the National University of Singapore [Project R-314-000-105-133]. The fourth author is supported by the NSF [Grant CMMI 1363314] and is a co-principle investigator on the first author's grant [Grant CMMI-1538451].

Supplemental Material: The online appendix is available at https://doi.org/10.1287/mnsc.2018.3053.

Keywords: inventory centralization • cooperative games • population monotonicity • log-concavity • duality

1. Introduction

Farmers in New York State are surely aware that cooperation by pooling demand and coordinating production can help them all. As has been reported in the *New York Times* (see Shattuck 2014), the state has been "a haven for dairy cooperatives," and more recently, witnessed the growth of Adriondack Grazers, a grass-fed beef cooperative started in 2012, to more than 40 farms across the states boundaries. These cooperatives enable farmers to leverage economies of scale to weather market uncertainties without giving up individual ownerships. The benefits are evidently not the best-kept secrets that are only known to farming communities: similar practices are also observed in other places like the pharmaceutical and retailing industries. For example, Good Neighbor Pharmacy is a cooperative of more than 3,200 pharmacies in America; Affiliated Foods Midwest supplies more than 850 independent retailers in the 12 Midwestern states with a full line of grocery products.

Cooperative game theory lays out a theoretical framework for analyzing cooperation among independent producers. The newsvendor model captures fundamental tradeoffs between the costs of excessive supply and shortage, inevitably caused by demand uncertainty. Hence, to develop a general understanding of cooperative relationships in broadly defined supply chain systems exemplified by the aforementioned cooperatives, the *newsvendor game* is a natural first step. In this game, cooperatives are referred to as coalitions, who solve a single newsvendor problem for setting a production quantity to serve demands of all participants. An essential issue of the game is to find an adequate allocation to distribute the expected cost of the coalition to individual participants. Substantial progress has been made on the study of this problem but many important questions remain to be answered.

In a cooperative game, the core for a coalition is defined as the set of allocations under which no participant can derive a better payoff from leaving the coalition, either individually or as a subgroup. Determining whether the core for the grand coalition (i.e., the coalition of all potential participants) is empty is a primary focus in cooperative game theory. It has been shown in the literature that for newsvendor games, the core for the grand coalition is not empty. Thus, it is possible to form a stable cooperative relationship of all producers. Nevertheless, for this possibility to become a reality, the existence of allocations in the core is not sufficient.

First, one needs to discover what these allocations actually are. The dual-based allocation rule formulated in Chen and Zhang (2009) fits this purpose. Their result is of special importance not only because the dual-based allocation rule results in a simple closed-form solution, but also because it is shown in Montrucchio and Scarsini (2007) that for a continuum of players, the dual-based allocation is the only element in the core under a nonatomic condition on the demand. Moreover, under some conditions the core of the newsvendor game with finite players shrinks to this single allocation as the number of players increases.

Second, as in the case of Adriondack Grazers, large cooperatives usually do not form overnight but expand gradually. As another example, Ocean Spray, a leading cooperative in the cranberry industry, started off with only three berry growers in 1930, and took many years to reach more than 700 members across North America as of today (Jesse and Rogers 2006). When a coalition is governed by consensus, its expansion has to be agreed upon by each of its existing members. An allocation in the core gives no incentive for any subset of members to leave a coalition, but it does not guarantee that none of them will be negatively affected by admitting a new member, and thus may not lead to a unanimous agreement for expansion.

To study incentives for coalition expansion, it is suitable to consider the notion of a population monotonic allocation scheme (PMAS) defined by Sprumont (1990). A PMAS is an allocation scheme that applies to all coalitions in a cooperative game. Under a PMAS, each time a coalition adds a new member, every current member will be allocated a (weakly) larger benefit, or in cases like the newsvendor game, a (weakly) smaller cost. Obviously, allocations under such a scheme must be in the core of the game that keeps the grand coalition stable. Given the essential role played by the newsvendor model in production and inventory theory, to develop a fundamental understanding of the formation of growing cooperative relationships in supply chains, it is critically important to investigate conditions for the existence of a PMAS in newsvendor games and identify widely applicable instances of such schemes. Nevertheless, to the best of our knowledge, these issues have not been directly addressed in the literature. The purpose of our paper is to fill this gap by developing a broad set of pertinent results and discussing underlying insights.

Our investigation shows that when demands are independent, in a broad set of instances of the newsvendor game, a PMAS exists and the dual-based allocation rule results in a PMAS, ensuring not only the stability of the grand coalition but also a smooth path toward its formation. When demands are dependent, the monotonic property of the dual-based allocation rule is largely determined by the degree to which demands of individual participants depend on each other. These messages are conveyed by our derivation of the following major results.

1. When demands are independent and continuous random variables, then a PMAS exists under mild conditions. In particular, if the demands of all players have log-concave probability density functions, then the game is convex, and we can further show that the dual-based allocation rule leads to a PMAS. It is worth noting that our convexity result includes several sufficient conditions in the newsvendor game literature (see Section 2 for more details) as special cases.

2. When demands are independent and discrete random variables, the log-concavity of the probability mass function does not imply the game is convex, but still guarantees that the dual-based allocation rule results in a PMAS. Log-concavity, a property satisfied by many commonly used distributions such as normal, exponential, uniform, Poisson, and so forth, together with independence ensures the existence of a PMAS regardless of whether demands are continuous or discrete.

3. When demands are dependent, the newsvendor game is not convex, except for a few special cases; for example, when demands follow a permutation symmetric multivariate normal distribution. We develop a sufficient condition under which a PMAS can be derived from the dual-based allocation rule. The condition is based on the copula of the demand distribution, which can be reduced to properties of correlation coefficients when specialized to normally distributed demands. We also complement these general results with studies on some special cases, and we provide a counterexample to show that, in certain cases, there is no PMAS.

In the rest of the paper, we give a brief review of related work in Section 2, formally define relevant concepts and models in Section 3, analyze the game with independent and dependent demands in Sections 4 and 5, respectively, and conclude the paper in Section 6.

2. Literature Review

Eppen (1979) is the first work that quantifies cost savings from the risk pooling effect when newsvendors at different locations consolidate their demands. Hartman et al. (2000) later formalize the situation into a newsvendor game and show that the core of such a game is nonempty for various special cases of the demands. This result has been generalized in a variety of ways: Müller et al. (2002) and Slikker et al. (2001) independently show that a newsvendor game always has a nonempty core if all demands have finite means. Montrucchio and Scarsini (2007) prove that the core is nonempty for newsvendor games with infinitely many players. Slikker et al. (2005) extend the results under the basic newsvendor setting to a general newsvendor situation by allowing nonidentical costs and transshipment costs. Özen et al. (2008) consider a distribution system with multiple warehouses and show that the associated cooperative game has a nonempty core. Based on duality theory, Chen and Zhang (2009) develop a unified approach for a class of inventory centralization games. They prove the nonemptiness of the core in these games and present a dual-based allocation rule to obtain an element in the core. This allocation rule has been applied to many other inventory problems (see Chen 2009, Zhang 2009, Chen and Chen 2013, Toriello and Uhan 2014, Chen and Zhang 2016, among others).

These papers focus exclusively on the nonemptiness of the core. However, it has been recognized in various contexts that an allocation should be able to accommodate variations in population. Thomson (1983, p. 319) emphasizes that "all of the original agents should share in the new responsibilities of the group" and introduces an axiom of monotonicity for fair division problems with a changing number of agents. In an analysis of quasi-linear social choice problems, Chun (1986) develops a solidarity axiom which requires that all agents in a coalition to be affected in the same direction when a new agent enters.

The concept of a PMAS for cooperative games follows the same spirit. As is shown in Sprumont (1990), a convex game always has a PMAS. Therefore, one may establish the existence of a PMAS in a newsvendor game by proving the game is convex. On the latter issue, Slikker et al. (2001) prove that a newsvendor game is convex if demands are independent and follow normal distributions. Özen et al. (2011) prove that a newsvendor game with three players is convex when demands are identical, independent, and uniformly distributed. They also show that a newsvendor game is convex when demands are independent and symmetric with unimodal distributions, and the optimal fractile is 1/2. In this paper, in the process of showing the existence of a PMAS, we establish that a newsvendor game is convex when demands are independent and continuous random variables with log-concave density functions. This condition is new and includes several existing results in the literature as special cases. As is exemplified by the discussion in Slikker et al. (2001) and Özen et al. (2011), a newsvendor game with dependent demands is not always convex. Montrucchio and Scarsini (2007) provide the only positive result we can find in the literature, which shows that a game is convex if demands follow a permutation symmetric multivariate normal distribution. In our paper, we show that for a large class of newsvendor games with dependent demands, a PMAS exists even though the game is not convex.

To put our work in a broader context, we briefly summarize relevant discussions on PMAS in other cooperative games in the operations management and economics literature. Potters and Sudhölter (1999) establish that a modified nucleolus in a class of airport games is a PMAS. Meca et al. (2004) show that in joint replenishment games, a PMAS can be reached through a proportional allocation rule. Norde et al. (2004) provide an algorithm that computes a PMAS for minimum cost spanning tree games. He et al. (2012) study joint replenishment games and characterize conditions under which the game is convex, and therefore a PMAS exists. Karsten and Basten (2014) consider a spare parts pooling game where each firm faces a Poisson demand process. They show that the proportional allocation of the total cost to different firms according to their demand rates is a PMAS. It turns out that our dual-based allocation rule reduces to this proportional allocation rule when demands are independent and Poisson distributed. Moulin and Shenker (2001) study a mechanism design problem in which a service is produced for a set of agents and costs are shared among them. They show that when the cost sharing method is population monotonic, then each agent is willing to truthfully reveal his willingness to pay.

Finally, it is helpful to distinguish a PMAS as discussed in this paper from the concept of farsighted stability defined in Chwe (1994). The latter notion also concerns coalition formation, and has found many applications in supply chain management (see Sošic 2006, Nagarajan and Sošic 2007, Nagarajan and Bassok 2008, Nagarajan and Sošic 2009). We refer the readers to Nagarajan and Sošić (2008) for a review. We emphasize here two key differences. First, farsighted stability is usually associated with the question of, given a coalition structure, whether a path of coalitional defections is possible. Population monotonicity, on the other hand, is concerned with the question of, when a growing path of coalitions is specified, whether the formation of coalitions along the path can be sustained. Second, it is commonly observed in various applications that even if an allocation is not in the core, the stability of a coalition may still be guaranteed from a farsighted perspective. Comparatively, the consideration of population monotonicity argues that even if an allocation *is* in the core, it may not guarantee the formation of the coalition.

3. Problem Formulation

A cooperative game is defined by a finite set of players $N = \{1, \ldots, n\}$ and a characteristic function $c(S)$, $S \subseteq N$. A subset of players S, $S \subseteq N$ is referred to as a coalition and the set of all players N is called the grand coalition. The characteristic function is a mapping from the set of all coalitions, 2^N, to the set of real numbers, $\mathbb{R}$. For a given game (N, c), we denote the size of coalition $S \subseteq N$ by $|S|$. Note that $|N| = n$. For any $x \in \mathbb{R}$, we denote $\max\{x, 0\}$ by x^+.

We consider a situation where there are N newsvendors selling an identical item, with the same per-unit holding cost h and shortage cost p. For simplicity, we assume zero ordering cost. Since the game with zero ordering cost is strategically equivalent[1] to the one with constant per-unit ordering cost (see p. 145 of Choi 2012), all our results can be extended to the latter case.

Let X_i with $E[X_i] < \infty$ be the random demand faced by player i, $i \in N$ and

$$X_S = \sum_{i \in S} X_i$$

be the total demand of coalition $S \subseteq N$. The newsvendor game is defined as (N, c), where the characteristic function $c(S)$ is the minimum expected cost that can be attained by coalition S when all newsvendors in coalition S jointly choose a supply quantity q to serve their demands X_S. In other words, $c(S)$ is the optimal value of the following newsvendor model with inputs p, h, and X_S:

$$c(S) = \min_{q \ge 0} \{E[h(q - X_S)^+ + p(X_S - q)^+]\}, \quad S \subseteq N. \quad (1)$$

Let $F_S(\cdot)$ be the cumulative distribution of X_S and denote $\tau := p/(p+h)$. The solution of the newsvendor problem is a classic result (Arrow et al. 1951). When X_S is a continuous random variable, the optimal solution to (1) is

$$q_S^* = F_S^{-1}(\tau). \quad (2)$$

When X_S is a discrete random variable (throughout the paper, we assume the demand modeled by a discrete random variable has nonnegative integer support), the optimal solution to (1) satisfies

$$F_{S-}(q_S^*) \le \tau \le F_S(q_S^*),$$

where $F_{S-}(x) := \lim_{m \to \infty} F_S(x - 1/m)$ with $F_{S-}(0) = 0$.

The core of a game is the set of real vectors $(l_i)_{i \in N} := (l_1, \ldots, l_n)$ such that

$$\sum_{i \in N} l_i = c(N) \quad \text{and} \quad \sum_{i \in S} l_i \le c(S) \quad \text{for any } S \subseteq N.$$

Under these conditions, the optimal cost of the grand coalition is fully allocated to individual players in a way that no subgroup of players can lower their total allocated cost by forming their own smaller coalition. A core allocation, which may not exist in some games, is necessary to keep the grand coalition stable.

As is defined in Sprumont (1990), a population monotonic allocation scheme (PMAS) is a general scheme that determines the cost distribution not just for the grand coalition, but also for every possible coalition in a given game. We denote by $l_{i,S}$ the amount of cost allocated to player i in coalition S. An allocation scheme[2] $(l_{i,S})_{i \in S, S \subseteq N}$ is a PMAS if and only if

$$\sum_{i \in S} l_{i,S} = c(S) \quad \text{for all } S \subseteq N \quad (3)$$

and

$$l_{i,S} \ge l_{i,T} \quad \text{for all } i \in S \text{ and } T, \text{ where } S \subseteq T \subseteq N. \quad (4)$$

Equation (3) mandates full cost allocation in every coalition, and Equation (4) requires that no member of an existing coalition can benefit individually from forming a smaller coalition. Observe that (3) and (4) together ensure $(l_{i,S})_{i \in S}$ is in the core of the game (S, c) for any $S \subseteq N$. Under a PMAS, no member of an existing coalition can benefit individually from forming a smaller coalition with others, which is a more stringent requirement than being in the core. The latter only requires the smaller coalition will not be better off as a whole. As mentioned earlier, a PMAS gives every player an incentive to expand his coalition to an ever-larger one.

One way of establishing the existence of a PMAS for a game (N, c) is to show that the game is convex. In other words, the characteristic function satisfies

$$c(S \cup \{j\}) - c(S) \ge c(T \cup \{j\}) - c(T), \quad \forall S \subseteq T \subseteq N \backslash \{j\},\ j \in N. \quad (5)$$

This means that the marginal cost of adding a new player to a coalition decreases as the coalition grows. Sprumont (1990) demonstrates that a convex game always has a PMAS, and specifically, the Shapley value as well as all extreme points of the core result in PMASes. Unfortunately, newsvendor games are not convex in general (see Hartman et al. 2000, Slikker et al. 2001, Özen et al. 2011).

An alternative and more direct approach is to verify (3) and (4) for a specific allocation scheme. For this approach, we focus our study on the dual-based allocation rule formulated in Chen and Zhang (2009). This rule has two appealing properties. First, the dual-based allocation can be constructed by solving the dual of a stochastic linear program and admits a simple closed-form solution. Other well-known allocation rules such as the nucleolus (which also lies in the core) or the Shapley value (which may not lie in the core if the game is not convex) are both analytically and computationally complicated. Second, the dual-based allocation is the only element in the core in a nonatomic newsvendor game where there is a continuum of players and each one of them has a negligible weight. Moreover, under some conditions the core of the newsvendor game with finite players shrinks to this single allocation as the number of players increases (Montrucchio and Scarsini 2007).

Given a newsvendor game, the dual-based allocation rule results in the following allocation scheme (which we will simply refer to as the dual-based allocation scheme),

$$l_{i,S} = E[\pi_S(\omega) X_i(\omega)], \quad i \in S,$$

for each $S \subseteq N$. Here $\omega \in \Omega$, the sample space of demands $(X_1, \ldots, X_n)$, and

$$\pi_S(\omega) = \begin{cases} -h, & \text{if } X_S(\omega) < q_S^*, \\ p - \eta, & \text{if } X_S(\omega) = q_S^*, \\ p, & \text{if } X_S(\omega) > q_S^*, \end{cases}$$

where

$$\eta = \frac{p - (p+h)F_{S-}(q_S^*)}{F_S(q_S^*) - F_{S-}(q_S^*)}, \quad \text{if } F_S(q_S^*) > F_{S-}(q_S^*),$$

and the choice of η is immaterial when $F_S(q_S^*) = F_{S-}(q_S^*)$.

4. Independent Demands

In this section, we consider newsvendor games in which all demands are independent. It turns out that log-concave random variables play an important role in our results. We start by defining continuous and discrete log-concave random variables.

Definition 1. A continuous random variable X is log-concave if the logarithm of its density function $f(x)$ is concave.

Definition 2. Let X be a random variable with support on the set of non-negative integers. Denote the probability mass function $p_i = P(X = i)$, $i \ge 0$. Then X is log-concave if $\{i \ge 0 \colon p_i > 0\}$ is a set of consecutive integers[3] and $p_i^2 \ge p_{i-1}p_{i+1}$ for all $i \ge 1$.

By Definition 1, if a continuous random variable is log-concave, its support must be connected. As is shown in Ibragimov (1956), a continuous random variable X is log-concave if and only if the corresponding distribution function $F(x)$ is strongly unimodal, that is, a convolution of $F(x)$ with any unimodal distribution function is unimodal. Clearly this means $F(x)$ itself is unimodal. For discrete random variables, Keilson and Gerber (1971) establish a similar equivalence between log-concavity of the probability mass function and strong unimodality of the distribution function. Log-concave probability distributions constitute a broad class of distributions including many commonly used ones such as normal, exponential, uniform, logistic, and so forth, and have wide applications in economics (see, e.g., Bagnoli and Bergstrom 2005, Moulin and Shenker 2001).

While Definitions 1 and 2 impose similar constraints on the shape of the distribution, they are quite different in the sense that one cannot use the limit of continuous log-concave distributions to approximate a discrete case nor vice versa, and hence a separate analysis is necessary. We first present our results for continuous random variables.

Proposition 1. *Suppose the players' demands $X_1, \ldots, X_n$ are independent continuous random variables with log-concave distribution functions. Then the newsvendor game is a convex game.*[4]

Instead of checking the inequality (5) directly, we employ a perturbation argument in the proof of Proposition 1. Let $\psi(X_S) = c(S)$ represent the total cost for coalition S when the total demand is X_S. Then for any $S \subseteq T \subseteq N\backslash\{j\}$ the inequality (5) to ensure (N, c) is a convex game can be rewritten as $\psi(X+Z) - \psi(X) \ge \psi(X+Y+Z) - \psi(X+Y)$, where $X = X_S$, $Y = X_T - X_S$, $Z = X_j$. The inequality holds if $\psi(X + \alpha Y + \beta Z)$ is submodular in α, β for $\alpha, \beta \ge 0$.

Remark 1. Slikker et al. (2001) prove that a newsvendor game is convex if demands are independent and follow normal distributions. Özen et al. (2011) prove that a newsvendor game with three players is convex when demands are identical, independent, and uniformly distributed. Proposition 1 includes the previous conditions as special cases. Özen et al. (2011, p. 39) also conjecture that "newsvendor games with three players all having independent uniform demands should be convex." Note that Proposition 1 not only confirms this conjecture, but also shows that the convexity holds for much more general distributions beyond uniform and for any number of players.

Remark 2. By Proposition 3 in Sprumont (1990), all extreme points of the core (referred to as "extended vectors of marginal contribution" in Sprumont 1990) lead to PMASes. Therefore, the allocation under Shapley value, which is a convex combination of the extreme points, also results in a PMAS for the newsvendor games that satisfy the condition in Proposition 1.

Interestingly, the condition in Proposition 1 is also sufficient to guarantee that the dual-based allocation scheme is a PMAS. When $X_1, \ldots, X_n$ are all independent and continuous random variables, $X_S = \sum_{i \in S} X_i$ for $S \subseteq N$ is also a continuous random variable. This allows us to simplify the dual solution $\pi_S(\omega)$ as

$$\pi_S(\omega) = \begin{cases} -h, & \text{if } X_S(\omega) < q_S^*, \\ p, & \text{if } X_S(\omega) \ge q_S^*. \end{cases}$$

Therefore, for any $i \in S \subseteq N$, the allocated cost is given as

$$l_{i,S} = E[\pi_S(\omega)X_i(\omega)] = -hE[X_i 1_{\{X_S < q_S^*\}}] + pE[X_i 1_{\{X_S \geq q_S^*\}}]. \quad (6)$$

We have the following result.

Proposition 2. *Suppose the players' demands $X_1, \dots, X_n$ are independent continuous random variables with log-concave distribution functions. Then the dual-based allocation scheme is a PMAS.*

Similar to the proof of Proposition 1, we prove Proposition 2 by a perturbation analysis. It is sufficient to show that for any subset $S \subseteq N$, any player $i \in S$, and any player j, $j \notin S$, the inequality $l_{i,S} \geq l_{i,S\cup\{j\}}$ holds. We denote $l_i(D)$ as the cost allocated to a player $i \in N$ when the player is in a coalition with total demand D. Then the inequality $l_{i,S} \geq l_{i,S\cup\{j\}}$ is the same as $l_i(X+Y) \geq l_i(X+Y+Z)$, where $X = X_i$, $Y = X_S - X_i$, $Z = X_j$. Define $g(\delta) = l_i(X+Y+\delta Z)$. We then prove the proposition by showing $g(0) \geq g(1)$.

When demands are discrete random variables, it becomes difficult to reach the existence of a PMAS through the path of convexity. In fact, an example from Slikker et al. (2001) shows that the newsvendor game is not convex even when demands follow identical Bernoulli distributions, the simplest log-concave discrete distributions. In their example, the Shapley value still leads to a PMAS. Our next example demonstrates that the Shapley value need not result in a PMAS when demands follow independent Bernoulli distributions.

Example 1. Let $N = \{1,2,3\}$, $p = 0.6$, $h = 0.4$, X_1, X_2, X_3 can take values 0 or 1 with probabilities $P(X_1 = 0) = 0.3$, $P(X_2 = 0) = 0.7$, $P(X_3 = 0) = 0.7$.

In Table 1, the first column lists all possible coalitions. The second and third columns, respectively, give the optimal order quantity q_S^* and the minimum cost $c(S)$ for each coalition. Columns 4–6 show the costs allocated to all players according to the Shapley value, and columns 7–9 present these costs under the dual-based allocation rule. Figures in Table 1 are rounded such that the allocations add up to the total cost. From this table we can observe that even though all demands follow log-concave distributions, the game is not convex because

$$c(\{1,3\}) - c(\{1\}) < c(\{1,2,3\}) - c(\{1,2\}).$$

Table 1. Calculations for Example 1

			Shapley value			Dual-based		
S	q_S^*	$c(S)$	$\varphi_{1,S}$	$\varphi_{2,S}$	$\varphi_{3,S}$	$l_{1,S}$	$l_{2,S}$	$l_{3,S}$
{1,2,3}	1	0.327	0.0756	0.1257	0.1257	0.0888	0.1191	0.1191
{1,2}	1	0.21	0.075	0.135		0.0905	0.1195	
{2,3}	1	0.25		0.125	0.125		0.125	0.125
{1,3}	1	0.21	0.075		0.135	0.0905		0.1195
{1}	1	0.12	0.12			0.12		
{2}	0	0.18		0.18			0.18	
{3}	0	0.18			0.18			0.18

Since the game is not convex, the Shapley value does not necessarily result in a PMAS. Indeed, in this example the allocation under the Shapley value is not even in the core because $\varphi_{2,\{1,2,3\}} + \varphi_{3,\{1,2,3\}} > c(\{2,3\})$, and the allocated cost for each player does not always decrease when a new member joins since $\varphi_{1,\{1,2,3\}} > \varphi_{1,\{1,2\}}$. However, one can easily verify that in this example the dual-based allocation specifies a PMAS, and Proposition 3 shows that this is not a coincidence.

Proposition 3. *Suppose $X_1, \dots, X_n$ are independent discrete random variables and are log-concave. Then the dual-based allocation scheme is a PMAS.*

When demands are discrete, the dual-based allocation scheme is given by

$$\begin{aligned} l_{i,S} = &-hE[X_i 1_{\{X_S < q_S^*\}}] + pE[X_i 1_{\{X_S > q_S^*\}}] \\ &+ (p-\eta)E[X_i 1_{\{X_S = q_S^*\}}], \quad i \in S,\ S \subseteq N, \end{aligned} \quad (7)$$

where

$$\eta = \frac{p - (p+h)F_-(q_S^*)}{F(q_S^*) - F_-(q_S^*)}.$$

Comparing Equation (7) with (6), there is an additional term in (7) corresponding to the case where there is a strictly positive probability that the total demand of the coalition is equal to the optimal ordering quantity.

Different from the proof of Proposition 2, $l_i(X+Y+\delta Z)$ is no longer differentiable in δ when X, Y and Z are discrete random variables. Therefore, when we analyze the monotonicity of $l_i(X+Y+\delta Z)$ with respect to δ, we demonstrate that for any $\delta \geq 0$, there exists $\bar{\epsilon} > 0$ such that for any $\epsilon \in [0, \bar{\epsilon}]$

$$l_i(X+Y+(\delta+\epsilon)Z) \leq l_i(X+Y+\delta Z).$$

As a corollary to Proposition 3, when demands follow independent Poisson distributions, the dual-based allocation rule reduces to a simple rule that fully allocates the cost of a coalition in proportion to the mean demands of its participants.

Corollary 1. *Suppose $X_1, \dots, X_n$ are independent Poisson random variables. Then the dual-based allocation scheme is given by*

$$l_{i,S} = \frac{\lambda_i}{\sum_{k\in S}\lambda_k} c(S), \quad \forall\, i \in S,\ S \subseteq N,$$

where $X_i \sim Poisson(\lambda_i)$, for $i \in N$, and it is a PMAS.

In the case discussed in the corollary, the cost per unit of expected demand of coalition S, $c(S)/\sum_{k\in S}\lambda_k$,

Table 2. Calculations for Example 2

S	q_S^*	$c(S)$	$l_{1,S}$	$l_{2,S}$	$l_{3,S}$
$\{1,2,3\}$	2	0.5	0.0625	0.375	0.0625
$\{1,2\}$	2	0.45	0.05	0.4	
$\{2,3\}$	2	0.45		0.4	0.05
$\{1,3\}$	1	0.25	0.125		0.125
$\{1\}$	1	0.2	0.2		
$\{2\}$	2	0.4		0.4	
$\{3\}$	1	0.2			0.2

always decreases with the addition of a new member. Under the proportional allocation rule, the cost of each existing member of S is reduced uniformly at the same rate. The corollary is consistent with an earlier result of Karsten and Basten (2014), who show that the proportional allocation rule results in a PMAS in a similar game with Poisson demands. We demonstrate that the proportional rule can originate from a more general allocation rule, which in many other cases may not preserve proportionality but sustains population monotonicity.

Propositions 2 and 3 assure that when demands are log-concave, regardless whether the distribution is continuous or discrete, the dual-based allocation scheme is a PMAS. Naturally one may ask whether this allocation scheme is always population monotonic, even without log-concave distributions. Example 2 gives a negative answer.

Example 2. Consider three players with independent demands X_1, X_2 and X_3. Suppose that X_1 can take values 0 or 1 with equal probability, X_2 can take values 0 or 2 with equal probability, and X_3 can be 0 or 1 with equal probability. Notice that X_2 is not log-concave since its support is not an interval of consecutive integers. Assume $p = 0.6$, $h = 0.4$. For each coalition S, we list the optimal ordering quantity q_S^*, the total cost $c(S)$, and the dual-based allocation $l_{i,S}$ for each player $i \in S$ in Table 2.

In this case, when players 1 and 2 cooperate, the cost allocated to player 1 is 0.05. If player 3 joins the coalition, player 1's allocated cost increases to 0.0625. Therefore, the dual-based allocation scheme in this example is not a PMAS.

5. Dependent Demands

When players in a newsvendor game have dependent demands, establishing the existence of a PMAS by proving convexity is a very narrow path. It is not clear how prevalent convexity is among games with dependent demands, and characterizing conditions for such an occurrence is also difficult. To the best of our knowledge, the only known sufficient condition for the convexity of newsvendor games with dependent demands is given in Montrucchio and Scarsini (2007), which requires all players' demands to be permutation symmetric (see Definition 3), and follow a multivariate normal distribution.

In this section, we follow an alternative path to identify a large family of newsvendor games in which it is feasible to implement a PMAS. In particular, we focus on developing conditions for the dual-based allocation scheme to be population monotonic. We start by considering the aforementioned permutation symmetric demand distribution, defined as follows Hartman et al. (2000).

Definition 3. A joint distribution F of n random variables is permutation symmetric if, for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and any permutation π of $(1, 2, \ldots, n)$, $F(x_1, x_2, \ldots, x_n) = F(x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_n})$.

With permutation symmetric demand distributions, individual identities play no role in determining the total cost of any coalition. One thus may naturally consider allocating the cost of a coalition evenly to each of its members. Proposition 4 shows that this equal allocation rule is a specialization of the dual-based allocation rule to this type of demand distributions, and is population monotonic.

Proposition 4. *If $X_1, \ldots, X_n$ have a permutation symmetric joint distribution, then the dual-based allocation scheme is given by*

$$l_{i,S} = \frac{c(S)}{|S|}, \quad \forall\, i \in S,\, S \subseteq N,$$

and it is a PMAS.

Slikker et al. (2001) show that permutation symmetry does not guarantee the convexity of a newsvendor game if demands are not normally distributed. Therefore, Proposition 4 extends the existence of a PMAS beyond the domain of convex newsvendor games. To address a more general family of games, we need to consider the degree of dependence among the demands. Our discussion will be restricted to continuous demands and we use the concept of copula to measure the dependency among random variables.

Consider a random vector $(X_1, \ldots, X_n)$. Let F denote the joint cumulative distribution and F_i denote the marginal cumulative distribution function for random variable X_i, $i = 1, \ldots, n$. Then the random vector $(U_1, \ldots, U_n) = (F_1(X_1), \ldots, F_n(X_n))$ has uniformly distributed marginal distributions. The copula of $(X_1, \ldots, X_n)$ is defined as the joint cumulative distribution function of $(U_1, \ldots, U_n)$.

Definition 4. The copula $C\colon [0,1]^n \mapsto [0,1]$ for a random vector $(X_1, \ldots, X_n)$ is defined as the function such that

$$C(u_1, \ldots, u_n) = F(x_1, \ldots, x_n),$$

where $u_i = F_i(x_i)\ \forall\, i = 1, \ldots, n$.

The copula of two random variables X and Y measures the degree of their dependence. The larger the copula is, the higher the positive dependency between X and Y. If C_1 and C_2 are copulas, we say that C_1 is smaller than C_2 and write $C_1 \prec C_2$ if $C_1(u,v) \le C_2(u,v)$ for all $(u,v) \in [0,1]^2$. This partial order is called the concordance ordering (Nelsen 2006). The following proposition reveals a connection between population monotonicity of the dual-based allocation scheme and the degree of dependence between a player's demand, X_i, $i \in S$, and the total demand of a coalition, X_S, $S \subseteq N$.

Proposition 5. *Let $C_{i,S}(\cdot,\cdot)$ denote the copula of X_i and X_S, for $i \in S$ and $S \subseteq N$. The dual-based allocation scheme is a PMAS for the newsvendor game if for any $S \subseteq T$,*

$$C_{i,T}(u,\tau) \le C_{i,S}(u,\tau), \quad \forall\, u \in [0,1].$$

The dual-based allocation scheme is not a PMAS if there exists $S \subseteq T$ such that

$$C_{i,T}(u,\tau) > C_{i,S}(u,\tau), \quad \forall\, u \in [0,1].$$

Proposition 5 shows that the dual-based allocation scheme is a PMAS if $C_{i,T} \prec C_{i,S}$ for all $S \subseteq T$. This general condition based on the concordance ordering of copulas reduces to more specific conditions for certain special cases, which we discuss next.

In the first case, demands are comonotonic. A set $A \subseteq \mathbb{R}^n$ is comonotonic if for any $x, y \in A$, either $x \le y$ or $y \le x$, where $x \le y$ represents $x_i \le y_i$ for all $i = 1, 2, \ldots, n$. A random vector $X = (X_1, \ldots, X_n)$ is comonotonic if it has comonotonic support. Comonotonicity represents a strong positive dependence. When two random variables are comonotonic, their copula is $C(u,v) = \min\{u,v\}$. The comonotonicity of the demands then implies $C_{i,S}(u,v) = C_{i,T}(u,v) = \min\{u,v\}$ for all $S \subseteq T$, giving rise to the following result.

Corollary 2. *In newsvendor games with comonotonic demands, the dual-based allocation scheme is a PMAS.*

In our second case, demands follow a multivariate normal distribution. We will refer to this type of newsvendor game as a "Gaussian Newsvendor Game." We denote by μ_i, σ_i, ρ_{ij} with $i, j \in N$, $i \ne j$ the means, standard deviations, and correlation coefficients of the demand distributions, respectively.

The total demand of a coalition $S \subseteq N$ and that of any one of its members have a bivariate normal distribution with correlation coefficient

$$\rho(X_i, X_S) = \frac{\mathrm{Cov}(X_i, X_S)}{\sigma_i \sigma_S} = \frac{\sigma_i + \sum_{j \ne i, j \in S} \rho_{ij}\sigma_j}{\sqrt{\sum_{j\in S}\sigma_j^2 + 2\sum_{j<k,\, j\in S,\, k\in S}\rho_{jk}\sigma_j\sigma_k}}, \quad i \in S. \tag{8}$$

Interestingly, the concordance ordering of the bivariate normal copulas, which is the copula of the bivariate standard normal distributions, is equivalent to the ordering of the correlation coefficient as the Lemma 1 shows.

Lemma 1 (Meyer (2013)). *Let $C(u,v;\rho)$ denote the bivariate normal copula with correlation coefficient ρ. We have $C(u,v;\rho_1) \prec C(u,v;\rho_2)$ if and only if $\rho_1 \le \rho_2$.*

Applying Lemma 1 to Proposition 4 leads to the following proposition.

Proposition 6. *Let $S \subset N$ be a proper subset of players in a Gaussian Newsvendor Game, and $j \in N$, $j \notin S$. Then under the dual-based allocation scheme, $l_{i,S} \ge l_{i,S\cup\{j\}}$, i.e., player $i \in S$ is allocated a lower expected cost in coalition $S \cup \{j\}$ than in coalition S if and only if*

$$\rho(X_i, X_{S\cup\{j\}}) \le \rho(X_i, X_S). \tag{9}$$

Hence the dual-based allocation scheme is a PMAS if and only if

$$\rho(X_i, X_T) \le \rho(X_i, X_S), \quad \text{for all } i \in S \text{ and } S \subset T \subseteq N. \tag{10}$$

Proposition 6 completely characterizes the class of Gaussian Newsvendor Games in which the dual-based allocation scheme is a PMAS. Observe that the normal distribution characterized by (10) is not necessarily permutation symmetric. So the proposition suggests the possibility that a PMAS exists in some Gaussian Newsvendor Games that are not convex. Example 3 demonstrates such a case.

Example 3. Suppose $N = \{1,2,3\}$, $p = h = 1$, and

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 100 & 10 & -10 \\ 10 & 100 & -20 \\ -10 & -20 & 100 \end{pmatrix} \right).$$

We conclude that the dual-based allocation scheme is a PMAS in this game by verifying that (9) holds for each coalition $S \subset \{1,2,3\}$, player $i \in S$, and $j \notin S$. Here values of $\rho(X_i, X_S)$, $S \subseteq N$ and $i \in S$ are determined by (8), using the covariance matrix of the example as inputs.

To show the game is not convex, observe that the total costs of different coalitions are optimal values of newsvendor models with the same cost parameters and different Gaussian demand distributions. It follows that $c(S) = \mathcal{K}\sigma_S$ where $\mathcal{K}$ is a constant independent of S (see Zipkin 2000, Hartman and Dror 2005). One can compute that

$$c(\{1,3\}) - c(\{3\}) = \mathcal{K}(\sqrt{180} - \sqrt{100}) < \mathcal{K}(\sqrt{260} - \sqrt{160}) = c(\{1,2,3\}) - c(\{2,3\}),$$

which violates the definition of a convex game.

To show the effectiveness of applying Proposition 6 to Gaussian Newsvendor Games, we develop the following analysis on a couple of representative special cases.

***Special Case* 1.** We have shown that either independent log-concave or permutation symmetric demand distributions are sufficient for the dual-based allocation scheme to be population monotonic. In Gaussian Newsvendor Games, independence implies $\rho_{ij} = 0$ for all $i, j \in N$ $(i \neq j)$ and permutation-symmetry can be attained by letting $\sigma_i = \sigma$ for all $i \in N$ and $\rho_{i,j} = \rho$ for all $i, j \in N$, $i \neq j$. We combine these two conditions to define a class of Gaussian Newsvendor Games as follows.

The set of players N is divided into two nonempty and mutually exclusive subsets N_r and N_s. All players' demands are normally distributed with the same standard deviation $\sigma_i = 1$ $(i \in N)$. Demand correlation only exists between players in set N_r, referred to as the set of "relatives." The demand of a player in set N_s, referred to as the set of "strangers," is uncorrelated with that of any other players. The correlation coefficient is

$$\rho_{ij} = \begin{cases} \rho, & i, j \in N_r,\ i \neq j; \\ 0, & i \in N_s,\ j \in N,\ i \neq j, \end{cases}$$

where $\rho \neq 0$ and for the covariance matrix to be positive semi-definite, $\rho \geq -1/(|N_r| - 1)$.

Proposition 7. *Let $S \subset N$ be a coalition in the Gaussian Newsvendor Game, and $j \notin S$.*

If $j \in N_s$, then

$$\rho(X_i, X_{S\cup\{j\}}) \leq \rho(X_i, X_S) \quad \textit{for all } i \in S. \tag{11}$$

Hence player $i \in S$ is always allocated a lower cost in coalition $S \cup \{j\}$ than in S.

If $j \in N_r$, then

- *for each $i \in S \cap N_s$, the inequality (11) holds and the allocated cost to player i is lower in coalition $S \cup \{j\}$, if and only if $\rho \geq \underline{\rho}(S)$, where $\underline{\rho}(S) = -1/(2|N_r \cap S|) < 0$.*
- *for each $i \in S \cap N_r$, the inequality (11) holds and the allocated cost to player i is lower in coalition $S \cup \{j\}$ if and only if $\tilde{\rho}(S) \leq \rho \leq \bar{\rho}(S)$, where $\tilde{\rho}(S) \in (-1/(|N_r \cap S| - 1), \underline{\rho}(S))$ and $\bar{\rho}(S) \in (0, 1)$ are values depending only on $|S|$ and $|N_r \cap S|$.*

Proposition 7 shows that if a stranger joins the coalition, all existing players' allocated costs will not increase for any ρ; if a relative joins the coalition, additional requirements on ρ need to be imposed to ensure that all existing players are not worse off. To understand this discrepancy, first note that admitting a new player allows more risk pooling in a coalition. This benefit is shared equally among existing members of the coalition if the newcomer is a stranger with its demand uncorrelated with that of any other player. However, if the newcomer is a relative, then the benefit will not be uniformly shared because of the asymmetry of demand correlation between different types of coalition members and the new entrant.

The previous discussion implies that in this type of game, the dual-based allocation scheme is not a PMAS when the relatives have a strong positive correlation or a strong negative correlation. Hence one should not expect that players will cooperate fully on *every* path of growing coalitions. Nevertheless, the grand coalition may still be formed along specific paths. For instance, consider a path that starts with a coalition of all players $i \in N_r$ in phase 1 and admits players in N_s in phase 2. The dual-based allocation scheme is population monotonic during the first phase since X_i, $i \in N_r$ have a permutation symmetric distribution. The allocation scheme remains a PMAS in the second phase since (11) applies to all players $i \in N_s$. The specific path along which a grand coalition can be formed is referred to as a population monotonic path scheme (PMPS) in the literature (see Çiftçi et al. 2010). The existence of a PMPS is a weaker condition than the existence of a PMAS. The concept of a PMPS is suitable for the situation where an effective way to establish the grand coalition can be specified (usually by a third party). For example, Cruijssen et al. (2010) utilize this solution concept to study a game in which shippers outsource their logistics to a service provider who can choose the most suitable sequence to propose offers of allocating generated savings to shippers. On the other hand, a PMAS guarantees the formation of a grand coalition without the guidance of a third party, which is the main focus of our paper.

***Special Case* 2.** Another interesting case is Gaussian Newsvendor Games with "systematic risk." In this game, the demand of every player $i \in N$ can be decomposed into

$$X_i = Y_i + Z, \quad i \in N, \tag{12}$$

where Y_i, $i \in N$ and Z are mutually independent normal random variables. The randomness of Z poses a "systematic risk," since the quantity is a common part of every player's demand. Without loss of generality, we normalize the variance of Z to unity and denote those of Y_i by σ_i^2. We also denote $s = |S|$. Then the variances and correlation coefficients of the demands are

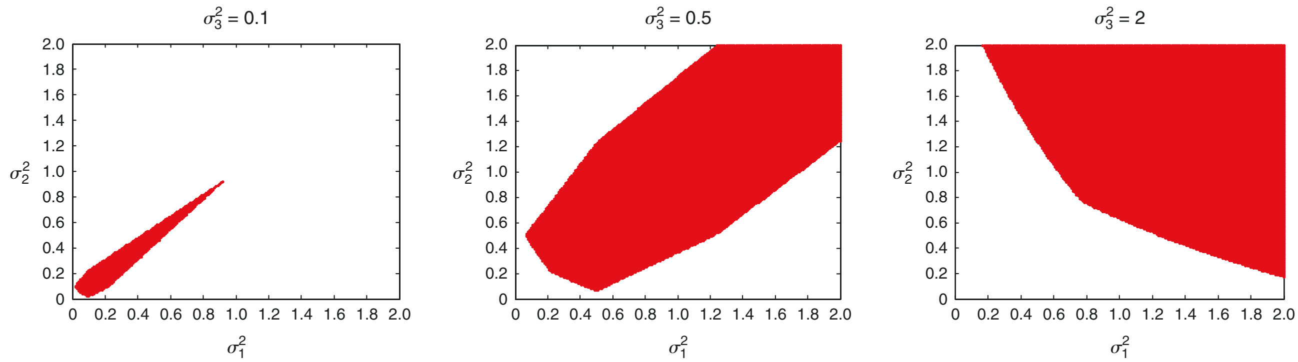
$$\mathrm{var}(X_i) = \sigma_i^2 + 1, \quad i \in N \quad \text{and}$$

$$\rho(X_i, X_S) = \frac{\sigma_i^2 + s}{\sqrt{\sigma_i^2 + 1}\sqrt{\sum_{j\in S}\sigma_j^2 + s^2}}, \quad S \subseteq N,\ i \in S.$$

Proposition 8. *In a Gaussian Newsvendor Game with demands defined by (12), for any coalition $S \subset N$, let $i \in S$ and $k \notin S$. If $\rho(X_i, X_S) \geq \rho(X_i, X_{S\cup\{k\}})$, then*

$$\rho(X_j, X_S) \geq \rho(X_j, X_{S\cup\{k\}}) \quad \textit{for any } j \in S \textit{ such that } \sigma_i^2 \leq \sigma_j^2. \tag{13}$$

Figure 1. (Color online) Range of Variances to Ensure a PMAS



Moreover, there exists a threshold $\bar{\sigma}_{i,S} \geq 0$ *such that*

$$\rho(X_i, X_S) \geq \rho(X_i, X_{S\cup\{k\}}) \quad \text{if and only if } \sigma_i \geq \bar{\sigma}_{i,S}.$$

The first part of Proposition 8 shows that in this game with "systematic risk," if it ever happens that demand correlation between a member and the coalition increases after adding a new member, it will happen first to the member with the least variance in his demand. This suggests that members with more "stable" demands are usually bottlenecks on the path of growing a coalition. The second part indicates that the dual-based allocation scheme is a PMAS only in games where each player's individual variance exceeds a certain threshold.

Figure 1 shows a game with three players, where the variance of player 3's demand is fixed at 0.1, 0.5, and 2, corresponding to the left, middle, and right panels, respectively. The red region in each panel represents combinations of demand variances of players 1 and 2 under which the dual-based allocation scheme is a PMAS. When the variance of player 3 is small, as is the case with the leftmost panel, variances of players 1 and 2 also have to be small, or player 3 will have a higher cost in coalition $\{1,2,3\}$ than in coalitions $\{1,3\}$ and/or $\{2,3\}$. Similarly, the rightmost panel shows that when player 3's demand has a high variance, those of players 1 and 2 must also be high, or they will also face a cost increase. The middle panel shows the intermediate case.

6. Open Problems, Initial Explorations, and Future Challenges

We have studied newsvendor games with a focus on the population monotonicity of allocation rules. We identified sufficient conditions for a newsvendor game to be convex and for the dual-based allocation scheme to be a PMAS, respectively. By discussing applications of these conditions, we shed interesting insights on players' incentives to expand a cooperative relationship.

While substantial new progress has been made here, our work also gives rise to a host of interesting new problems, ranging from algorithmic challenges to theoretical possibilities. Solving these problems is well beyond the scope of this paper. Nevertheless, to inspire future research, we highlight a few difficult issues and share initial findings.

Computational Complexity. In theory, the conditions developed in this paper can be applied to determine if the dual-based allocation scheme is a PMAS of a newsvendor game. However, in many cases, especially in the presence of demand dependency, computational complexity can be a major obstacle to testing these conditions.

Take the Gaussian Newsvendor Game as an example. For the general case, Proposition 6 requires testing (9) for all $S \subseteq N$ and $i \in S$. The number of inequalities grows exponentially with the number of players. Finding a uniformly applicable procedure to cut through the complexity is desirable but difficult. Our progress so far has been limited to a few interesting cases with particular exploitable structures.

Proposition 9. *Consider a Gaussian Newsvendor Game with n players.*

- *If the game is of the form of Special Case* 1, *the complexity of verifying the dual-based allocation scheme is a PMAS is* $O(n)$;
- *If the game is of the form of Special Case* 2, *the complexity of verifying the dual-based allocation scheme is a PMAS is* $O(n^2)$.

Going beyond Gaussian Newsvendor Game, we may rely on comparisons of copulas, suggested in Proposition 5, to test population monotonicity, an even harder problem that calls for more innovative approaches.

Beyond the Convex Game and the Dual-Based Allocation Scheme. Convex games ensure the existence of a PMAS. When a newsvendor game is not convex, we provide conditions to ensure that the dual-based allocation scheme is a PMAS. One may then ask the following: When a newsvendor game is not convex and

Table 3. Calculations for Example 4

S	X_S	q_S^*	$c(S)$
{1}, {2}, {3}	$\mathcal{U}(0,1)$	1/2	1/4
{1,2,3}	$\mathcal{U}(1,2)$	3/2	1/4
{1,2}	$\mathcal{U}(0,2)$	1	1/2
{1,3}, {2,3}	1	1	0

the dual-based allocation scheme is not a PMAS, is it still possible that a PMAS exists? The answer, as we demonstrate by the following example, is yes.

Example 4. Consider a newsvendor game of three players. Let $p = h = 1$. Assume that player 1's demand $X_1 \sim \mathcal{U}(0,1)$, where $\mathcal{U}(a,b)$ represents the uniform distribution over $[a,b]$. Let $X_2 = X_1$ and $X_3 = 1 - X_1$. For each $S \subseteq N$, X_S, q_S^*, and $c(S)$ can be determined as shown in Table 3:

The game is not convex because $c(\{2,3\}) - c(\{3\}) < c(\{1,2,3\}) - c(\{1,3\})$. The dual-based allocation scheme cannot be a PMAS in this case. Any coalition with two players is permutation symmetric, and thus as Proposition 4 concludes, the scheme always allocates an equal amount of the total cost to each player. Hence

$$l_{i,S} = c(S)/2 = 0, \quad i = 1,2,3 \quad \text{and} \quad S = \{1,3\}, \{2,3\}.$$

Since $c(\{1,2,3\}) > 0$, some player has to incur a positive cost in the grand coalition. Thus, expanding either coalition {1,3} or {2,3} will cause at least one player's cost to go up.

In this case, permutation symmetry in a coalition leads to allocation symmetry because any potential participants that have not joined are ignored. Deviating from this basic feature of the dual-based allocation scheme may save population monotonicity. Note that in this game, it makes no difference to player 3 which of the other two players to cooperate with first. If player 3 exercises his bargaining power that comes with this flexibility to minimize his cost in a two-player coalition, then the allocation scheme specified in Table 4 will emerge as a PMAS.

Example 4 makes it imperative to develop generally applicable PMASes other than the dual-based allocation scheme or relying on the notion of a convex game.

Table 4. A Population Monotonic Allocation Scheme

S	$c(S)$	$l_{1,S}$	$l_{2,S}$	$l_{3,S}$
{1,2,3}	1/4	1/4	1/4	−1/4
{1,2}	1/2	1/4	1/4	
{2,3}	0		1/4	−1/4
{1,3}	0	1/4		−1/4
{1}	1/4	1/4		
{2}	1/4		1/4	
{3}	1/4			1/4

Existence of a PMAS. So far we have identified various cases of newsvendor games in which a PMAS exists. But a PMAS is not guaranteed to exist for all newsvendor games. We provide an example with no PMAS.

The game has four players with demands $X_1 \sim \mathcal{U}(0,1)$, $X_2 = X_1$, $X_3 = X_4 = 1 - X_1$. Assume $p > 0$, $h > 0$. Since $c(\{1,3\}) = 0$ and $c(\{1,2,3\}) > 0$, player 2 has to bear a strictly positive cost in the latter coalition to keep it stable. For the same reason, in coalition {2,3,4}, player 3's cost should also be strictly positive. Thus to meet the definition of PMAS, both players 2 and 3 have to pay a strictly positive cost in coalition {2,3}, which is a part of coalitions of {1,2,3} and {2,3,4}. This is not possible since $c(\{2,3\}) = 0$.

It might be unsettling to learn that a PMAS is not guaranteed for some newsvendor games. This fact implies that there are natural limits on the growth of cooperative relationships in supply chains. It is important to understand those limits and thus to characterize those games that do have a PMAS and those that do not. In addition, interesting and surprising technical results may surface from such exercises.

Acknowledgments

The authors would like to thank the department editor, the associate editor, and three anonymous referees for their constructive and insightful comments. The authors are grateful to Paul Zipkin for his careful reading of the paper and his constructive feedbacks.

Appendix A

Proof of Proposition 1

Recall the condition (5) to ensure convex games and

$$c(S) = \sum_{i \in S} l_{i,S} = -hE[X_S 1_{\{X_S < q_S^*\}}] + pE[X_S 1_{\{X_S \geq q_S^*\}}]$$
$$= -hE[X_S] + (h+p)E[X_S 1_{\{X_S \geq q_S^*\}}],$$

with $q_S^* = F_S^{-1}(\tau)$. Let $\psi(X_S) = c(S)$ represent the total cost for coalition S when the total demand is X_S. Then for any $S \subseteq T \subseteq N \backslash \{j\}$ the convex game condition (5) can be rewritten as $\psi(X+Z) - \psi(X) \geq \psi(X+Y+Z) - \psi(X+Y)$, where $X = X_S$, $Y = X_T - X_S$, $Z = X_j$. The inequality holds if $\psi(X + \alpha Y + \beta Z)$ is submodular in α, β for $\alpha, \beta \geq 0$.

Notice that $\psi(X + \alpha Y + \beta Z) = -hE[X + \alpha Y + \beta Z] + (h+p) \cdot E[(X + \alpha Y + \beta Z)1_{\{(X+\alpha Y+\beta Z) \geq q(\alpha,\beta)\}}]$, where $q(\alpha, \beta)$ is the optimal ordering quantity when the total demand is $X + \alpha Y + \beta Z$. Clearly, the first term $-hE[X + \alpha Y + \beta Z]$ is submodular in (α, β). Therefore, we only need to show that $g(\alpha,\beta) = E[(X + \alpha Y + \beta Z)1_{\{(X+\alpha Y+\beta Z) \geq q(\alpha,\beta)\}}]$ is submodular. Let $f(x,y,z)$ be the joint p.d.f of X, Y and Z, and $L(y,z) = f(q(\alpha,\beta) - \alpha y - \beta z, y, z)$. Then

$$g(\alpha,\beta) = \int \left[\int_{x \geq q(\alpha,\beta) - \alpha y - \beta z} (x + \alpha y + \beta z) f(x,y,z)\, dx \right] dy\, dz, \tag{A.1}$$

and

$$\int \left[\int_{x \geq q(\alpha,\beta) - \alpha y - \beta z} f(x,y,z)\, dx \right] dy\, dz = 1 - \tau. \tag{A.2}$$

From (A.1),

$$\frac{\partial g}{\partial \alpha}(\alpha,\beta)=\int\left[\int_{x\ge q(\alpha,\beta)-\alpha y-\beta z} y f(x,y,z)\,dx - q(\alpha,\beta)\cdot\left(\frac{\partial q}{\partial \alpha}(\alpha,\beta)-y\right)L(y,z)\right]dy\,dz. \quad (A.3)$$

Differentiating on both sides of (A.2) with respect to α results in

$$-\int\left(\frac{\partial q}{\partial \alpha}(\alpha,\beta)-y\right)L(y,z)\,dy\,dz=0. \quad (A.4)$$

By applying (A.4) to (A.3), it follows $(\partial g/\partial\alpha)(\alpha,\beta)=\int[\int_{x\ge q(\alpha,\beta)-\alpha y-\beta z} y f(x,y,z)\,dx]\,dy\,dz$. Hence,

$$\frac{\partial^2 g}{\partial\alpha\partial\beta}(\alpha,\beta)=-\int y\left(\frac{\partial q}{\partial \beta}(\alpha,\beta)-z\right)L(y,z)\,dy\,dz. \quad (A.5)$$

By differentiating on both sides of (A.2) with respect to β, $-\int((\partial q/\partial\beta)(\alpha,\beta)-z)L(y,z)\,dy\,dz=0$, which leads to

$$\frac{\partial q}{\partial \beta}(\alpha,\beta)=\frac{\int zL(y,z)\,dy\,dz}{\int L(y,z)\,dy\,dz}. \quad (A.6)$$

By the right hand side of (A.5) with (A.6), $(\partial^2 g/(\partial\alpha\partial\beta))\cdot(\alpha,\beta)=-\Delta/(\int L(y,z)\,dy\,dz)$, where $\Delta=(\int yL(y,z)\,dy\,dz)\cdot(\int zL(y,z)\,dy\,dz)-(\int L(y,z)\,dy\,dz)(\int yzL(y,z)\,dy\,dz)$.

Note that

$$\left(\int yL(y,z)\,dy\,dz\right)\left(\int zL(y,z)\,dy\,dz\right)=\frac{1}{2}\left[\left(\int z_1L(y_1,z_1)\,dy_1dz_1\right)\left(\int y_2L(y_2,z_2)\,dy_2dz_2\right)+\left(\int z_2L(y_2,z_2)\,dy_2dz_2\right)\left(\int y_1L(y_1,z_1)\,dy_1dz_1\right)\right];$$

$$\left(\int yzL(y,z)\,dy\,dz\right)\left(\int L(y,z)\,dy\,dz\right)=\frac{1}{2}\left[\left(\int y_1z_1L(y_1,z_1)\,dy_1dz_1\right)\left(\int L(y_2,z_2)\,dy_2dz_2\right)+\left(\int y_2z_2L(y_2,z_2)\,dy_2dz_2\right)\left(\int L(y_1,z_1)\,dy_1dz_1\right)\right].$$

Thus, we can rewrite Δ as

$$\begin{aligned}\Delta&=\frac{1}{2}\int -(y_2-y_1)(z_2-z_1)L(y_1,z_1)L(y_2,z_2)\,dy_1dz_1dy_2dz_2\\&=\frac{1}{2}\int_{(y_2-y_1)(z_2-z_1)\le 0}-(y_2-y_1)(z_2-z_1)L(y_1,z_1)\cdot L(y_2,z_2)\,dy_1dz_1dy_2dz_2+\frac{1}{2}\int_{(y_2-y_1)(z_2-z_1)\ge 0}-(y_2-y_1)\cdot(z_2-z_1)L(y_1,z_1)L(y_2,z_2)\,dy_1dz_1dy_2dz_2\\&=\frac{1}{2}\int_{(y_2-y_1)(z_2-z_1)\le 0}-(y_2-y_1)(z_2-z_1)L(y_1,z_1)\cdot L(y_2,z_2)\,dy_1dz_1dy_2dz_2+\frac{1}{2}\int_{(y_1-y_2)(z_2-z_1)\ge 0}-(y_1-y_2)\cdot(z_2-z_1)L(y_2,z_1)L(y_1,z_2)\,dy_2dz_1dy_1dz_2,\end{aligned}$$

where in the last equality we simply switched the notations y_1 and y_2 in the second term. Hence, combining the two terms we have

$$\Delta=\frac{1}{2}\int_{(y_2-y_1)(z_2-z_1)\le 0}-(y_2-y_1)(z_2-z_1)\cdot[L(y_1,z_1)L(y_2,z_2)-L(y_2,z_1)L(y_1,z_2)]\,dy_1dz_1dy_2dz_2.$$

Now, we see that a sufficient condition for $\Delta\ge 0$ is for $(y_2-y_1)(z_2-z_1)\le 0$, $L(y_1,z_1)L(y_2,z_2)-L(y_2,z_1)L(y_1,z_2)\ge 0$. In other words, $\Delta\ge 0$ if $\log L(y,z)$ is submodular in (y,z). Indeed, by independence of X,Y,Z, $\log L(y,z)=\log f_X(q(\alpha,\beta)-\alpha y-\beta z))+\log f_Y(y)+\log f_Z(z)$. Since when all demands have log-concave distributions, the sum of demands is also log-concave, i.e., X is log-concave. Hence, $\log f_X(q(\alpha,\beta)-\alpha y-\beta z))$ is submodular in (y,z) for $\alpha,\beta\ge 0$. Therefore, $\Delta\ge 0$ and $g(\alpha,\beta)$ is submodular in α and β for $\alpha,\beta\ge 0$, which completes our proof.

Proof of Proposition 2

Similar to the idea of the proof for Proposition 1, we employ a perturbation analysis. It is sufficient to show that for any $S\subseteq N$, $i\in S$, $j\notin S$, the inequality $l_{i,S}\ge l_{i,S\cup\{j\}}$ holds. We use $l_i(D)$ to denote the cost allocated to player i when the total demand is D. Then the inequality $l_{i,S}\ge l_{i,S\cup\{j\}}$ is the same as $l_i(X+Y)\ge l_i(X+Y+Z)$, where $X=X_i$, $Y=X_S-X_i$, $Z=X_j$. We establish the inequality by showing that $l_i(X+Y+\delta Z)$ is a decreasing function of δ for $\delta\ge 0$. Let $F(x,y,z)$ and $f(x,y,z)$ denote the joint c.d.f. and p.d.f. of X, Y and Z. Then, the optimal order quantity for the coalition $S\cup\{j\}$, denoted as $q(\delta)$, should satisfy

$$\int_{x+y+\delta z\le q(\delta)} dF(x,y,z)=\tau. \quad (A.7)$$

Meanwhile, the cost allocated to player i according to the dual-based allocation scheme is $l_i(X+Y+\delta Z)=-hE[X1_{\{X+Y+\delta Z\le q(\delta)\}}]+pE[X1_{\{X+Y+\delta Z\ge q(\delta)\}}]=pE[X]-(p+h)\cdot E[X1_{\{X+Y+\delta Z\le q(\delta)\}}]$. Equivalently, we show that $g(\delta):=E[X1_{\{X+Y+\delta Z\le q(\delta)\}}]$ is increasing in δ. Since

$$g(\delta)=\int_{x+y+\delta z\le q(\delta)} xf(x,y,z)\,dx\,dy\,dz=\int x\left[\int_{y\le q(\delta)-\delta z-x} f(x,y,z)\,dy\right]dx\,dz,$$

we then have

$$\frac{dg}{d\delta}=\int x\left(\frac{dq}{d\delta}-z\right)f(x,q(\delta)-\delta z-x,z)\,dx\,dz. \quad (A.8)$$

To check $dg/d\delta\ge 0$, observe from (A.7) that $\int(dq/d\delta-z)\cdot f(x,q(\delta)-\delta z-x,z)\,dx\,dz=0$, and thus $dq/d\delta=(\int zf(x,q(\delta)-\delta z-x,z)\,dx\,dz)(\int f(x,q(\delta)-\delta z-x,z)\,dx\,dz)$. Denote $L(x,z)=f(x,q(\delta)-\delta z-x,z)$, we then obtain

$$dg/d\delta=\int x\left[\left(\int zL(x,z)\,dx\,dz\right)\Big/\left(\int L(x,z)\,dx\,dz\right)-z\right]L(x,z)\,dx\,dz=\Delta\Big/\left(\int L(x,z)\,dx\,dz\right),$$

where $\Delta := (\int zL(x,z)\,dxdz)(\int xL(x,z)\,dxdz) - (\int xz \cdot L(x,z)\,dxdz)(\int L(x,z)\,dxdz)$. In the following, we show that $\Delta \geq 0$, which then implies $dg/d\delta \geq 0$. Note that

$$\left(\int zL(x,z)\,dxdz\right)\left(\int xL(x,z)\,dxdz\right) = \frac{1}{2}\left[\left(\int z_1L(x_1,z_1)\,dx_1dz_1\right)\left(\int x_2L(x_2,z_2)\,dx_2dz_2\right) + \left(\int z_2L(x_2,z_2)\,dx_2dz_2\right)\left(\int x_1L(x_1,z_1)\,dx_1dz_1\right)\right];$$

$$\left(\int xzL(x,z)\,dxdz\right)\left(\int L(x,z)\,dxdz\right) = \frac{1}{2}\left[\left(\int x_1z_1L(x_1,z_1)\,dx_1dz_1\right)\left(\int L(x_2,z_2)\,dx_2dz_2\right) + \left(\int x_2z_2L(x_2,z_2)\,dx_2dz_2\right)\left(\int L(x_1,z_1)\,dx_1dz_1\right)\right].$$

Following similar steps as in the proof of Proposition 1, we can rewrite Δ as $\Delta = (1/2)\int_{(x_2-x_1)(z_2-z_1)\leq 0} -(x_2-x_1)\cdot(z_2-z_1)[L(x_1,z_1)L(x_2,z_2) - L(x_2,z_1)L(x_1,z_2)]\,dx_1dz_1dx_2dz_2$.

Now, we see that a sufficient condition for $\Delta \geq 0$ is $(x_2-x_1)(z_2-z_1) \leq 0$, $L(x_1,z_1)L(x_2,z_2) - L(x_2,z_1)L(x_1,z_2) \geq 0$. In other words, $\Delta \geq 0$ if $\log L(x,z)$ is submodular. When demands of all players are independent, then X, Y and Z are independent. Consequently, we have $\log L(x,z) = \log f_X(x) + \log f_Y(q(\delta) - \delta z - x) + \log f_Z(z)$, where $f_X(\cdot)$, $f_Y(\cdot)$ and $f_Z(\cdot)$ are the p.d.f for X, Y and Z, respectively. Requiring $\log L(x,z)$ to be submodular is equivalent to requiring $(\partial^2 \log f_Y(q(\delta) - \delta z - x))/(\partial x \partial z) \leq 0$, i.e., $f_Y(\cdot)$ is log-concave. Since when all demands have log-concave distributions, the sum of demands is also log-concave, i.e., Y is log-concave, this completes our proof.

Proof of Proposition 3

In this proof, we first consider the case where demand has finite support. Later we will extend our proof to include the case where demands can have infinite support. Similar to the proof of Proposition 2, let X denote the demand faced by the player under study, i.e., $X := X_i$ for some $i \in S$, $S \subseteq N$, Y denote the sum of demands faced by the rest of players in the coalition, i.e., $Y := \sum_{k\neq i, k\in S} X_k$ and Z be the demand faced by the new incoming player, i.e., $Z := X_j$ for some $j \notin S$. We use $l_i(D)$ to denote the cost allocated to player i when the total demand is D. Employing a perturbation analysis, we are going to show that for any $\delta \geq 0$, there exists $\bar{\epsilon} > 0$ such that for any $\epsilon \in (0, \bar{\epsilon}]$, $l_i(X + Y + (\delta+\epsilon)Z) \leq l_i(X + Y + \delta Z)$. In other words, for any $\delta \geq 0$, the function $l_i(X + Y + \delta Z)$ is locally decreasing in δ.

Suppose that random variables X and Y can take values in $\{x_m: m = 1, \ldots, M\}$ and $\{y_n: n = 1, \ldots, N\}$, respectively. With a slight abuse of notation, suppose Z can take values in $\{z_j: j = 1, \ldots, J\}$ and $X + Y$ can take values in $\{u_i: i = 1, \ldots, I\}$. We assume, without loss of generality, that the sequences x_m, y_n, u_i, z_j are all ordered from the smallest to the largest.

Given $\delta \geq 0$, we define $\bar{\epsilon} > 0$ as follows. Let $\bar{\epsilon}_1 = \min_{j,j'\in\{1,\ldots,J\}, j>j'} \kappa_1(z_j, z_{j'})/(z_j - z_{j'})$, where

$$\kappa_1(z_j, z_{j'}) = \begin{cases} \lceil \delta(z_j - z_{j'})\rceil - \delta(z_j - z_{j'}), & \text{if } \lceil \delta(z_j - z_{j'})\rceil - \delta(z_j - z_{j'}) > 0, \\ 1, & \text{if } \lceil \delta(z_j - z_{j'})\rceil - \delta(z_j - z_{j'}) = 0, \end{cases}$$

and $\lceil x \rceil$ represents the smallest integer that is greater than or equal to x. Let

$$\bar{\epsilon}_2 = \min_{i,i'\in\{1,\ldots,I\}, j,j'\in\{1,\ldots,J\}} \frac{\kappa_2(u_i, u_{i'}, z_j, z_{j'})}{z_J - z_1},$$

where

$$\kappa_2(u_i, u_{i'}, z_j, z_{j'}) = \begin{cases} |(u_i + \delta z_j) - (u_{i'} + \delta z_{j'})|, & \text{if } |(u_i + \delta z_j) - (u_{i'} + \delta z_{j'})| > 0, \\ +\infty, & \text{if } |(u_i + \delta z_j) - (u_{i'} + \delta z_{j'})| = 0. \end{cases}$$

Note that the objective functions in the definitions of $\bar{\epsilon}_1$ and $\bar{\epsilon}_2$ are strictly positive over a finite set of points. Therefore, $\bar{\epsilon}_1, \bar{\epsilon}_2 > 0$ and we can choose $\bar{\epsilon} > 0$ such that $\bar{\epsilon} < \min\{\bar{\epsilon}_1, \bar{\epsilon}_2\}$.

We explain why we need $\bar{\epsilon} < \bar{\epsilon}_1$ here. The reason why we need $\bar{\epsilon} < \bar{\epsilon}_2$ will become clear later. If $u_i + (\delta+\epsilon)z_j = u_{i'} + (\delta+\epsilon)z_{j'}$ for some $\epsilon \in (0, \bar{\epsilon}]$, then we must have $u_i = u_{i'}$ and $z_j = z_{j'}$ (or equivalently $i = i'$ and $j = j'$). To see this, without loss of generality assume $z_j > z_{j'}$. Because $\epsilon < \bar{\epsilon}_1$, $(\delta+\epsilon)\cdot(z_j - z_{j'})$ cannot be an integer. Since $u_i - u_{i'}$ is an integer, there does not exist any u_i, $u_{i'}$, z_j, $z_{j'}$ satisfying $(\delta+\epsilon)(z_j - z_{j'}) - (u_i - u_{i'}) = 0$.

Let q and q_ϵ denote the optimal order quantity when the total demand is $X + Y + \delta Z$ and $X + Y + (\delta+\epsilon)Z$, respectively, where $\epsilon \in (0, \bar{\epsilon}]$. Then there exists a unique pair (i^*, j^*), such that $q_\epsilon = u_{i^*} + (\delta+\epsilon)z_{j^*}$, where $u_{i^*} = x_m + y_n$ for some m, n. Notice that $\{(i,j): u_i + \delta z_j = u_{i^*} + \delta z_{j^*}\}$ need not be a singleton. Let $\mathscr{K} = \{(\hat{i}, \hat{j}): u_{\hat{i}} + \delta z_{\hat{j}} = u_{i^*} + \delta z_{j^*}, u_{\hat{i}} > u_{i^*}, z_{\hat{j}} < z_{j^*}\}$, $\mathscr{L} = \{(i', j'): u_{i'} + \delta z_{j'} = u_{i^*} + \delta z_{j^*}, u_{i'} < u_{i^*}, z_{j'} > z_{j^*}\}$. We index the elements in $\mathscr{K}$ by $k \in \{1, \ldots, K\}$ and the elements in $\mathscr{L}$ by $l \in \{1, \ldots, L\}$.

To facilitate the analysis and simplify the comparison of allocated costs $l_i(X + Y + (\delta+\epsilon)Z)$ and $l_i(X + Y + \delta Z)$, we define $U^* = \{(m,n,j): x_m + y_n = u_{i^*}, z_j = z_{j^*}\}$, $\hat{U}_k = \{(m,n,j): x_m + y_n = u_{\hat{i}_k}, z_j = z_{\hat{j}_k}\}$, $(\hat{i}_k, \hat{j}_k) \in \mathscr{K}$, $k = 1, \ldots, K$, $U'_l = \{(m,n,j): x_m + y_n = u_{i'_l}, z_j = z_{j'_l}\}$, $(i'_l, j'_l) \in \mathscr{L}$, $l = 1, \ldots, L$. Moreover, let $\hat{U} = \bigcup_{k=1}^K \hat{U}_k$, $U' = \bigcup_{l=1}^L U'_l$, $U = U^* \cup \hat{U} \cup U'$. Notice that U consists of all possible (m,n,j) such that $x_m + y_n + \delta z_j = u_{i^*} + \delta z_{j^*}$. And U can be divided into three scenarios, namely U^*, $\hat{U}$ and U', based on whether $z_j = z_{j^*}$, $z_j < z_{j^*}$ or $z_j > z_j^*$, respectively. Then $\hat{U}$ can be further divided into $\hat{U}_k$ for $k \in \mathscr{K}$, and U' is divided into U'_l for $l \in \mathscr{L}$.

Let $P_{m,n,j} := P(X = x_m, Y = y_n, Z = z_j)$. We then define some probabilities corresponding to the sets U^*, $\hat{U}_k$, and U'_l: $P^* = P(X + Y = u_{i^*}, Z = z_{j^*}) = \sum_{(m,n,j)\in U^*} P_{m,n,j}$, $\hat{P}_k = P(X + Y = u_{\hat{i}_k}, Z = z_{\hat{j}_k}) = \sum_{(m,n,j)\in\hat{U}_k} P_{m,n,j}$, $k = 1, \ldots, K$, $P'_l = P(X + Y = u_{i'_l}, Z = z_{j'_l}) = \sum_{(m,n,j)\in U'_l} P_{m,n,j}$, $l = 1, \ldots, L$. Similarly, we define the following expectations: $E^* = E[X1_{\{X+Y=u_{i^*}, Z=z_{j^*}\}}] = \sum_{(m,n,j)\in U^*} x_m P_{m,n,j}$, $\hat{E}_k = E[X1_{\{X+Y=u_{\hat{i}_k}, Z=z_{\hat{j}_k}\}}] = \sum_{(m,n,j)\in\hat{U}_k} x_m P_{m,n,j}$, $k = 1, \ldots, K$, $E'_l = E[X1_{\{X+Y=u_{i'_l}, Z=z_{j'_l}\}}] = \sum_{(m,n,j)\in U'_l} x_m P_{m,n,j}$, $l = 1, \ldots, L$. Since $\hat{U}_k$, $k = 1, \ldots, K$ and U'_l, $l = 1, \ldots, L$ are disjoint sets, then we have $\hat{P} = \sum_{(m,n,j)\in\hat{U}} P_{m,n,j} = \sum_{k=1}^K \hat{P}_k$, $P' = \sum_{(m,n,j)\in U'} P_{m,n,j} = \sum_{l=1}^L P'_l$; $\hat{E} = \sum_{(m,n,j)\in\hat{U}} x_m P_{m,n,j} = \sum_{k=1}^K \hat{E}_k$, $E' = \sum_{(m,n,j)\in U'} x_m P_{m,n,j} = \sum_{l=1}^L E'_l$. Define $P^0_<, P^0_\leq, P^0_=$ as the probability that the total demand $X + Y + \delta Z$ is strictly less than, less than or equal to, and equal to $u_{i^*} + \delta z_{j^*}$, respectively. In other words, $P^0_< = P(X + Y + \delta Z < u_{i^*} + \delta z_{j^*})$, $P^0_\leq = P(X + Y + \delta Z \leq u_{i^*} + \delta z_{j^*})$, $P^0_= = P(X + Y + \delta Z = u_{i^*} + \delta z_{j^*})$. Similarly, define $P^{\epsilon}_{<}, P^{\epsilon}_{\leq}, P^{\epsilon}_{=}$ as $P^{\epsilon}_{<} = P(X+Y+(\delta+\epsilon)Z < u_{i^*} + (\delta+\epsilon)z_{j^*})$, $P^{\epsilon}_{\leq} = P(X+Y+(\delta+\epsilon)Z \leq u_{i^*} + (\delta+\epsilon)z_{j^*})$, $P^{\epsilon}_{=} = P(X+Y+(\delta+\epsilon)Z = u_{i^*} + (\delta+\epsilon)z_{j^*})$. Before we compare the costs, we need the following lemma:

Lemma 2. *$P^{\epsilon}_{<} - P^0_{<} = \hat{P}, P^0_{\leq} - P^{\epsilon}_{\leq} = P'$. In addition, $q = u_{i^*} + \delta z_{j^*}$.*

Proof of Lemma 2. we first show that $\{X+Y+\delta Z < u_{i^*} + \delta z_{j^*}\} \subseteq \{X+Y+(\delta+\epsilon)Z < u_{i^*} + (\delta+\epsilon)z_{j^*}\}$. This is equivalent to

$$\{(i,j): u_i + \delta z_j < u_{i^*} + \delta z_{j^*}\} \subseteq \{(i,j): u_i + (\delta+\epsilon)z_j < u_{i^*} + (\delta+\epsilon)z_{j^*}\} \quad (A.9)$$

For any (i,j) satisfying $u_i + \delta z_j < u_{i^*} + \delta z_{j^*}$, we have $u_i + \delta z_j < u_{i^*} + \delta z_{j^*} + \epsilon(z_{j^*} - z_j)$. To see this, if $z^*_j > z_j$, then clearly $u_i + \delta z_j < u_{i^*} + \delta z_{j^*} + \epsilon(z_{j^*} - z_j)$ since $\epsilon > 0$. If $z^*_j < z_j$, then by definition of $\bar{\epsilon}_2$, we have $\epsilon < \bar{\epsilon}_2 \leq (u^*_i + \delta z^*_j - (u_i + \delta z_j))/(z_j - z^*_j)$. Therefore, $u_i + \delta z_j < u_{i^*} + \delta z_{j^*} + \epsilon(z_{j^*} - z_j)$. This proves (A.9). Similarly if $u_i + \delta z_j > u_{i^*} + \delta z_{j^*}$, then $u_i + \delta z_j > u_{i^*} + \delta z_{j^*} + \epsilon(z_{j^*} - z_j)$. Therefore, we have

$$\begin{aligned}
&\{(i,j): u_i + (\delta+\epsilon)z_j < u_{i^*} + (\delta+\epsilon)z_{j^*}\} \\
&\quad - \{(i,j): u_i + \delta z_j < u_{i^*} + \delta z_{j^*}\} \\
&= \{(i,j): u_i + (\delta+\epsilon)z_j < u_{i^*} + (\delta+\epsilon)z_{j^*},\ u_i + \delta z_j \geq u_{i^*} + \delta z_{j^*}\} \\
&= \{(i,j): u_i + (\delta+\epsilon)z_j < u_{i^*} + (\delta+\epsilon)z_{j^*},\ u_i + \delta z_j = u_{i^*} + \delta z_{j^*}\} \\
&= \{(i,j): z_j < z_{j^*},\ u_i + \delta z_j = u_{i^*} + \delta z_{j^*}\}
\end{aligned}$$

Notice that $P^{\epsilon}_{=} = P^*$, $P^0_{=} = P^* + \hat{P} + P'$, together with $P^{\epsilon}_{<} - P^0_{<} = \hat{P}$ we obtain $P^0_{\leq} - P^{\epsilon}_{\leq} = P'$. Since the optimal order quantity when the total demand is $X+Y+(\delta+\epsilon)Z$ satisfies $q_{\epsilon} = u_{i^*} + (\delta+\epsilon)z_{j^*}$, we have $P^{\epsilon}_{<} \leq \tau \leq P^{\epsilon}_{\leq}$. Recall that $\tau = p/(p+h)$ is the critical fractile. Hence, $P^0_{<} \leq \tau \leq P^0_{\leq}$. Therefore, the optimal ordering quantity when the total demand is $X+Y+\delta Z$ is given by $q = u_{i^*} + \delta z_{j^*}$. This completes the proof of Lemma 2. □

Now we are ready to compare the allocated costs. When the demand is $X+Y+\delta Z$, the cost allocated to player i according to the dual-based allocation scheme is $l_i(X+Y+\delta Z) = pE(X) - (p+h)E[X1_{\{X+Y+\delta Z<q\}}] - \eta E[X1_{\{X+Y+\delta Z=q\}}]$, where $\eta = (p - (p+h)P(X+Y+\delta Z < q))/P(X+Y+\delta Z = q)$. Similarly, we can obtain the cost allocated to player X when the demand changes to $X+Y+(\delta+\epsilon)Z$ as follows: $l_i(X+Y+(\delta+\epsilon)Z) = pE(X) - (p+h)E[X1_{\{X+Y+(\delta+\epsilon)Z<q_{\epsilon}\}}] - \eta_{\epsilon}E[X1_{\{X+Y+(\delta+\epsilon)Z=q_{\epsilon}\}}]$, where $\eta_{\epsilon} = (p - (p+h)P(X+Y+(\delta+\epsilon)Z < q_{\epsilon}))/P(X+Y+(\delta+\epsilon)Z = q_{\epsilon})$. Using the notation introduced earlier and the fact $q = u_{i^*} + \delta z_{j^*}$ proved in Lemma 2, we have the following observations: $\eta/(p+h) = (\tau - P^0_{<})/(P^* + \hat{P} + P')$, $\eta_{\epsilon}/(p+h) = (\tau - P^{\epsilon}_{<})/P^*$, $E[X1_{\{X+Y+\delta Z=q\}}] = E^* + \hat{E} + E'$, $E[X1_{\{X+Y+(\delta+\epsilon)Z=q_{\epsilon}\}}] = E^*$, $E[X1_{\{X+Y+(\delta+\epsilon)Z<q_{\epsilon}\}}] - E[X1_{\{X+Y+\delta Z<q\}}] = \hat{E}$.

Therefore, the cost difference is

$$\begin{aligned}
&l_i(X+Y+(\delta+\epsilon)Z) - l_i(X+Y+\delta Z) \\
&= -(p+h)\Bigg\{E[X1_{\{X+Y+(\delta+\epsilon)Z<q_{\epsilon}\}}] - E[X1_{\{X+Y+\delta Z<q\}}] \\
&\quad + \frac{\eta_{\epsilon}}{p+h}E[X1_{\{X+Y+(\delta+\epsilon)Z=q_{\epsilon}\}}] - \frac{\eta}{p+h}E[X1_{\{X+Y+\delta Z=q\}}]\Bigg\} \\
&= -(p+h)\left\{\hat{E} + \frac{\tau - P^{\epsilon}_{<}}{P^*}E^* - \frac{\tau - P^0_{<}}{P^* + \hat{P} + P'}(\hat{E} + E^* + E')\right\} \\
&= -(p+h)\left\{\hat{E} + \frac{\xi}{P^*}E^* - \frac{\xi + \hat{P}}{P^* + \hat{P} + P'}(\hat{E} + E^* + E')\right\} \\
&:= -(p+h)g(\xi),
\end{aligned}$$

where $g(\xi) = \hat{E} + (\xi/P^*)E^* - ((\xi + \hat{P})/(P^* + \hat{P} + P'))(\hat{E} + E^* + E')$, $\xi = \tau - P^{\epsilon}_{<}$, and $\xi + \hat{P} = \tau - P^0_{<}$ since $P^{\epsilon}_{<} - P^0_{<} = \hat{P}$ due to Lemma 2. We also have $\xi \in [0, P^*]$ since $0 \leq \tau - P^{\epsilon}_{<} \leq P^{\epsilon}_{\leq} - P^{\epsilon}_{<} = P^*$. Because $g(\xi)$ is linear in ξ, we only need to show that $g(0) \geq 0$ and $g(P^*) \geq 0$. Notice that $g(0) \geq 0$ is equivalent to

$$\hat{E}(P^* + P') - \hat{P}(E^* + E') \geq 0, \quad (A.10)$$

and $g(P^*) \geq 0$ is equivalent to

$$P'(\hat{E} + E^*) - E'(\hat{P} + P^*) \geq 0. \quad (A.11)$$

In order to prove inequalities (A.10) and (A.11), we present an important property of log-concave random variables.

Lemma 3. *If X and Y are independent log-concave random variables, then $u \mapsto E[X \mid X+Y=u]$ is increasing.*

Lemma 3 is a special case of Theorem 1 in Efron (1965). Using Lemma 3, we can prove the following lemma.

Lemma 4. *$\hat{E}P' - \hat{P}E' \geq 0$, $E^*P' - P^*E' \geq 0$, $\hat{E}P^* - \hat{P}E^* \geq 0$.*

Proof of Lemma 4. we only show that $\hat{E}P' - \hat{P}E' \geq 0$, the other two inequalities follow similarly.

$$\begin{aligned}
\hat{E}P' - \hat{P}E' &= \sum_{k=1}^{K}\sum_{l=1}^{L}(\hat{E}_k P'_l - \hat{P}_k E'_l) \\
&= \sum_{k=1}^{K}\sum_{l=1}^{L}\hat{P}_k P'_l\left(\frac{\hat{E}_k}{\hat{P}_k} - \frac{E'_l}{P'_l}\right) \\
&= \sum_{k=1}^{K}\sum_{l=1}^{L}\hat{P}_k P'_l(E[X \mid X+Y=u_{\hat{i}_k}] - E[X \mid X+Y=u_{i'_l}]) \\
&\geq 0.
\end{aligned}$$

The third equality is due to $\hat{E}_k/\hat{P}_k = E[X \mid X+Y=u_{\hat{i}_k}, Z = z_{\hat{j}_k}] = E[X \mid X+Y=u_{\hat{i}_k}]$ since X and Z are independent of each other. Similarly we have $E'_l/P'_l = E[X \mid X+Y=u_{i'_l}]$. The last inequality is because of Lemma 3. This completes the proof of Lemma 4. □

Inequalities (A.10) and (A.11) follow immediately from Lemma 4. This completes the proof when every demand has a finite support.

In the following we extend our result to the case where demands can have infinite support. The idea of the proof is to use a sequence of finite support random variables to approximate the one with infinite support. We start with the case where only one of the random variables $X_1, \ldots, X_n$ has infinite support. Without loss of generality, we assume $X := X_i$ has infinite support for some $i \in N$. The truncation of a nonnegative random variable X at level $k > 0$, denoted as X^k, is defined with the probability mass function

$$f_{X^k}(x) = \begin{cases} \dfrac{P(X=x)}{M_k}, & \text{if } x \leq k, \\ 0, & \text{if } x > k, \end{cases}$$

where the normalizing constant $M_k = \sum_{0 \leq x \leq k} P(X=x)$.

With this truncation, if X is log-concave, then X^k is also log-concave (see Theorem 7 of Bagnoli and Bergstrom 2005), and since $M_k \uparrow 1$ as $k \to \infty$, we have $\lim_{k\to\infty} f_{X^k}(x) = P(X=x) = f_X(x)$, for any $x \geq 0$.

Let X^k_S be the total demands faced by S after one replaces X_i by X^k. If $i \notin S$, clearly $X^k_S = X_S$, for any $k > 0$. Denote

$Y = \sum_{j\in S, j\neq i} X_j$ if $i \in S$. Then for $q \geq 0$, $P(X_S^k = q) = \sum_{x=0}^{q} P(X^k = x)P(Y = q - x) = (1/M_k)\sum_{x=0}^{\min\{q,k\}} P(X = x) \cdot P(Y = q - x)$. Note that when $k > q$, we have $P(X_S^k = q) = (1/M_k)\sum_{x=0}^{q} P(X = x)P(Y = q - x) = P(X_S = q)/M_k$. Consequently, $F_{X_S^k}(q) \to F_{X_S}(q)$ for any $q > 0$ and when $k > q$, $F_{X_S^k}(q)$ is monotonically decreasing in k.

We denote the optimal order quantities by coalition S before and after the truncation of X by q_S^* and q_S^k. We have the following lemma.

Lemma 5. *There exists K, such that for* $k > K$, $q_S^k = q_S^*$.

Proof of Lemma 5. For simplicity, we suppress the dependence on S and let $F(\cdot) := F_{X_S}(\cdot)$, $F^k(\cdot) := F_{X_S^k}(\cdot)$, $q^* := q_S^*$ and $q^k := q_S^k$. The optimal order quantities q^* and q^k then must satisfy the inequalities

$$F_-(q^*) \leq \tau \leq F(q^*), \tag{A.12a}$$

$$F_-^k(q^k) \leq \tau \leq F^k(q^k), \quad \forall k. \tag{A.12b}$$

It is possible that q^* (or q^k) satisfying (A.12a) (or (A.12b)) is not unique. In fact, there can be at most two consecutive integers satisfying (A.12a) (or (A.12b)) under which the optimal costs are the same. We assume here that when there are multiple solutions, we pick the smallest one.

We first prove that the sequence of solutions $\{q^k: k > 0\}$ is bounded above by q^*. Suppose, on the contrary, $q^k > q^*$ for some k. Note that we must have $k \geq q^k$, since $F^k(k) = 1 > \tau$. As a result, $\tau \geq F_-^k(q^k) \geq F_-^{k+1}(q^k) \geq \cdots \geq F_-(q^k)$. On the other hand, $F_-(q^*) < F(q^*) \leq F_-(q^k) \leq F_-^k(q^k) \leq \tau$, where the second inequality is by the assumption $q^k > q^*$. Clearly, this contradicts (A.12a). Thus, $q^k \leq q^*$ for all k.

Next, we show that q^k is monotonically increasing in k. For convenience, we assume without loss of generality that $k > q^*$ so that $F^k(q)$ is decreasing in k for any $q \leq q^*$. Suppose $q^k > q^{k+1}$ for some k. Then, by a similar argument, we have

$$F_-^{k+1}(q^{k+1}) < F^{k+1}(q^{k+1}) \leq F_-^{k+1}(q^k) \leq F_-^k(q^k) \leq \tau,$$

which contradicts with (A.12b). Therefore, q^k is increasing in k. By boundedness and the fact that q^k can only take integers, there must be some K and $\bar{q} \leq q^*$ such that $q^k = \bar{q}$ for any $k > K$.

Finally, we show that $\bar{q} = q^*$. In fact, for any $k > K$, we have $F_-^k(\bar{q}) \leq \tau \leq F^k(\bar{q})$. Taking limit with respect to k on both sides, we have $F_-(\bar{q}) \leq \tau \leq F(\bar{q})$, which shows $\bar{q} = q^*$. This completes the proof of Lemma 5. □

For the rest of the proof, we simply assume that $k > K$ such that $q_S^k = q_S^*$. Similar to before, we denote the cost allocated to player i in S before and after truncation as $l_{i,S}$ and $l_{i,S}^k$, respectively, which are given by $l_{i,S} = pE[X] - (h+p)E[X1_{\{X+Y\leq q_S^*\}}] - ((p-(p+h)P(X+Y<q_S^*))/P(X+Y = q_S^*))E[X1_{\{X+Y=q_S^*\}}]$, and $l_{i,S}^k = pE[X^k] - (h+p)E[X^k 1_{\{X^k+Y\leq q_S^k\}}] - ((p-(p+h)P(X^k+Y<q_S^k))/P(X^k+Y=q_S^k))E[X^k 1_{\{X^k+Y=q_S^k\}}]$. We next show that $\lim_{k\to\infty} l_{i,S}^k = l_{i,S}$. First note that $E[X] - E[X^k] = (1 - 1/M_k)\sum_{x=0}^{k} xP(X = x) + \sum_{x\geq k+1} xP(X = x)$ Since $E[X]$ is finite, we have $\sum_{x\geq k+1} xP(X=x) \to 0$. On the other hand, since $\sum_{x=0}^{k} xP(X = x) \to E[X] < \infty$, we have $(1-1/M_k)\sum_{x=0}^{k} xP(X = x) \to 0$. Therefore, $\lim_{k\to\infty} E[X^k] = E[X]$.

Similarly, we prove convergence of the rest of the terms one by one as follows:

$$\begin{aligned}
&P(X^k + Y = q_S^k) \\
&\quad = \frac{1}{M_k}\sum_{x=0}^{q_S^*} P(X = x)P(Y = q_S^* - x) \to P(X + Y = q_S^*); \\
&P(X^k + Y < q_S^k) \\
&\quad = \frac{1}{M_k}\sum_{q=0}^{q_S^*-1}\sum_{x=0}^{q} P(X = x)P(Y = q - x) \to P(X + Y < q_S^*); \\
&E[X^k 1_{\{X^k+Y=q_S^k\}}] \\
&\quad = \frac{1}{M_k}\sum_{x=0}^{q_S^*} xP(X = x)P(Y = q_S^* - x) \to E[X1_{\{X+Y=q_S^*\}}]; \\
&E[X^k 1_{\{X^k+Y\leq q_S^k\}}] \\
&\quad = \frac{1}{M_k}\sum_{q=0}^{q_S^*}\sum_{x=0}^{q} xP(X = x)P(Y = q - x) \to E[X1_{\{X+Y\leq q_S^*\}}].
\end{aligned}$$

Therefore, $\lim_{k\to\infty} l_{i,S}^k = l_{i,S}$. We can show, in the same way, that $\lim_{k\to\infty} l_{j,S}^k = l_{j,S}$ for any $j \neq i$, $j \in S$, and $\lim_{k\to\infty} c^k(S) := \lim_{k\to\infty}(\sum_{j\in S} l_{j,S}^k) = c(S)$. By the result for the finite support case, we know $(l_{j,S}^k)_{j\in S, S\subseteq N}$ is a PMAS. Consequently,

$$l_{j,S} = \lim_{k\to\infty} l_{j,S}^k \geq \lim_{k\to\infty} l_{j,T}^k = l_{j,T}, \quad \forall\, S \subseteq T \subseteq N,$$

$$\sum_{j\in S} l_{j,S} = \lim_{k\to\infty}\sum_{j\in S} l_{j,S}^k = \lim_{k\to\infty} c^k(S) = c(S), \quad \forall\, S \subseteq N.$$

Therefore, $(l_{j,S})_{j\in S, S\subseteq N}$ is a PMAS as well.

We remark that in the previous proof, the finite support condition of X_j, $j \neq i$, $j \in N$ is not used until the last step. As a result, when there are more than one demands having infinite support, we can prove this result via induction—truncating one of the random variables with infinite support following the previous process and then invoking the induction hypothesis at the last step. This completes the proof of Proposition 3.

Proof of Corollary 1

The total demand for the coalition S is $X_S = \sum_{i\in S} X_i \sim Poisson(\lambda_S)$, where $\lambda_S = \sum_{i\in S}\lambda_i$. According to the dual-based allocation scheme, we can write the cost allocated to player i under the coalition S as $l_{i,S} = pE[X_i] - (h+p)E[X_i 1_{\{X_S<q_S^*\}}] - \eta E[X_i 1_{\{X_S=q_S^*\}}]$, where q_S^* denotes the optimal ordering quantity for the coalition S and $\eta = (p - (p+h)P(X_S \leq q_S^* - 1))/P(X_S = q_S^*)$. Now we can write the cost as

$$\begin{aligned}
l_{i,S} &= pE[X_i] - (h+p)E[X_i \mid X_S < q_S^*]P(X_S < q_S^*) \\
&\quad - \eta E[X_i \mid X_S = q_S^*]P(X_S = q_S^*) \\
&= pE[X_i] - (h+p)\sum_{j=0}^{q_S^*-1} E[X_i \mid X_S = j]P(X_S = j) \\
&\quad - \eta E[X_i \mid X_S = q_S^*]P(X_S = q_S^*).
\end{aligned}$$

Note that the total cost can be written in a similar way: $c(S) = pE[X_S] - (h+p)E[X_S \mid X_S < q_S^*]P(X_S < q_S^*) - \eta E[X_S \mid X_S = q_S^*]P(X_S = q_S^*)$. With X_i being Poisson, X_i conditioned on

$X_S = j$ follows the binomial distribution: $Binomial(j, \lambda_i/\lambda_S)$. Thus, $E[X_i \mid X_S = j] = j\lambda_i/\lambda_S$ and

$$
\begin{aligned}
l_{i,S} &= pE[X_i] - (h+p)\sum_{j=0}^{q_S^*-1} j\frac{\lambda_i}{\lambda_S}P(X_S = j) - q_S^*\frac{\lambda_i}{\lambda_S}\eta P(X_S = q_S^*)\\
&= p\lambda_i + \frac{\lambda_i}{\lambda_S}\left[-(h+p)\sum_{j=0}^{q_S^*-1} jP(X_S = j) - q_S^*\eta P(X_S = q_S^*)\right]\\
&= p\lambda_i + \frac{\lambda_i}{\lambda_S}[c(S) - pE[X_S]]\\
&= \frac{\lambda_i}{\lambda_S}c(S).
\end{aligned}
$$

In order to show that $l_{i,T} \le l_{i,S}$, for any $S \subseteq T$, it is sufficient to show that $c(T)/\sum_{k\in T}\lambda_k \le c(S)/\sum_{k\in S}\lambda_k$. Recall that $c(S) = \min_q E[\phi(q, X_S)] := \min_q E[h(q - X_S)^+ + p(X_S - q)^+]$. The function defined by $\psi(X_S) = \min_q E[\phi(q, X_S)]$ can be shown to be positive homogeneous (see Müller et al. 2002). Consequently, $c(S)/\sum_{k\in S}\lambda_k = \min_q E[\phi(q, X_S/\sum_{k\in S}\lambda_k)]$, and it is equivalent to verify $\min_q E[\phi(q, X_T/\sum_{k\in T}\lambda_k)] \le \min_q E[\phi(q, X_S/\sum_{k\in S}\lambda_k)]$. From Example 3.A.32 in Shaked and Shanthikumar (2007), it follows that $X_T/\sum_{k\in T}\lambda_k \le_{cx} X_S/\sum_{k\in S}\lambda_k$, where $\le_{cx}$ denotes the convex order of random variables. Since $\phi(q,\cdot)$ is convex, we then have for any q, $E[\phi(q, X_T/\sum_{k\in T}\lambda_k)] \le E[\phi(q, X_S/\sum_{k\in S}\lambda_k)]$, and our claim holds.

Proof of Proposition 4

For any $S \subseteq N$ and $i, j \in S$, since X_i and X_j are permutation symmetric, we have $l_{i,S} = E[X_i(\omega)\pi_S(\omega)] = E[X_j(\omega)\pi_S(\omega)] = l_{j,S}$. By $\sum_{i\in S} l_{i,S} = c(S)$, $l_{i,S} = c(S)/|S|$.

For $S \subseteq T \subseteq N$, let $l_{i,T} = c(T)/|T|$. We know that $l_{i,T}$ is in the core of T. That is, for any $S \subseteq T$, $\sum_{i\in S} l_{i,T} = |S|(c(T)/|T|) \le c(S)$. We then have for any $i \in S$ $l_{i,T} = c(T)/|T| \le c(S)/|S| = l_{i,S}$. This proves that $(l_{i,S})_{i\in S, S\subseteq N}$ is a PMAS.

Proof of Proposition 5

We firstly present the following lemma, which will be used in the proof of Proposition 5.

Lemma 6. *The dual-based allocation for player i in coalition S is given by*

$$(p+h)\int_0^1 C_{i,S}\left(u, \frac{p}{p+h}\right)\bigg/\phi(u)\,du + H_i,$$

where

$$\phi(u) = \frac{du}{dx} = \frac{dF_{X_i}}{dx}(F_{X_i}^{-1}(u)) > 0,$$

and

$$H_i = (p+h)\int_0^1 \frac{(1-\tau)1_{\{u\ge F_{X_i}(0)\}} - u}{\phi(u)}\,du - hE[X_i].$$

Proof of Lemma 6. Consider any coalition S. By Definition 4 we have $C_{i,S}(u,\tau) = P(X_i \le x, X_S \le q)$, where $u = F_{X_i}(x)$, $\tau = F_{X_S}(q) = p/(p+h)$, and q is the optimal ordering quantity. The dual-based allocation for player i is $-hE[X_i] + (p+h)\cdot E[X_i 1_{\{X_S\ge q\}}]$. Notice that adding more players into the coalition only affects the term $E[X_i 1_{\{X_S\ge q\}}]$ in the cost allocation. Define $G(x) = \int_{-\infty}^{\infty} 1_{\{y\ge q\}} f_{X_i,X_S}(x,y)\,dy$. Then

$$
\begin{aligned}
E[X_i 1_{\{X_S\ge q\}}] &= \int_{-\infty}^{\infty} xG(x)\,dx\\
&= \int_{-\infty}^{0} xG(x)\,dx + \int_{0}^{\infty} xG(x)\,dx\\
&= \int_{-\infty}^{0} x\,d\left(\int_{-\infty}^{x} G(z)\,dz\right) - \int_{0}^{\infty} x\,d\left(\int_{x}^{\infty} G(z)\,dz\right)\\
&= x\int_{-\infty}^{x} G(z)\,dz\bigg|_{-\infty}^{0} - \int_{-\infty}^{0}\int_{-\infty}^{x} G(z)\,dz\,dx\\
&\qquad - x\int_{x}^{\infty} G(z)\,dz\bigg|_{0}^{\infty} + \int_{0}^{\infty}\int_{x}^{\infty} G(z)\,dz\,dx\\
&= -\int_{-\infty}^{0}\int_{-\infty}^{x} G(z)\,dz\,dx + \int_{0}^{\infty}\int_{x}^{\infty} G(z)\,dz\,dx\\
&= -\int_{-\infty}^{0} P(X_i \le x, X_S \ge q)\,dx\\
&\qquad + \int_{0}^{\infty} P(X_i \ge x, X_S \ge q)\,dx.
\end{aligned}
$$

And we have $P(X_i \le x, X_S \ge q) = P(X_i \le x) - P(X_i \le x, X_S \le q) = u - C_{i,S}(u,\tau)$, and $P(X_i \ge x, X_S \ge q) = 1 - u - \tau + C_{i,S}(u,\tau)$. Let $\phi(u) = du/dx = (dF_{X_i}/dx)(F_{X_i}^{-1}(u))$. We have $E[X_i 1_{\{X_S\ge q\}}] = \int_0^{F_{X_i}(0)}((C_{i,S}(u,\tau) - u)/\phi(u))\,du + \int_{F_{X_i}(0)}^1((1 - u - \tau + C_{i,S}(u,\tau))/\phi(u))\,du = \int_0^1(((1-\tau)1_{\{u\ge F_{X_i}(0)\}} + C_{i,S}(u,\tau) - u)/\phi(u))\,du$. Therefore, the allocated cost for player i is $(p+h)\cdot\int_0^1 C_{i,S}(u, p/(p+h))/\phi(u)\,du + (p+h)\int_0^1(((1-\tau)1_{\{u\ge F_{X_i}(0)\}} - u)/\phi(u))\,du - hE[X_i]$. The last two terms do not depend on coalition S. □

Proposition 5 follows easily from Lemma 6. Consider a player i and two coalitions S and T where $i \in S$, $i \in T$ and $S \subseteq T$. If $C_{i,T}(u, p/(p+h)) \le C_{i,S}(u, p/(p+h))$, $\forall\, u \in [0,1]$, then according to Lemma 6 the allocated cost for player i in coalition T is smaller than or equal to that in coalition S. Similarly, if $C_{i,T}(u, p/(p+h)) > C_{i,S}(u, p/(p+h))$, $\forall\, u \in [0,1]$, then the cost allocated for player i in coalition T is greater than that in coalition S.

Proof of Corollary 2

Notice that when two random variables U_1 and U_2 are comonotonic, the copula is given by $C(u_1, u_2) = P(F_1(U_1) \le u_1, F_1(U_2) \le u_2) = \min(u_1, u_2)$. When all the demands are comonotonic, the demand of any player of interest and the total demand of any coalition are also comonotonic (Müller et al. 2002). Then it follows from Proposition 5 that the allocated cost does not change when adding more players, ensuring that the dual-based allocation scheme is a PMAS.

Proof of Proposition 6

Consider any player i and coalition S with $i \in S$, we have that X_i and X_S are both normally distributed. Let $Z = (X_i - \mu_{X_i})/\sigma_{X_i}$ and $\hat{Z} = (X_S - \mu_{X_S})/\sigma_{X_S}$, where μ_X and σ_X denote the mean and standard deviation of a random variable X, respectively. Then both Z and $\hat{Z}$ are standard normal random variables and the correlation coefficients satisfy $\rho_{X_i,X_S} = \rho_{Z,\hat{Z}}$. It then follows from Proposition 5 and Lemma 1 that the allocated cost of the player with demand X decreases when adding new members if and only if the resulting correlation coefficient does not increase.

Proof of Proposition 7

If $j \in N_s$, one can easily verify that (11) always holds.

In the following let $s = |S|$ and $m = |N_r \cap S|$.

If $j \in N_r, i \in S \cap N_s$, then $\mathrm{Cov}(X_i, X_S) = \mathrm{Cov}(X_i, X_{S\cup\{j\}}) = 1$. And $\sigma_S^2 = s + \rho(m-1)m$, $\sigma_{S\cup\{j\}}^2 = s + 1 + \rho m(m+1)$. We need $\sigma_{S\cup\{j\}}^2 \geq \sigma_S^2$, which is equivalent to $\rho \geq -1/2m = \underline{\rho}(S)$.

If $j \in N_r, i \in S \cap N_r$, then

$$\rho(X_i, X_S) = (1 + (m-1)\rho)/\sqrt{s + m(m-1)\rho}, \quad \text{and}$$
$$\rho(X_i, X_{S\cup\{j\}}) = (1 + m\rho)/\sqrt{s + 1 + m(m+1)\rho}.$$

We need $(1 + (m-1)\rho)/\sqrt{s + m(m-1)\rho} \geq (1 + m\rho)/\sqrt{s + 1 + m(m+1)\rho}$. After rearranging the terms we have

$$f(\rho) := (1 + (m-1)\rho)^2(s + 1 + m(m+1)\rho) - (1 + m\rho)^2(s + m(m-1)\rho) \geq 0. \tag{A.13}$$

Observe that $f(\rho)$ is a polynomial function of degree 3, and $\lim_{\rho\to\infty} f(\rho) = -\infty$, $\lim_{\rho\to-\infty} f(\rho) = \infty$, $f(0) = 1$, $f(1) = (2m+1)(m-s) < 0$, $f(-1/(m-1)) = -(s-m)/(m-1)^2 < 0$, $f(-1/(2m)) = ((2m+1)(2s-m+1))/(8m^2) > 0$, $-1/(m-1) < -1/(2m)$. Therefore, there exist $\underline{\rho}(S) \in (-1/(m-1), -1/(2m))$ and $0 < \bar{\rho}(S) < 1$ such that (A.13) holds if and only if $\underline{\rho}(S) \leq \rho \leq \bar{\rho}(S)$. Finally, we remark that even though $f(\rho)$ will become positive when ρ is sufficiently small, recall that we have restricted $\rho \geq -1/(|N_r| - 1)$ to guarantee the covariance matrix to be positive semi-definite and because $-1/(m-1) \leq -1/(|N_r| - 1)$ we do not need to consider the region below $-1/(m-1)$.

Proof of Proposition 8

Given any $i \in S \subset N$,

$$\rho(X_i, X_S) - \rho(X_i, X_{S\cup\{k\}}) = \frac{1}{\sqrt{\sigma_i^2 + 1}}\left(\frac{\sigma_i^2 + s}{\sqrt{\sum_{l\in S}\sigma_l^2 + \sigma_k^2 + (s+1)^2}}\right) \cdot \left(\sqrt{\frac{\sum_{l\in S}\sigma_l^2 + \sigma_k^2 + (s+1)^2}{\sum_{l\in S}\sigma_l^2 + s^2}} - 1 - \frac{1}{\sigma_i^2 + s}\right).$$

The inequality $\rho(X_i, X_S) \geq \rho(X_i, X_{S\cup\{k\}})$ is equivalent to $1/(\sigma_i^2 + s) \leq \sqrt{(\sum_{l\in S}\sigma_l^2 + \sigma_k^2 + (s+1)^2)/(\sum_{l\in S}\sigma_l^2 + s^2)} - 1$. Clearly if $\rho(X_i, X_S) \geq \rho(X_i, X_{S\cup\{k\}})$, then for any $j \in S$ such that $\sigma_i^2 \leq \sigma_j^2$, we have $1/(\sigma_j^2 + s) \leq 1/(\sigma_i^2 + s) \leq \sqrt{(\sum_{l\in S}\sigma_l^2 + \sigma_k^2 + (s+1)^2)/(\sum_{l\in S}\sigma_l^2 + s^2)} - 1$. Therefore, (13) holds.

For the second part of the proposition, we first prove by contradiction that if $\sigma_i^2 \geq \max\{\sigma_j^2 : j \in S, j \neq i\}$, then we must have $\rho(X_i, X_{S\cup\{k\}}) \leq \rho(X_i, X_S)$. If not, then the proof of (13) implies that $\rho(X_j, X_{S\cup\{k\}}) > \rho(X_j, X_S)$ for any $j \in S$. By Proposition 6 we know that $\sum_{i\in S} l_{i,S\cup\{k\}} > \sum_{i\in S} l_{i,S} = c(S)$, which means that the dual-based allocation scheme for $S \cup \{k\}$ is not in the core since all players in S are worse off after player k joins and will want to deviate from $S \cup \{k\}$ and form S. This contradicts the fact that the dual-based allocation is always in the core. Therefore, for the rest of the proof we only need to consider the case where $\sigma_i^2 < \max\{\sigma_j^2 : j \in S, j \neq i\}$.

In the following we show that there exists a threshold $\bar{\sigma}_{i,S}$ such that $\rho(X_i, X_{S\cup\{k\}}) \leq \rho(X_i, X_S)$ if and only if $\sigma_i \geq \bar{\sigma}_{i,S}$. Notice that

$$\rho(X_i, X_S) - \rho(X_i, X_{S\cup\{k\}}) = \frac{1}{\sqrt{\sigma_i^2 + 1}}\left(\frac{\sigma_i^2 + s}{\sqrt{\sum_{j\in S}\sigma_j^2 + s^2}} - \frac{\sigma_i^2 + s + 1}{\sqrt{\sum_{j\in S}\sigma_j^2 + \sigma_k^2 + (s+1)^2}}\right).$$

Let

$$f(\sigma_i^2) = (\sigma_i^2 + s)\Bigg/\sqrt{\sum_{j\in S}\sigma_j^2 + s^2} - (\sigma_i^2 + s + 1)\Bigg/\sqrt{\sum_{j\in S}\sigma_j^2 + \sigma_k^2 + (s+1)^2}.$$

Then we only need to show that $f(\sigma_i^2)$ is increasing in $\sigma_i^2 \geq 0$. To further simplify notation, let $\eta = \sigma_i^2 + s$, $a = \sum_{j\in S, j\neq i}\sigma_j^2 + s^2 - s$, $b = \sum_{j\in S, j\neq i}\sigma_j^2 + \sigma_k^2 + s + s^2$. Then we only need to show that $\eta/\sqrt{\eta + a} - (\eta + 1)/\sqrt{\eta + b + 1}$ has a positive derivative with respect to η. The derivative is given by $(\eta + 2a)/(2(\eta + a)^{3/2}) - (\eta + 1 + 2b)/(2(\eta + 1 + b)^{3/2})$. We need to show that $(\eta + 1 + 2b)/(\eta + 1 + b)^{3/2} \leq (\eta + 2a)/(\eta + a)^{3/2}$. Since $\sigma_i^2 < \max\{\sigma_j^2 : j \in S, j \neq i\}$, we have $\eta < 2a$. Hence, the derivative of $(\eta + 2x)/(\eta + x)^{3/2}$ with respect to x is negative when $a \leq x \leq b$. Therefore, $(\eta + 2b)/(\eta + b)^{3/2} \leq (\eta + 2a)/(\eta + a)^{3/2}$. It is easy to see that $(\eta + 1 + 2b)(\eta + b) \leq (\eta + 2b)(\eta + 1 + b)$. Hence, $(\eta + 1 + 2b)(\eta + b)^{3/2} \leq (\eta + 2b)(\eta + 1 + b)^{3/2}$, which is equivalent to $(\eta + 1 + 2b)/(\eta + 1 + b)^{3/2} \leq (\eta + 2b)/(\eta + b)^{3/2}$. In conclusion, $(\eta + 1 + 2b)/(\eta + 1 + b)^{3/2} \leq (\eta + 2b)/(\eta + b)^{3/2} \leq (\eta + 2a)/(\eta + a)^{3/2}$. This completes the proof.

Proof of Proposition 9

Firstly consider Special Case 1. Recall from the proof of Proposition 7 that the largest possible value of $|N_r \cap S|$ is $|N_r| - 1$, since $j \in N_r$ and $j \notin S$. Hence, we only need $\rho \geq -0.5/(|N_r| - 1)$ to ensure that in any coalition any player who is a stranger will not have increased cost after adding a relative. For the case where $j \in N_r, i \in N_r$, notice that the left hand side of (A.13) is decreasing in s. Hence, we only need to check whether (A.13) holds for any $m \leq s = n$. Since there are at most n inequalities to be checked, the overall computational complexity is $O(n)$.

Then we consider Special Case 2. To ensure that the dual-based allocation scheme is a PMAS, it is sufficient to show that for any $i \in S$ and $S \cup \{k\}, k \notin S$, we have $\rho(X_i, X_{S\cup\{k\}}) \leq \rho(X_i, X_S)$, which is equivalent to

$$\frac{\sigma_i^2 + s + 1}{\sqrt{\sum_{j\in S}\sigma_j^2 + \sigma_k^2 + (s+1)^2}} \leq \frac{\sigma_i^2 + s}{\sqrt{\sum_{j\in S}\sigma_j^2 + s^2}}. \tag{A.14}$$

In the following we provide a method to check the inequalities (A.14) in $O(n^2)$ time. After rearranging (A.14) we have

$$\frac{\sigma_i^2 + s + 1}{\sigma_i^2 + s} \leq \sqrt{\frac{\sum_{j\in S}\sigma_j^2 + \sigma_k^2 + (s+1)^2}{\sum_{j\in S}\sigma_j^2 + s^2}}. \tag{A.15}$$

Notice that the right hand side is increasing in σ_k^2 and decreasing in σ_j^2. We can sort $\{\sigma_j^2, j = 1, \ldots, n\}$. For any player i, we use $\sigma_{(1)}^2 \geq \sigma_{(2)}^2 \geq \cdots \geq \sigma_{(n-1)}^2$ to denote the sorted elements of $\{\sigma_j^2 \mid 1 \le j \le n, j \ne i\}$. Given any $s = 2, \ldots, n-1$, choose $S = \{\sigma_i^2\} \cup \{\sigma_{(1)}^2, \sigma_{(2)}^2, \ldots, \sigma_{(s-1)}^2\}$ and $\sigma_k^2 = \sigma_{(n-1)}^2$, then (A.15) holds if and only if $(\sigma_i^2 + s + 1)/(\sigma_i^2 + s) \le \sqrt{(\sum_{j=1}^{s-1} \sigma_{(j)}^2 + \sigma_i^2 + \sigma_{(n-1)}^2 + (s+1)^2)/(\sum_{j=1}^{s-1} \sigma_{(j)}^2 + \sigma_i^2 + s^2)}$. Therefore, for each $i \in N$, it is sufficient to verify $O(n)$ inequalities and the overall computational complexity is $O(n^2)$.

Appendix B

We are able to extend the result in Proposition 1 to the setting where there is a fixed ordering cost and a linear ordering cost. In this setting, the newsvendor game is (N, c_K) and the characteristic function is given by $c_K(S) = \min_{q \ge 0}\{K1_{\{q>0\}} + E[h(q - X_S)^+ + p(X_S - q)^+]\}, S \subseteq N$, where K is the fixed ordering cost. We assume zero linear ordering cost without loss of generality.

Proposition 10. *Suppose the players' demands $X_1, \ldots, X_n$ are independent continuous random variables with log-concave distribution functions, and $X_i \ge 0$ for all $i \in N$. Then the newsvendor game (N, c_K) is a convex game.*

Proof of Proposition 10. In the following, we use $c_0(\cdot)$ to denote the characteristic cost function of the game with zero ordering cost. Proposition 1 has established that under the condition of independent, continuous and log-concave demands, one has

$$c_0(S \cup \{j\}) - c_0(S) - c_0(T \cup \{j\}) + c_0(T) \ge 0, \quad \forall S \subseteq T,\ j \notin T. \tag{B.1}$$

We claim here that under the same condition and an additional technical requirement that $X_i \ge 0$ for all $i \in N$, the inequalities

$$c_K(S \cup \{j\}) - c_K(S) - c_K(T \cup \{j\}) + c_K(T) \ge 0, \quad \forall S \subseteq T,\ j \notin T, \tag{B.2}$$

hold for any $K \ge 0$, which in turn implies the convexity of the game (N, c_K).

The requirement of $X_i \ge 0, i \in N$ implies that when no order is placed, the cost for coalition S is $pE[X_S]$. Hence, we can write $c_K(S) = \min\{pE[X_S], K + c_0(S)\}$. To further simplify the exposition, define $c^{\mathcal{N}}(S) = pE[X_S]$, $c^{\mathcal{Y}}(S) = K + c_0(S)$, and let q_S^*, q_T^* be the optimal ordering quantities of coalitions S and T. The superscripts $\mathcal{N}, \mathcal{Y}$ are used to emphasize the costs corresponding to the scenarios with no ordering and a positive ordering, respectively.

Our proof of (B.2) proceeds by discussing whether there is a positive amount of order placed in each of the coalitions S, $S \cup \{j\}$, T and $T \cup \{j\}$. To reduce the number of possible cases, we use Lemma 1 from Chen (2009) which states that $q_S^* \le q_T^*$ for any $S \subseteq T$. It is then impossible that, say, coalition S orders a positive amount but coalition $S \cup \{j\}$ chooses not to order. We summarize all possible cases in Table B.1. In the following, we prove that (B.2) holds for each case in Table B.1.

***Case* 1.** Since for each of the coalitions S, $S \cup \{j\}$, T and $T \cup \{j\}$ the ordering quantity is positive, (B.2) can be reduced to $c^{\mathcal{Y}}(S \cup \{j\}) - c^{\mathcal{Y}}(S) - c^{\mathcal{Y}}(T \cup \{j\}) + c^{\mathcal{Y}}(T) \ge 0$, which holds due to the definition that $c^{\mathcal{Y}}(S) = K + c_0(S)$ and (B.1).

***Case* 2.** Since the optimal order quantity for coalition S is zero, we have $c^{\mathcal{N}}(S) \le c^{\mathcal{Y}}(S)$. It follows that $c_K(S \cup \{j\}) - c_K(S) - c_K(T \cup \{j\}) + c_K(T) = c^{\mathcal{Y}}(S \cup \{j\}) - c^{\mathcal{N}}(S) - c^{\mathcal{Y}}(T \cup \{j\}) + c^{\mathcal{Y}}(T) \ge c^{\mathcal{Y}}(S \cup \{j\}) - c^{\mathcal{Y}}(S) - c^{\mathcal{Y}}(T \cup \{j\}) + c^{\mathcal{Y}}(T) \ge 0$, where the last inequality is because of (B.1).

Table B.1. The List of 6 Possible Cases

	$S \cup \{j\}$	S	$T \cup \{j\}$	T
Case 1	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$	$\mathcal{Y}$
Case 2	$\mathcal{Y}$	$\mathcal{N}$	$\mathcal{Y}$	$\mathcal{Y}$
Case 3	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{Y}$	$\mathcal{Y}$
Case 4	$\mathcal{Y}$	$\mathcal{N}$	$\mathcal{Y}$	$\mathcal{N}$
Case 5	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$
Case 6	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{Y}$	$\mathcal{N}$

***Case* 3.** By definition $c_K(S \cup \{j\}) - c_K(S) - c_K(T \cup \{j\}) + c_K(T) = c^{\mathcal{N}}(S \cup \{j\}) - c^{\mathcal{N}}(S) - c^{\mathcal{Y}}(T \cup \{j\}) + c^{\mathcal{Y}}(T) = pE[X_j] - c^{\mathcal{Y}}(T \cup \{j\}) + c^{\mathcal{Y}}(T) = pE[X_j] - c_0(T \cup \{j\}) + c_0(T)$. It remains to show that $pE[X_j] - c_0(T \cup \{j\}) + c_0(T) \ge 0$. Recall q_T^* is the optimal ordering quantity for coalition T. Hence, $c_0(T) = hE[(q_T^* - X_T)^+] + pE[(X_T - q_T^*)^+]$ and $c_0(T \cup \{j\}) \le hE[(q_T^* - X_T - X_j)^+] + pE[(X_T + X_j - q_T^*)^+]$. Notice that $hE[(q_T^* - X_T)^+] - hE[(q_T^* - X_T - X_j)^+] \ge 0$ and $pE[(X_T - q_T^*)^+] + pE[X_j] - pE[(X_T + X_j - q_T^*)^+] \ge 0$. Therefore, $pE[X_j] - c_0(T \cup \{j\}) + c_0(T) \ge 0$.

***Case* 4.** The proof is similar to Case 3, we omit the details for brevity.

***Case* 5.** Since all demands are independent, we have $c_K(S \cup \{j\}) - c_K(S) - c_K(T \cup \{j\}) + c_K(T) = c^{\mathcal{N}}(S \cup \{j\}) - c^{\mathcal{N}}(S) - c^{\mathcal{N}}(T \cup \{j\}) + c^{\mathcal{N}}(T) = pE[X_S + X_j - X_S - X_T - X_j + X_T] = 0$.

***Case* 6.** Since the optimal order quantity for coalition $T \cup \{j\}$ is positive, we have $c^{\mathcal{Y}}(T \cup \{j\}) \le c^{\mathcal{N}}(T \cup \{j\})$. It follows that $c_K(S \cup \{j\}) - c_K(S) - c_K(T \cup \{j\}) + c_K(T) = c^{\mathcal{N}}(S \cup \{j\}) - c^{\mathcal{N}}(S) - c^{\mathcal{Y}}(T \cup \{j\}) + c^{\mathcal{N}}(T) \ge c^{\mathcal{N}}(S \cup \{j\}) - c^{\mathcal{N}}(S) - c^{\mathcal{N}}(T \cup \{j\}) + c^{\mathcal{N}}(T) = 0$. □

One may ask the following question: For the game (N, c_K) which satisfies the conditions in Proposition 10, does the dual-based allocation result in a PMAS? While Chen and Zhang (2009) have constructed a dual-based allocation that is in the core for newsvendor games with a fixed ordering cost, unlike the case of linear ordering cost, the dual-based allocation may not be unique. When comparing the dual-based allocations with another one in a larger coalition, it is unclear which allocation one should choose. This creates difficulties even for numerical explorations as the nonuniqueness issue can be present in all of the coalitions. Therefore, no definite answer can be drawn at this point.

Endnotes

[1] One can refer to Peleg and Sudhölter (2007) for the definition of strategic equivalence. In our paper, the strategic equivalence ensures that the game with zero ordering cost is convex/has a PMAS if and only if the game with constant per-unit ordering cost is convex/has a PMAS.

[2] We use the word "allocation" to refer to a cost vector that specifies the cost allocated to each player given a particular coalition; we use "allocation scheme" to refer to a cost vector that specifies the cost allocated to each player for every coalition; we use "allocation rule" to refer to a function that assigns a cost vector to a game.

[3] The definition on p. 109 of Dharmadhikari and Joag-dev (1988) omits this condition and therefore is not complete.

[4] This result can be generalized to allow a fixed ordering cost. For details, see Appendix B.

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