This article was downloaded by: [192.17.150.33] On: 28 September 2019, At: 14:50
Publisher: Institute for Operations Research and the Management Sciences (INFORMS)
INFORMS is located in Maryland, USA

Operations Research

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Technical Note—Stochastic Optimization with Decisions Truncated by Positively Dependent Random Variables

Xin Chen, Xiangyu Gao

To cite this article:
Xin Chen, Xiangyu Gao (2019) Technical Note—Stochastic Optimization with Decisions Truncated by Positively Dependent Random Variables. Operations Research 67(5):1321-1327. https://doi.org/10.1287/opre.2018.1815

Full terms and conditions of use: https://pubsonline.informs.org/Publications/Librarians-Portal/PubsOnLine-Terms-and-Conditions

This article may be used only for the purposes of research, teaching, and/or private study. Commercial use or systematic downloading (by robots or other automatic processes) is prohibited without explicit Publisher approval, unless otherwise noted. For more information, contact permissions@informs.org.

The Publisher does not warrant or guarantee the article's accuracy, completeness, merchantability, fitness for a particular purpose, or non-infringement. Descriptions of, or references to, products or publications, or inclusion of an advertisement in this article, neither constitutes nor implies a guarantee, endorsement, or support of claims made of that product, publication, or service.

Copyright © 2019, INFORMS

Please scroll down for article—it is on subsequent pages

informs

With 12,500 members from nearly 90 countries, INFORMS is the largest international association of operations research (O.R.) and analytics professionals and students. INFORMS provides unique networking and learning opportunities for individual professionals, and organizations of all types and sizes, to better understand and use O.R. and analytics tools and methods to transform strategic visions and achieve better outcomes.
For more information on INFORMS, its publications, membership, or meetings visit http://www.informs.org

informs
http://pubsonline.informs.org/journal/opre

OPERATIONS RESEARCH
Vol. 67, No. 5, September–October 2019, pp. 1321–1327
ISSN 0030-364X (print), ISSN 1526-5463 (online)

Contextual Areas

Technical Note—Stochastic Optimization with Decisions Truncated by Positively Dependent Random Variables

Xin Chen,[a] Xiangyu Gao[b]

[a] Department of Industrial and Enterprise Systems Engineering, University of Illinois at Urbana–Champaign, Urbana, Illinois 61801; [b] Department of Decision Sciences and Managerial Economics, CUHK Business School, Chinese University of Hong Kong, Shatin, Hong Kong

Contact: xinchen@illinois.edu, https://orcid.org/0000-0002-5168-4823 (XC); xiangyu@cuhk.edu.hk, http://orcid.org/0000-0002-7126-7330 (XG)

Received: May 10, 2018
Revised: May 20, 2018; August 13, 2018
Accepted: August 19, 2018
Published Online in Articles in Advance: June 19, 2019

Subject Classifications: inventory/production: stochastic uncertainty; mathematics: convexity
Area of Review: Operations and Supply Chains

https://doi.org/10.1287/opre.2018.1815

Copyright: © 2019 INFORMS

Abstract. We study stochastic optimization problems with decisions truncated by random variables. This paper extends existing results in the literature by allowing positively dependent random variables and a two-part fee structure. We develop a transformation technique to convert the original nonconvex problems to equivalent convex ones. We apply our transformation technique to an inventory substitution model with random supply capacities and a two-part fee cost structure. In addition, we extend our results to incorporate the decision maker's risk attitude.

Funding: This research of X. Chen is partly supported by the Division of Civil, Mechanical and Manufacturing Innovation of the National Science Foundation [Grants CMMI-1363261, CMMI-1538451, and CMMI-1635160], the National Science Foundation of China (NSFC) [Grant 71520107001], and a gift fund from JD.com.
Supplemental Material: The online appendix is available at https://doi.org/10.1287/opre.2018.1815.

Keywords: stochastic optimization • dependent supply capacity uncertainty • two-part fee structure • inventory management • risk aversion

1. Introduction

Consider the following stochastic optimization problem:

$$\inf_{u \in \mathcal{U}} l(u) + E[f(u \wedge \Xi)], \tag{1}$$

where u is the decision vector, Ξ is a random vector, and $\wedge$ is a componentwise minimum operator. This problem has broad applications in both inventory management and revenue management (see Chen et al. 2018 and the references therein). Chen et al. (2018) studied a special case of problem (1) where $l(u) = 0$, and Ξ has independent components. They show that the objective function may not be convex in u even if the function f is jointly convex, and they develop a transformation technique to convert the original problem to an equivalent convex optimization problem. In this paper, we generalize the results in Chen et al. (2018) along several directions. Instead of restricting to the case where Ξ has independent components, we allow the components of Ξ to be positively dependent, which extends the applications of our model but at the same time creates significant difficulties for the analysis. We include an additional term, $l(u)$, in the objective function, which allows us to incorporate a two-part fee cost structure. Furthermore, we can incorporate the decision maker's risk attitude into the model. We demonstrate the application of the transformation technique through an inventory substitution model with dependent random supply capacities and a two-part fee cost structure. We conduct numerical studies based on the transformed problem to show that ignoring capacity dependence can be very costly for the firm.

To formally set up the mathematical model, in this paper we use $\mathfrak{R}$ to denote the real space and $\mathfrak{L}$ to denote the set of integers. We use $\mathcal{F}$ to denote either $\mathfrak{R}$ or $\mathfrak{L}$ for convenience. Define $\bar{\mathfrak{R}} = \mathfrak{R} \cup \{+\infty\}$. Componentwise minimum and maximum operators are denoted by $\wedge$ and $\vee$, respectively. The indicator function of any set $\mathcal{V} \subseteq \mathcal{F}^n$, denoted by $\delta_{\mathcal{V}}$, is defined as $\delta_{\mathcal{V}}(x) = 0$ for $x \in \mathcal{V}$ and $+\infty$ otherwise. We use uppercase letters (e.g., Ξ) to denote random vectors and lowercase letters (e.g., ξ) for their realizations. Given a random vector $\Xi = (\Xi_1, \ldots, \Xi_n)$, we use $\mathcal{X} = Supp(\Xi)$ to denote the support of this random vector. Throughout this paper, we use "decreasing" and "increasing" in a weak sense.

2. Transformation Technique

We start with the following unconstrained optimization problem:

$$\tau^* = \inf_{u \in \mathcal{F}^n} l(u) + E[f(u \wedge \Xi)], \tag{2}$$

where $l : \mathcal{F}^n \to \bar{\mathfrak{R}}, f : \mathcal{F}^n \to \bar{\mathfrak{R}}$, Ξ is a random vector with support $\mathcal{X} = \prod_{j=1}^n \mathcal{X}_j$, and $\mathcal{X}_j \in \mathcal{F}$ for all $j = 1, \ldots, n$. We assume that when $\mathcal{F} = \mathfrak{R}, f$ is Borel measurable on

$\mathfrak{R}^n$ and $\mathscr{X}_j$ is a Borel measurable subset of $\mathfrak{R}$ for all $j = 1, \ldots, n$. One important application of problem (2) (with an additional nonnegative constraint of the decision vector) is an inventory management problem with random supply capacities. A firm wants to minimize the expected total cost by choosing the ordering quantities before the random capacities are realized. The effective inventory level after receiving the orders is the minimum of the ordering quantities and the realized supply capacities. Different from Chen et al. (2018), we include the cost term $l(u)$ in the objective function. This term allows a more general cost structure. In practice, the firm's ordering cost may depend on the received quantity as well as the quantity she initially ordered, so that the ordering cost has a two-part fee structure, which allows the cost consequence of the supply uncertainties to be shared between the supplier and the firm who orders. This two-part fee cost structure has been studied for inventory control models with random yield (see Henig and Gerchak 1990 and Federgruen and Yang 2011). Interestingly, to the best of our knowledge, no paper studying inventory control models with random capacities has considered this two-part fee structure. Moreover, we do not need to assume that the random vector Ξ has independent components. It is quite common in reality that the capacities of different suppliers can depend on each other.

One technical challenge of problem (2) is that even though the function $l(\cdot)$ and $f(\cdot)$ are jointly convex, the objective function may not be convex in u. Therefore, the main purpose of this section is to develop a transformation technique to convert the original problem to an equivalent convex minimization problem.

Our transformation technique requires that the function $l(\cdot)$ be increasing (by "increasing," we mean componentwise increasing) and the components of random vector Ξ be positively dependent. We begin by providing the definition of stochastically increasing functions, which will be used to define positively dependent random variables.

Definition 1 (Topkis 1998, p. 159). Let $\{F_t(w) : t \in T\}$ be a collection of distribution functions on $\mathfrak{R}^n$ that are indexed by a parameter t, with t contained in a subset T of $\mathfrak{R}^m$. If $\int h(w)dF_t(w)$ is increasing in t on T for each increasing real-valued function $h(w)$ on $\mathfrak{R}^n$, then $F_t(w)$ is stochastically increasing in t on T.

Definition 2. Random variables $\Xi_1, \Xi_2, \ldots, \Xi_n$ are positively dependent if for all $j = 1, \ldots, n$, the joint distribution of $\Xi_1, \ldots, \Xi_{j-1}, \Xi_{j+1}, \ldots \Xi_n$ conditioned on $\Xi_j = \xi_j$, denoted by $\tilde{F}_j(\xi_{-j}|\xi_j)$, is stochastically increasing in $\xi_j \in \mathscr{X}_j$.

It is well known that a collection of distribution functions $\{F_t(w) : t \in T\}$ on $\mathfrak{R}^1$ is stochastically increasing in t on a subset T of $\mathfrak{R}^m$ if and only if $1 - F_t(w)$ is increasing in t on T for each w in $\mathfrak{R}^1$. Therefore, if there are only two random variables Ξ_1 and Ξ_2, then they are positively dependent if for $j = 1, 2$, the distribution of Ξ_j conditioned on $\Xi_{3-j} = \xi_{3-j}$ is decreasing in ξ_{3-j} for any ξ_j. Similar concepts are also used in Li et al. (2013) and Feng et al. (2019) when studying multisourcing problems. In the following lemma, we provide an example where the random vector has positively dependent components.

Lemma 1. *Let $X_1, \ldots, X_n$ be independent log-concave random variables,*[1] *and $X = (X_1, \ldots, X_n)$. Let $A = (a_{ij})$ be an $m \times n$ matrix with nonnegative entries. Then the random vector $\Xi = AX$ has positively dependent components.*

The proofs of all the results can be found in the supplemental material (Appendix EC.1). Many commonly used continuous and discrete random variables are log-concave, such as normal, exponential, uniform, logistics, binomial, Poisson, etc.

Theorem 1. *Suppose that* (T1.a) *the objective function of* (2) *is lower semicontinuous and goes to $+\infty$ when $|u| \to +\infty$,* (T1.b) *f is componentwise convex (componentwise discrete convex if $\mathscr{F} = \mathscr{Z}$), and* (T1.c) *$l(u)$ is increasing.*[2]

(i) *If the random vector Ξ has independent components, then problem* (2) *has the same optimal objective value as the following problem*:

$$\begin{aligned} &\inf\ l(u) + E[f(v(\Xi))] \\ &\text{s.t. } v(\xi) = (v_1(\xi_1), \ldots, v_n(\xi_n)) \in \mathscr{F}^n, \\ &\qquad v(\xi) \le \xi\ \ \forall \xi \in \mathscr{X}, \\ &\qquad v(\xi) \le u\ \ \forall \xi \in \mathscr{X}. \end{aligned} \tag{3}$$

(ii) *If $\Xi_1, \Xi_2, \ldots, \Xi_n$ are positively dependent, and the function f is supermodular, then problem* (2) *has the same optimal objective value as the following problem*:

$$\begin{aligned} &\inf\ l(u) + E[f(v(\Xi))] \\ &\text{s.t. } v(\xi) = (v_1(\xi_1), \ldots, v_n(\xi_n)) \in \mathscr{F}^n, \\ &\qquad v(\xi) \le \xi\ \ \forall \xi \in \mathscr{X}, \\ &\qquad v(\xi) \le u\ \ \forall \xi \in \mathscr{X}, \\ &\qquad v_j(\xi_j) \text{ is increasing } \forall j = 1, \ldots, n. \end{aligned} \tag{4}$$

In the above theorem, assumption (T1.a) ensures that the optimization problem admits finite optimal solutions. Consider the inventory application with random supply capacities Ξ, where the firm wants to minimize the total expected inventory cost by choosing the optimal ordering quantities u. Assumption (T1.a) requires that the total cost goes to infinity if the ordering quantity goes to infinity. Assumption (T1.b) imposes some convexity property of the inventory cost function, which is commonly assumed in the literature. Assumption (T1.c) ensures that the ordering cost function is increasing in the ordering quantity, which usually holds in practical scenarios. In part (i) we present the equivalent transformation when Ξ has independent components. Notice that we omit the requirement that $v(\cdot)$ is measurable in the formulation for brevity. For the rest of the paper, we require that $v(\cdot)$ is measurable in all of our formulations and therefore omit it for brevity. In part (ii) we consider the case where the components of Ξ are positively dependent. In this case, we need to further assume that the function f is supermodular. Supermodularity of inventory costs usually occurs when products or sourcing channels are substitutes for each other, such as the inventory transshipment model in Hu et al. (2008) and the dual sourcing model in Chen et al. (2018). Later we will present an inventory substitution problem with random supply capacities as an application.

We provide intuition why $l(\cdot)$ needs to be increasing. Consider an unconstrained problem with $n = 1$. The original optimization problem is

$$\min_u l(u) + E[f(u \wedge \Xi)].$$

Because the function $l(\cdot)$ is increasing, we know that the optimal solution $u^* \le \hat{u}$, where $\hat{u} \in \arg\min_u f(u)$. The transformed problem is

$$\begin{aligned} \min \quad & l(u) + E[f(v(\Xi))] \\ \text{s.t.} \quad & v(\xi) \le \xi \ \forall \xi \in \mathscr{X}, \\ & v(\xi) \le u \ \forall \xi \in \mathscr{X}. \end{aligned}$$

We want $v(\xi)$ to be as close to $\hat{u}$ as possible. For given $u \le \hat{u}$, it will push $v(\xi)$ to be equal to $u \wedge \xi$. On the other hand, if the function $l(\cdot)$ is not increasing, then the optimal solution u^* may be greater than $\hat{u}$. In this case the optimal solution for the transformed problem may be $v(\xi) = \hat{u} \wedge \xi$ for some $\xi \in \mathscr{X}$, which results in a smaller objective value.

Given Theorem 1, if functions $l(\cdot)$ and $f(\cdot)$ are jointly convex, then the transformed problem (4) is a convex minimization problem. Note that if $l(u) = 0$, the optimal solution of the transformed problem (4) may not directly provide us with an optimal solution of the original problem (2). Nevertheless, we can construct an optimal solution of the original problem based on an optimal solution (u^*, v^*) of the transformed problem according to a similar procedure in Chen et al. (2018). For the first component, let $S = \{\xi_1 | v_1^*(\xi_1) < \xi_1, \xi_1 \in \mathscr{X}_1\}$. If $\Pr(S) > 0$, then randomly pick $\hat{\xi}_1 \in S$ and define $\hat{u}_1 = v_1^*(\hat{\xi}_1)$. If $\Pr(S) = 0$, let $\hat{u}_1 = \bar{\xi}_1$. Repeating this process for each component $j = 1, \ldots, n$, we can obtain an optimal solution $\hat{u} = (\hat{u}_1, \ldots, \hat{u}_n)$ for the original problem.

Theorem 1, part (ii) requires that the random vector has positively dependent components. We provide an example showing that if this condition does not hold, then the transformation may not work. Consider $\min_{(u_1,u_2)\in\mathfrak{R}^2} E[f(u_1 \wedge \Xi_1, u_2 \wedge \Xi_2)]$, where

$$f(u_1, u_2) = 2u_1^2 + 2u_1u_2 + 2u_2^2 - 8u_1 - 2u_2,$$

which is a convex and supermodular function. The random variable Ξ_1 can take values 0 or 2, and Ξ_2 can take values 1 or 3. The probability mass function is $\Pr(\Xi_1 = 0, \Xi_2 = 3) = 0.5, \Pr(\Xi_1 = 2, \Xi_2 = 1) = 0.5$. Hence, Ξ_1 and Ξ_2 are not positively dependent. The optimal solution of the original problem is $u_1^* = 2, u_2^* = 0$, and the optimal objective value is -4. However, for the transformed problem, the optimal solution is $v_1(\Xi_1 = 0) = 0$, $v_1(\Xi_1 = 2) = 2, v_2(\Xi_2 = 1) = -0.5, v_2(\Xi_2 = 3 \ = 0.5$, and the optimal objective value is -4.5. In this example, the transformed problem does not yield the same optimal value as the original problem. It remains an open question whether we can derive a convex reformulation for a more general dependence structure.

In the following we introduce constraints into problem (2) and consider a more general optimization problem as follows:

$$\inf_{u\in\mathscr{U}} l(u) + E[f(u \wedge \Xi)], \tag{5}$$

where $f : \mathscr{F}^n \to \bar{\mathfrak{R}}$ and $\mathscr{U} \subseteq \mathscr{F}^n$. Define $\underline{\xi}_j =$ ess $\inf\{\xi_j | \xi \in \mathscr{X}\}, \xi_j =$ ess $\sup\{\xi_j | \xi \in \mathscr{X}\}, \underline{\xi} = [\underline{\xi}_1, \ldots, \underline{\xi}_n]^\intercal, \xi = [\bar{\xi}_1, \ldots, \bar{\xi}_n]^\intercal$, and

$$\mathscr{V} = \{u \wedge \xi : u \in \mathscr{U}, \underline{\xi} \le \xi \le \bar{\xi}, \xi \in \mathscr{F}^n\}. \tag{6}$$

The following assumption identifies conditions under which the equivalent transformation in Theorem 1 can be generalized to constrained optimization problems.

Assumption 1.

(A1.a) *For any $u \in \mathscr{F}^n$ such that $u \wedge \xi \in \mathscr{V}$ $\forall \xi \in \mathscr{X}$, there exists $u' \in \mathscr{U}, u' \le u$ such that $u' \wedge \xi = u \wedge \xi$ $\forall \xi \in \mathscr{X}$.*

(A1.b) *The indicator function of the set $\mathscr{V}$ is componentwise convex (componentwise discrete convex if $\mathscr{F} = \mathscr{Z}$).*

(A1.c) *The indicator function of the set $\mathscr{V}$ is supermodular.*

Parts (A1.a) and (A1.b) are similar to those from assumption 1 in Chen et al. (2018). Part (A1.c) is needed for our transformation when random variables are positively dependent. We provide a lemma with conditions under which the indicator function of $\mathscr{V}$ is supermodular.

Lemma 2. *If the set $\mathscr{V}$ can be represented as $\{v : \psi_i(v) \le 0\}$, where functions $\psi_i : \mathscr{F}^n \to \bar{\mathfrak{R}}$ are all monotone, then the indicator function of the set $\mathscr{V}$ is supermodular.*

The transformation technique for constrained problems is summarized in the following theorem.

Theorem 2. *Consider the optimization problem (5), where $f : \mathscr{F}^n \to \bar{\mathfrak{R}}, l : \mathscr{F}^n \to \bar{\mathfrak{R}}$ satisfy assumptions (T1.a), (T1.b), and (T1.c) in Theorem 1.*

(i) *Suppose that parts (A1.a) and (A1.b) of Assumption 1 are satisfied. If the random vector Ξ has independent* *components, then problem* (5) *and the following optimization problem have the same optimal objective value*:

$$\begin{aligned} &\inf\ l(u) + E[f(v(\Xi))] \\ &\text{s.t. } v(\xi) = (v_1(\xi_1), \ldots, v_n(\xi_n)) \in \mathcal{F}^n, \\ &\qquad v(\xi) \le \xi\ \forall \xi \in \mathcal{X}, \\ &\qquad v(\xi) \le u\ \forall \xi \in \mathcal{X}. \end{aligned} \tag{7}$$

(ii) *Suppose that Assumption* 1 *is satisfied. If* $\Xi_1, \Xi_2, \ldots, \Xi_n$ *are positively dependent, and the function* f *is supermodular, then problem* (5) *and the following optimization problem have the same optimal objective value*:

$$\begin{aligned} &\inf\ l(u) + E[f(v(\Xi))] \\ &\text{s.t. } v(\xi) = (v_1(\xi_1), \ldots, v_n(\xi_n)) \in \mathcal{V}\ \forall \xi \in \mathcal{X}, \\ &\qquad v(\xi) \le \xi\ \forall \xi \in \mathcal{X}, \\ &\qquad v(\xi) \le u\ \forall \xi \in \mathcal{X}, \\ &\qquad v_j(\xi_j) \text{ is increasing } \forall j = 1, \ldots, n. \end{aligned} \tag{8}$$

Notice that in Theorem 2 we do not include the original constraint set $u \in \mathcal{U}$ in the transformed problem (7) or (8). Instead, we require that the added decisions $v(\xi) \in \mathcal{V}\ \forall \xi \in \mathcal{X}$. If we explicitly include $u \in \mathcal{U}$ in the transformed problem, then we can relax the assumption on the function l by requiring that $l(u)$ is increasing only for $u \in \mathcal{U}$.

The conditions in Assumption 1 may not be straightforward to check. Therefore, in the following lemma we provide an example with linear constraints that satisfies Assumption 1. Later we will demonstrate an application of our transformation technique, and its constraints satisfy the conditions we present below.

Lemma 3. *Assume that* $\mathcal{U} = \{u \mid Au \le b, u \ge \underline{u}\}$, *where* $b, \underline{u}$ *are given constant vectors;* $A = (a_{ij})$ *is a matrix with nonnegative entries. In addition,* $\xi \ge \underline{u}\ \forall \xi \in \mathcal{X}$. *Then Assumption* 1 *is satisfied.*

Preservation of Structural Properties

Similar to Chen et al. (2018), we can show that some structural properties can be preserved when considering the following parameterized optimization problem:

$$g(x, z) = \inf_{u:(x,z,u) \in \mathcal{A}} l(x, z, u) + E[f(x, u \wedge (z + \Xi))], \tag{9}$$

where $f(\cdot, \cdot) : \mathcal{F}^m \times \mathcal{F}^n \to \bar{\mathfrak{R}}$, $l(\cdot, \cdot, \cdot) : \mathcal{F}^m \times \mathcal{F}^n \times \mathcal{F}^n \to \bar{\mathfrak{R}}$, $x \in \mathcal{F}^m, z \in \mathcal{F}^n$, and a set $\mathcal{A} \subseteq \mathcal{F}^m \times \mathcal{F}^n \times \mathcal{F}^n$ is nonempty. Our transformation technique can be used to establish preservation properties of function g and the monotonicity property of the optimal solution. Define a set

$$\mathcal{A}^{\Xi} = \{(x, z, w) \mid w = u \wedge (z + \xi), (x, z, u) \in \mathcal{A}, \xi \in \mathcal{X}\}.$$

Similar to Assumption 1, we specify the following condition.

Assumption 2.

(A2.a) *For any* (x, z, u) *such that* $(x, z, u \wedge (z + \xi)) \in \mathcal{A}^{\Xi}\ \forall \xi \in \mathcal{X}$, *there exists* $(x, z, u') \in \mathcal{A}, u' \le u$ *such that* $u' \wedge (z + \xi) = u \wedge (z + \xi)\ \forall \xi \in \mathcal{X}$.

(A2.b) *The indicator function of the set* $\mathcal{A}^{\Xi}$ *is componentwise convex in* w (*componentwise discrete convex if* $\mathcal{F} = \mathcal{Z}$).

(A2.c) *The indicator function of the set* $\mathcal{A}^{\Xi}$ *is supermodular in* w.

Note that when Ξ has positively dependent components, we require f to be supermodular in Theorems 1 and 2. In this case, we restrict our attention to $n = 2$ (i.e., $u \in \mathcal{F}^2, z \in \mathcal{F}^2$) when studying the preservation and monotonic properties for dependent Ξ. (The result cannot be generalized to cases with $n \ge 3$.) To facilitate the presentation, define $\tilde{u} = (\tilde{u}_1, \tilde{u}_2) = (u_1, -u_2)$, $\tilde{z} = (\tilde{z}_1, \tilde{z}_2) = (z_1, -z_2)$, $\tilde{\Xi} = (\tilde{\Xi}_1, \tilde{\Xi}_2) = (\Xi_1, -\Xi_2)$, $\tilde{\mathcal{A}} = \{(x, z_1, z_2, u_1, u_2) \mid (x, z_1, -z_2, u_1, -u_2) \in \mathcal{A}\}$, $\tilde{\mathcal{A}}^{\Xi} = \{(x, z_1, z_2, w_1, w_2) \mid (x, z_1, -z_2, w_1, -w_2) \in \mathcal{A}^{\Xi}\}$. Define new functions $\tilde{f}(x, u_1, u_2) = f(x, u_1, -u_2)$, $\tilde{l}(x, z_1, z_2, u_1, u_2) = l(x, z_1, -z_2, u_1, -u_2)$, and $\tilde{g}(x, z_1, z_2) = g(x, z_1, -z_2)$. Then problem (9) can be equivalently reformulated as

$$\begin{aligned} \tilde{g}(x, \tilde{z}) = &\inf_{\tilde{u}:(x,\tilde{z},\tilde{u}) \in \tilde{\mathcal{A}}} \tilde{l}(x, \tilde{z}, \tilde{u}) \\ &+ E[\tilde{f}(x, \tilde{u}_1 \wedge (\tilde{z}_1 + \tilde{\Xi}_1), \tilde{u}_2 \vee (\tilde{z}_2 + \tilde{\Xi}_2))]. \end{aligned} \tag{10}$$

The following theorem summarizes the preservation properties of the optimal value function.

Theorem 3. *Consider the optimization problem* (9), *where* f *and* l *satisfy assumptions* (T1.a), (T1.b), *and* (T1.c) *in Theorem* 1 *for any given* x.

(i) *Suppose that parts* (A2.a) *and* (A2.b) *of Assumption* 2 *are satisfied. If the random vector* Ξ *has independent components, then*

(ia) *if functions* f, l, *and the set* $\mathcal{A}^{\Xi}$ *are convex, then* g *is also convex; and*

(ib) *if functions* f *and* l *are submodular, and* $\mathcal{A}^{\Xi}$ *is a lattice, then* g *is submodular.*

(ii) *Suppose that* $n = 2$, *and Assumption* 2 *is satisfied. If* Ξ_1, Ξ_2 *are positively dependent, and the function* $f(x, u_1, u_2)$ *is supermodular in* (u_1, u_2) *for any* x, *then we have the following results:*

(iia) *If functions* f, l, *and the set* $\mathcal{A}^{\Xi}$ *are convex, then* g *is also convex.*

(iib) *If functions* $\tilde{f}$ *and* $\tilde{l}$ *are submodular, and* $\tilde{\mathcal{A}}^{\Xi}$ *is a lattice, then* $\tilde{g}$ *is submodular.*

The following theorem characterizes the monotonicity properties of the solution set. We omit the proof because it is similar to that for theorem 4 of Chen et al. (2018).

Theorem 4. *Consider the optimization problem* (9), *where* f *and* l *satisfy the assumptions* (T1.a), (T1.b), *and* (T1.c) *in Theorem* 1 *for any given* x. *Let* $\mathcal{U}^*(x, z)$ *and* $\tilde{\mathcal{U}}^*(x, \tilde{z})$ *denote the optimal solution sets of* (9) *and* (10), *respectively. If* $\mathcal{A}$,

$\mathcal{A}^\Xi$ are closed, and, in addition, $u_j \le z_j + \bar{\xi}_j, j = 1, 2$, then we have the following results:

(i) *Suppose that parts (A2.a) and (A2.b) of Assumption 2 are satisfied. If the random vector Ξ has independent components, functions f and l are submodular, and $\mathcal{A}, \mathcal{A}^\Xi$ are lattices, then $\mathcal{U}^*(x,z)$ is increasing in (x,z). There exist a greatest element and a least element in $\mathcal{U}^*(x,z)$, which are increasing in (x,z).*

(ii) *Suppose that $n = 2$, and Assumption 2 is satisfied. If Ξ_1, Ξ_2 are positively dependent, the function f is supermodular, $\tilde{f}$ and $\tilde{l}$ are submodular, and $\tilde{\mathcal{A}}, \tilde{\mathcal{A}}^\Xi$ are lattices, then $\tilde{\mathcal{U}}^*(x,\tilde{z})$ is increasing in $(x,\tilde{z})$. There exist a greatest element and a least element in $\tilde{\mathcal{U}}^*(x,\tilde{z})$, which are increasing in $(x,\tilde{z})$.*

Incorporating Risk Attitude

Our transformation technique above depends on the assumption that the decision maker's objective is to minimize the expected cost (or, equivalently, to maximize the expected profit). However, not all decision makers are risk neutral in real life. To capture the decision maker's risk attitude, it is suitable to incorporate a risk measure into the model. Introduced by Rockafellar and Uryasev (2000), conditional value-at-risk (CVaR) is a commonly used risk measure in practice. There are a number of studies that address operations management problems using CVaR (see Chen et al. 2009 and the references therein).

The CVaR of a random variable with confidence level α is defined as the mean of the generalized α-tail distribution. In the following we present an equivalent definition through a convex optimization problem that is more convenient to work on:

$$\mathrm{CVaR}_\alpha(X) = \inf_{\lambda \in \mathfrak{R}} \{\lambda + \frac{1}{1-\alpha} E[(x-\lambda)^+]\},$$

where $\alpha \in [0,1)$ is the degree of risk aversion. The larger α is, the more risk averse the decision maker is. Under the CVaR criterion, the optimization problem becomes

$$\inf_{u \in \mathcal{U}} \mathrm{CVaR}_\alpha[l(u) + f(u \wedge \Xi)]. \tag{11}$$

CVaR is a special case of a more general class of risk measure, called the distortion risk measure, which is commonly used in portfolio optimization and risk allocation in finance as the decision criterion (see McNeil et al. 2015). A distortion risk measure $\rho(\cdot)$ can be represented as a weighted average of CVaRs with different degrees of risk aversion; that is,

$$\rho(X) = \int_0^1 \mathrm{CVaR}_\alpha(X) d\mu(\alpha),$$

where $\mu(\cdot)$ is a probability measure function. A decision maker with a distortion measure faces the following optimization problem:

$$\inf_{u \in \mathcal{U}} \rho[l(u) + f(u \wedge \Xi)]. \tag{12}$$

Notice that when $\mu(\cdot)$ is concentrated at one α, problem (12) reduces to (11).

Theorem 5. *Consider the optimization problem (12), where functions f and l satisfy assumptions (T1.a), (T1.b), and (T1.c) in Theorem 1. Let $g(u,\lambda,\alpha) = \lambda + \frac{1}{1-\alpha}(f(u)-\lambda)^+$. Define the constraint set $\mathcal{V}$ according to (6).*

(i) *Suppose that parts (A1.a) and (A1.b) of Assumption 1 are satisfied. If the random vector Ξ has independent components, then problem (12) and the following optimization problem have the same optimal objective value:*

$$\begin{aligned} \inf\ & l(u) + E\left[\int_0^1 g(v(\Xi), \lambda(\alpha), \alpha) d\mu(\alpha)\right] \\ \text{s.t. } & v(\xi) = (v_1(\xi_1), \dots, v_n(\xi_n)) \in \mathcal{V}\ \forall \xi \in \mathcal{X}, \\ & v(\xi) \le \xi\ \forall \xi \in \mathcal{X}, \\ & v(\xi) \le u\ \forall \xi \in \mathcal{X}, \\ & \lambda(\alpha) \in \mathfrak{R}\ \forall \alpha \in [0,1). \end{aligned} \tag{13}$$

(ii) *Suppose that Assumption 1 is satisfied. If the random vector Ξ has positively dependent components, f is continuously differentiable, and $\frac{\partial f}{\partial u_i}\frac{\partial f}{\partial u_j} \ge 0\ \forall u \in \mathcal{U}$, then problem (12) and the following optimization problem have the same optimal objective value:*

$$\begin{aligned} \inf\ & l(u) + E\left[\int_0^1 g(v(\Xi), \lambda(\alpha), \alpha) d\mu(\alpha)\right] \\ \text{s.t. } & v(\xi) = (v_1(\xi_1), \dots, v_n(\xi_n)) \in \mathcal{V}\ \forall \xi \in \mathcal{X}, \\ & v(\xi) \le \xi\ \forall \xi \in \mathcal{X}, \\ & v(\xi) \le u\ \forall \xi \in \mathcal{X}, \\ & v_j(\xi_j) \text{ is increasing } \forall \xi_j \in \mathcal{X}_j, \\ & \lambda(\alpha) \in \mathfrak{R}\ \forall \alpha \in [0,1). \end{aligned} \tag{14}$$

In the above formulations, $\lambda(\cdot)$ is required to be measurable. In part (ii) we require that $\frac{\partial f}{\partial u_i}\frac{\partial f}{\partial u_j} \ge 0\ \forall u \in \mathcal{U}$. One example is that the function f is monotone. Another example is that $f(u_1, \dots, u_n) = \phi(\sum_{i=1}^n \psi_i(u))$, where functions $\psi_i : \mathfrak{R}^n \to \mathfrak{R}$ are either all increasing or all decreasing.

3. An Application: Inventory Substitution with Random Supply Capacities

Inventory models with substitution have been studied in Bassok et al. (1999), Netessine et al. (2002), Rao et al. (2004), Shumsky and Zhang (2009), and Yu et al. (2015), among others. But none of these papers considered random capacities. In this section, we study a single-period multiproduct inventory model with downward substitution and random capacities.

The firm manages N types of products to satisfy customer demands. The products are indexed by $i = 1, \dots, N$ with product 1 having the highest quality. There is a demand class corresponding to each product, indexed by $j = 1, \dots, N$. If any demand class j cannot be satisfied, products with higher quality ($i \ge j$) can be used for substitution. At the beginning of each period, the firm observes the initial inventory level, denoted by $x = (x_1, \ldots, x_N)^\intercal$. Then the firm decides the target order-up-to inventory levels $y = (y_1, \ldots, y_N)^\intercal$. This target inventory level y is not necessarily achieved because of random supply capacities. We model the random supply capacity of product i as a random variable K_i. Let $K = (K_1, \ldots, K_N)^\intercal$. Therefore, after the capacity is realized and the order is received, the actual inventory level of product i is $y_i \wedge (x_i + k_i)$, where k_i denotes the realization of K_i.

Next, the demands $D = (D_1, \ldots, D_N)^\intercal$ are observed, and the firm makes the substitution decision. We use w_{ij} to denote the amount of substitution of product i to demand j, u_i to denote the leftover inventory of product i, and u_j to denote the shortage of demand j. The unit substitution cost to use product i to satisfy demand j is s_{ij}, the unit shortage cost for demand j is p_j, and the unit holding cost for product i is h_i. We allow h_i to be negative when the unit salvage value exceeds the unit holding cost. We assume that the demands are independent of capacities, whereas the random capacities of different products can depend on each other. The firm's objective is to minimize the expected total costs. The problem formulation is as follows:

$$\min_{y \geq x} E[c(x, y, K) + L(y \wedge (x + K)|D)], \tag{15}$$

where

$$\begin{aligned} L(y|d) = \min & \sum_{i=1}^{N} h_i u_i + \sum_{j=1}^{N} p_j u'_j + \sum_{i=1}^{N} \sum_{j=i}^{N} s_{ij} w_{ij}, \\ \text{s.t.} & \sum_{i=1}^{j} w_{ij} + u'_j = d_j, \ \forall j = 1, \ldots, N, \\ & \sum_{j=i}^{n} w_{ij} + u_i = y_i, \ \forall i = 1, \ldots, N, \\ & w_{ij}, u_i, u'_j \geq 0, \ \forall i = 1, \ldots, N, j = 1, \ldots, N. \end{aligned} \tag{16}$$

In (15), $c(x, y, k)$ is the total ordering cost, which depends on the initial inventory x, the target inventory level y, and the realized capacity k. We assume that the ordering cost has a two-part fee structure. The first part of the ordering cost is proportional to the quantity actually received, which is $y \wedge (x + k) - x$, whereas the second part is proportional to the quantity initially ordered, which is $y - x$. Hence, $c(x, y, k) = c_e^\intercal (y \wedge (x + k) - x) + c_o^\intercal (y - x)$. The two-part cost structure includes, as special cases, that setting where the firm pays only for the effective units ($c_o = 0$) or where it pays exclusively for all ordered units ($c_e = 0$), and it allows the cost consequence of capacity uncertainties to be shared between the supplier and the retailer. The second term $L(y|d)$ represents the inventory holding and shortage costs as well as the substitution costs given the inventory level and realized demands. Define $g(y) = c_e^\intercal y + E[L(y|D)]$; then (15) is equivalent to

$$\min_{y \geq x} E[g(y \wedge (x + K))] + c_o^\intercal y - (c_o^\intercal + c_e^\intercal) x. \tag{17}$$

The next theorem presents an equivalent transformation of problem (17).

Theorem 6. *Problem* (17) *is equivalent to the following problem* (*we remove the term* $(c_o^\intercal + c_e^\intercal)x$ *in the objective because it does not affect the optimal solution*):

$$\begin{aligned} \min \ & E[g(v(K))] + c_o^\intercal y \\ \text{s.t.} \ & v(k) = (v_1(k_1), \ldots, v_n(k_n)) \ \forall k \in \mathcal{K}, \\ & v_i(k_i) \geq x_i, \ \forall k_i \in \mathcal{K}_i, \forall i = 1, \ldots, N, \\ & v_i(k_i) \leq x_i + k_i \ \forall k_i \in \mathcal{K}_i, \forall i = 1, \ldots, N, \\ & v_i(k_i) \leq y_i, \ \forall k_i \in \mathcal{K}_i, \forall i = 1, \ldots, N, \\ & v_i(k_i) \text{ is increasing}, \ \forall i = 1, \ldots, N. \end{aligned} \tag{18}$$

The transformed problem (18) can be formulated as a three-stage stochastic program and solved via various approximation methods, such as scenario approximation or the piecewise linear decision rule approximation (Georghiou et al. 2011). We have conducted numerical studies to show that ignoring the capacity dependence can be very costly for the firm. The details are relegated to the Online Appendix EC.2.

Acknowledgments

The authors thank Tava Olsen, the associate editor, and two anonymous referees for helpful comments and suggestions that have greatly improved the paper.

Endnotes

[1] A continuous random variable X is log-concave if the logarithm of its density function $f(x)$ is concave. A discrete random variable X with support on the set of nonnegative integers is log-concave if $\{i \geq 0 : p_i = \Pr(X = i) > 0\}$ is a set of consecutive integers and $p_i^2 \geq p_{i-1} p_{i+1}$ for all $i \geq 1$.

[2] During the revision of this paper, we become aware of Feng et al. (2019), who independently studied a special case of our Theorem 1 with $l(u) = 0$.

References

Bassok Y, Anupindi R, Akella R (1999) Single-period multi-product inventory models with substitution. *Oper. Res.* 47(4):632–642.

Chen X, Gao X, Pang Z (2018) Preservation of structural properties in optimization with decisions truncated by random variables and its applications. *Oper. Res.* 66(2):340–357.

Chen Y, Xu M, Zhang ZG (2009) A risk-averse newsvendor model under the CVaR criterion. *Oper. Res.* 57(4):1040–1044.

Federgruen A, Yang N (2011) Procurement strategies with unreliable suppliers. *Oper. Res.* 59(4):1033–1039.

Feng Qi, Jia J, ShanthikumarJG (2019) Dynamic multisourcing with dependent supplies. *Management Sci.* Forthcoming.

Georghiou A, Wiesemann W, Kuhn D (2011) The decision rule approach to optimisation under uncertainty: Methodology and applications in operations management. Working paper, Imperial College London, London.

Henig M, Gerchak Y (1990) The structure of periodic review policies in the presence of random yield. *Oper. Res.* 38(4):634–643.

Hu X, Duenyas I, Kapuscinski R (2008) Optimal joint inventory and transshipment control under uncertain capacity. *Oper. Res.* 56(4): 881–897.

Li T, Sethi SP, Zhang J (2013) Supply diversification with responsive pricing. *Production Oper. Management* 22(2):447–458.

McNeil AJ, Frey R, Embrechts P (2015) *Quantitative Risk Management: Concepts, Techniques and Tools* (Princeton University Press, Princeton, NJ).

Netessine S, Dobson G, Shumsky R (2002) Flexible service capacity: Optimal investment and the impact of demand correlation. *Oper. Res.* 50(2):376–388.

Rao U, Swaminathan J, Zhang J (2004) Multi-product inventory planning with downward substitution, stochastic demand and setup costs. *IIE Trans.* 36(1):59–71.

Rockafellar RT, Uryasev S (2000) Optimization of conditional value-at-risk. *J. Risk* 2(3):21–42.

Shumsky RA, Zhang F (2009) Dynamic capacity management with substitution. *Oper. Res.* 57(3) 671–684.

Topkis DM (1998) *Supermodularity and Complementarity* (Princeton University Press, Princeton, NJ).

Yu Y, Chen X, Zhang F (2015) Dynamic capacity management with general upgrading. *Oper. Res.* 63(6):1372–1389.

Xin Chen is a professor, an Abel Bliss Faculty Scholar, and the Jerry S. Dobrovolny Faculty Scholar at the University of Illinois at Urbana–Champaign. His research interests lie in optimization, data analytics, revenue management, and supply chain management. He received the INFORMS Revenue Management and Pricing Section Prize in 2009, and is the coauthor of the book *The Logic of Logistics: Theory, Algorithms, and Applications for Logistics and Supply Chain Management*.

Xiangyu Gao is an assistant professor in the Department of Decision Sciences and Managerial Economics at the CUHK Business School, the Chinese University of Hong Kong. His research interests include inventory and supply chain management, dynamic pricing and revenue management, and data-driven online learning algorithms.